Optimal design of composite shells based on minimum weight and maximum feasibility robustness

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Abstract A robust design optimization (RDO) approach for minimum weight and safe shell composite structures with minimal variability into design constraints under uncertainties is proposed. A new concept of feasibility robustness associated to the variability of design constraints is considered. So, the feasibility robustness is defined through the determinant of variance–covariance matrix of constraint functions introducing in this way the joint effects of the uncertainty propagations on structural response. A new framework considering aleatory uncertainty into RDO of composite structures is proposed. So, three classes of variables and parameters are identified: deterministic design variables, random design variables and random parameters. The bi-objective optimization search is performed using on a new approach based on two levels of dominance denoted by Co-Dominance-based Genetic Algorithm (CoDGA). The use of evolutionary concepts together sensitivity analysis based on adjoint variable method is a new proposal. The examples with different sources of uncertainty show that the Pareto front definition depends on random design variables and/or random parameters considered in RDO. Furthermore, the importance to control the uncertainties on the feasibility of constraints is demonstrated. CoDGA approach is a powerfully tool to help designers to make decision establishing the priorities between performance and robustness.

Keywords Bi-objective optimization · Composite structures · Feasibility robustness · Uncertainty sources · Sensitivity · Co-dominance

1 Introduction

The robust design optimization (RDO) of composite structures is currently a very important area of research. Indeed, the principal objective of robust design is to improve product quality by minimising the uncertainty effects or stabilising variations in structural response without eliminating their causes. RDO applied to composite structures under probabilistic constraints is a very important field due to uncertainties associated with physical properties of fibre-reinforced composites. Adali et al. (2003) developed a model aimed at the optimal design of composite laminates under buckling load uncertainty. In this model loads belong to a given uncertainty domain. Walker and Hamilton (2005) described a procedure to design symmetric laminates for maximum buckling load under manufacturing uncertainty in ply angle design variable. Gumbert and Newman (2005) analysed the effect of geometric uncertainty in shape parameters in a 3-D flexible wing. Choi et al. (2008) proposed an approach based on searching the stacking sequence of laminated composite structures which, corresponds to the less sensitive performance properties relatively to uncertainties in the input parameters. This perspective follows RDO concepts where the objective is to minimize the effects of uncertainty on optimal design. The same strategy based on considering the statistical data in objective and constraint functions is also used by António and Hoffbauer (2009, 2010) combining reliability and robustness.

Other perspective of RDO used in structural applications but not applied in composite structures is based on the optimization of mean performance commonly known as optimality, and the minimization of the variability of the performance function known as robustness (Huang and Du 2007; Zaman et al. 2011; Ragavajhala and Mahadevan 2013). Nevertheless, another concept of robustness can be defined as the maximization size of the deviations from the target design that can be tolerated, whereby the product satisfies all requirements (Salazar and Rocco 2007). This design rule is based on the concept such as the response variability does not necessarily have to be minimised but rather that it be bounded. So, the design with largest tolerance to the input uncertainty is considered as the robust design.

In this paper, the proposed approach introduces a new concept of RDO based on feasibility robustness together performance optimization. The feasibility robustness is associated to design constraints instead on the variability of the performance function as suggested previously. In particular, the feasibility robustness is associated to the variability of critical displacement and to the variability of critical Tsai number for stress integrity analysis. Furthermore, a new framework aiming to consider the aleatory uncertainty into RDO of composite structures is proposed. So, three classes of variables and parameters identified in robust design of composite structures are considered in the approach: the deterministic design variables, the random design variables and the random parameters.

The use of evolutionary concepts together sensitivity analysis based on adjoint variable method is a new concept used in proposed RDO approach. In general evolutionary methods as genetic algorithms only use zero order information considered in the fitness definition avoiding in this way the calculation of derivatives. In this approach the use of gradients are necessary to define the components of variance–covariance matrix. However, since the adjoint variable method is preferred when the number of design variables or parameters is greater than the number of functions, the additional computational cost is not dramatically increased. Indeed, in the proposed approach only two functionals, the critical displacement and the critical Tsai number are considered.

In the proposed multi-objective optimization approach the weight and the determinant of the variance–covariance matrix of the response of composite structures are considered as performance and robustness functions, respectively. The Pareto front is built using a genetic algorithm with co-evolution of two populations denoted by Co-Dominance-based Genetic Algorithm (CoDGA). The idea of co-evolution of populations is generalized in the literature on genetic algorithms. However, this concept has not been applied to multi-objective optimization of composite structures. In this approach the concept of co-evolution enables to use the elitism and dominance at short population and to use only dominance at enlarged population. The paper is organized as follows: the measures of uncertainty for composite structures are introduced in Sect. 2, the RDO of composites is formulated and the proposed approach are presented in Sect. 3, the results and the discussion are presented in Sect. 4 and the conclusions are established in Sect. 5.

2 Uncertainty analysis for robustness definition

The important parts of the RDO are the uncertainty and sensitivity analysis in studying complex systems such as composite laminated structures, for robustness assessment. Specifically, uncertainty analysis refers to the determination of the uncertainty in the response as a result of uncertainties in random variables, and sensitivity analysis refers to the evaluation of the contributions of individual uncertainties of random variables to the uncertainty in response results.

The uncertainty can be classified as epistemic or aleatory. The epistemic uncertainty comes from a lack of knowledge of the appropriate value to consider for a quantity that is assumed to have a fixed value used in a particular analysis. Epistemic uncertainty is related to imprecise probabilistic information (fuzzy) and is generally taken to be distinct from aleatory uncertainty under the conceptual and modeling point of view. Aleatory uncertainty arises from inherent randomness in the behavior of the system under study. RDO of composite structures is commonly based on aleatory uncertainty (Adali et al. 2003; Rais-Rohani and Singh 2004; Carbillet et al. 2009).

In this work the quantification of response uncertainties of composite structures due to uncertainty in the mechanical properties and loads of the structural model is implemented based on linear statistical analysis. This methodology uses a Taylor's series expansion to obtain a linear relationship between the response random variables—displacements and stresses, and the random structural input parameters (Cacuci 2003; Helton and Davis 2006; Saltelli et al. 2006; Rocquigny et al. 2008; António and Hoffbauer 2008). The adjoint variable method is used to obtain the sensitivity matrix (António 1995). This method is appropriated for composite structures due to the large number of random input parameters.

The almost totality of sensitivity analyses in applications with composite structures used local importance measures of uncertainty on design parameters (Rais-Rohani and Singh 2004; Carbillet et al. 2009). António and Hoffbauer (2008, 2010) studied the dominant effects on the stochastic characteristics and analyze the influence of different random parameters using a global analysis based on an Artificial Neural Network and a Monte Carlo Simulation approach. In particular in António and Hoffbauer (2008), the uncertainty propagation on structural response of composite laminated structures are analysed using three different approaches: a first-order local method, a Global Sensitivity Analysis supported by a variance-based method and an extension of local variance to estimate the global variance over the domain of inputs. The needs for global variance methods are discussed by comparing the results obtained from proposed methodologies. The results show that a first order local method is acceptable to analyse the uncertainty propagation on response for angle-ply laminates. An obvious advantage of local methods in robustness assessment is the reduced computational costs of the associated uncertainty analysis.

2.1 Propagation of uncertainties

Lets consider the system response ψ to be a real-valued function of n system parameters denoted as $\mathbf{x} = (x_1, \ldots, x_n)$. The true values of these parameters are not known and so, only the *nominal values* $\mathbf{x}^0 = (x_1^0, \ldots, x_n^0)$ and their uncertainties $\delta\mathbf{x} = (\delta x_1, \ldots, \delta x_k)$ are available. Assuming the system parameters as random variables, the nominal values are taken to be the expected values and the associated uncertainties are given by the corresponding standard deviations. Commonly, the relative uncertainties $\delta x_i / x_i^0$ are symmetrically distributed in the neighborhood of x_i^0 and they are smaller than unity. The true parameter value is defined in vector form as

$$\mathbf{x} = \mathbf{x}^0 + \delta\mathbf{x} = (x_1^0 + \delta x_1, \ldots, x_n^0 + \delta x_n) \tag{1}$$

The response is related to the parameters using the equation of the computational model written in close form as

$$\psi = \psi(x_1, \ldots, x_n) = \psi(x_1^0 + \delta x_1, \ldots, x_n^0 + \delta x_n) \tag{2}$$

In the above functional relationship ψ is used in both senses as random function and as its numerical realization. The expansion in Taylor's series of functional in Eq. (2) around the nominal value $\mathbf{x}^0 = (x_1^0, \ldots, x_n^0)$ considering only up to the first order terms is the following:

$$\begin{aligned}\psi(x_1, \ldots, x_n) &= \psi(\mathbf{x}^0) + \sum_{i=1}^{n} \left(\frac{\partial \psi}{\partial x_i}\right)_{\mathbf{x}^0} \delta x_i \\ &= \psi^0 + \sum_{i=1}^{n} S_i \delta x_i\end{aligned} \tag{3}$$

being $\psi^0 \equiv \psi(\mathbf{x}^0)$ and $S_i = (\partial\psi/\partial x_i)_{\mathbf{x}^0}$ the response sensitivity to parameter x_i. The mean value and the variance of the response is obtained respectively from Eq. (3) as

$$E(\psi) \equiv \psi^0 \tag{4}$$

$$\begin{aligned}\operatorname{var}(\psi) &\equiv E\left(\left(\psi - \psi^0\right)^2\right) \\ &= \sum_{i=1}^{n} S_i^2 \operatorname{var}(x_i) + 2 \sum_{i \neq j = 1}^{n} S_i S_j \operatorname{cov}(x_i, x_j)\end{aligned} \tag{5}$$

The last equation can be written in matrix form as

$$\operatorname{var}(\psi) = \mathbf{S}\mathbf{C}_x\mathbf{S}^T \tag{6}$$

where the superscript "T" denotes the transposition, $\mathbf{C}_x$ is the covariance matrix for parameters $(x_1, \ldots, x_n)$ with components defined as

$$(\mathbf{C}_x)_{ij} = \begin{cases} \mathrm{cov}(x_i, x_j) = \rho_{ij}\sigma_i\sigma_j, & i \neq j, \rho_{ij} \equiv \textit{correlation coefficient} \\ \mathrm{var}(x_i) = \sigma^2, & i = j \end{cases} \tag{7}$$

and the column vector $\mathbf{S} = (S_1, \ldots, S_n)$ has components $S_i = (\partial\psi/\partial x_i)_{\mathbf{x}^0}$.

If the system parameters $(x_1, \ldots, x_n)$ are uncorrelated then Eq. (5) can be reduced to

$$\mathrm{var}(\psi) = \sum_{i=1}^{n} S_i^2 \mathrm{var}(x_i) = \sum_{i=1}^{n} S_i^2 \sigma_i^2 \tag{8}$$

The previous concepts can be extended to the case of m response functions all of them depending on parameters $(x_1, \ldots, x_n)$. Firstly, considering vector notation the m responses can be presented as

$$\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_m) \tag{9}$$

and the corresponding equivalent equations to Eq. (3) are the following first order Taylor expansion of $\boldsymbol{\varphi}(\mathbf{x})$:

$$\boldsymbol{\varphi}(\mathbf{x}^0 + \delta\mathbf{x}) = \boldsymbol{\varphi}(\mathbf{x}^0) + \delta\boldsymbol{\varphi} \cong \boldsymbol{\varphi}(\mathbf{x}^0) + \mathbf{S}\delta\mathbf{x} \tag{10}$$

where $\mathbf{S}$ is a rectangular matrix of order $m \times n$ with components representing the sensitivity of the *j-th* response to the *i-th* system parameter such as

$$(\mathbf{S})_{ji} = \partial\varphi_j/\partial x_i \tag{11}$$

The expectation $E(\boldsymbol{\varphi})$ of $\boldsymbol{\varphi}$ is obtained using the same procedure adapted to Eq. (4):

$$E(\boldsymbol{\varphi}) = \boldsymbol{\varphi}^0 \tag{12}$$

Finally the covariance matrix $\mathbf{C}_{\boldsymbol{\varphi}}$ for $\boldsymbol{\varphi}$ is obtained by a similar procedure applied to Eq. (6) and this is

$$\mathbf{C}_{\boldsymbol{\varphi}} = E\left(\mathbf{S}\delta\mathbf{x}(\mathbf{S}\delta\mathbf{x})^T\right) = \mathbf{S}E\left(\delta\mathbf{x}\delta\mathbf{x}^T\right)\mathbf{S}^T = \mathbf{S}\mathbf{C}_x\mathbf{S}^T \tag{13}$$

Equations for the *propagation of higher-order moments* become very complex and are avoided in practice (Cacuci 2003; Helton and Davis 2006; Saltelli et al. 2006; Rocquigny et al. 2008). From Eq. (13) for the propagation of uncertainties it is observed the dependence of the covariance matrix $\mathbf{C}_{\boldsymbol{\varphi}}$ relatively to the sensitivity matrix $\mathbf{S}$. The components of this matrix are evaluated using the adjoint variable method (António 1995).

2.2 Sensitivity analysis using adjoint variable method

The objective of sensitivity analysis is to analyze the behavior of the response of the system and to evaluate the sensitivities of the system response to variations in the system input parameters around their nominal values. The methodology presented here is based on the adjoint variable method. The methodology was developed in connection with structural analysis of composite structures (António 1995). The structural analysis of laminated composite structures is based on a displacement formulation of the Finite Element Method (FEM), in particular using the shell finite element model developed by Ahmad (1969) and further improvements (Figueiras 1983). This shell element is obtained from a 3-D finite element using a degenerative procedure. It is an isoparametric element with eight nodes and five freedom degrees per node based on the Mindlin shell theory. The shell consists of a number of perfectly bonded plies. Each individual ply is assumed homogeneous and anisotropic. A shortly description of Ahmad element can be found in paper published by António and Hoffbauer (2008). In this work it is considered the linear behavior of structural systems with the equilibrium equation set established as

$$\mathbf{K}(\mathbf{x})\ \mathbf{u} = \mathbf{F} \tag{14}$$

where $\mathbf{K}$ is the stiffness matrix, $\mathbf{u}$ is the displacement vector, $\mathbf{x}$ is the vector of the system parameters and $\mathbf{F}$ are the applied external loads.

In the adjoint variable method, an augmented Lagrangian is defined in terms of adjoint variable fields in order to eliminate the implicit derivatives. Following the method proposed by Arora and Cardoso (1992), considering a given functional $\boldsymbol{\varphi} = \boldsymbol{\varphi}(x_1, \ldots, x_n)$ and writing the response equation of the system in the following form

$$\boldsymbol{\Psi}(\mathbf{u}, \mathbf{x}) = \mathbf{K}(\mathbf{x})\mathbf{u} - \mathbf{F} \tag{15}$$

the augmented functional can be written as

$$L(\mathbf{u}, \mathbf{x}, \boldsymbol{\phi}) = \boldsymbol{\varphi}(\mathbf{u}, \mathbf{x}) - \boldsymbol{\phi}^{\mathbf{T}}\boldsymbol{\Psi}(\mathbf{u}, \mathbf{x}) \tag{16}$$

The vector of adjoint variables $\boldsymbol{\varphi}$ is assumed as Lagrange multipliers selected to make stationary the functional L relatively to the displacement vector u. This condition can be formulated as

$$\frac{\partial L}{\partial \mathbf{u}} = \frac{\partial \boldsymbol{\varphi}(\mathbf{u}, \mathbf{x})}{\partial \mathbf{u}} - \boldsymbol{\phi}^{\mathbf{T}} \frac{\partial \boldsymbol{\Psi}(\mathbf{u}, \mathbf{x})}{\partial \mathbf{u}} = \mathbf{0} \tag{17}$$

Considering the independence of $\mathbf{F}$ to the displacements $\mathbf{u}$ and Eq. (15), the adjoint set of equations is obtained

$$\mathbf{K}(\mathbf{x})\boldsymbol{\phi} = \frac{\partial \boldsymbol{\varphi}(\mathbf{u}, \mathbf{x})}{\partial \mathbf{u}} \tag{18}$$

being the tangent stiffness matrix defined for the equilibrium solution

$$\boldsymbol{\Psi}(\mathbf{u}, \mathbf{x}) = \mathbf{K}(\mathbf{x})\mathbf{u} - \mathbf{F} = 0 \tag{19}$$

On the other hand taking into account that in an equilibrium situation the functional in Eq. (15) is stationary, it proofs (Arora and Cardoso 1992) that

$$\frac{d\boldsymbol{\phi}}{d\mathbf{x}} = \frac{\partial L}{\partial \mathbf{x}} \tag{20}$$

Differentiating Eq. (16) to variables $\mathbf{x}$ it is obtained

$$\begin{aligned}\frac{\partial L}{\partial \mathbf{x}} = {} & \frac{\partial \boldsymbol{\varphi}(\mathbf{u}, \mathbf{x})}{\partial \mathbf{x}} + \frac{\partial \boldsymbol{\varphi}(\mathbf{u}, \mathbf{x})}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \\ & - \boldsymbol{\phi}^{\mathbf{T}} \left[\frac{\partial \boldsymbol{\Psi}(\mathbf{u}, \mathbf{x})}{\partial \mathbf{x}} + \frac{\partial \boldsymbol{\Psi}(\mathbf{u}, \mathrm{x})}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right]\end{aligned} \tag{21}$$

that can be simplified using equality in Eq. (17) yielding to

$$\frac{\partial L}{\partial \mathbf{x}} = \frac{\partial \boldsymbol{\varphi}(\mathbf{u}, \mathbf{x})}{\partial \mathbf{x}} - \boldsymbol{\phi}^{\mathbf{T}} \frac{\partial \boldsymbol{\Psi}(\mathbf{u}, \mathbf{x})}{\partial \mathbf{x}} \tag{22}$$

Considering the independence of $\mathbf{F}$ to variables $\mathbf{x}$ and using Eq. (15) it gives

$$\frac{d\boldsymbol{\varphi}}{d\mathbf{x}} = \frac{\partial L}{\partial \mathbf{x}} = \frac{\partial \boldsymbol{\varphi}(\mathbf{u}, \mathbf{x})}{\partial \mathbf{x}} - \boldsymbol{\phi}^{\mathbf{T}} \frac{\partial \mathbf{K}(\mathbf{x})}{\partial \mathbf{x}} \mathbf{u} \tag{23}$$

The adopted methodology for sensitivity analysis is twofold (António 1995; Arora and Cardoso 1992):

1st: Solve the adjoint set of equations, defined in Eq. (18);
2nd: Get the sensitivities from Eq. (23).

Using the Eq. (23), the components of the matrix $\mathbf{S}$ in Eq. (13) can be calculated and further to obtain the variance–covariance matrix $\mathbf{C}_{\boldsymbol{\varphi}}$ associated with the variability of the structural response.

2.3 Response functions for composite structures

Two functional are considered in the sensitivity-uncertainty analysis, one related with the maximum displacement on the composite structure,

$$\bar{u} = Max(u_1, \ldots, u_r), \quad r = 1, \ldots, N_{dis} \tag{24}$$

and the second one related with the most critical Tsai number,

$$\bar{R} = Max(R_1, \ldots, R_j), \quad j = 1, \ldots, N_{str} \tag{25}$$

being N_{dis} the total number of displacements and N_{str} the total number of points where the stress vector is evaluated on the composite structure.

The stress analysis is performed using the strength parameter R_j known as *Tsai number* and calculated as the ratio between the failure (or maximum allowable) stress and the actual stress at the *j-th* point of the structure where the stress vector is evaluated (Tsai 1987). The *Tsai number* R_j is a function of the actual stresses and it is obtained by solving the interactive quadratic failure criterion of Tsai-Wu (Tsai 1987) as follows

$$(F_{ik}s_i s_k)R_j^2 + (F_i s_i)R_j = 1 \quad i, k = 1, 2, 6 \tag{26}$$

where s_i is the *i-th* component of the stress vector, F_{ik} and F_i are strength parameters associated with unidirectional reinforced laminate defined from the macro-mechanical point of view (Tsai 1987). The vector response can be presented as $\boldsymbol{\varphi} = (\bar{u}, \bar{R})$ depending on input random parameters $(x_1, \ldots, x_n)$.

2.4 Joint effects of uncertainties

The above analysis was performed considering an independent analysis for each input parameters in the Eq. (13) of propagation of uncertainties. This analysis is important in order to evaluate the individual influence of each input parameter. However, the joint effects of the propagation of uncertainties on the response play an important role in structural reliability analysis. The Eq. (13) of propagation of uncertainties is

$$\mathbf{C}_{\boldsymbol{\varphi}} = \mathbf{S}\mathbf{C}_x\mathbf{S}^T \tag{27}$$

where each component of matrix $\mathbf{C}_x$ denoted by $(\mathbf{C}_x)_{ij}$ is defined in Eq. (7) and the each component of sensitivity matrix, $(\mathbf{S})_{ji} = \partial \boldsymbol{\varphi}_j / \partial x_i$ is referring to the sensitivity of the *j-th* response functional relatively to the *i-th* system parameter such as defined in Eq. (11). If the input parameters are uncorrelated then matrix $\mathbf{C}_x$ is diagonal and the above equation gives

$$\mathbf{C}_{\varphi} = \begin{bmatrix} \mathrm{var}\,(\bar{u}) & \mathrm{cov}(\bar{u}, \bar{R}) \\ \mathrm{cov}(\bar{u}, \bar{R}) & \mathrm{var}(\bar{R}) \end{bmatrix} \tag{28}$$

The evaluation of the response uncertainty is done in a simple and systematic way using the variance–covariance matrix $\mathbf{C}_{\varphi}$ of structural response defined in Eq. (28).

In the mathematical formulation of the RDO problem, the design constraints define the design space to be considered along the optimization process. The feasibility of the solutions during the optimization is continuously checked through the design constraints analysis. However, the entities defining the design constraints are not determinist values due to the uncertainties propagation from input parameters or design variables to structural response. So, the variability of design constraints is associated with feasibility robustness. In the proposed approach the feasibility robustness is defined through the determinant of variance–covariance matrix of constraint functions introducing in this way the joint effects of the uncertainty propagations on structural response of composite structures.

The variance–covariance matrix $\mathbf{C}_{\varphi}$ represents the joint effects of the propagation of uncertainties (Salazar and Rocco 2007; Ragavajhala and Mahadevan 2013). Its components are associated with the variability on critical values of displacement and stress fields of structural response of composite structures in the proposed approach. Since the constraints of RDO problem depend on those critical values it can be concluded that the feasibility robustness can be represented trough the determinant of the variance–covariance matrix $\mathbf{C}_{\varphi}$.

3 Robust design optimization of composite shells

3.1 Bi-objective optimization based on robustness feasibility

The fundamental objective of robust design is to improve the structural performance and to stabilise response performances by minimising the effects of the propagation of uncertainties. Although some formulations are proposed in the literature for RDO, their advantages applied to composite structures, in terms of accuracy and efficiency, are not yet fully known. Those formulations are based on the robustness of a performance associated with the dispersion around its mean (Adali et al. 2003; Walker and Hamilton 2005; Gumbert and Newman 2005; Choi et al. 2008; António and Hoffbauer 2009, 2010). However, in composite plate/shell structures the variability of both the maximum displacement in Eq. (24) and of the most critical Tsai number in Eq. (25), both of them included in the vector $\boldsymbol{\varphi} = \left(\overline{u}, \overline{R}\right)$, are measures of the structural response variability. Since the displacement and stress constraints must be considered on optimal design formulation defining the feasibility of design space, the variability of both the critical values $\overline{u}$ and $\overline{R}$ are measures of feasibility robustness. So, in this work the evaluation of the response uncertainty is done in a simple and systematic way using the determinant of variance–covariance matrix $\mathbf{C}_{\varphi}$ of structural response defined in Eq. (28).

In the proposed approach for RDO of composite structures, the feasibility robustness of the system is searched together the minimization process of performance/cost. The goal is to minimise the sensitivity of the optimal performance/cost of the system associated with the response to the uncertainty on the feasibility of constraints. A bi-objective optimization is performed by considering the following objective functions: (a) a function describing the performance/cost of the structural composite structure and (b) a function describing the feasibility robustness of constraints related to the variability of the structural response.

The design and uncertainty rules of the proposed RDO approach are controlled by following classes of variables and parameters: the vector of deterministic design variables, $\mathbf{d} \in \mathbf{R}^k$, the vector of random design variables, $\mathbf{z} \in \mathbf{R}^m$, and the vector of random parameters, $\boldsymbol{\pi} \in \mathbf{R}^p$. The nominal values of random design and random parameters are taken to be the expected values $\boldsymbol{\mu}_{\mathbf{z}}$ and $\boldsymbol{\mu}_{\boldsymbol{\pi}}$, respectively, and the associated uncertainties are given by the corresponding standard deviations. No probability distribution functions are considered in the present analysis.

The design variables intervening in the optimization procedure are the deterministic design variables,

$\mathbf{d}$, and the nominal/expected values $\boldsymbol{\mu}_{\mathbf{z}}$ of the random design variables, $\mathbf{z}$. The standard deviation of $\mathbf{z}$ is kept constant during the optimization procedure. The performance/cost of the composite structure is given by its weight $W(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{z}})$. The functional $V(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{z}}, var(\overline{u}), var(\overline{R}), cov(\overline{u}, \overline{R}))$ is a measure of feasibility robustness, which is concerned with ensuring that the constraints are adequately satisfied under uncertainty. The bi-objective optimization problem can then be established as

$$\underset{\text{over } \mathbf{d}, \boldsymbol{\mu}_{\mathbf{z}}}{\text{Minimise}} \quad OBJ\left(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{z}}, \mathbf{C}_{\boldsymbol{\varphi}}\right) = (f_1, f_2) \tag{29}$$

with

$$f_1 = W(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{z}}) \quad \text{and}$$
$$f_2 = V(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{z}}, var(\overline{u}), var(\overline{R}), cov(\overline{u}, \overline{R})) = det\, \mathbf{C}_{\boldsymbol{\varphi}}$$

$$\text{subject to} \quad \begin{aligned} g_1(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{z}}) &= \frac{\overline{u}(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{z}})}{u_a} - 1 \leq 0 \\ g_2(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{z}}) &= 1 - \frac{\overline{R}(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{z}})}{R_a} \leq 0, \end{aligned} \tag{30}$$

and

$$\begin{aligned} &d_j^l \leq d_j \leq d_j^u, \quad j = 1, \ldots, \bar{N}_{\mathbf{d}} \\ &\mu_{z_j}^l \leq \mu_{z_j} \leq \mu_{z_j}^u, \quad j = 1, \ldots \bar{N}_{\mathbf{z}} \end{aligned} \tag{31}$$

being $\overline{u}$ and $\overline{R}$ the critical displacement and critical Tsai number both of them defined by Eqs. (24) and (25), respectively. These critical values are compared with the allowable values u_a and R_a for displacement and Tsai number, respectively. In this approach the feasibility robustness of composite structures is associated with the variability of the structural response, V defined as the determinant of variance–covariance matrix $\mathbf{C}_{\boldsymbol{\varphi}}$ of the system defined on Eq. (28) of propagation of uncertainties. In the inequalities (30) $\bar{N}_{\mathbf{d}}$ and $\bar{N}_{\mathbf{z}}$ are the number of deterministic and random design variables, respectively.

The performance/cost $W(\mathbf{d}, \boldsymbol{\mu}_z)$ depends on deterministic design variables and/or random design variables (throughout their nominal/expected values). The feasibility robustness associated with the variability of the structural response, $V(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{z}}, var(\overline{u}), var(\overline{R}), cov(\overline{u}, \overline{R}))$ depends on both deterministic/random design variables and also on random parameters of the system.

Uncertainties in different groups of random variables and/or random parameters show distinct behaviours and importance on structural response variability during RDO search (António and Hoffbauer 2009, 2010). In particular, the definition of feasibility robustness depends on the groups of random design variables and/or random parameters considered on optimization process loop. This aspect will be studied for different random variables/parameters used for feasibility robustness definition. At the end of the RDO optimization process, the Pareto front representing the frontier of the trade-off between the "performance" and the "robustness" functions is obtained.

3.2 Multi-objective evolutionary algorithm

The use of multiple objective evolutionary algorithms (MOEAs) in robust design of systems has been reported by few publications found in literature (Konak et al. 2006; Salazar and Rocco 2007; Taboada et al. 2007). Most of the referred approaches are based on dominance concepts to build the Pareto front proposed by Deb (2001). In the proposed approach the multi-objective optimization search is performed using on a new proposed approach based on dominance concepts applied in two populations exchanging data during the evolutionary process. The Pareto front is built by this co-evolutionary procedure denoted by Co-Dominance-based Genetic Algorithm (CoDGA). A self-adaptive genetic search incorporating Pareto dominance and an elitist strategy storing the non-dominated solutions found during the evolutionary process is considered (António 2009, 2013).

The problem of stacking sequence design of composite structures is well known for having many local optima, and so, dominated solutions are expected. The approach proposed in this work uses a mixture of developed techniques (António 2013) and new techniques in order to find multiple Pareto-optimal solutions in parallel using two populations (short and enlarged). The principal aspects are: (1) the use of the concept of Pareto dominance in order to assign scalar fitness values to individuals; (2) the clustering through the co-evolution of a short population (SP) to reduce the number of non-dominated solutions stored without destroying the characteristics of the Pareto-optimal front; and (3) the storage of the obtained Pareto-optimal solutions in an enlarged population (EP); (4) exchange of information between short and enlarged populations through the crossover operator.

The proposed CoDGA performs according to the flow diagram presented in Fig. 1. The algorithm performs using the concept of local dominance at short population (SP) and storing the new generated non-dominated individuals/solutions (rank 1) from SP sorting, into an enlarged population (EP). The enlarged population is continuously updated based on global dominance concepts and has two principal functionalities: to build the global Pareto front and to transmit its best member's genetic properties to the next populations of the evolutionary process.

Three important aspects must be considered for the proposed approach CoDGA: 1. *Local dominance definition*; 2. *Fitness assignment based on local dominance*; and 3. *Building of global Pareto front at enlarged population*.

1. *Local dominance definition* At isolation stage of SP defined here as set $\mathbf{Q} \subseteq \Re^n$ the individuals are sorted and ranked according to local non-constrained dominance. Following the definition by Deb (2001), an individual $\mathbf{v}_i \in \mathbf{Q}$ is said to constrain-dominate an individual $\mathbf{v}_j \in \mathbf{Q}$, if any of the following conditions are verified:

(1) $\mathbf{v}_i$ and $\mathbf{v}_j$ are feasible, with

  (i) $\mathbf{v}_i$ is no worse than $\mathbf{v}_j$ for all objectives, and

  (ii) $\mathbf{v}_i$ is strictly better than $\mathbf{v}_j$ in at least one objective,

(2) $\mathbf{v}_i$ is feasible while individual $\mathbf{v}_j$ is not,

(3) $\mathbf{v}_i$ and $\mathbf{v}_j$ are both infeasible, but $\mathbf{v}_i$ has smaller constraint violation.

The constraint violation of an individual $\mathbf{v}$ is defined to be equal to the sum of the violated constraint function values in the multi-objective optimization problem formulated from Eqs. (29) to (31):

$$\xi(\mathbf{d},\ \boldsymbol{\mu}_\mathbf{z}) = \sum_{i=1}^{2} \boldsymbol{\Gamma}_i(\mathbf{d},\ \boldsymbol{\mu}_\mathbf{z}) \tag{32}$$

with

$$\boldsymbol{\Gamma}_i[g_i(\mathbf{d},\boldsymbol{\mu}_\mathbf{z})] = \begin{cases} 0 & \text{if} \quad g_i(\mathbf{d},\boldsymbol{\mu}_\mathbf{z}) \le 0 \\ g_i(\mathbf{u},\boldsymbol{\pi}) & \text{if} \quad g_i(\mathbf{d},\boldsymbol{\mu}_\mathbf{z}) > 0 \end{cases} \tag{33}$$

The constraint violation defined in Eq. (32) are referred to $\bar{u}$ and $\overline{R}$ the critical displacement and critical Tsai number. The side constraints defined in Eq. (31) are considered when the phenotype of design variables (deterministic or random) are converted to genotype using the binary code format. The concept of constrain-domination enables to compare two individuals in problems having multiple objectives and constraints, since if $\mathbf{v}_i$ constrain-dominates $\mathbf{v}_j$, then $\mathbf{v}_i$ is better than $\mathbf{v}_j$. If none of the three conditions referred above are verified, then $\mathbf{v}_i$ does not constrain-dominate $\mathbf{v}_j$.

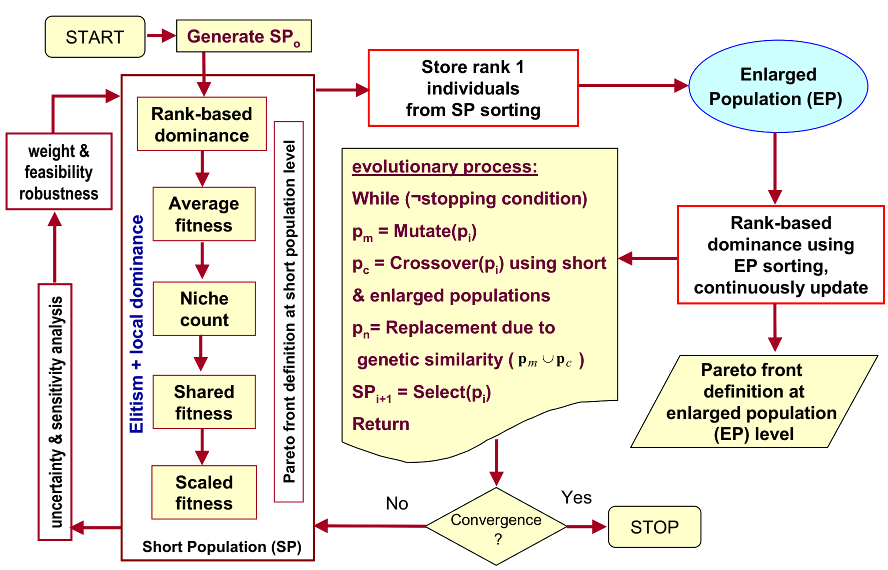


Fig. 1 Flowchart of the proposed CoDGA

2. *Fitness assignment based on local dominance* The dominance concept is applied only to the restricted set of individuals of SP being so denoted by local dominance. The definition of the fitness of each individual no longer depends on an absolute value related to the individual's fitness but on the concept of dominance. The individual fitness is calculated according to the niche occupied by the individual in the short population and also depending on the number of individuals with the same level of dominance in its neighbourhood. So, the concept of shared fitness is adopted (Deb 2001; António 2013). This aims to obtain a balanced distribution of solutions along the constructed local Pareto front and updated during evolutionary process. The elitist strategy adopted at that stage is based on fitness as also on the concept of dominance albeit implicitly (António 2013). The Fig. 1 describes briefly the procedure to assign the fitness at SP level. A sharing function is used to improve the distribution of rank's 1 solutions (non-dominated) along the Pareto front at SP level. More details can be found in references (Deb 2001; António 2013).

3. *Building of global Pareto front* The enlarged population (EP) is used to store the best ranked solutions (rank 1) from sorting of individuals at short population. The EP is organized based on the concept of global dominance applied in each generation of the evolutionary process. To do this the same concepts of dominance previously described are applied to individuals stored at enlarged population. Given the size and history of this population, the dominance is applied in the global sense, allowing the progressive construction of global Pareto front. As the process is continuously applied at every generation, it is possible that an individual with non-dominated status will be subsequently dominated and consequently does not will intervene in the evolutionary process. This leads to an increased historical record of global rank 1 individuals/non-dominated solutions in the course of the evolutionary process obtaining finally the global Pareto front. The enlarged population is continuously updated during the evolutionary process.

The evolutionary process of CoDGA is performed by four genetic operators: mutation, crossover, replacement due to genetic similarity and selection as shown in Fig. 1. The binary code format is used to encoding the phenotype of design variables. The stopping criterion is based on reaching the minimum number of generations without improvement of Pareto front of enlarged population.

3.3 Genetic operators of CoDGA

The genetic operators used in CoDGA are applied according the scheme shown in Fig. 2. The linkage between short population (SP) and enlarged population (EP) is made through the crossover operator. Three mechanisms of recombination are identified in this operator (Herrera et al. 2003; António 2009): *Mating selection mechanism* (MSM), *Offspring generation mechanism* (OGM) and *Offspring selection mechanism* (OSM).

The MSM of proposed crossover is described in Fig. 2 and is composed by two schemes applied in alternative way from t-*th* to (t + 2)-*th* generations:

First scheme of MSM: The short population is divided in two groups, the first one having best assigned fitness denoted by elite **U** and the other one grouping the set **L** with the worst assigned fitness. The couple of parents ($\mathbf{p}_1$,$\mathbf{p}_2$) is obtained using two independent selection processes in **U** and **L** sets.
Second scheme of MSM: One parent of the couple, denoted by $\mathbf{p}_1$, comes from the elite group of short population. The other parent, denoted by $\mathbf{p}_2$, having ranking score less than "rank 3" of dominance comes from enlarged population sorting (non-dominated solutions = rank 1) after dominance updating.

The MSM process is repeated until the necessary couples ($\mathbf{p}_1$,$\mathbf{p}_2$) (one per each offspring) are found. Both above MSM schemes applied in alternative way are elitist.

Production of new chromosomes from a set of parents selected by MSM is carried out by an appropriate recombination scheme denoted by offspring generation mechanism (OGM). This mechanism enables the genetic material to be transferred from parents to offspring and performs a multipoint combination of genes from both parents' chromosomes. The genetic material exchange of OGM is based on the technique "Parameterised Uniform Crossover" proposed by Spears and DeJong and following the version presented by António (2002, 2009).

Departing from the offspring generated for each set of parents the offspring selection mechanism (OSM)

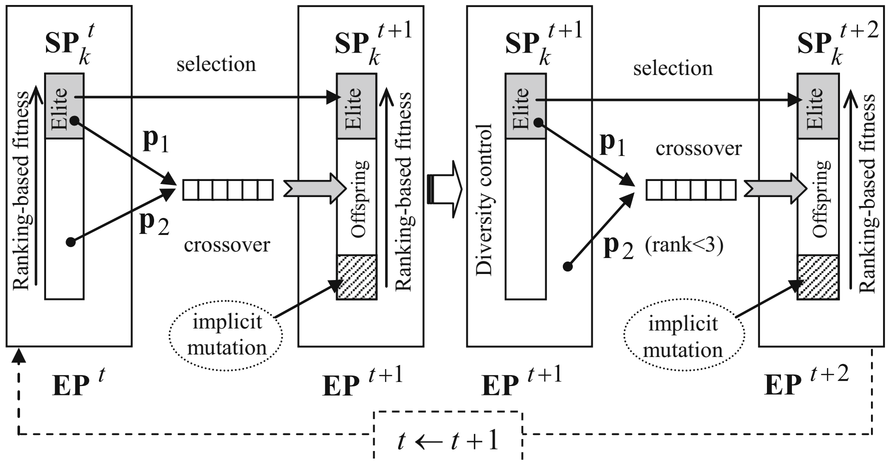


Fig. 2 Concurrent recombination schemes between short population (SP) and the enlarged population (EP) used for CoDGA

of crossover operator chooses the individuals that will become SP members at next generation. In this case OSM chooses a core of best offspring to form the next short population (António 2009).

The *implicit mutation* operator (António 2002, 2009, 2013) is considered in genetic search as shown in Fig. 2. In this kind of mutation a set of new chromosomes generated randomly is inserted into the sub-population. In general since these new individuals have fair fitness their influence is neither explicit nor immediate in the current generation. However, their effects are widely shown in future generations as they provoke a refreshing of the genetic material of the population through the combination with other older chromosomes.

The *genetic similarity control* operator is implemented during the evolutionary process taking the most representative bits of each design variable for all individuals/solutions belonging to short population (SP). This is followed by elimination of solutions with similar genetic properties and their replacement with new solutions/chromosomes with genes randomly generated. This ensures the genetic diversity of the population.

4 Applications to composite structures

4.1 Problem definition

To study the capability of the proposed approach for bi-objective optimization based on feasibility robustness, a clamped cylindrical shell laminated structure is considered as shown in Fig. 3. Nine vertical loads of mean value $P_k = 7\,kN$ are applied along the free linear side (AB) of the structure. This free linear side (AB) is constrained in the y-axis direction. The structure is divided into four macro-elements, grouping all elements, and there is one laminate per each macro-element. The laminate distribution of the structure is shown in Fig. 3. The balanced angle-ply laminates with five layers and the stacking sequence $[+a/+a/-a/-a]_s$ are considered in the symmetric composite construction. Ply angle, a, is a design variable and is referenced to the x-axis of the reference axis, as detailed in Fig. 3. The design variable h_i, denotes the laminate thickness and four laminates are considered in this example. A smoothing procedure is followed at the boundary of laminates to guarantee the continuity of structure.

The structural analysis of laminated composite structures is based on the shell finite element model as previously referred in Sect. 2.2. A composite material built with the carbon/epoxy system denoted by T300/N5208 (Tsai 1987), is used in the presented analysis. This is a unidirectional carbon long fibres aggregated in a epoxy matrix. The macro mechanics mean values of the elastic and strength properties of the ply material used in the symmetric laminate construction of the composite structure are presented in Table 1. The elastic constants of the orthotropic ply are the longitudinal elastic modulus E_1, the transversal elastic modulus E_2, the in-plane shear modulus G_{12}, and the in-plane Poisson's ratio ν_{12}. The ply strength properties are the longitudinal strength in tensile, X, and in

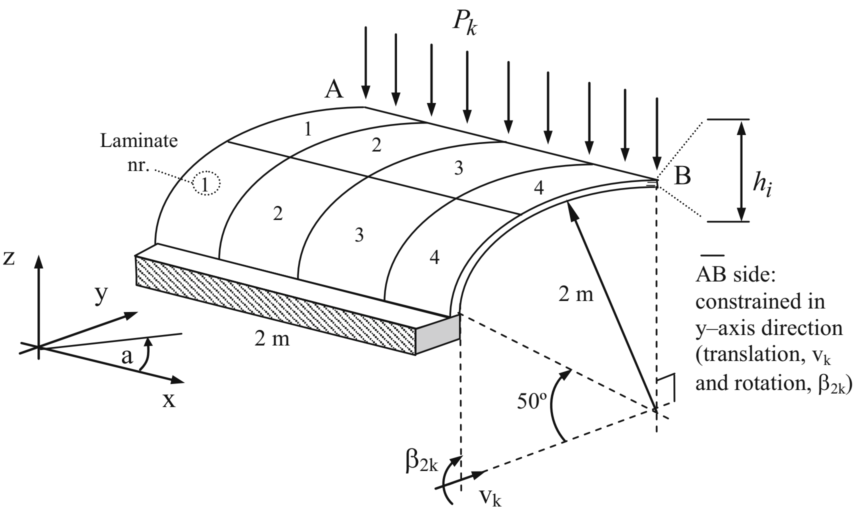


Fig. 3 Geometric definition of the cylindrical shell structure and composite laminates distribution

Table 1 Mean values of mechanical properties of composite layers

Material	E_1 (GPa)	E_2 (GPa)	G_{12} (GPa)	ν_{12}
T300/N5208	181.00	10.30	7.17	0.28
	X; X′ (MPa)	Y; Y′ (MPa)	S (MPa)	ρ (kg/m^3)
T300/N5208	1500; 1500	40; 246	68	1600

compression, X', the transversal strength in tensile, Y, and in compression, Y', and the shear strength, S.

To investigate the influence of uncertainty analysis on the proposed multi-objective design optimization of composite structures four case studies are considered. The uncertainty of the system is considered through the vector of random design variables, $\mathbf{z} \in \mathbf{R}^m$, and the vector of random parameters, $\boldsymbol{\pi} \in \mathbf{R}^p$. The nominal values of random design variables and random parameters are taken to be the expected values $\boldsymbol{\mu}_\mathbf{z}$ and $\boldsymbol{\mu}_{\boldsymbol{\pi}}$, respectively. The corresponding standard deviations are considered in robustness feasibility definition as established in Sect. 2.

The design variables intervening in the optimization procedure are the deterministic design variables,**d**, and the nominal/expected values $\boldsymbol{\mu}_\mathbf{z}$ of the random design variables, **z**. The standard deviation of **z** is kept constant during the optimization procedure. The design variables are encoded using a binary code format with different number of digits. The genetic parameters used at short population evolution and the design variables constraint intervals are defined in Table 2.

The RDO problem based on weight minimization and feasibility robustness maximization formulated from Eqs. (29) to (31) is solved using the CoDGA approach proposed in Sect. 3. The optimization process evolves along 300 generations. The allowable values in the constraints on displacement and Tsai number are $u_a = 8.0 \times 10^{-2}$ m and $R_a = 1$, respectively.

The use of joint feasibility robustness through the determinant of variance–covariance matrix does not show the partial effects of the variability of $\overline{u}$ and $\overline{R}$ the critical displacement and critical Tsai number defined by Eqs. (24) and (25), respectively. The coefficient of variation of each structural response parameters $\overline{u}$ and $\overline{R}$, weighted by prescribed values u_a and R_a are used to analyse the partial effects of the variability on the obtained optimal Pareto front. These weighted coefficients of variation are defined as follows:

$$CV^*(R) = \frac{\sqrt{var(\overline{R})}}{R_a} \times 100\ (\%) \tag{34}$$

Table 2 Genetic parameters and design variables constraint intervals

Population size	30
Elite group size (%)	33.33
Mutation group size (%)	20
Number of generations	300
Code format (digits nr.)/size constraint interval, for ply angle a	4/[0°, 90°]
Code format (digits nr.)/size constraint interval, for laminate thickness, $h_i, i = 1, \ldots, 4$	5/[0.005 m, 0.040 m]

$$CV^*(u) = \frac{\sqrt{var(\overline{u})}}{u_a} \times 100\,(\%) \tag{35}$$

4.2 Case study 1: RDO based on mixed randomness properties

In this studied case, the variance properties of the response of composite plate/shell structures are associated with two sources of uncertainty: on random design variables and on random parameters of the structural system. They are organized in following four groups with allowable tested variations:

Group 1 of the mechanical properties ($\mathbf{m}$), defined as random parameters;
Group 2 of the ply angle (a) on laminates, defined as random design variable;
Group 3 of the laminate thicknesses ($\mathbf{h}$), defined as random design variable;
Group 4 of the point loads ($\mathbf{P}$), defined as random parameters.

The mechanical properties group, $\mathbf{m}$, includes the following random parameters: longitudinal Young's modulus $E_{1,j}$, transversal modulus $E_{2,j}$, transversal tensile strength Y_j, and shear strength S_j, where subscript j denotes the laminate number. Sixteen mechanical properties are considered as random parameters with uncertainty in this analysis: $E_{1,j}$, $E_{2,j}$, Y_j, S_j, j = 1, …, 4. This random parameters are aggregated in vector $\boldsymbol{\pi}$.

Five random design variables are considered in vector $\mathbf{z}$ for this case study: one ply angle a for all symmetric laminates with the stacking sequence $[+a/+a/-a/-a]_s$, and the laminate thicknesses $h_i, i = 1, \ldots, 4$. So, it can be written,

$$\mathbf{z} = (a, h_1, \ldots, h_4) \tag{36}$$

The variability is referred to the expected values $\boldsymbol{\mu}_\mathbf{z}$ corresponding to the design solution value obtained at each generation of the optimization procedure. However, a prescribed and fixed standard deviation is allowed for these random design variables. Since the expected values $\boldsymbol{\mu}_\mathbf{z}$ are not fixed during the optimization process, prescribed fixed standard deviations are used to consider the uncertainty in random design variables $\mathbf{z}$. On contrary, the coefficients of variation $\mathrm{CV}(\boldsymbol{\pi})$ are used to prescribe the uncertainty of the random parameters $\boldsymbol{\pi}$ having means and standard deviations fixed at the beginning of the optimization process. Thus, the variability in input variables/parameters are prescribed as follows:

- Group 1: The mechanical properties group ($\mathbf{m}$), with the prescribed coefficient of variation, $\mathrm{CV}(\mathrm{m_i}) = 6\,\%, \quad i = 1, \ldots 16$;
- Group 2: The ply angle group (a), with the prescribed standard deviation, $\sigma(\alpha) = 5°$;
- Group 3 The laminate thickness group ($\mathbf{h}$), with the prescribed standard deviation, $\sigma(h_i) = 5 \times 10^{-4}\,\mathrm{m}, \quad i = 1, \ldots, 4$;
- Group 4: The point load group ($\mathbf{P}$), with the prescribed coefficient of variation, $\mathrm{CV}(P_k) = 6\,\%, \quad k = 1, \ldots, 9$

The RDO problem formulated from Eqs. (29) to (31) is solved using the proposed CoDGA approach. In this case the RDO problem is formulated as:

$$\underset{\text{over } \boldsymbol{\mu}_\mathbf{z}}{\text{Minimise}} \quad OBJ\left(\boldsymbol{\mu}_\mathbf{z}, \mathbf{C}_\boldsymbol{\varphi}\right) = (f_1, f_2) \tag{37}$$

with

$$f_1 = W(\boldsymbol{\mu}_\mathbf{z}) \quad \text{and}$$
$$f_2 = V(\boldsymbol{\mu}_\mathbf{z}, var(\overline{u}), var(\overline{R}), cov(\overline{u}, \overline{R})) = \det \mathbf{C}_\boldsymbol{\varphi}$$

$$\begin{aligned} \text{subject to} \quad g_1(\boldsymbol{\mu}_\mathbf{z}) &= \frac{\overline{u}(\boldsymbol{\mu}_\mathbf{z})}{u_a} - 1 \le 0 \\ g_2(\boldsymbol{\mu}_\mathbf{z}) &= 1 - \frac{\overline{R}(\boldsymbol{\mu}_\mathbf{z})}{R_a} \le 0 \end{aligned} \tag{38}$$

and

$$\mu_{z_j}^l \le \mu_{z_j} \le \mu_{z_j}^u, \quad j = 1, \ldots, \bar{N}_\mathbf{z} \tag{39}$$

From the Eq. (13) of propagation of uncertainties, the robustness feasibility functional depends on the expected values of random design variables vector $\boldsymbol{\mu}_\mathbf{z}$, and on the derivatives of $\boldsymbol{\varphi} = \left(\overline{u},\ \overline{R}\right)$ in order to random design variables and random parameters also calculated at expected value vector $\boldsymbol{\mu}_{\mathbf{z}}$, as follows:

$$det\,\mathbf{C}_{\boldsymbol{\varphi}} = det\left(\mathbf{S}\mathbf{C}_x\mathbf{S}^T\right) = f_2\left(\boldsymbol{\mu}_{\mathbf{z}}, \partial\overline{u}/\partial\mathbf{z}|_{\boldsymbol{\mu}_{\mathbf{z}}}, \partial\overline{R}/\partial\mathbf{z}|_{\boldsymbol{\mu}_{\mathbf{z}}}, \partial\overline{u}/\partial\boldsymbol{\pi}|_{\boldsymbol{\mu}_{\mathbf{z}}}, \partial\overline{R}/\partial\boldsymbol{\pi}|_{\boldsymbol{\mu}_{\mathbf{z}}}\right) \tag{40}$$

Figure 4 shows the evolution of the construction of optimal Pareto front showing rank1 solutions on three generations of CoDGA procedure application. The bi-objective optimization problem based on minimizations of weight and variability appears to have contradictory objectives. Also it is evident the influence of the sharing function applied in fitness assignments as referred in Sect. 3.2, on the good distribution of solutions along the Pareto front at the end of optimization process (300th generation).

The weighted coefficients of variation $CV^*(R)$ and $CV^*(u)$ are used to analyse the partial components of variability of the solutions located on optimal Pareto front as shown in Fig. 5. The weighted coefficient of variation for critical displacement, $CV^*(u)$ follows the same behaviour of the system variance measured by $det\,\mathbf{C}_{\boldsymbol{\varphi}}$, with same increasing order. Although the coefficient $CV^*(R)$ is large it shows few changes for most of the optimal points along Pareto front. So, it can be concluded that the partial effects of the variability of critical displacement $\overline{u}$ are greater than the effects of the variability of critical Tsai number $\overline{R}$ on the feasibility robustness measured by $det\,\mathbf{C}_{\boldsymbol{\varphi}}$.

The establishment of a preference function could be formulated using the results compared in Fig. 5 together the analysis of the solutions belonging to optimal Pareto front. Since the changes on variability of the critical Tsai number are not relevant the decision can be associated with the changes on variability of critical displacement measured by $CV^*(u)$.

The random design variable ply angle a, has the same value $a = 90°$ for all solutions along the Pareto front. However, the random design variables laminate thickness h_i with $i = 1, \ldots, 4$, take the optimal solutions shown in Fig. 6. The solutions for the random design variable h_4 have a similar shape profile of Pareto front. The remaining random design variables have values around the lower limit of the design interval defined in Table 2. These solutions obtained for Pareto optima front of RDO problem are equivalent to minimum structural weight with reinforcement of the laminate number 4 of the shell composite structure defined in Fig. 3.

4.3 Case study 2: RDO based on mechanical properties uncertainty

In this case study the vector of deterministic design variables $\mathbf{d}$, and the vector of random parameters $\boldsymbol{\pi}$, are considered in RDO problem. Five deterministic design variables are considered in vector $\mathbf{d}$ for this case study: one ply angle a for all symmetric laminates with the stacking sequence $[+a/+a/-a/-a]_s$, and four laminate thicknesses h_i, $i = 1, \ldots,$ 4. Only mechanical properties group $(\mathbf{m})$ are considered as random parameters with the coefficient of variation, $CV(m_i) = 6\,\%$, $i = 1, \ldots 16$. These mechanical properties are the same considered in previously studied case and are aggregated in vector $\boldsymbol{\pi}$.

The mathematical formulation of RDO problem for this second case is

$$\underset{\text{over } \mathbf{d}}{\text{Minimise}} \quad OBJ\left(\mathbf{d}, \mathbf{C}_{\boldsymbol{\varphi}}\right) = (f_1, f_2) \tag{41}$$

with

$$f_1 = W(\mathbf{d}) \quad \text{and}$$
$$f_2 = V(\mathbf{d}, var(\overline{u}), var(\overline{R}), cov(\overline{u}, \overline{R})) = det\,\mathbf{C}_{\boldsymbol{\varphi}}$$

$$\text{subject to} \quad g_1(\mathbf{d}) = \frac{\overline{u}(\mathbf{d})}{u_a} - 1 \le 0 \qquad g_2(\mathbf{d}) = 1 - \frac{\overline{R}(\mathbf{d})}{R_a} \le 0, \tag{42}$$

and

$$d_j^l \le d_j \le d_j^u, \quad j = 1, \ldots, \bar{N}_{\mathbf{d}}, \tag{43}$$

The robustness feasibility functional depends on the current values of deterministic design variables $\mathbf{d}$ and on the derivatives of $\boldsymbol{\varphi} = \left(\overline{u},\ \overline{R}\right)$ in order to random parameters $\boldsymbol{\pi}$ calculated at the current values of $\mathbf{d}$, as follows:

$$det\,\mathbf{C}_{\boldsymbol{\varphi}} = det\left(\mathbf{S}\mathbf{C}_x\mathbf{S}^T\right) = f_2\left(\mathbf{d}, \partial\overline{u}/\partial\boldsymbol{\pi}|_{\mathbf{d}}, \partial\overline{R}/\partial\boldsymbol{\pi}|_{\mathbf{d}}\right) \tag{44}$$

Figure 7 shows the evolution of the Pareto along the optimization process. From the analysis of this

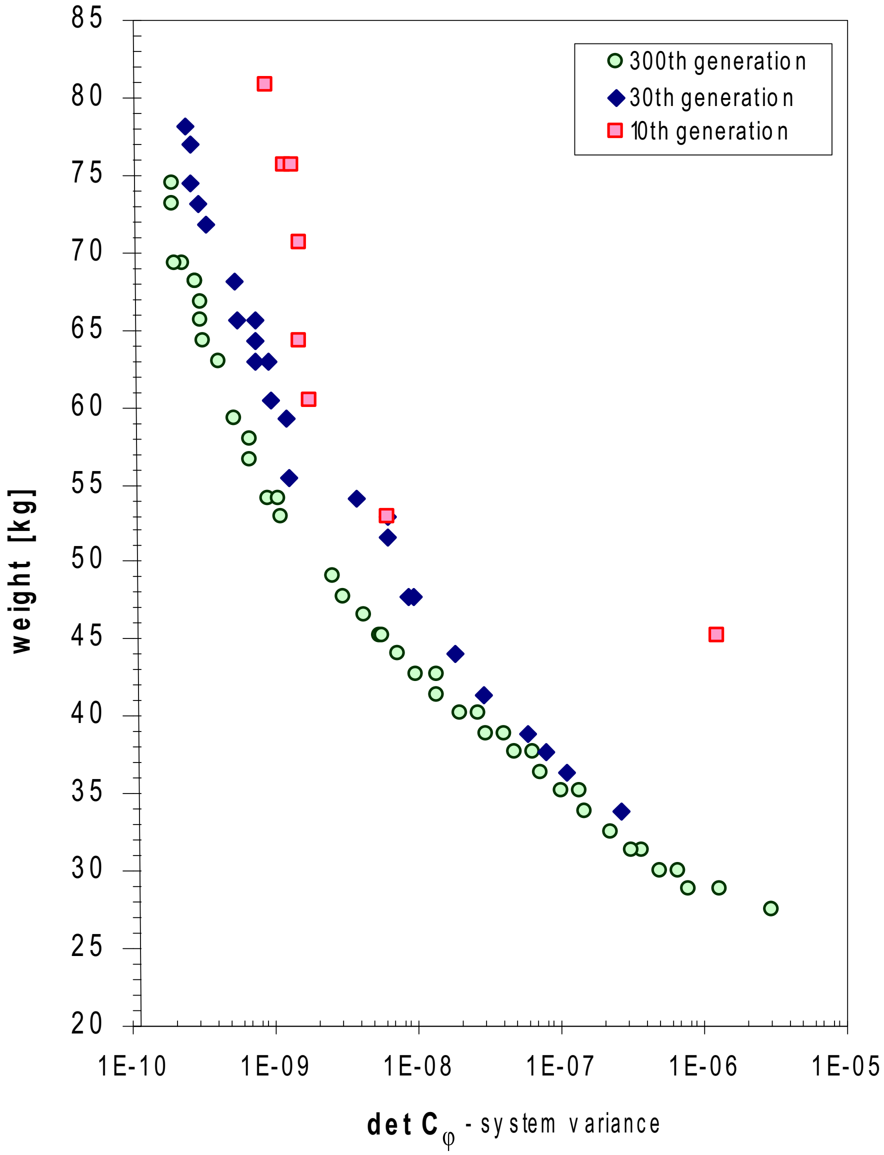


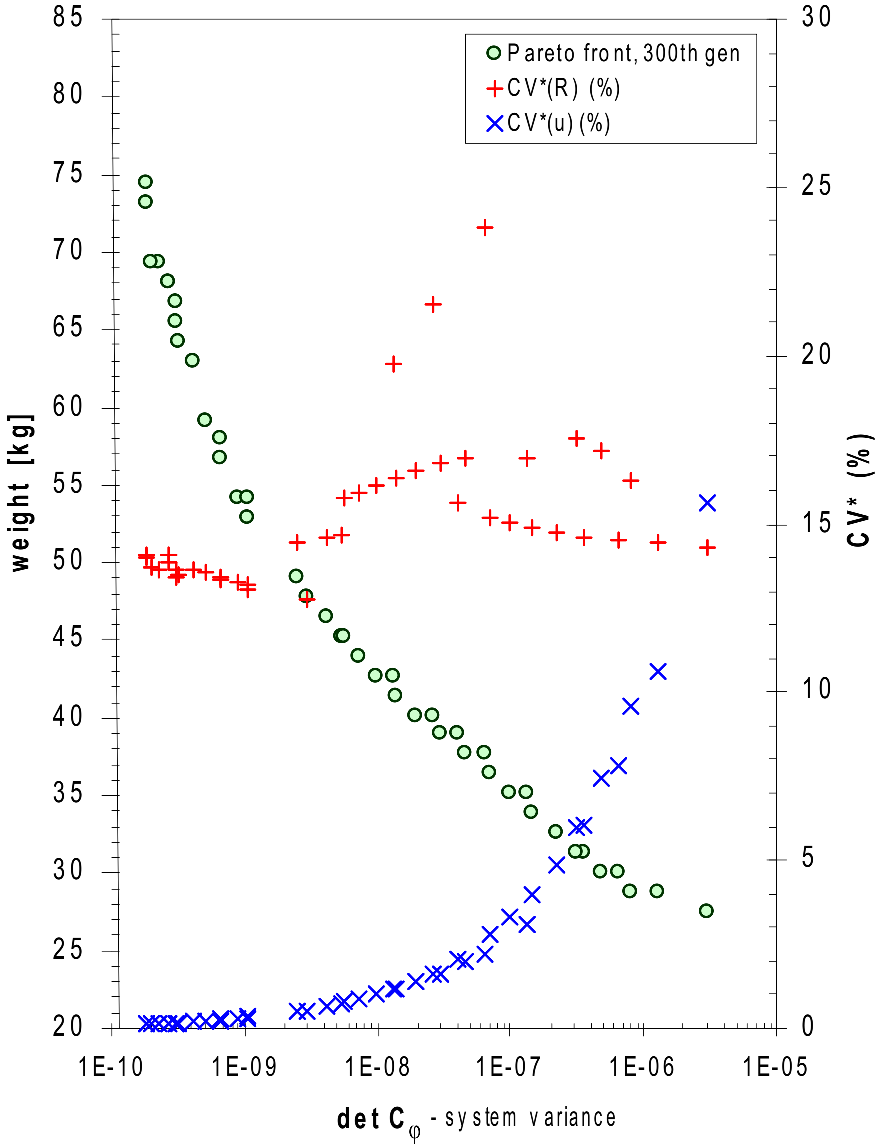
Fig. 4 Pareto front evolutions considering mixed randomness properties (Groups 1, 2, 3 and 4)

figure it can be conclude on the efficiency of proposed CoDGA approach to obtain shared solutions along Pareto front.

In Fig. 8 the weighted coefficients of variation $CV^*(R)$ and $CV^*(u)$ of solutions located on optimal Pareto front are compared. The weighted coefficient of variation for critical displacement $CV^*(u)$ increases such as the variance measured by $det\,\mathbf{C}_\varphi$, when the structural weight are decreasing. The weighted coefficient of variation for critical Tsai number $CV^*(R)$ is larger for solutions located on left side of Pareto front and lower for the remaining. So, it can be concluded that the partial effects of the variability of the critical Tsai number $\overline{R}$ influences the feasibility robustness for solutions with larger structural weight. On other hand, analysing $CV^*(u)$ it is demonstrated that the effects of variability of the critical displacement $\overline{u}$ are more important on solutions of Pareto front with smaller structural weight.

A comparison of $CV^*(R)$ represented in Figs. 5 and 8 shows that the uncertainty response effects due to mechanical properties is two (left side of Pareto front)

Fig. 5 Structural response variability of both critical Tsai number and critical displacement considering system variance (all groups of mixed randomness properties)



times until five times (right side of Pareto front) lower than the values considering system variance. The similar comparison for $CV^*(u)$ shows that the uncertainty response effects due to mechanical properties variance are five times less than the uncertainty response effects due to system variance (all groups of mixed randomness properties).

The deterministic design variables laminate thickness h_i (with $i = 1, \ldots, 4$), take the optimal solutions on Pareto front presented in Fig. 9a). The solutions for the design variable h_4 have a similar profile of Pareto front as the results shown in first studied case. The remaining deterministic design variables of laminate thicknesses have values around the lower limit of the design interval defined in Table 2. Although assuming different magnitudes for this case, the optimal design values for laminate thicknesses h_i (with $i = 1, \ldots, 4$), have similar shape profiles when compared with previous studied case considering system variance in Fig. 6. The deterministic design variable ply angle a, has the solutions along the Pareto front presented in Fig. 9b). The distributions of solutions along the optimal Pareto front are very different of the ones in previous studied case.

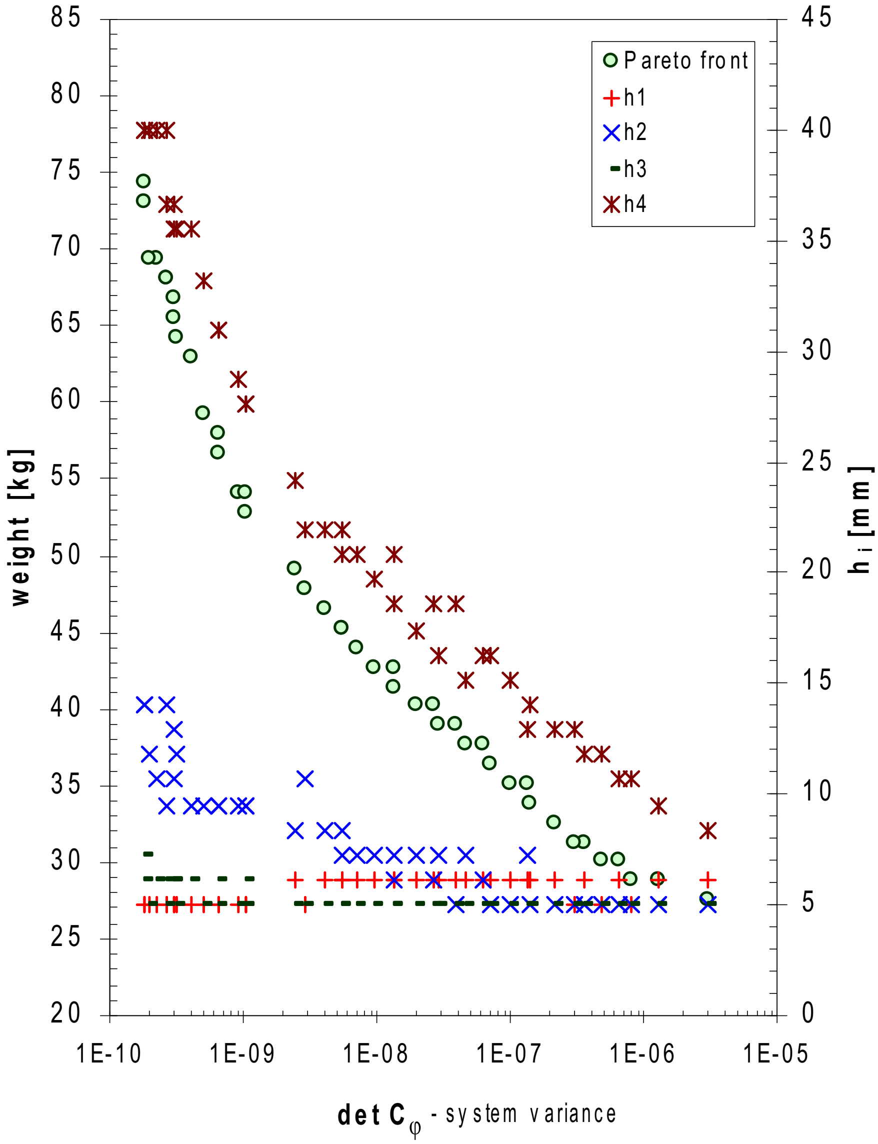


Fig. 6 Optimal laminate thickness solution h_i along Pareto front considering system variance (all groups of mixed randomness properties)

4.4 Case study 3: RDO based on ply angle uncertainty

Only the uncertainty in ply angle a is considered in the third case study shown in Fig. 10. In this case, the ply angle a is a random design variable with the standard deviation, $\sigma(\alpha) = 5°$. The variability is referred to the design expected values for ply angle μ_a obtained at each generation of the bi-objective optimization process. Furthermore, four deterministic design variables are considered in vector $\mathbf{d}$ for this case study: the laminate thickness variables $h_i, i = 1, \ldots, 4$.

In this case the RDO problem is formulated as

$$\underset{\text{over } \mathbf{d},\mu_a}{\text{Minimise}} \quad OBJ\left(\mathbf{d}, \mu_a, \mathbf{C}_{\boldsymbol{\varphi}}\right) = (f_1, f_2) \tag{45}$$

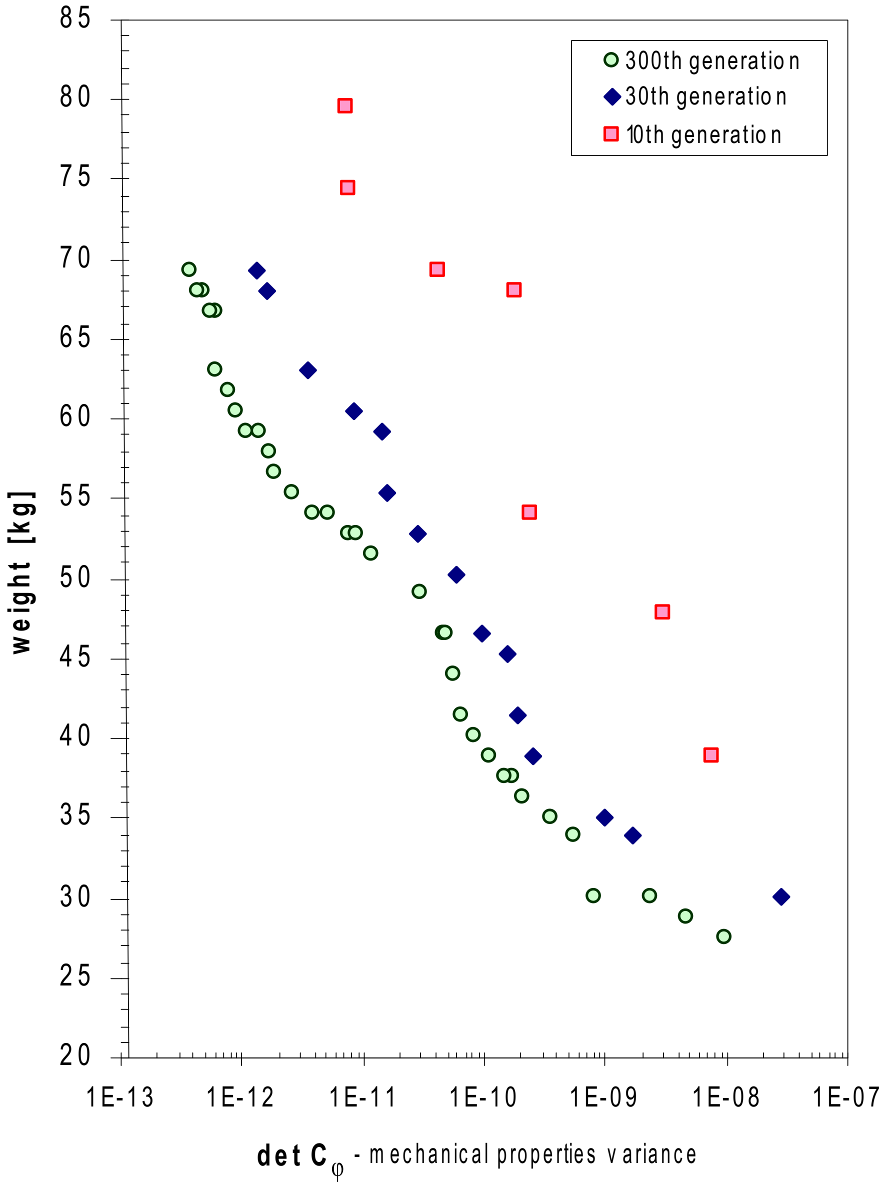
with

$$f_1 = W(\mathbf{d}) \quad \text{and}$$
$$f_2 = V(\mathbf{d}, \mu_a, var(\overline{u}), var(\overline{R}), cov(\overline{u}, \overline{R})) = det\,\mathbf{C}_{\boldsymbol{\varphi}}$$

$$\text{subject to} \quad \begin{aligned} g_1(\mathbf{d}, \mu_a) &= \frac{\overline{u}(\mathbf{d}, \mu_a)}{u_a} - 1 \leq 0 \\ g_2(\mathbf{d}, \mu_a) &= 1 - \frac{\overline{R}(\mathbf{d}, \mu_a)}{R_a} \leq 0 \end{aligned} \tag{46}$$

and

Fig. 7 Pareto front evolutions considering only mechanical properties variance



$$d_j^l \leq d_j \leq d_j^u, \quad j = 1, \ldots, \bar{N}_{\mathbf{d}},$$
$$\mu_a^l \leq \mu_a \leq \mu_a^u \tag{47}$$

In particular, the robustness feasibility functional depends on the current values of deterministic design variables vector $\mathbf{d}$, on the design expected values of ply angle μ_a, and on the derivatives of $\boldsymbol{\varphi} = \left(\overline{u}, \overline{R}\right)$ in order to ply angle a, calculated at the current values of $\mathbf{d}$ and at the expected value μ_a, as follows:

$$\begin{aligned} det\,\mathbf{C}_{\boldsymbol{\varphi}} &= det\left(\mathbf{S}\mathbf{C}_x\mathbf{S}^T\right) \\ &= f_2\left(\mathbf{d}, \mu_a, \partial\overline{u}/\partial a\big|_{\mathbf{d},\mu_a}, \partial\overline{R}/\partial a\big|_{\mathbf{d},\mu_a}\right) \end{aligned} \tag{48}$$

The proposed CoDGA approach considering weight minimization and feasibility robustness

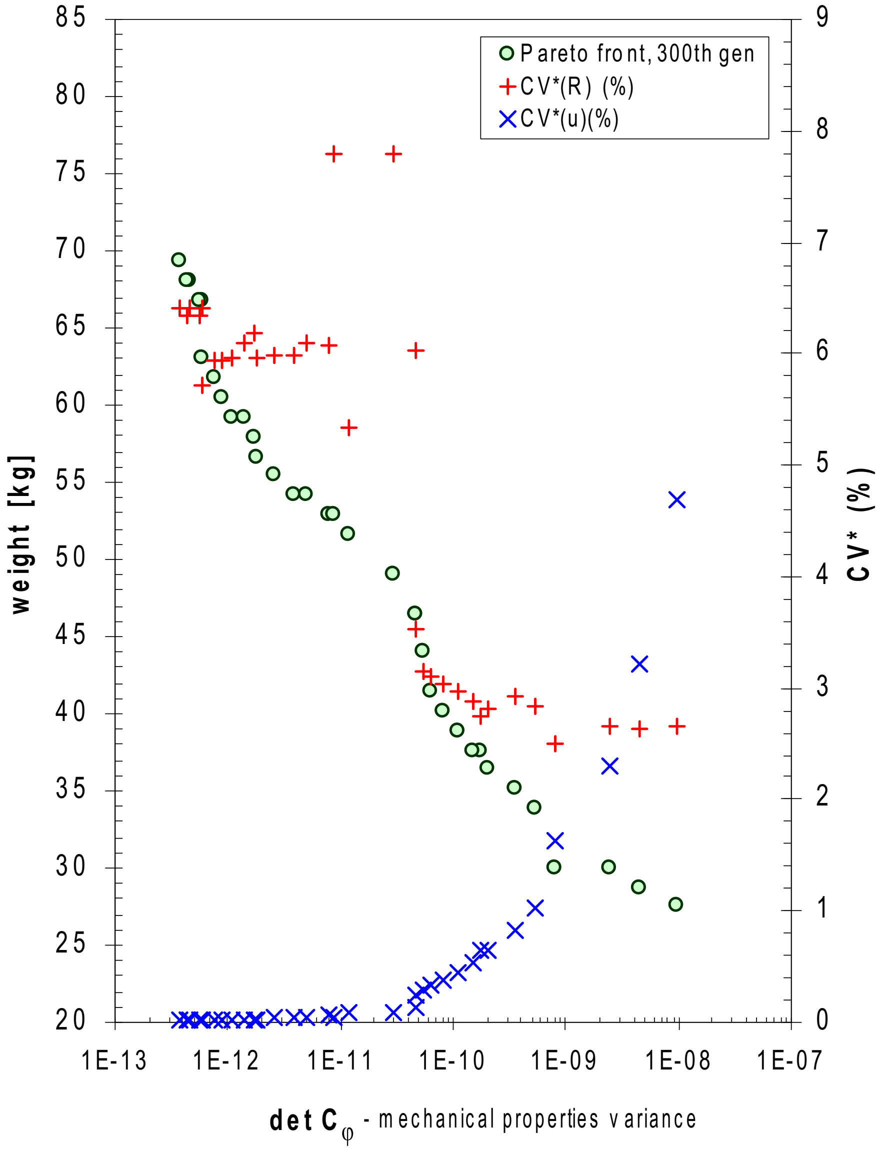


Fig. 8 Structural response variability of both critical Tsai number and critical displacement considering only mechanical properties variance

maximization (minimum variability) show again its effectiveness, with the solutions shared along the Pareto front as shown in Fig. 10.

Assuming the weighted coefficient of variations $CV^*(R)$ and $CV^*(u)$ as variability measures, it can be conclude from analysis of Fig. 10 that those coefficients follow the same profile of the structural response variance measured by *det* $\mathbf{C}_\varphi$ for all solutions located on the Pareto front. The variability of the critical displacement response is low when only ply angle uncertainty is considered with $CV^*(u) \in [0, 2]$ (%).

4.5 Case study 4: RDO based on laminate thickness uncertainty

Only the uncertainty in laminate thickness group ($\mathbf{h}$) is considered in the fourth case study. In this case, the laminate thicknesses aggregated in vector

$$\mathbf{z} = (h_1, \ldots, h_4) \tag{49}$$

are random design variables with fixed standard deviations, $\sigma(h_i) = 5 \times 10^{-4}$ m, $\quad i = 1, \ldots, 4$. This variability is referred to the expected value vector $\boldsymbol{\mu}_\mathbf{z}$, obtained at each generation of the bi-objective

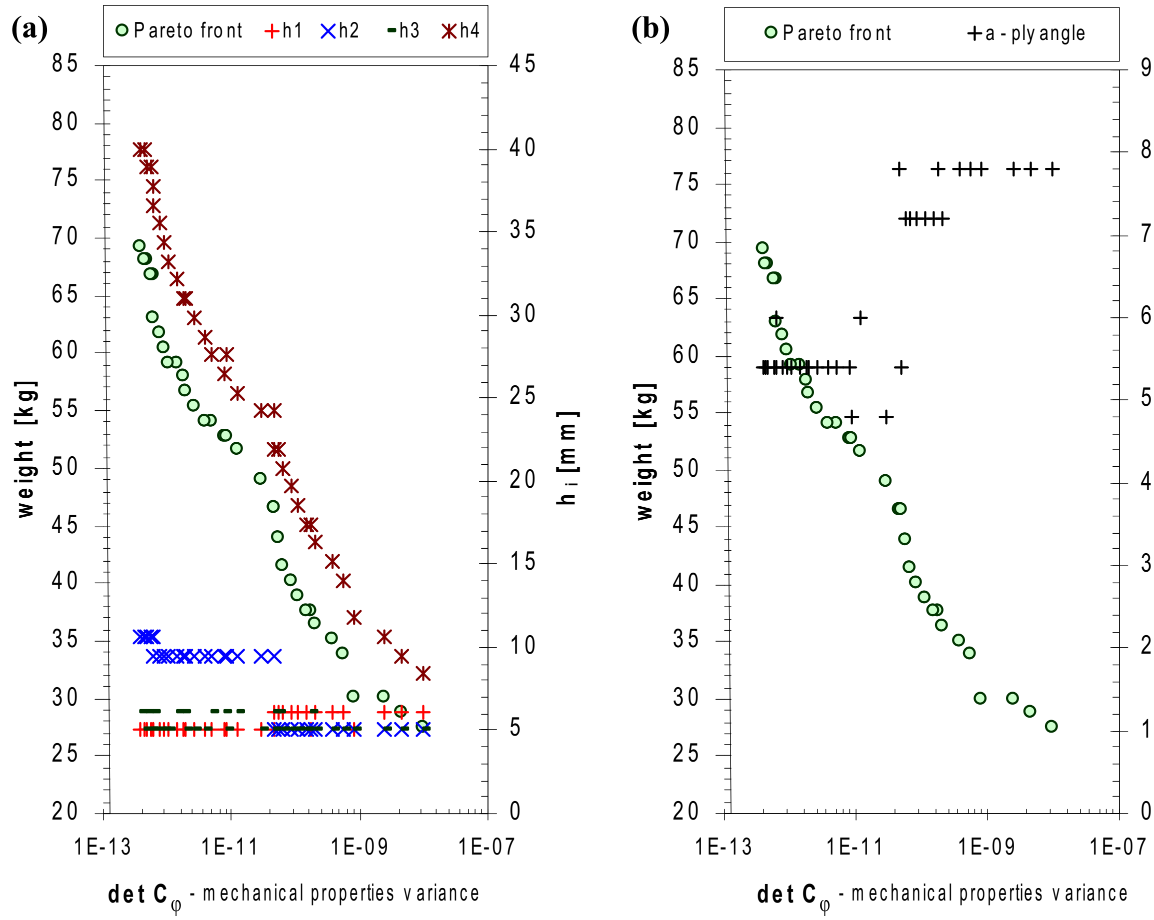


Fig. 9 Optimal design solutions along Pareto front considering only mechanical properties variance (Group 1): **a** laminate thickness h_i, **b** ply angle a

optimization process. Furthermore, the ply angle a is a deterministic design variable. In this case the RDO problem is formulated as

$$\underset{\text{over } a,\, \boldsymbol{\mu}_{\mathbf{z}}}{\text{Minimise}} \quad OBJ\left(a, \boldsymbol{\mu}_{\mathbf{z}}, \mathbf{C}_{\boldsymbol{\varphi}}\right) = (f_1, f_2) \tag{50}$$

with

$$f_1 = W(\boldsymbol{\mu}_{\mathbf{z}}) \quad \text{and}$$
$$f_2 = V(a, \boldsymbol{\mu}_{\mathbf{z}}\, var(\overline{u}), var(\overline{R}), cov(\overline{u}, \overline{R})) = det\, \mathbf{C}_{\boldsymbol{\varphi}}$$

$$\text{subject to} \quad g_1(a, \boldsymbol{\mu}_{\mathbf{z}}) = \frac{\overline{u}(a, \boldsymbol{\mu}_{\mathbf{z}})}{u_a} - 1 \leq 0 \qquad g_2(a, \boldsymbol{\mu}_{\mathbf{z}}) = 1 - \frac{\overline{R}(a, \boldsymbol{\mu}_{\mathbf{z}})}{R_a} \leq 0 \tag{51}$$

and

$$a^l \leq a \leq a^u, \qquad \mu^l_{z_j} \leq \mu_{z_j} \leq \mu^u_{z_j}, \quad j = 1, \ldots, \bar{N}_{\mathbf{z}} \tag{52}$$

In particular, the robustness feasibility functional depends on the current values of deterministic design variables $\mathbf{d}$, on the expected values of ply angle μ_a, and on the derivatives of $\boldsymbol{\varphi} = \left(\overline{u}, \overline{R}\right)$ in order to ply angle a, calculated at a and μ_a, as follows:

$$det\, \mathbf{C}_{\boldsymbol{\varphi}} = det\left(\mathbf{S}\mathbf{C}_x\mathbf{S}^T\right) = f_2\left(a, \boldsymbol{\mu}_{\mathbf{z}}, \partial\overline{u}/\partial\mathbf{z}|_{a,\boldsymbol{\mu}_{\mathbf{z}}}, \partial\overline{R}/\partial\mathbf{z}|_{a,\boldsymbol{\mu}_{\mathbf{z}}}\right) \tag{53}$$

From analysis of RDO results presented in Fig. 11 it can be concluded that the weighted coefficient of variance $CV^*(u)$ for critical displacement increases when the structural weight decreases while the

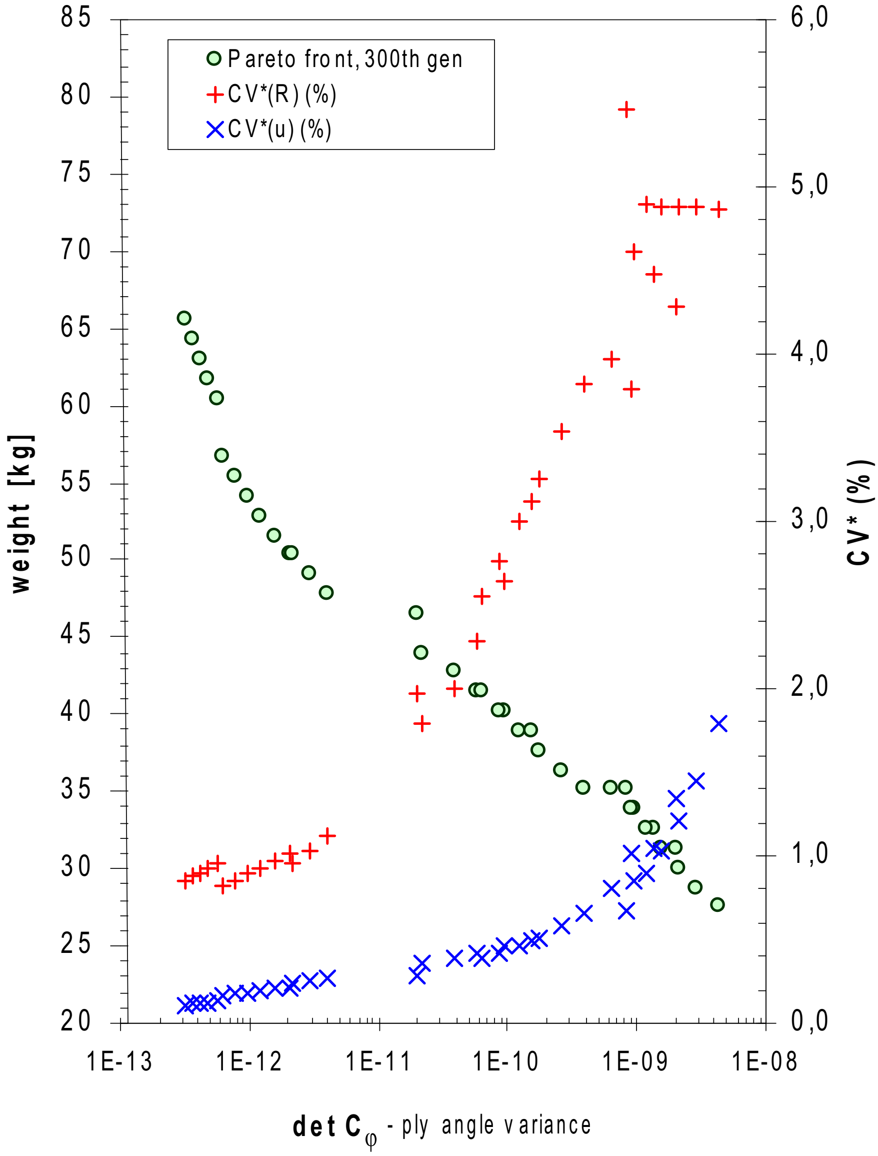


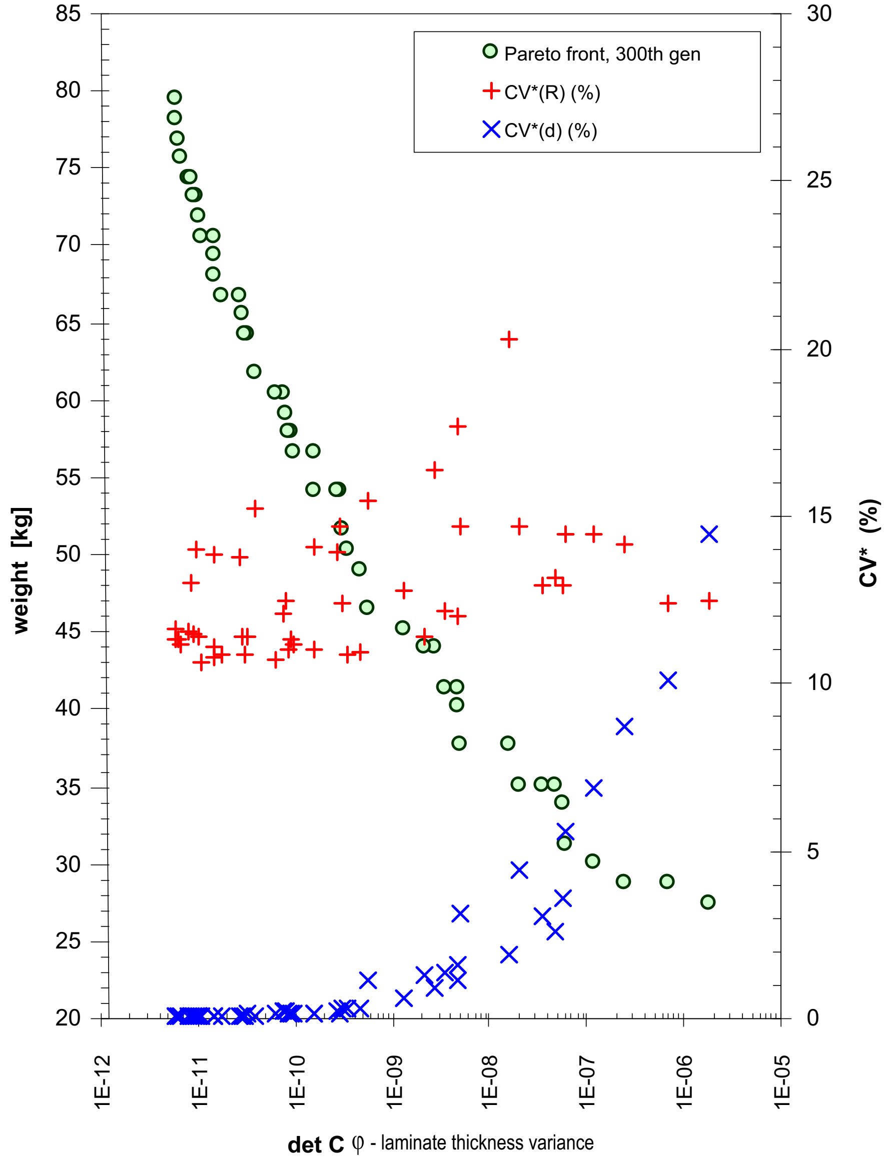
Fig. 10 Optimal Pareto front and structural response variability of both critical Tsai number and critical displacement considering only ply angle variance

weighted coefficient of variance $CV^*(R)$ for critical Tsai number does not change very much for the most of points located along Pareto front although their variance values are high. A comparison magnitude of weighted coefficients of variation $CV^*(R)$ and $CV^*(u)$ represented in Figs. 5 and 11 show that the most uncertainty effects in structural response measured considering all random variable groups (first case studied) are due to laminate thickness variance. Thus, the uncertainties in laminate thicknesses of composite structures are very important for RDO based on feasibility robustness.

4.6 Comparison of RDO studied cases: final remarks

Since feasibility robustness is a new concept applied specifically to composite structures there are important challenges behind the study of the influence of different groups of random variables and/or random parameters on RDO. This aspect is very important in aeronautical and industrial applications of composite structures. Pareto front depends on random design variables and/or random parameters considered in the uncertainty analysis for the same design variables used

Fig. 11 Optimal Pareto front and structural response variability of both critical Tsai number and critical displacement considering only laminate thickness variance



in RDO procedure. The optimal Pareto fronts of the four studied cases are shown in Fig. 12. So, it is possible to compare the effects of different sources of uncertainties and their influence onto structural response variability. Since, the design variables are the same in the four studied cases the differences come from the feasibility robustness measures.

The synergetic effects are important as is shown when the results of first case study are compared with the other cases. The combination of uncertainty sources is very important for design rules established from optimal Pareto front as shown in Fig. 12. In particular, for a fixed weight/cost the best minimum system variability measured by *det* $\mathbf{C}_{\varphi}$ can increases around times 1000. The uncertainty on laminate thickness plays the most important role on the structural shell composite structures. In real scenarios this uncertainty source is related to dimensional stability of composites shown the importance to control the uncertainty influence on the feasibility of constraints. So, this can be reached through the RDO based on feasibility robustness.

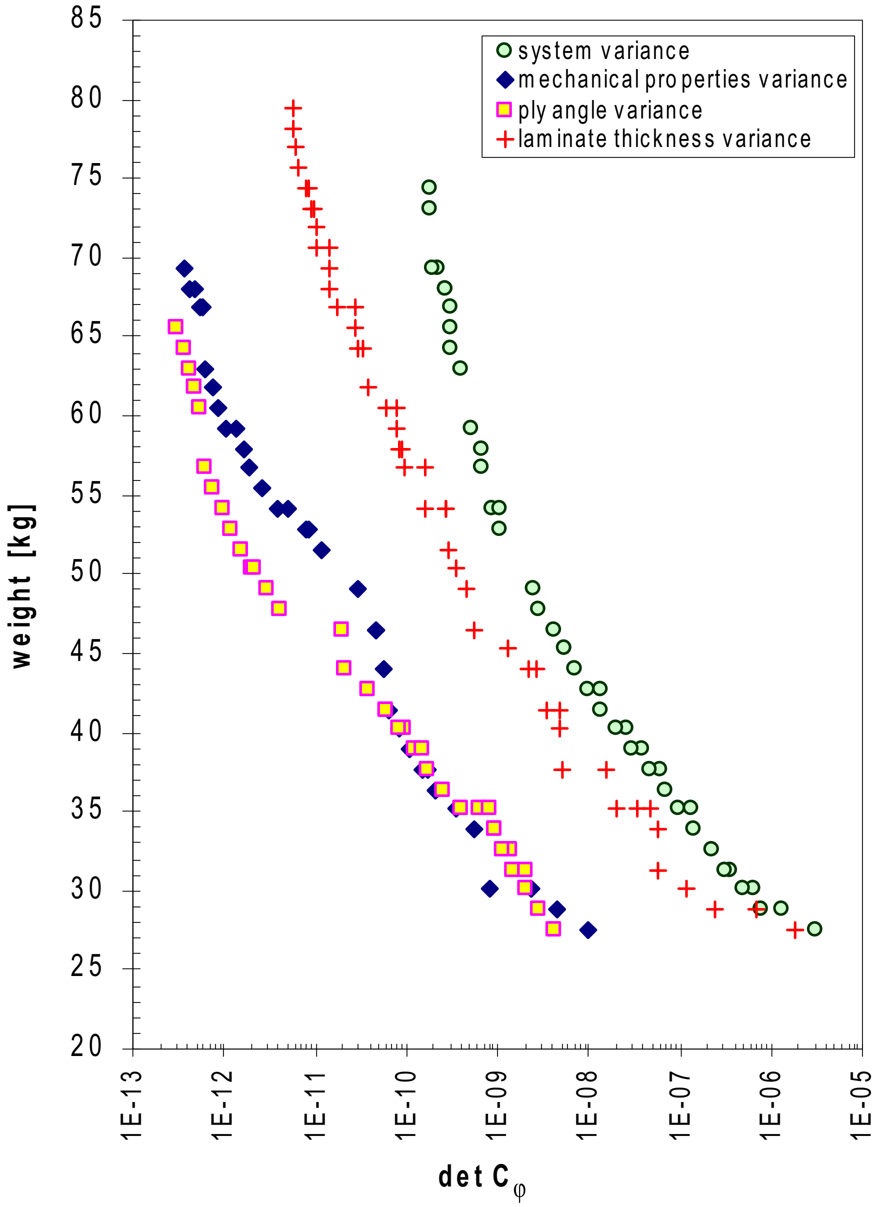


Fig. 12 Comparison of optimal Pareto fronts for different groups of random design variables and/or random parameters used in uncertainty analysis

5 Conclusions

The evaluation of the response uncertainty is done in a simple and systematic way using the variance–covariance matrix of structural response of composite shell structures. Uncertainties in different groups of random design variables and/or random parameters show distinct behaviours and importance on structural response during RDO search of composite structures. RDO searches for minimum weight (performance) and safe structural systems with minimal variability in the response defined as feasibility robustness, when subjected to uncertainties at the input design variables and/or input parameters. The multi-objective optimization search is based on the proposed Co-Dominance-based Genetic Algorithm (CoDGA), which uses two levels of dominance concepts and two populations with exchange of data. At the end of the optimization process the Pareto front representing the frontier of the trade-off between the "performance" and the "feasibility robustness" functions is obtained. The most important innovative aspects of the CoDGA supported by the proposed approach are:

- Fitness assessment based on dominance concepts;
- Fitness definition depending on zero order information (weight objective) and first-order information (feasibility robustness objective by determinant of variance–covariance matrix);
- Co-evolution of a short population with overlapping of elitism and of local dominance together an enlarged population structured according global dominance;
- The enlarged population only receives rank 1 individuals/solutions (non-dominated) from short population and only dominance concepts are applied in this enlarged population;
- Continuous updating of enlarged population based on only dominance concepts;
- One crossover operator linking evolution of both populations with selective mating selection of parents considering the dominance (rank < 3) at enlarged population (MSM). Furthermore, the elite group survives into next generation (OSM);
- The Pareto front is built inside the enlarged population during the evolutionary process. Here the updating process is only controlled by dominance concept.

The numerical tests with different sources of uncertainty show that the Pareto front definition depends on random design variables and/or random parameters considered in the uncertainty analysis for the same design variables used in RDO procedure. The synergetic effects are important as is shown when the results of first case study considering all uncertainty sources are compared with the other cases of partial uncertainty contribution. The combination of uncertainty sources is very important for design rules established from optimal Pareto front. In particular, for a fixed weight/cost the best minimum system variability can increases in several orders of magnitude when combining the uncertainty sources. The uncertainty on laminate thickness plays the most important role on the structural shell composite structures. The relationship between uncertainty in laminate thickness and the dimensional stability of composites showed the importance to control the uncertainty influence on the feasibility of constraints. Finally, the analysis shows that the proposed CoDGA approach is a powerfully tool to help designers to make decision establishing the priorities between performance and robustness.

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