Ján Lovíšek

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OPTIMAL DESIGN OF CYLINDRICAL SHELL WITH A RIGID OBSTACLE

JÁN LOVÍŠEK

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Summary. The aim of the present paper is to study problems of optimal design in mechanics, whose variational form are inequalities expressing the principle of virtual power in its inequality form. We consider an optimal control problem in which the state of the system (involving an elliptic, linear symmetric operator, the coefficients of which are chosen as the design — control variables) is defined as the (unique) solution of stationary variational inequalities. The existence result proved in Section 1 is applied in Section 2 to the optimal design of an elastic cylindrical shell subject to unilateral constraints. We assume that the bending of the shell is limited by a rigid obstacle. The role of the design variable is played by the thickness of the shell.

Keywords: Optimal control, variational inequality, convex set, cylindrical shell, thickness-function, obstacle.

AMS Classification: 49A29, 49A27, 49A34.

1. EXISTENCE AND UNIQUENESS OF SOLUTION

Let $V(\Omega)$ be a real Hilbert space and $V^*(\Omega)$ its dual space, the pairing between $V(\Omega)$ and $V^*(\Omega)$ being denoted by $\langle\cdot,\cdot\rangle_{V(\Omega)}$. Next, $H(\Omega)$ is a separable real Hilbert space such that $V(\Omega)$ is dense in $H(\Omega)$ and the injection of $V(\Omega)$ is completely continuous. Let $U(\Omega)$ be a Hilbert space of controls, $U_{ad}(\Omega) \subset U(\Omega)$ a set of admissible controls ($U_{ad}(\Omega)$ is compact in $U(\Omega)$). By $L(V(\Omega), H(\Omega))$ we denote the family of bounded linear operators from $V(\Omega)$ to $H(\Omega)$.

Let $A(e)$: $V(\Omega) \to V^*(\Omega)$ for every $e \in U_{ad}(\Omega)$ be a family of linear symmetric operators $\{A(e)\}$ with the following properties:

(HO) 1^0 For any $e \in U_{ad}(\Omega)$ the operator $A(e) \in L(V(\Omega), V^*(\Omega))$ ($\{A(e)\}$ is uniformly bounded, i.e.

$$\|e\|_{U(\Omega)} \leqq c_1\,, \quad \|v\|_{V(\Omega)} \leqq c_2 \Rightarrow \|A(e)\,v\|_{V^*(\Omega)} \leqq c(c_2))\,.$$

2° For any $e \in U_{ad}(\Omega)$, the operator $A(e)$ satisfies the uniform coercivity condition:

$$\langle A(e)\,v, v\rangle_{V(\Omega)} \geqq \alpha\|v\|^2_{V(\Omega)}\,(\alpha > 0) \quad \text{for all} \quad v \in V(\Omega)\,,$$

$e \in U_{ad}(\Omega)$ where α is independent of e.

3° For every $v \in V(\Omega)$ the operator $A(\cdot)\, v$: $U_{ad}(\Omega) \to V^*(\Omega)$ is weakly–strongly continuous: $e_n \to e_0$ in $U(\Omega)$ (weakly) for $n \to \infty \Rightarrow A(e_n)\, v \to A(e_0)\, v$ (strongly) in $V^*(\Omega)$.

Suppose that $\mathfrak{K}(e, \Omega)$ is a closed, convex, nonempty subset of $V(\Omega)$.

Next, let $j(\cdot)$: $V(\Omega) \to \bar{R}$ be a proper, convex and Lipschitz continuous fuctional on $V(\Omega)$ with $D(j) = \{v \in \mathfrak{K}(e, \Omega): j(v) < +\infty$ for any $e \in U_{ad}(\Omega)\} \neq \emptyset$. We have a sequence $\{\mathfrak{K}(e_n, \Omega)\}_n$, $e_n \in U_{ad}(\Omega)$, of convex subsets $\mathfrak{K}(e_n, \Omega) \subset V(\Omega)$, which converges to $\mathfrak{K}(e, \Omega)$ in the sense of Mosco, i.e.:

(H 1) 1° any $v \in \mathfrak{K}(e, \Omega)$ is the strong limit of a sequence $\{v_n\}_n$ such that $v_n \in \mathfrak{K}(e_n, \Omega)$ for every $n \in N$;

2° for all $v_{n_k} \in \mathfrak{K}(e_{n_k}, \Omega)$ ($\{\mathfrak{K}(e_{n_k}, \Omega)\}_{n_k}$ being a subsequence of $\{\mathfrak{K}(e_n, \Omega)\}_n$) satisfying $v_{n_k} \to v$ (weakly), it follows that $v \in \mathfrak{K}(e, \Omega)$ for $e_n \to e$ (strongly) in $U(\Omega)$ ($e_n \in U_{ad}(\Omega)$).

Let $\mathfrak{L}$: $H(\Omega) \to R$, $\mathfrak{P}$: $U(\Omega) \to \bar{R}$ and $\mathfrak{L}^\wedge$: $U_{ad}(\Omega) \times V(\Omega) \to R$ be given functions satisfying the following conditions – assumptions:

(E 0) 1° $\mathfrak{L}(u)$ is locally Lipschitz and non-negative on $H(\Omega)$;

2° $\mathfrak{P}(e) = \begin{cases} 0 & \text{if } e \in U_{ad}(\Omega)\,, \\ +\infty & \text{otherwise}\,; \end{cases}$

3° $\mathfrak{L}^\wedge(e, u(e))$ is lower semicontinuous in $U_{ad}(\Omega)$ – weak $\times$ $V(\Omega)$ – (i.e., for weak sequential topology of $U(\Omega)$ and strong topology of $V(\Omega)$), and $\mathfrak{L}^\wedge(e, \cdot)$ is continuous in $V(\Omega)$.

Let an operator $B \in L(U(\Omega), V^*(\Omega))$ be given such that

(E 1) B is completely continuous from $U_{ad}(\Omega)$ to $V^*(\Omega)$. (Hypothesis (E 1) is satisfied in particular if the injection of $V(\Omega)$ into $H(\Omega)$ is completely continuous and $B \in L(U(\Omega), H(\Omega))$.)

SETTING THE PROBLEM ($\mathfrak{B}$).

Minimize the function

$$(\mathfrak{B}) \quad \begin{cases} \mathfrak{L}(u(e)) + \mathfrak{P}(e) \\ \text{or} \\ \mathfrak{L}^\wedge(e, u(e)) \end{cases}$$

over all $u(e) \in \mathfrak{K}(e, \Omega)$ and $e \in U_{ad}(\Omega)$ subject to the state system

$$(1.1) \qquad \langle A(e)\, u(e), v - u(e)\rangle_{V(\Omega)} + j(v) - j(u(e)) \geqq \langle f + Be, v - u(e)\rangle_{V(\Omega)} \quad \text{for all} \quad v \in \mathfrak{K}(e, \Omega)\,.$$

The parameter $e \in U_{ad}(\Omega)$ is called a control, and the corresponding solution $u(e)$ is called the state of the system (1.1). For every $f \in V^*(\Omega)$ and for every $e \in U_{ad}(\Omega)$ the variational inequality (1.1) has a unique solution (see [3]).

A pair $[e_0, u(e_0)] \in U_{ad}(\Omega) \times V(\Omega)$ for which the infimum in problem ($\mathfrak{B}$) is attained is called the optimal pair, and the corresponding control e_0 is called the optimal control.

Theorem 1. *Under assumptions* (H 0), (H 1) *and* (E 0), (E 1), *problem* ($\mathfrak{B}$) *has at least one optimal pair.*

Proof. Let $\{e_n\}_n \subset U_{ad}(\Omega)$ be weakly convergent in $U(\Omega)$ to e_0. By assumption (E 1) it follows that

$$Be_n \to Be_0 \quad \text{(strongly) in} \quad V^*(\Omega)\,. \tag{1.2}$$

We set $u_n = u(e_n) \in \mathfrak{K}(e_n, \Omega)$, $n = 1, 2, \dots$ and we can write

$$\langle A(e_n)(u_n - v), u_n - v\rangle_{V(\Omega)} \leqq j(v) - j(u_n) -$$
$$- \langle f + Be_n, v - u_n\rangle_{V(\Omega)} - \langle A(e_n) v, u_n - v\rangle_{V(\Omega)}$$

for any $v \in \mathfrak{K}(e_n, \Omega)$.

Then by (**Proposition** 1.7 ([3])) we have int $D(j) \subset D(\partial j)$ and if

$$\bigcap_{e \in U_{ad}(\Omega)} \mathfrak{K}(e, \Omega) \cap \text{int } D(j) \neq \emptyset \quad (\text{int } C \equiv \text{interior of } C) \tag{1.3}$$

then there exists an element $v_0 \in \bigcap_{e \in U_{ad}(\Omega)} \mathfrak{K}(e, \Omega) \cap \text{int } D(j)$ such that

$$j(v_0) - j(w) \leqq \langle p, v_0 - w\rangle_{V(\Omega)} \quad \text{where} \quad p \in \partial j(v_0)\,, \quad w \in \mathfrak{K}(e_n, \Omega)\,.$$

This means that the function $\theta(w) = (j(v_0) - j(w))/\|v_0 - w\|_{V(\Omega)}$ is bounded (θ: $\mathfrak{K}(e_n, \Omega) \to R$). Then by assumption ((H 0), 2) we get $\alpha\|u_n - v_0\|_{V(\Omega)} \leqq \theta(u_n) + (\|f\|_{V^*(\Omega)} + \|Be_n\|_{V^*(\Omega)}) + |\langle A(e_n) v_0; u_n - v_0\rangle_{V(\Omega)}|/\|u_n - v_0\|_{V(\Omega)}$.

Thus we have $\|u_n\|_{V(\Omega)} \leqq C$ (using the assumption ((H 0), 1°)). We can extract a subsequence $\{u_{n_k}\}_{n_k} (\subset \{u_n\}_n)$ such that

$$u_{n_k} \rightharpoonup u \quad \text{(weakly) in} \quad V(\Omega)\,. \tag{1.4}$$

Since $u_{n_k} \in \mathfrak{K}(e_{n_k}, \Omega)$ by assumption ((H 1), 2°), we have $u \in \mathfrak{K}(e_0, \Omega)$ as well. For any $w \in V(\Omega)$ we have by assumption ((H 0), 3°)

$$\lim_{n_k \to \infty} \langle A(e_{n_k}) u(e_{n_k}), w\rangle_{V(\Omega)} = \lim_{n_k \to \infty} \langle A(e_{n_k}) w, u_{n_k}\rangle_{V(\Omega)} =$$
$$\langle A(e_0) w, u\rangle_{V(\Omega)} = \langle A(e_0) u, w\rangle_{V(\Omega)}$$

and therefore

$$A(e_{n_k}) u(e_{n_k}) \rightharpoonup A(e_0) u \quad \text{(weakly) in} \quad V^*(\Omega) \tag{1.5}$$

if $u_{n_k} \rightharpoonup u$ is weakly convergent in $V(\Omega)$. Further, in virtue of the monotonicity of $A(e_n)$ (by assumption ((H 0), 2°) we can write

$$\langle A(e_{n_k}) u_{n_k}, u_{n_k} - u\rangle_{V(\Omega)} \geqq \langle A(e_{n_k}) u, u_{n_k} - u\rangle_{V(\Omega)} \quad n_k = 1, 2, \dots\,.$$

Hence we have (by passing to the limit)

$$\lim_{n_k \to \infty} 2\langle A(e_{n_k}) u, u_{n_k}\rangle_{V(\Omega)} \leqq \liminf_{n_k \to \infty} \langle A(e_{n_k}) u_{n_k}, u_{n_k}\rangle_{V(\Omega)} + \lim_{n_k \to \infty} \langle A(e_{n_k}) u, u\rangle_{V(\Omega)}\,.$$

This yields (by (1.4) and (H 0), 3°))

(1.6) $$\liminf_{n_k\to\infty} \langle A(e_{n_k})\, u_{n_k}, u_{n_k}\rangle_{V(\Omega)} \geqq \langle A(e_0)\, u, u\rangle_{V(\Omega)}\,.$$

Let $v \in \mathfrak{K}(e_0, \Omega)$ be an arbitrary element and $\{v_{n_k}\}_{n_k}$ such a sequence that

(1.7) $$v_{n_k} \to v \quad \text{(strongly) in} \quad V(\Omega),\, v_{n_k} \in \mathfrak{K}(e_{n_k}, \Omega)\,, \quad n_k = 1, 2, \ldots$$

(The existence of $\{v_{n_k}\}_{n_k}$ is ensured by ((H 1), 1°).) Next, we can write

$$\langle A(e_{n_k})\, u_{n_k}, v_{n_k} - u_{n_k}\rangle_{V(\Omega)} + j(v_{n_k}) - j(u_{n_k}) \geqq \langle (f + Be_{n_k}), v_{n_k} - u_{n_k}\rangle_{V(\Omega)}$$
$$\text{for any} \quad v_{n_k} \in \mathfrak{K}(e_{n_k}, \Omega)\,, \quad e_{n_k} \in U_{ad}(\Omega)\,.$$

This yields

$$\langle A(e_{n_k})\, u_{n_k}, u_{n_k}\rangle_{V(\Omega)} - j(v_{n_k}) \leqq \langle A(e_{n_k})\, u_{n_k}, v_{n_k}\rangle_{V(\Omega)} -$$
$$- \langle (f + Be_{n_k}), v_{n_k} - u_{n_k}\rangle_{V(\Omega)} - j(u_{n_k})\,,$$

and we get

$$\limsup_{n_k\to\infty} \langle A(e_{n_k})\, u_{n_k}, u_{n_k}\rangle_{V(\Omega)} - \lim_{n_k\to\infty} j(v_{n_k}) \leqq$$
$$\leqq \lim_{n_k\to\infty} (\langle A(e_{n_k})\, u_{n_k}, v_{n_k}\rangle_{V(\Omega)} - \langle (f + Be_{n_k}), v_{n_k} - u_{n_k}\rangle_{V(\Omega)}) - \liminf_{n_k\to\infty} j(u_{n_k})\,.$$

Hence by (1.4), (1.5) and in virtue of the continuity of $j(v)$ on $V(\Omega)$ one has the following relations. (Since proper convex functional lower semicontinuity in the strong topology is the same as sequencial lower semicontinuity in the weak topology.)

(1.8) $$\limsup_{n_k\to\infty} \langle A(e_{n_k})\, u_{n_k}, u_{n_k}\rangle_{V(\Omega)} - j(v) \leqq \langle A(e_0)\, u, v\rangle_{V(\Omega)} -$$
$$- \langle f + Be_0, v - u\rangle_{V(\Omega)} - j(u) \quad \text{for any} \quad v \in \mathfrak{K}(e_0, \Omega)\,,$$

and therefore (we take $v = u$ in (1.8))

(1.9) $$\limsup_{n_k\to\infty} \langle A(e_{n_k})\, u_{n_k}, u_{n_k}\rangle_{V(\Omega)} \leqq \langle A(e_0)\, u, u\rangle_{V(\Omega)}\,.$$

This means (by (1.6) and (1.9)) that

(1.10) $$\lim_{n_k\to\infty} \langle A(e_{n_k})\, u_{n_k}, u_{n_k}\rangle_{V(\Omega)} = \langle A(e_0)\, u, u\rangle_{V(\Omega)}\,.$$

From (1.4), (1.5), (1.7) and (1.10), (E 1) we obtain

$$\langle A(e_0)\, u, u - v\rangle_{V(\Omega)} + j(u) \leqq \lim_{n_k\to\infty} [\langle A(e_{n_k})\, u_{n_k}, u_{n_k} - v_{n_k}\rangle_{(\Omega)} + j(u_{n_k})] \leqq$$
$$\leqq \lim_{n_k\to\infty} [\langle f + Be_{n_k}, u_{n_k} - v_{n_k}\rangle_{V(\Omega)} + j(v_{n_k})] = \langle f + Be_0, u - v\rangle_{V(\Omega)} + j(v)$$

for any $v \in \mathfrak{K}(e_0, \Omega)$ (by continuity of $j(v)$ on $V(\Omega)$). In other words, we have proved that $u = u(e_0)$, $u(e_n) \rightharpoonup u(e_0)$ (weakly) in $V(\Omega)$. Next, by virtue of ((H 0), 2°) we can write

$$\alpha \lim_{n_k \to \infty} \|u(e_{n_k}) - u(e_0)\|^2_{V(\Omega)} \leqq$$

$$\leqq \lim_{n_k \to \infty} \langle A(e_{n_k}) (u(e_{n_k}) - u(e_0)), u(e_{n_k}) - u(e_0) \rangle_{V(\Omega)} =$$

$$= \lim_{n_k \to \infty} \{ \langle A(e_{n_k}) u(e_{n_k}), u(e_{n_k}) \rangle_{V(\Omega)} + \langle A(e_{n_k}) u(e_0), u(e_0) \rangle_{V(\Omega)} -$$

$$- \langle A(e_{n_k}) u(e_{n_k}), u(e_0) \rangle_{V(\Omega)} - \langle A(e_{n_k}) u(e_0), u(e_{n_k}) \rangle_{V(\Omega)} \} = 0$$

(using (1.4), (1.5), ((H 0), 3°) and (1.10)). This means that

$$\lim_{n_k \to \infty} u(e_{n_k}) = u(e_0)$$

(in the strong topology of $V(\Omega)$). Thus we have shown that the map $e \to u(e)$ is weakly – strongly continuous from $U_{\text{ad}}(\Omega)$ to $V(\Omega)$. Let $d = \inf \{ \mathfrak{L}(u(e)), e \in U_{\text{ad}}(\Omega) \}$. By the assumption (E 0, 1°) we see that $0 \leqq d < +\infty$. Now let $\{e_n\} \subset U_{\text{ad}}(\Omega)$ be such that $\mathfrak{L}(u(e_n)) \to d$. Since $U_{\text{ad}}(\Omega)$ is compact there exists a subsequence $\{e_{n_k}\}_{n_k} \subset \{e_n\}_n$ such that $e_{n_k} \to e_0$ (strongly) in $U(\Omega)$, and $u(e_{n_k}) \to u(e_0) \equiv u_0$ (strongly) in $V(\Omega)$. Since $\mathfrak{L}$ is continuous on $V(\Omega)$, we have ($\mathfrak{L}$ is locally Lipschitz and non-negative from $V(\Omega)$ into R) $\mathfrak{L}(u(e_0)) = d$.

On the other hand, it follows from ((E 0), 3°) that $\mathfrak{L}^\wedge(e_0, u(e_0)) \leqq \leqq \liminf_{n_k \to \infty} \mathfrak{L}^\wedge(e_{n_k}, u(e_{n_k})) = \inf_{e \in U_{\text{ad}}(\Omega)} \mathfrak{L}^\wedge(e, u(e))$, which completes the proof of Theorem 1.

In other words, e_0 is an optimal control of problem ($\mathfrak{B}$).

2. THE CYLINDRICAL SHELL

The geometry of cylindrical shell

Let R^3 be the usual Euclidean space with a fixed orthonormal system $(0, \boldsymbol{i}_x, \boldsymbol{i}_\varphi, \boldsymbol{i}_z)$, and let Ω be a bounded open subset in a plane R^2 with a boundary $\partial\Omega$. Then the middle surface $\mathfrak{S}$ of a cylindrical shell is the image in R^3 of the set $\bar{\Omega}$ by the mapping $\boldsymbol{\Phi}$:

$$\boldsymbol{\Phi}: (\xi_x, \xi_\varphi) \in \bar{\Omega} \subset R^2 \to \boldsymbol{\Phi}(\xi_x, \xi_\varphi) \in R^3 .$$

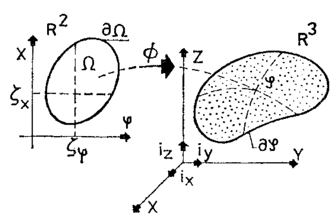

Fig. 1.

We assume that the boundary $\partial\Omega$ and the function $\boldsymbol{\Phi}$ are sufficiently smooth.

A cylindrical shell is an elastic body $\mathfrak{C}$ defined in the space R^3 by

$$\mathfrak{C} = \{M \in R^3 : OM = \Phi(x, \varphi) + z\nu(\varphi), (x, \varphi) \in \Omega - e(x, \varphi)/2 \leqq z \leqq e(x, \varphi)/2\}$$

where $e: \bar{\Omega} \to R^+$ is the thickness of the shell, ν is the normal vector for the middle surface $\mathfrak{S}$, and we assume

$$\Omega = [-H, H] \times [\alpha, \beta],$$

$$\Phi(x, \varphi) = x\boldsymbol{i}_x + a \cos \varphi \boldsymbol{i}_\varphi + a \sin \varphi \boldsymbol{i}_z \quad \text{where} \quad a = \text{const}.$$

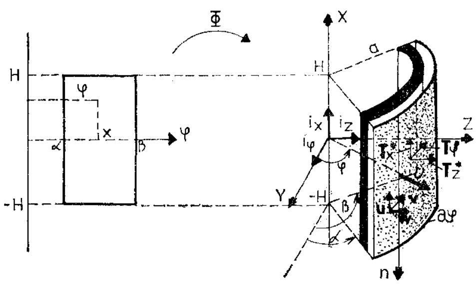

Fig. 2.

In what follows, this geometry of the cylindrical shell $\mathfrak{C}$ is used as a reference configuration. Indeed, we study stationary problems falling into the following category. Let $\mathfrak{C}$ be the shell configuration before deformation. We assume (for simplicity) that the shell is clamped on the boundary and loaded with a distribution of volume and surface forces. These act on the upper and the lower faces. Under the action of these forces the shell deforms to a new configuration $\mathfrak{C}^*$. Then assuming the physical characteristics of the material of the cylindrical shell, the initial configuration $\mathfrak{C}$, the distribution of the forces applied, and the boundary conditions are known, the problem is to determine displacement of the points of $\mathfrak{C}$. From the knowledge of the displacements, we are able to determine the strains and the stresses at any point of $\mathfrak{C}^*$. The Kirchhoff theory is based on complementary hypotheses which permit us to derive an approximation of the displacement field of the particles of the cylindrical shell $\mathfrak{C}$ only from the knowledge of the displacement field $\boldsymbol{u}$ of the particles of the middle surface.

The kinematic homogeneous boundary conditions on $\partial\Omega$ are given by $u = v = w = \partial w/\partial n = 0$ where $\boldsymbol{n}$ is the normal vector to the surface

$$\{\Phi(x, \varphi) + z\,\nu(\varphi): (x, \varphi, z) \in \partial\Omega x[(-e/2), (e/2)]\},$$

$\boldsymbol{u} = [u, v, w]$ is the displacement vector of the points on the shell middle surface. (Thus the three function $u, v, w: (x, \varphi) \in \bar{\Omega} \to u(x, \varphi), v(x, \varphi), w(x, \varphi)$ are the (principal) unknowns of the state system.)

We denote by $L_2(\Omega)$ the space of all measurable square integrable functions with respect to the Lebesgue measure $d\Omega = a\, dx\, d\varphi$.

Let us denote: $C^m(\bar{\Omega})$: the space of m-times continuously differentiable real valued functions whose all derivatives up to order m are continuous in $\bar{\Omega}$;

$$H_0^1(\Omega) = \{v \in H^1(Q) : v = 0 \text{ on } \partial\Omega\} ;$$
$$H_0^2(\Omega) = \{v \in H^2(\Omega) : v = \partial v/\partial n = 0 \text{ on } \partial\Omega\} ;$$
$$H^0(\Omega) = L_2(\Omega) ,$$

where $H^1(\Omega)$, $H^2(\Omega)$ are Sobolev spaces. The space of virtual displacements of the middle surface of the shell is the space

$$V(\Omega) = H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^2(\Omega) .$$

Then, $V(\Omega)$ is a Hilbert space equipped with the scalar product

(2.1)
$$(\mathbf{u}, \mathbf{v})_{V(\Omega)} = (u, p)_{H^1(\Omega)} + (v, q)_{H^1(\Omega)} + (w, \theta)_{H^2(\Omega)} \quad \text{for} \quad \mathbf{u} = [u, v, w] ; \quad \mathbf{v} = [p, q, \theta] ,$$

and the norm

$$\|\mathbf{v}\|_{V(\Omega)} = (\mathbf{v}, \mathbf{v})_{V(\Omega)}^{1/2} , \quad \mathbf{v} \in V(\Omega) .$$

Next, we define

$$U(\Omega) = H^1(\Omega) \cap C^0(\bar{\Omega}) ,$$
$$U_{ad}(\Omega) = \{e \in H^2(\Omega) ; 0 < e_{min} \leqq e(x, y) \leqq e_{max} \text{ for all } (x, y) \in \Omega , \quad \|e\|_{H^2(\Omega)} \leqq c_1 , \quad \int_\Omega e(x, y) \, d\Omega = c_2\}$$

where the constants e_{min}, e_{max}, c_1, c_2 are such that $U_{ad}(\Omega) \neq \emptyset$.

We shall use the linear theory of shells ([11, 12]), and formulate the equilibrium in terms of the displacement vector $\mathbf{u}(e)$.

Let us define the following system of strains:

$$\{\mathcal{N}_i(\mathbf{v})\}_i , \quad i = 1, 2, \ldots, 6$$

where

(2.2)
$$\mathcal{N}_1(\mathbf{v}) = \partial p/\partial x , \quad \mathcal{N}_2(v) = (1/a)(\partial q/\partial\varphi - \theta) ,$$
$$\mathcal{N}_3(\mathbf{v}) = (1/2a \, \partial p/\partial\varphi + \partial q/\partial x) , \quad \mathcal{N}_4(\mathbf{v}) = \partial^2\theta/\partial x^2 ,$$
$$\mathcal{N}_5(\mathbf{v}) = (1/a^2)(\partial^2\theta/\partial\varphi^2 + \theta) ,$$
$$\mathcal{N}_6(\mathbf{v}) = (1/2a)((-1/a) \, \partial p/\partial\varphi + \partial q/\partial x + 2 \, \partial^2\theta/\partial x \, \partial\varphi) .$$

(Thus we have the system of six deformation operators, where $\mathcal{N}_i \in L(V(\Omega), L_2(\Omega))$ are linear continuous operators from $V(\Omega)$ to $L_2(\Omega)$.)

Further, let us define a matrix

$$\mathbf{K}(e) = \begin{bmatrix} S(e) & S(e) & 0 & 0 & 0 & 0 \\ B(e) & S(e) & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\,S(e)(1-\mu) & 0 & 0 & 0 \\ 0 & 0 & 0 & D(e) & D(e)\mu & 0 \\ 0 & 0 & 0 & D(e)\,\mu & D(e) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\,D(e)(1-\mu) \end{bmatrix}_{(6,6)}$$

where

$$S(e) = (Ee)/(1 - \mu^2), \quad D(e) = (Ee^3)/(12(1 - \mu^2)),$$

E is the so called Young modulus of elasticity and μ is Poisson's ratio ($0 \leqq \mu < 1/2$).

We will consider physical situations such as those in Fig. 3 in which the transverse displacement of a thin cylindrical shell is constrained by the presence of a foundation (rigid, frictionless shells) located at a distance b under the middle surface of the shell, for which all admissible transverse displacements satisfy

(2.3) $$\theta(x, \varphi) + \Delta(e) \geqq 0 \quad \text{for any} \quad (x, \varphi) \in \Omega$$

where

$$\Delta(e) = b - (e/2), \quad b = \text{const.}$$

Physically, if $\theta(x, \varphi) > -\Delta(e)$, then the cylindrical shell does not come in contact with the rigid frictionless shells (which are parallel to the original configuration of the middle surface of the deformable shell), and no reactive force is developed on the surface of the rigid shell. On the other hand, if $\theta(x, \varphi) = -\Delta(e)$ at some point $(x, \varphi) \in \Omega$, then the shell is in contact with the rigid shell and a transverse reactive force $p^c(e)$ is developed on the cylindrical shell. Thus,

(2.4) $$p^c(e) = 0 \quad \text{if} \quad \theta(x, \varphi) > -\Delta(e) \quad \text{and} \quad p^c(e) \geqq 0 \quad \text{if} \quad \theta(x, \varphi) = -\Delta(e),$$
$$\text{or} \quad \theta(x, \varphi) + \Delta(e) \geqq 0, \quad p^c(e) \geqq 0 \quad p^c(e)(\theta + \Delta(e)) = 0 \quad \text{in} \quad \Omega.$$

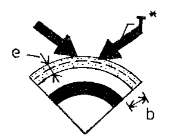

Fig. 3.

The last condition in (2.4) is a version of the complementarity condition of mathematical programming in which the reactive force $p^c(e)$ is interpreted as a Lagrange multiplier associated with the constraint (2.3).

The governing linear, symmetric operator $\mathscr{R}(e)$ has the form

$$\mathscr{R}(e) = \begin{bmatrix} L_{11}(e) & L_{12}(e) & L_{13}(e) \\ L_{21}(e) & L_{22}(e) & L_{23}(e) \\ L_{31}(e) & L_{32}(e) & L_{33}(e) \end{bmatrix}_{(3,3)}$$

where

$$L_{11}(e) = -\partial\{(Ee)/(1 - \mu^2)[(1/a^4)(1 + (e^2/3a^2))\,\partial/\partial x]\}/\partial x - \\ - \partial\{(Ee)/(1 + \mu)[(1/2a^2)(1 + (e^2/3a^2))\,\partial/\partial\varphi]\}/\partial\varphi,$$
$$L_{12}(e) = -\partial\{(Ee)/(1 - \mu^2)[(\mu/a^2)\,\partial/\partial x]\}/\partial\varphi - \\ - \partial\{(Ee)/(1 + \mu)[(1/2a^2)\,\partial/\partial\varphi]\}/\partial x \equiv L_{21}(e),$$

$$L_{13}(e) = - \partial\{(Ee)/(1-\mu^2)[(1/a^3)(1+(e^2/6a^2)) - (e^2/6a^5)\,\partial^2/\partial\varphi^2 - (\mu e^2)/(6a^3)\,\partial^2/\partial x^2]\}/\partial\varphi - \partial\{(Ee)/(1+\mu)[-e^2/(6a^3)\,\partial^2/\partial x\,\partial\varphi]\}/\partial x \equiv L_{31}(e)\,,$$

$$L_{22}(e) = -\partial\{(Ee)/(1+\mu)(1/2a^2)(-\partial/\partial\varphi)\}/\partial\varphi - \partial\{(Ee)(1-\mu^2)(\partial/\partial x)\}/\partial x\,,$$

$$L_{23}(e) = - \partial\{(Ee)(1-\mu^2)(\mu/a)\}/\partial x = L_{32}(e)\,,$$

$$L_{33}(e) - (Ee)/(1-\mu^2)[(1/a^2)(1+(e^2/12a^2)) - (e^2/12a^4)\,\partial^2/\partial\varphi^2 - (\mu e^2)/(12a^2)\,\partial^2/\partial x^2] + \partial^2\{(Ee^2)/(12a^5(1-\mu^2)) \times [-a + a\,\partial^2/\partial\varphi^2 + \mu a^3\,\partial^2/\partial x^2]\}/\partial\varphi^2 - \partial^2\{(Ee^3)/((6a^3)(1-\mu)) \times a\,\partial^2/\partial x\,\partial\varphi]\}/\partial x\,\partial\varphi + \partial^2\{(Ee^2)/((12a^3)(1-\mu^2)) \times [-\mu a + \mu a\,\partial^2/\partial\varphi^2 + a^3\,\partial^2/\partial x^2]\}/\partial x^2\,.$$

Then we can write

(2.5) $$\mathcal{R}_x(e)\,\mathbf{u} = L_{11}(e)\,u + L_{12}(e)\,v + L_{13}(e)\,w\,,$$
$$\mathcal{R}_\varphi(e)\,\mathbf{u} = L_{12}(e)\,u + L_{22}(e)\,v + L_{23}(e)\,w\,,$$
$$\mathcal{R}_z(e)\,\mathbf{u} = L_{13}(e)\,u + L_{23}(e)\,v + L_{33}(e)\,w\,,$$

(at least formally).

We will concentrate on the following model of state problem:

Unilateral problem for a clamped cylindrical shell

Given $[T_x^*, T_\varphi^*, T_z^*]$, $S(e)$, $D(e)$, μ, $\Delta(e)$ and the operator $B(H_0^2(\Omega) \to L_2(\Omega))$ (where T_x^*, T_φ^*, T_z^* are the applied distributed loads per unit area of the middle surface of the cylindrical shell, see Fig. 2), define

(2.6) $$[0, 0, p^c] \equiv [\mathcal{R}_x(e)\,\mathbf{u}(e), \mathcal{R}_\varphi(e)\,\mathbf{u}(e)\ \mathcal{R}_z(e)\,\mathbf{u}(e)] - [T_x^*, T_\varphi^*, (T_z^* + Be)]\,.$$

Find $\mathbf{u}(e)$ for any $e \in U_{ad}(\Omega)$ such that

(2.6$_1$) $$p^c((x,\varphi), e) \geqq 0\,, \quad w((x,\varphi), e) \geqq -\Delta(e)\,,$$
$$p^c(w((x,\varphi), e) + \Delta(e)) = 0 \quad \text{in} \quad \Omega\,,$$
$$u = v = 0\,, \quad w = \partial w/\partial n = 0 \quad \text{on} \quad \partial\Omega\,,$$

This system describes the deformation of a clamped cylindrical shell under the load $[T_x^*, T_\varphi^*, (T_z^* + Be)]$, the transverse displacement w being constrained by the presence of a rigid frictionless surface located at a distance $\Delta(e)$ under the lower surface of the shell, and $p^c(e)$ is the contact pressure.

Further, we introduce the set of kinematically admissible displacements by

(2.7) $$\mathfrak{K}(e, \Omega) = \{\mathbf{v} = [p, q, \theta] \in V(\Omega) : \theta + \Delta(e) \geqq 0 \text{ on } \Omega\}$$

where $\Delta(e) > 0$ for any $e \in U_{ad}(\Omega)\ (\Leftrightarrow)\ e_{max} \leqq 2b$).

Lemma 1. *The set $\mathfrak{K}(e, \Omega)$ is non-empty, convex and closed in $V(\Omega)$ for fixed $e \in U_{ad}(\Omega)$.*

Proof. The condition (2.3) ensures that the set $\mathfrak{K}(e, \Omega)$ is non-empty for every $e \in U_{ad}(\Omega)$ (the element $[p, q, 0] \in \mathfrak{K}(e, \Omega)$). The convexity of $\mathfrak{K}(e, \Omega)$ can be seen directly from definition. Let us now consider such a sequence $\{\mathbf{v}_n\}_n$ ($\mathbf{v}_n \in \mathfrak{K}(e, \Omega)$, $n = 1, 2, \ldots$) that $\mathbf{v}_n \to \mathbf{v}$ (strongly) in $V(\Omega)$. Then, if $\mathbf{v} = [p, q, \theta]$, $\mathbf{v}_n = [p_n, q_n, \theta_n]$, we have $\theta_n \to \theta$ (strongly) in $H_0^2(\Omega)$.

Next, due to the imbedding theorem for the space $H_0^2(\Omega)$ ([1]) we get $\lim_{n\to\infty} \theta_n(x, \varphi) = \theta(x, \varphi)$ for every point $(x, \varphi) \in \Omega$. Thus, as $\theta_n(x, \varphi) + \Delta(e) \geqq 0$ for all $(x, \varphi) \in \Omega$, we obtain $\theta(x, \varphi) + \Delta(e) \geqq 0$ in Ω and hence $\mathbf{v} \in \mathfrak{K}(e, \Omega)$ as claimed.

Lemma 2. *The system $\{\mathfrak{K}(e, \Omega)\}$ defined by (2.7) fulfils the conditions ((H1), 1°, 2°).*

Proof. Indeed, if $\lim_{n\to\infty} e_n = e_0$ in $U(\Omega)$ $(= H^1(\Omega) \cap C(\Omega))$, $e_n \in U_{ad}(\Omega)$, then there exists a subsequence $\{e_{n_k}\}_{n_k}$ $(\subset \{e_n\}_n)$ weakly convergent in $H^2(\Omega)$ to the element $e_0 \in U_{ad}(\Omega)$. Let $\{h_n, q_n, \theta_n\} \to \{p, q, \theta\}$ ($[h_n, q_n, \theta_n] \in \mathfrak{K}(e_n, \Omega)$, $[p, q, \theta] \in V(\Omega)$) be weakly convergent in $V(\Omega)$. Then we have $\theta_n(x, \varphi) + \Delta(e_n) \geqq 0$ for all $(x, \varphi) \in \Omega$, which implies, with respect to the compact imbedding $H^2(\Omega) \hookrightarrow C(\bar{\Omega})$,

$$\theta + \Delta(e_0) \geqq 0 \quad \text{for all} \quad (x, \varphi) \in \Omega$$

and hence $[p, q, \theta] \in \mathfrak{K}(e_0, \Omega)$ ((H 1), 2°). If $[p, q, \theta] \in \mathfrak{K}(e_0, \Omega)$, then we put $[p_n, q_n, \theta_n] = [p, q, \theta] + [0, 0, (e_n - e_0)/2]$. The elements $[p_n, q_n, \theta_n]$ satisfy the conditions $[p_n, q_n, \theta_n] \in \mathfrak{K}(e_n, \Omega)$, $\lim_{n\to\infty} [p_n, q_n, \theta_n] = [p, q, \theta]$ (strongly) in $V(\Omega)$. Hence the condition ((H 1, 1°) holds. The subspace $R(\Omega) \subset V(\Omega)$ is the set of rigid body motions (representing virtual displacements of a rigid shell) given by $R(\Omega) = \{\mathbf{v} \in V(\Omega): P_{\mathbf{v}}: V(\Omega) \to R,\ P_{\mathbf{v}}\mathbf{v} = \mathcal{N}_1^2(\mathbf{v}) + \mathcal{N}_2^2(\mathbf{v}) + \mathcal{N}_3^2(\mathbf{v}) + \mathcal{N}_4^2(\mathbf{v}) + \mathcal{N}_5^2(\mathbf{v}) + \mathcal{N}_6^2(\mathbf{v}) = 0\}$.

Lemma 3. *Let $\mathbf{v} \in V(\Omega)$ and $P_{\mathbf{v}}\, \mathbf{v} = 0$. Then we have $R(\Omega) = \{\mathbf{0}\}$.*

Proof (Lemma 4.1 in [12]). On the open set Ω we now define a bilinear form

$$a(e, \mathbf{u}, \mathbf{v}) = \int_\Omega \mathcal{N}(\mathbf{u})\, \mathbf{K}(e)\, \mathcal{N}^{\mathrm{T}}(\mathbf{v})\, a \,\mathrm{d}x\, \mathrm{d}\varphi \quad \text{for any} \quad \mathbf{u}, \mathbf{v} \in V(\Omega) \quad \text{and} \quad e \in U_{ad}(\Omega)\,, \tag{2.8}$$

and a linear functional (the work of the external loads associated with a displacement $\mathbf{v} = [p, q, \theta]$):

$$\langle \mathbf{L}(e), \mathbf{v}\rangle_{V(\Omega)} = \int_\Omega [T_x^*(p - (e/2a)\, \partial\theta/\partial x) + T_\varphi^*(q - (e/2a)\, \partial\theta/\partial\varphi) + T_z^*\theta]\, a \,\mathrm{d}x\, \mathrm{d}\varphi + \langle Be, \theta\rangle_{H^2(\Omega)},\ \mathbf{v} \in V(\Omega),\ e \in U_{ad}(\Omega) \tag{2.9}$$

$$(\text{for } T_x^*, T_\varphi^*, T_z^* \in L_2(\Omega))\,,$$

$$\mathcal{N}(\mathbf{v}) = [\mathcal{N}_1(\mathbf{v}), \mathcal{N}_2(\mathbf{v}), \mathcal{N}_3(\mathbf{v}), \mathcal{N}_4(\mathbf{v}), \mathcal{N}_5(\mathbf{v}), \mathcal{N}_6(\mathbf{v})]\,,$$

$$B\colon H^2(\Omega) \to L_2(\Omega)\,, \quad K_{ij} \in C([e_{\min}, e_{\max}])\,, \quad i, j = 1, 2, \ldots, 6\,,$$
$$K_{ij} = K_{ji}\,.$$

(If we define $\langle Be, \theta\rangle_{H^2(\Omega)} = (-ke \sin \varphi, \theta)_{L_2(\Omega)}$, $k = \text{const.} > 0$, the operator $B\colon H^2(\Omega) \to L_2(\Omega)$ is continuous. It corresponds to the loading caused e.g. by the own weight of the shell.)

Next, we define a family of linear operators $\{A(e)\}$ by the equation

$$a(e, \mathbf{u}, \mathbf{v}) = \langle A(e)\, \mathbf{u}, \mathbf{v}\rangle_{V(\Omega)}\,. \tag{2.10}$$

($a(e, \mathbf{u}, \mathbf{v})\colon V(\Omega) \times V(\Omega) \to R$ is the Dirichlet form associated with $A(e)$ for any $e \in U_{\text{ad}}(\Omega)$), $A(e)\, u \in V^*(\Omega)$ and $A(e) \in L(V(\Omega), V^*(\Omega))$ is the canonical isometric operator (by $a(e, \mathbf{u}, \mathbf{v})$). If we define $a(e, \mathbf{u}, \mathbf{v})$ and $\langle \mathbf{L}(e), \mathbf{v}\rangle_{V(\Omega)}$ by the formulas (2.8), (2.9), (2.10) and $\mathfrak{K}(e, \Omega)$ by (2.7), then the state problem (1.1) corresponds to a unilateral problem for a shell the edge of which is clamped, under a load $\mathbf{T}^*([T_x^*, T_\varphi^*, T_z^*])$.

For $\mathbf{L}(e) \in V^*(\Omega)$ consider the following problem:

Find $\mathbf{u}(e) \in \mathfrak{K}(e, \Omega)$ such that

$$\langle A(e)\, \mathbf{u}(e), \mathbf{v} - \mathbf{u}(e)\rangle_{V(\Omega)} \geqq \langle \mathbf{L}(e), \mathbf{v} - \mathbf{u}(e)\rangle_{V(\Omega)}$$

for any $\mathbf{v} \in \mathfrak{K}(e, \Omega)$, $e \in U_{\text{ad}}(\Omega)$. (This is an abstract elliptic variational inequality associated with the symmetric bilinear form $a(e, \cdot, \cdot,)$.)

Theorem 2. *Let $\mathbf{u}(e)$ be a solution of ((2.6), (2.6$_1$)). Then*

$$\begin{aligned} &\langle A(e)\, \mathbf{u}(e), \mathbf{v} - \mathbf{u}(e)\rangle_{V(\Omega)} \geqq \qquad && \textit{for any } \mathbf{v} \in \mathfrak{K}(e, \Omega)\,, \\ &\geqq \langle \mathbf{L}(e), \mathbf{v} - \mathbf{u}(e)\rangle_{V(\Omega)} && e \in U_{\text{ad}}(\Omega)\,. \end{aligned} \tag{2.11}$$

Conversely, let $\mathbf{u}(e) \in \mathfrak{K}(e, \Omega)$ be a solution of (2.11) for $\mathbf{L}(e) \in [L_2(\Omega)]^3$. Then

(2.12)1° $$[0, 0, p^c(e)] = [(\mathcal{R}_x(e)\, \mathbf{u}(e) - T_x^*), (\mathcal{R}_\varphi(e)\, \mathbf{u}(e) - T_\varphi^*), (\mathcal{R}_z(e)\, \mathbf{u}(e) - (T_z^* + Be))]$$

in the distributional sense, where $\mathcal{R}_z(e)\, \mathbf{u}(e) - (T_z^* + Be) \geqq 0$;

2° *the following weak form of the complementarity conditions holds:*

$$\langle A(e)\, \mathbf{u}(e) - \mathbf{L}(e), \mathbf{u}(e) + [0, 0, \Delta(e)]\rangle_{V(\Omega)} = 0\,.$$

Moreover, if $\mathbf{u}(e) \in (H^2(\Omega) \times H^2(\Omega) \times H^4(\Omega)) \cap V(\Omega)$ *then* $p^c(e) \in L_2(\Omega)$ *and* $p^c(e)\,(w(e) + \Delta(e)) = 0$.

Remark 1. According to the Sobolev imbedding theorem ([1]) we have $H^2(\Omega) \subset\subset C^0(\bar{\Omega})$ (this imbedding is compact). Therefore, if $\theta \in H^2(\Omega)$, the condition $\theta \geqq \Delta(e)$ can be imposed pointwise in Ω.

Remark 2. The assertion "$p^c(e) \geqq 0$ in the distributional sense" has the following interpretation. Let $\mathcal{D}^+(\Omega) = \{\xi \in \mathcal{D}(\Omega)\colon \xi > 0\}$. Then $\mathcal{D}^+(\Omega)$ is a positive cone of test functions. We then define a distribution $p^c(e)$ to be non-negative, written $p^c(e) \geqq 0$, whenever $\langle p^c(e), \xi \rangle_{\mathscr{D}(\Omega)} \geqq 0$ for any $\xi \in \mathscr{D}^+(\Omega)$ (where $\langle \cdot, \cdot \rangle_{\mathscr{D}(\Omega)}$ denotes the duality pairing on $\mathscr{D}^*(\Omega) \times \mathscr{D}(\Omega)$).

Proof of Theorem 2. We first note that the constraint condition (2.3) implies

$$p^c(e)(\theta - w) \geqq 0 \quad \text{for any} \quad \theta \geqq -\Delta(e) \quad \text{and} \quad e \in U_{ad}(\Omega)\,. \tag{2.13}$$

Then by (2.6) and by integration by parts we get (for $\boldsymbol{u}(e) \in (H^2(\Omega) \times H^2\Omega \times H^4(\Omega)) \cap V(\Omega)$ and $\boldsymbol{v} \in V(\Omega)$)

$$\int_\Omega (\mathscr{R}(e)\, \boldsymbol{u}(e), \boldsymbol{v})_{R^3}\, d\Omega = a(e, \boldsymbol{u}(e), \boldsymbol{v}))\,,$$

$$\langle A(e)\, \boldsymbol{u}(e), \boldsymbol{v} - \boldsymbol{u}(e) \rangle_{V(\Omega)} = \int_\Omega p^c(e)(\theta - w(e))\, d\Omega + \langle \boldsymbol{L}(e), \boldsymbol{v} - \boldsymbol{u}(e) \rangle_{V(\Omega)}\,.$$

Using the relation (2.13) we obtain

$$\langle A(e)\, \boldsymbol{u}(e), \boldsymbol{v} - \boldsymbol{u}(e) \rangle_{V(\Omega)} \geqq \langle \boldsymbol{L}(e), \boldsymbol{v} - \boldsymbol{u}(e) \rangle_{V(\Omega)}\,.$$

Next, let $\boldsymbol{u}(e)$ be a solution of (2.11) and set $\theta(e) = (w(e) + \xi)$, $p = (u + \eta)$, $q = (v + \varkappa)$, $\xi \in \mathscr{D}^+(\Omega)$, $\eta, \varkappa \in \mathscr{D}(\Omega)$. Integration by parts yields

$$\begin{aligned}
&\langle (\mathscr{R}_z(e)\, \boldsymbol{u}(e) - (T_z^* + Be)), \xi \rangle_{\mathscr{D}(\Omega)} \geqq 0 && \text{for any} \quad \xi \in \mathscr{D}^+(\Omega)\,, && e \in U_{ad}(\Omega)\,,\\
&\langle (\mathscr{R}_x(e)\, \boldsymbol{u}(e) - T_x^*), \eta \rangle_{\mathscr{D}(\Omega)} = 0 && \text{for any} \quad \eta \in \mathscr{D}(\Omega)\,, && e \in U_{ad}(\Omega)\,,\\
&\langle (\mathscr{R}_\varphi(e)\, \boldsymbol{u}(e) - T_\varphi^*), \varkappa \rangle_{\mathscr{D}(\Omega)} = 0 && \text{for any} \quad \varkappa \in \mathscr{D}(\Omega)\,, && e \in U_{ad}(\Omega)\,.
\end{aligned}$$

Note that this result means that the distribution $\boldsymbol{p}^c(e) = [0, 0, p^c(e)] = [(\mathscr{R}_x(e)\, \boldsymbol{u}(e) - T_x^*), (\mathscr{R}_\varphi(e)\, \boldsymbol{u}(e) - T_\varphi^*), (\mathscr{R}_z(e)\, \boldsymbol{u}(e) - (T_z^* + Be))]$ is nonnegativ on Ω.

Next, we set $u = p$, $v = q$, $\theta(e) = (2w(e) + \Delta(e))$ and then $u = p$, $v = q$, $\theta(e) = -\Delta(e)$ in (2.11). It is obvious that in both cases $\theta(e) \in \mathfrak{K}(e, \Omega)$. From the pair of inequalities resulting from these choices we conclude that ((2.12), 2°) holds. On the other hand, if $\boldsymbol{u}(e) \in (H^2(\Omega) \times H^2(\Omega) \times H^4(\Omega)) \cap V(\Omega)$ then $[0, 0, p^c(e)] \in L_2(\Omega) \times L_2(\Omega) \times L_2(\Omega)$. Then again by integration of ((2.12), 2°) we obtain

$$\int_\Omega p^c(e)(w(e) - \Delta(e))\, d\Omega = 0\,.$$

This means (by $p^c(e) \geqq 0$ and $(w(e) - \Delta(e)) \geqq 0$) $p^c(e)(w(e) - \Delta(e)) = 0$ a.e. in Ω.

Lemma 4. *The family* $\{A(e)\}$, $e \in U_{ad}(\Omega)$ *of operators defined by* (2.10) *satisfies the assumptions* ((H 0), 1° *to* 3°).

Proof. By virtue of the definition of $U_{ad}(\Omega)$ we have

$$\|A(e)\, \boldsymbol{v}\|_{V^*(\Omega)} \leqq c\|\boldsymbol{v}\|_{V(\Omega)}\,. \tag{2.14}$$

(We obtain the estimate $|a(e, \boldsymbol{u}, \boldsymbol{v})| \leqq c\|\boldsymbol{u}\|_{V(\Omega)}\, \|\boldsymbol{v}\|_{V(\Omega)}$ after using the Schwarz inequality.) Here the positive constant c is independent of $(e, \boldsymbol{v})$. Now ((H 0), 1°) is an immediate consequence of (2.14).

To prove the assumption ((H 0), 2°) we first realize that $\mathbf{K}(e)$ is positive definite, i.e. $\xi^T \mathbf{K}(e)\, \xi \geqq c_0 \xi^T \xi$ for any $\xi \in R^6$ and $e \in U_{ad}(\Omega)$, where the positive constant c_0 is independent of e.

We may write (the energy deformation of the cylindrical shell is positive definite)

$$\langle A(e)\, \mathbf{v}, \mathbf{v} \rangle_{V(\Omega)} = a(e, \mathbf{v}, \mathbf{v}) \geqq c\big(\sum_{i=1}^{6} \|\mathcal{N}_i(\mathbf{v})\|^2_{L_2(\Omega)}\big)$$

(the positive constant c is independent of e). The system $\{\mathcal{N}_i(\mathbf{v})\}_{i=1}^6$ is coercive on $V(\Omega)$ (with respect to Lemma 3 and Lemma 11.3.2 ([12])). Thus we obtain

$$\langle A(e)\, \mathbf{v}, \mathbf{v})\rangle_{V(\Omega)} \geqq \alpha \|\mathbf{v}\|^2_{V(\Omega)} \quad \text{for any} \quad \mathbf{v} \in V(\Omega)\,, \quad e \in U_{ad}(\Omega)$$
$$\text{and} \quad \alpha > 0 \quad \text{independent of } e\,.$$

This completes the verification of ((H 0), 2°). Let $e_n \in U_{ad}(\Omega)$ such that $e_n \rightharpoonup e_0$ (weakly) in $U(\Omega)$. We may write (for fixed $\boldsymbol{\omega} \in V(\Omega)$)

$$(2.15) \qquad |\langle (A(e_n) - A(e_0))\, \mathbf{v}, \boldsymbol{\omega}\rangle_{V(\Omega)}| =$$
$$= |\textstyle\int_\Omega (\mathcal{N}^T(\mathbf{v}) [\mathbf{K}(e_n) - \mathbf{K}(e_0)]\, \mathcal{N}(\boldsymbol{\omega}))\, a\, \mathrm{d}x\, \mathrm{d}\varphi| \leqq$$
$$\leqq \textstyle\int_\Omega |(\mathcal{N}^T(\mathbf{v}) [\mathbf{K}(e_n) - \mathbf{K}(e_0)]\, \mathcal{N}(\boldsymbol{\omega})|\, a\, \mathrm{d}x\, \mathrm{d}\varphi \leqq$$
$$c(\|e_n^3 - e_0^3\|_{C(\bar\Omega)} + \|e_n - e_0\|_{C(\bar\Omega)}) [\textstyle\int_\Omega \sum_{i=1}^{6} \mathcal{N}_i^2(\mathbf{v}) \mathrm{d}\Omega]^{1/2} [\int_\Omega \sum_{i=1}^{6} \mathcal{N}_i^2(\boldsymbol{\omega}) \mathrm{d}\Omega]^{1/2} \leqq$$
$$\leqq c(\|e_n^3 - e_0^3\|_{C(\bar\Omega)} + \|e_n - e_0\|_{C(\bar\Omega)})\, \|\mathbf{v}\|_{V(\Omega)}\, \|\boldsymbol{\omega}\|_{V(\Omega)} \to 0$$

for every $\mathbf{v} \in V(\Omega)$. This proves ((H 0), 3°), which completes the proof of Lemma 4.

Now we define the cost functional in this case.

1° The desired thickness of the shell is given by the distribution $z_d(x, \varphi)$ of the deflection, and we look for a control parameter e subject to constraints, i.e. $e \in U_{ad}(\Omega)$, such that the system response $w(e_0)$ has a minimum deviation of $z_d(x, \varphi)$ in a certain sense. We define the cost functional $(\mathfrak{L}(\mathbf{v}) + \mathfrak{P}(e))$ by

$$(2.16) \quad \begin{aligned} &\mathfrak{L}(\mathbf{v}) = \textstyle\int_\Omega [\theta(e) - z_d]^2\, a\, \mathrm{d}x\, \mathrm{d}\varphi\,, \\ &\mathfrak{P}(e) = 0 \quad \text{for} \quad e \in U_{ad}(\Omega) \\ &\mathfrak{P}(e) = +\infty \quad \text{otherwise}\,, \\ &\text{where } z_d \in H_0^2(\Omega)\,. \end{aligned}$$

Lemma 5. *The cost functional* (2.16) *satisfies the condition* ((E 0), 1°, 2°).

Proof. We have $\mathfrak{P}(e) = I_{U_{ad}}(\Omega)$ (the indicator function of $U_{ad}(\Omega)$). Obviously the function $\mathfrak{L}(\mathbf{v})$ satisfies the assumption ((E 0)).

2° We define the cost functional $\mathfrak{L}^\wedge(e, \mathbf{v})$ in the following form:

$$(2.17) \qquad \mathfrak{L}^\wedge(e, \mathbf{v}) = (1/2) \langle A(e)\, \mathbf{v}, \mathbf{v}\rangle_{V(\Omega)} - \langle \mathbf{L}(e), \mathbf{v}\rangle_{V(\Omega)}$$

(the total potential energy of the cylindrical shell evaluated in the equilibrium state).

Lemma 6. *The cost functional* (2.17) *satisfies the condition* ((E 0), 3°).

Proof. Let us verify ((E 0), 3°). For any fixed $e \in U_{ad}(\Omega)$ the functional $\mathfrak{L}^\wedge(e, \cdot)$: $V(\Omega) \to R^+$ is weakly lower semicontinuous. Moreover, we have

$$\mathfrak{L}^\wedge(e_n, \mathbf{v}_n) = \mathfrak{L}^\wedge(e, \mathbf{v}_n) + (1/2) \langle (A(e_n) - A(e)) \mathbf{v}_n, \mathbf{v}_n \rangle_{V(\Omega)} + \\ + \langle \mathbf{L}(e) - \mathbf{L}(e_n), \mathbf{v}_n \rangle_{V(\Omega)} = \mathfrak{L}(e, \mathbf{v}_n) + I$$

where

$$|I| = |(1/2) \langle (A(e_n) - A(e_0)) \mathbf{v}_n, \mathbf{v}_n \rangle_{V(\Omega)} + \langle (\mathbf{L}(e_0) - \mathbf{L}(e_n)), \mathbf{v}_n \rangle_{V(\Omega)}| \leqq \\ \leqq (1/2) |\langle (A(e_n) - A(e_0)) \mathbf{v}_n, \mathbf{v}_n \rangle_{V(\Omega)}| + |\langle (\mathbf{L}(e_0) - \mathbf{L}(e_n)), \mathbf{v}_n \rangle_{V(\Omega)}| \leqq \\ \leqq (1/2) \|A(e_n) - A(e_0)\|_{L(V(\Omega), V^*(\Omega))} \|\mathbf{v}_n\|^2_{V(\Omega)} + \|\mathbf{L}(e_0) - \mathbf{L}(e_n)\|_{V^*(\Omega)} \|\mathbf{v}_n\|_{V(\Omega)} \to 0$$

since all norms $\|\mathbf{v}_n\|_{V(\Omega)}$ are bounded. By virtue of (2.15) we have

(2.18)
$$\|A(e_n) - A(e_0)\|_{L(V(\Omega), V^*(\Omega))} = \sup_{\mathbf{v} \in V(\Omega), \|\mathbf{v}\|_{V(\Omega)} = 1} \\ \|(A(e_n) - A(e_0)) \mathbf{v}\|_{V^*(\Omega)} = \sup_{\mathbf{v} \in V(\Omega), \|\mathbf{v}\|_{V(\Omega)} = 1} \\ \sup_{\omega \in V(\Omega), \|\omega\|_{V(\Omega)} = 1} |\langle (A(e_n) - A(e_0)) \mathbf{v}, \omega \rangle_{V(\Omega)}| \to 0$$

On the basis of Sobolev imbedding theorem, the functional $\mathbf{L}(e)$ is linear and continuous on $V(\Omega)$. Indeed, we have the estimate

$$|\langle (\mathbf{L}(e_n) - \mathbf{L}(e_0)), \mathbf{v} \rangle_{V(\Omega)}| \leqq \\ \leqq (1/2a) \int_\Omega |(e_0 - e_n) (\partial\theta/\partial x + \partial\theta/\partial\varphi)| \, a \, dx \, d\varphi + |\langle B(e_n - e_0), \theta \rangle_{H_0(\Omega)}| \leqq \\ \leqq c \|e_0 - e_n\|_{C(\bar{\Omega})} \|\mathbf{v}\|_{V(\Omega)} .$$

This means that

$$\|\mathbf{L}(e_0) - \mathbf{L}(e_n)\|_{V^*(\Omega)} = \sup_{\substack{\mathbf{v} \in V(\Omega) \\ \|\mathbf{v}\|_{V(\Omega)} \leqq 1}} |\langle (\mathbf{L}(e_n) - \mathbf{L}(e_0)), \mathbf{v} \rangle_{V(\Omega)}| \leqq c \|e_0 - e_n\|_{C(\bar{\Omega})} \to 0.$$

(Moreover, we can write $ke_n \sin \varphi \to ke_0 \sin \varphi$ (weakly) in $H^2(\Omega)$.) Thus

$$\mathbf{L}(e_n) \to \mathbf{L}(e_0) .$$

Hence we obtain

$$\liminf_{n \to \infty} \mathfrak{L}^\wedge(e_n, \mathbf{v}_n) \geqq \liminf_{n \to \infty} \mathfrak{L}^\wedge(e_0, \mathbf{v}_n) + \lim_{n \to \infty} I \geqq \mathfrak{L}^\wedge(e_0, \mathbf{v}) .$$

Consequently, ((E 0), 3°) is satisfied. From Theorem 1; Lemmas 4, 5, 6, and with regard to (E 1), (H 1) one conclude the following assertion (optimization of the thickness of a cylindrical shell): the optimal control problem ($\mathfrak{B}$), where the data are defined above has at least one solution.

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Súhrn

OPTIMÁLNE RIADENIE VALCOVEJ ŠKRUPINY S TUHOU PREKÁŽKOU

JÁN LOVÍŠEK

Je študovaná úloha optimálneho riadenia variačnou nerovnicou s riadeniami v koeficientoch operátora nerovnice v pravej strane a v konvexnej množine možných stavov. Dokazuje sa existencia optimálneho riadenia. Riešená je úloha optimálneho navrhovania pružnej valcovej škrupiny s prekážkou a premennou hrúbkou ako kontrolnou premennou.

Резюме

ОПТИМАЛЬНОЕ УПРАВЛЕНИЕ ДЛЯ ЦИЛИНДРИЧЕСКОЙ ОБОЛОЧКИ С ПРЕПЯТСТВИЕМ

JÁN LOVÍŠEK

В работе изучается задача оптимального управления вариационным неравенством с управлениями в коэффициентах оператора неравенства, в правой части и в выпуклом множетсве допустимых состояний.

Author's address: Doc. Ing. Dr. Ján Lovíšek, CSc., Stavebná fakulta SVŠT, Radlinskeho 11, 813 68 Bratislava.