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Optimal Design of Decentralized Event-triggered Controllers for Large-scale Systems with Contention-based Communication

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Abstract— **The design of large-scale networked control systems urges an efficient usage of available resources, such as communication, energy and computation. Recent results indicate substantial benefits of event-based control compared to conventional designs, when these resources are sparse. This paper considers multiple entities of heterogeneous control systems that are coupled through a common communication medium. Each control system may decide upon its available information, whether a state update shall be transmitted to the controller over a contention-based medium. The objective is to design an optimal decentralized control and transmission scheme that minimizes the aggregate quadratic cost function. A state aggregation technique is used to derive a decentralized event-triggering scheme, which is asymptotically optimal as the number of control units increases. Numerical simulations give a comparison of the optimal centralized, time-triggered and event-triggered schemes and corroborate the efficiency of the proposed design method.**

I. INTRODUCTION

Technological advances in embedded systems and digital communications have led to an increased interest in the design of networked control systems. A networked control system can be regarded as an aggregation of sensors, controllers, and actuators that form a network of self-contained entities exchanging information over a digital communication medium. In such complex and highly distributed systems, a successful control design depends highly on the choice of the communication scheme. There are many examples showing that common paradigms in the design of communication schemes do not apply. A prominent example is given by the fact that time-triggered information acquisition schemes, which are commonly used for digital control design, are outperformed by event-triggered exchange of information [1]–[6] in the presence of resource constraints in networked control systems. Such observations have been made in a diversity of problem settings, such as control over communications [1]–[3], multi-agent systems [4], distributed optimization algorithms [5] and control design in embedded real-time systems [6].

While the majority of these results deal with single feedback loop systems with communication constraints, the control and communication design for multiple loops sharing a medium is still widely unexplored. Exceptions can be found in [2], [7]–[10] that analyze the performance of event-triggered schemes in contention-based networked control systems. Depending on the communication model under consideration, different conclusions are derived. Using CDMA schemes with priority or randomized arbitration as proposed in [2], [7], event-triggered scheduling schemes for data transmission outperform significantly time division multiple access (TDMA) schemes. Unlike [2], it has been shown in [8], [9] that time-triggered scheduling outperforms event-triggered schemes for slotted and unslotted ALOHA transmission schemes. Under the assumption that collisions between transmissions can be modelled by a Bernoulli process, a condition has been derived in [10], where event-triggered scheduling yields better performance than its time-triggered counterpart.

These results majorly consider scalar integrator dynamics that are modelled by a controlled Brownian motion process. The control law is predefined by an impulse controller and the event-trigger is given by a level-triggering policy, where the event threshold is the design parameter which is to be set appropriately.

In contrast to the described work, this paper investigates the optimal synthesis of control and scheduling laws. We consider N subsystems whose feedback loops are closed over a contention-based network. The communication model is adopted to the framework in [2], [7] and assumes the presence of a randomized arbitration scheme, which can be implemented in the CAN-bus protocol. The subsystems may be heterogeneous and are modelled as stochastic linear discrete-time systems with arbitrary state dimension.

The contribution of this paper is to develop a methodology for the joint design of decentralized schedulers and controllers that share a common communication network. The design objective is to minimize the aggregate linear quadratic cost of all subsystems. Inspired by the concept of state aggregation in mean field theory, the approach assumes a large number of control loops closed over a contention-based network. In oder to conduct our analysis, we make use of recent results for single-loop control systems with communication constraints given by [11]–[15]. Under mild assumptions on the admissible policies, it is shown that the complex behavior of the shared communication system reduces asymptotically to a deterministic system, when the number of loops approaches infinity. This observation allows an efficient design of the optimal decentralized schedulers and controllers that can be derived by means of convex optimization and dynamic programming. A numerical comparison with TDMA scheduling and the optimal centralized arbitration mechanism is conducted. Most notably, the sim-

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ulation results show that our approach is close to optimality even for a moderate number of control loops.

The remaining part of this paper is structured as follows. In section II, we describe the system model and introduce the problem statement. Section III derives the asymptotically optimal decentralized event-triggered controller and section IV illustrates the efficiency of the proposed approach by numerical simulations.

Notation. In this paper, the operator $(\cdot)^{\mathsf{T}}$ denotes the transpose operator. The Euclidean norm and its induced matrix norm is denoted by $\|\cdot\|_2$. The variable P denotes the probability measure on the abstract sample space denoted by Ω. The expression F, P −a.s. denotes that the event F occurs almost surely w.r.t. probability measure P. The expectation operator is denoted by $\mathsf{E}[\cdot]$ and the conditional expectation is denoted by $\mathsf{E}[\cdot|\cdot]$. The relation $x \sim \mathcal{N}(0, I)$ denotes a Gaussian random variable with zero-mean and unity covariance. The operator $\mathbb{1}_{\{\cdot\}}$ denotes the indicator function.

II. PROBLEM STATEMENT

In this paper, we consider N independent control systems whose feedback loops are connected through a shared communication network. A control subsystem i consists of a process $\mathcal{P}_i$, a controller $\mathcal{C}_i$ that is implemented at the actuator and a sensor $\mathcal{S}_i$. The complete networked control system is depicted in Fig. 1. The process $\mathcal{P}_i$ is described by the following time-invariant difference equation

$$x_{k+1}^i = A_i x_k^i + B_i u_k^i + w_k^i, \tag{1}$$

where $A_i \in \mathbb{R}^{n_i \times n_i}, B_i \in \mathbb{R}^{n_i \times d_i}$. The variables x_k^i and u_k^i denote the state and the control input and are taking values in $\mathbb{R}^{n_i}$ and $\mathbb{R}^{d_i}$, respectively. The system noise w_k^i takes values in $\mathbb{R}^{n_i}$ at each k and is i.i.d. with $w_k^i \sim \mathcal{N}(0, I)$. The initial states x_0^i, $i \in \{1, \dots, N\}$ take an arbitrary distribution, but symmetric distribution with finite second moment.

Remark 1: It is straightforward to extend all results to arbitrary noise covariance matrices. The chosen restriction facilitates the illustration of results without loosing generality.

We assume that the statistics of a subsystem are known to its corresponding sensor and controller. At each time step k, the scheduler at the sensor station $\mathcal{S}_i$ may decide, whether a state update should be sent to the controller $\mathcal{C}_i$ over the contention-based network. We assume that there are N_{slot} transmission slots per time step. If the number of sensors that have decided to transmit information at time k exceeds N_{slot}, then an arbitrator chooses randomly N_{slot} sensors that are allowed to transmit. The remaining sensors are blocked and may try to send information at the next time instance. The random arbitration mechanism uses no prioritization of sensors and chooses its sensors according to a uniform distribution. We define the scheduling variable δ_k^i as follows.

$$\delta_k^i = \begin{cases} 1 & \text{update } x_k^i \text{ is sent} \\ 0 & \text{otherwise} \end{cases}$$

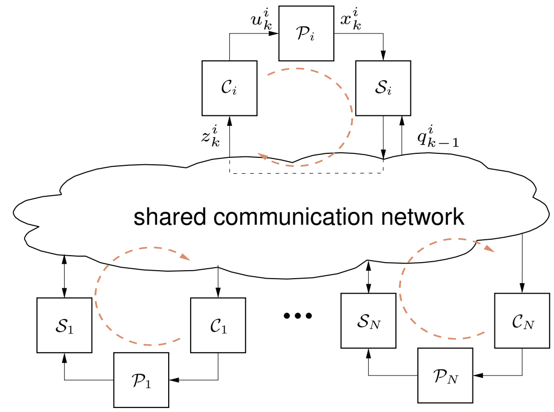

Fig. 1. System model of the networked control system with N control systems closed over a shared communication network with processes $\mathcal{P}_1, \dots, \mathcal{P}_N$, sensors $\mathcal{S}_1, \dots, \mathcal{S}_N$ and controllers $\mathcal{C}_1, \dots, \mathcal{C}_N$.

The random arbitrator is described by the binary random variable q_k^i defined as

$$q_k^i = \begin{cases} 1 & \text{allow to transmit} \\ 0 & \text{block transmission} \end{cases}$$

The conditional probability distribution of $[q_k^1, \dots, q_k^N]$ conditioned on the scheduling variables δ_k^i, $i \in \{1, \dots, N\}$ is time-invariant and satisfies

$$\mathsf{P}[q_k^i = 1 | \delta_k^i, i \in \{1, \dots, N\}] = \begin{cases} 1 & \sum_{i=1}^N \delta_k^i \leq N_{\text{slot}} \\ \frac{N_{\text{slot}}}{\sum_{i=1}^N \delta_k^i} & \text{otherwise} \end{cases} \tag{2}$$

for subsystems i with $\delta_k^i = 1$ and

$$q_k^1 + \dots + q_k^N = N_{\text{slot}}, \quad \mathsf{P}\text{ −a. s.}$$

if $\delta^1 + \dots + \delta_k^N \geq N_{\text{slot}}$. We consider an instantaneous acknowledgement channel that informs the scheduler, whether a transmission was successful or blocked. Therefore, the scheduler at sensor $\mathcal{S}_i$ knows the preceding variables q_m^i at time k for $m < k$. Let z_k^i denote the received data at controller $\mathcal{C}_i$ at time k.

Remark 2: The described arbitration scheme can be implemented in a CAN-bus protocol with time-varying, randomly assigned priorities.

The signal z_k^i can be described by the previously defined scheduling variable δ_k^i and arbitration variable q_k^i

$$z_k^i = \begin{cases} x_k^i, & \delta_k^i = 1 \wedge q_k^i = 1 \\ \emptyset, & \text{otherwise} \end{cases} \tag{3}$$

Each subsystem $i \in \{1, \dots, N\}$ has a cost function J_i given by the linear quadratic average-cost criterion

$$J_i = \lim_{T \to \infty} \frac{1}{T} \mathsf{E}\left[\sum_{k=0}^{T-1} x_k^{i,\mathsf{T}} Q_i x_k^i + u_k^{i,\mathsf{T}} S_i u_k^i\right]. \tag{4}$$

The weighting matrix Q_i is positive definite and S_i is positive semi-definite for each $i \in \{1, \dots, N\}$. We assume that the pair (A_i, B_i) is stabilizable and the pair $(A_i, Q_i^{\frac{1}{2}})$ is detectable with $Q_i = (Q_i^{\frac{1}{2}})^{\mathsf{T}} Q_i^{\frac{1}{2}}$.

We want to minimize the aggregate cost function V normed by the number of subsystems. It consists of the sum of individual costs J_i of each subsystem divided by N, i.e.

$$V = \frac{1}{N}\sum_{i=1}^{N} J_i. \tag{5}$$

The control law $\gamma^i = \{\gamma_0^i, \gamma_1^i, \ldots\}$ is described by admissible policies γ_k^i for each time k. These are defined as Borel-measurable functions of their past available data

$$u_k^i = \gamma_k^i(\mathcal{I}_k^{\mathcal{C}}),$$

where the information available at the controller is given by

$$\mathcal{I}_k^{\mathcal{C}_i} = \{z_0^i, u_0^i, \ldots, z_{k-1}^i, u_{k-1}^i\}.$$

The information $\mathcal{I}_k^{\mathcal{S}_i}$ available at the sensor station $\mathcal{S}_i$ to decide whether to transmit a state update for the decentralized scheduler is given by

$$\mathcal{I}_k^{\mathcal{S}_i} = \{x_0^i, \delta_0^i, q_0^i, x_1^i, \ldots, x_{k-1}^i, \delta_{k-1}^i, q_{k-1}^i, x_k^i\}.$$

Unlike the control policies γ^i, the policy of the scheduler may be randomized and is described by a sequence $\pi^i = \{\pi_1^i, \pi_2^i, \ldots\}$, which is given by the stochastic kernels $\pi_k^i(\cdot|\mathcal{I}_k^{\mathcal{S}_i})$ on the set $\{0, 1\}$ conditioned on the history $\mathcal{I}_k^{\mathcal{S}_i}$.

Apart from this class of schedulers, we will consider two other classes in section IV. These are on the one hand centralized scheduling schemes that globally decide which subsystems may transmit information and on the other hand TDMA schemes, where transmission timings are fixed before runtime for each subsystem.

III. MAIN RESULT

This section develops an algorithm to find approximatively optimal scheduling policies π and control policies γ that minimize the aggregate cost V given by equation (5). Although the coupling between subsystems in the optimization problem occurs only in the shared communication network, the underlying problem is difficult to solve. The reason for that is secondarily given by the fact that the number of subsystems might be large, but is rather grounded in the distributed information pattern. It is shown in [16] that solutions for optimization problems with a distributed information pattern are rather hard to obtain. This fact motivates us to search for a suitable approximation of the problem setting, where we can apply known results that lead to efficient algorithms. The idea is to let the number of subsystems grow to infinity and scale the system accordingly, so that the optimal solution for system with infinite subsystems will be a good approximation for the original system with a finite number of subsystems.

A. Approximative system

In the subsequent paragraph, we introduce the approximative system. A crucial parameter in describing the communication network is the variable R which is defined as the ratio between available transmission slots N_{slot} per time step and the number of subsystems N, i.e.

$$R = \frac{N_{\text{slot}}}{N}$$

While letting N approach infinity, the ratio R is kept constant, i.e. N_{slot} grows uniformly with N. On the other hand, the subsystems with same system parameter are replicated with increasing N. We define the 4-tuple $\mathcal{K}_i = (A_i, B_i, Q_i, S_i)$ to describe a subsystem. Then, if we double N, the number of subsystems with $\mathcal{K}_i$ is doubled for each $i \in \{1, \ldots, N\}$. Based on such approximative system description, we derive decentralized control and scheduling laws in the following subsections.

B. Assumptions

We first focus on homogeneous systems, i.e. systems with identical subsystems. The obtained results will then be applied to systems with heterogeneous subsystems in section III-F. In a homogeneous system setup, a subsystem i is described as the 4-tuple $\mathcal{K} = (A, B, Q, S)$, where we assume that $A_i = A$, $B_i = B$, $Q_i = Q$ and $S_i = S$ for every $i \in \{1, \ldots, N\}$. Subsequently, we drop the sub- or superscript i for notational convenience, whenever it is not needed. We introduce the following assumptions for the admissible control and scheduling polices given a subsystem $\mathcal{K}$. These assumptions will enable a simplified design approach.

(A1) The scheduling policy and control policy are identical for every subsystem.

(A2) The control law γ stabilizes the subsystem $\mathcal{K}$ in a bounded moment sense for all dropout probabilities $\mathsf{P}[q_k = 0] \leq 1 - R$, when assuming $\delta_k = 1$ for all $k \geq 0$ and q_k being i.i.d..

(A3) The scheduling policy and control policy are stationary. The resulting closed-loop system with $q_k = 1$, $k \geq 1$ is ergodic with

$$\lim_{k\to\infty} \mathsf{P}[\delta_k = 1] = R, \quad \mathsf{P}-\text{a.s.}$$

and future scheduler outputs δ_m, $m > k$ are independent of $\mathcal{I}_k^{\mathcal{S}}$ in case of $\delta_k = 1$.

Assumption (A2) will guarantee that the closed-loop process converges to a stationary process for every finite N and ensures that the obtained solution leads to bounded moment stability. It should be remarked that assumptions (A2) and (A3) do not need to take into account the complex behavior between subsystems due to the shared communication network. Instead, the network is modelled as an i.i.d. Bernoulli distributed packet dropout process for assumption (A2), which has been studied extensively [17]. With respect to assumption (A3), subsystems can be viewed as isolated entities having their own dedicated feedback channel.

The following paragraph is concerned with the question, how restrictive the taken assumptions are with respect to optimality. Assumption (A1) can be reasoned by results in subsection III-F on heterogeneous subsystems that show the underlying global optimization problem is a resource allocation problem with concave utility functions. The fairness property of this optimization problem implies that identical subsystems attain the same solution. Assumption (A2) may introduce some conservatism, as it poses additional stability conditions. On the other hand, the assumption leads to a robust design with respect to other subsystem that do not comply to the global design procedure. Assumption (A2) ensures that all subsystems remain stable, even for an arbitrary number of subsystems transmitting persistently. Assumption (A3) states that the average transmission rate of a subsystem is R, when considered as an isolated control system with perfect communication. The reasoning behind this heuristic assumption is that R is the unique rate that fully utilizes the communication network while avoiding collisions in the limit $k \to \infty$ for the approximated system with $N \to \infty$. The assumption that the scheduler does not take into account the information $\mathcal{I}_k^{\mathcal{S}}$ in case of $\delta_k = 1$ for its future decisions is motivated by the fact that the data of the scheduler and controller are synchronized for $\delta_k = 1$ and therefore the information $\mathcal{I}_k^{\mathcal{S}}$ is outdated.

C. Design approach for identical subsystems

This subsection proposes a control and scheduling design, whose optimality and stability properties are derived in the subsequent subsections III-D and III-E. The control law is given by a certainty equivalence controller

$$u_k = \gamma_k^{\mathrm{CE}}(\mathcal{I}_k^{\mathcal{C}}) = -L\,\mathsf{E}[x_k|\mathcal{I}_k^{\mathcal{C}}], \tag{6}$$

where L is the control gain of the linear quadratic problem, i.e.

$$\begin{aligned} L &= (B^{\mathsf{T}}PB + S)^{-1}B^{\mathsf{T}}PA, \\ P &= A^{\mathsf{T}}(P - PB(B^{\mathsf{T}}PB + S)^{-1}B^{\mathsf{T}}P)A + Q. \end{aligned}$$

The least-squares estimate $\mathsf{E}[x_k|\mathcal{I}_k^{\mathcal{C}}]$ can be computed by a Kalman-like estimator given by

$$\mathsf{E}[x_k|\mathcal{I}_k^{\mathcal{C}}] = \begin{cases} x_k & \delta_k = 1 \wedge q_k = 1 \\ (A - BL)\,\mathsf{E}[x_{k-1}|\mathcal{I}_{k-1}^{\mathcal{C}}] & \text{otherwise} \end{cases}$$

with $\mathsf{E}[x_0|\mathcal{I}_0^{\mathcal{C}}] = 0$ for $\delta_0 = 0$ or $\delta_0 = 1$ and $q_0 = 0$. By defining the estimation error

$$e_k = x_k - \mathsf{E}[x_k|\mathcal{I}_{k-1}^{\mathcal{C}}, \delta_k = 0],$$

the optimal scheduling policy is the solution of a constrained Markov decision process [18] with state e_k that evolves by

$$e_{k+1} = g(e_k, \delta_k, w_k) = (1 - \delta_k)Ae_k + w_k \tag{7}$$

with initial condition $e_0 = x_0 - \mathsf{E}[x_0]$. Besides, we define the average transmission rate as

$$r = \lim_{T\to\infty} \frac{1}{T}\,\mathsf{E}[\sum_{k=0}^{T-1} \delta_k].$$

When substituting γ^{CE} into J_i defined by (4), the objective is to find the optimal scheduling law π^* that minimizes

$$J^{\mathcal{S}} = \lim_{T\to\infty} \frac{1}{T}\,\mathsf{E}\left[\sum_{k=0}^{T-1}(1 - \delta_k)e_k^{\mathsf{T}}\Gamma e_k\right], \text{s.t. } r \le R, \tag{8}$$

where $\Gamma = L^{\mathsf{T}}(R + B^{\mathsf{T}}SB)L$. It can be observed that the optimization problem (8) guarantees that assumption (A3) is satisfied. Rather than solving (8) directly, we first determine the Pareto frontier of feasible pairs $[J^{\mathcal{S}}, r]$ and then choose the pair with minimal $J^{\mathcal{S}}$ satisfying $r \le R$. The calculation of the Pareto frontier is performed by a scalarization approach described in the next subsection.

D. Asymptotic optimality property

Based on the assumptions (A1)-(A3), this subsection shows that the proposed design approach is optimal for the approximative system, i.e. it is asymptotically optimal for $N \to \infty$ for constant R. First, we make the following observation by considering the ratio of subsystems that decide to transmit information. The ratio is given by the term $\frac{1}{N}\sum_{i=1}^{N}\delta_k^i$. Assume that the initial state distribution of each subsystem is given by the stationary distribution resulting from an isolated subsystem fulfilling assumption (A3). Then, the ratio $\frac{1}{N}\sum_{i=1}^{N}\delta_k^i$ is R for the approximative system for $N \to \infty$ for every time step k. As $\frac{1}{N}\sum_{i=1}^{N}\delta_k^i$ is a random variable with a zero-one law for $N \to \infty$, this result holds P-almost surely. It also implies that no collisions occur at any time step, i.e. $q_k^i = 1$ for $\delta_k^i = 1$, P-almost surely. Assumption (A2) ensures that the aggregate system with the shared communication network converges to a stationary distribution for finite N. Further, it is conjectured that the stationary distribution resulting from the isolated subsystems is also attained for the approximative system from any arbitrary initial distribution. As a consequence of the fact that the limiting transmission rate is R, P-almost surely and under assumptions (A1)-(A3), we are able to reduce the initial optimization problem for the approximative system with $N \to \infty$ to the following local optimization problem considering only one subsystem.

$$\min_{\pi,\gamma} J, \quad \text{s.t. } r \le R. \tag{9}$$

Above statements for reducing the optimization problem are also valid for an average transmission rate smaller than R. Therefore, we have replaced the equality constraint by an inequality constraint in (9). Based on the ergodicity assumption in (A3), we can replace $\mathsf{P}[\delta_k = 1]$ by r.

Several works [11]–[14] have already addressed optimization problems that are related to (9). Similarly as in [13], [14], it can be shown that the optimal control law is given by γ^{CE} defined in (6). This is mainly due to the stationarity assumption of the policies in (A3) and the nestedness property of the information pattern. The information pattern is nested, since the information available at the controller is a subset of the information available at the scheduler.

Taking the obtained results into account, the remaining task is to find the optimal event-triggering law π^* that minimizes (8). In order to solve above optimization problem, we first consider the corresponding vector-valued optimization problem, where we drop the inequality constraint and consider the average transmission rate r as our second objective besides $J^{\mathcal{S}}$. In order to calculate feasible points in the cost region $[J^{\mathcal{S}}, r] \in \mathbb{R}^2$ that are Pareto-optimal with a corresponding scheduling policy, we use a scalarization approach. The scalarization approach takes the following form

$$\min_{\pi} J^{\mathcal{S}} + \lambda r, \tag{10}$$

where λ is a non-negative weighting term penalizing transmissions. The unconstrained Markov decision process given by (10) has been studied in [11]. In the following, we adopt the following assumption made in [11].

(A4) The scheduling policy is $\pi_k(\delta_k = 1|e_k) = 1$ for $\|e_k\|_2 > M$ for some arbitrary M.

Remark 3: This assumption does not put severe restrictions on the design, as M may be chosen arbitrarily large. When Γ is not positive definite, but only non-negative definite, optimal solutions of (10) generally violate assumption (A4) for any M as is shown in [14]. But as M can be arbitrarily large, there always exists an ϵ-optimal scheduling law taking assumption (A4) into account.

Based on assumption (A4), it is shown in [11] that the optimal scheduling policy is deterministic, stationary and takes the form of a threshold policy. Due to continuity of the Pareto points $[J^{\mathcal{S}}, r]$ in λ, optimization problem (8) and (10) correspond to each other for an appropriately chosen λ. Besides, this result implies that the Pareto frontier is described by a convex and non-increasing function $J^{\mathcal{S}}(r)$. Summarizing this subsection, we have shown that the design approach in previous subsection is optimal for the approximative system. The Pareto frontier for obtaining the optimal scheduling law π^* can be calculated efficiently by using a scalarization approach.

E. Stability

In this subsection we address the question whether the original system with finite N and the shared network together with the optimal solution of (9) is stable. Every closed-loop subsystem can be described by the augmented state $[\mathsf{E}[x_k|\mathcal{I}_k^{\mathcal{C}}], e_k]$. This system has a triangular structure due to (7), i.e. the evolution of the estimation error is independent of the $\mathsf{E}[x_k|\mathcal{I}_K^{\mathcal{C}}]$ given the current estimation error e_k. The evolution of $\mathsf{E}[x_k|\mathcal{I}_k^{\mathcal{C}}]$ can be viewed as a stable system disturbed by the estimation error e_k. Therefore, it suffices to analyze the stability properties of the estimation error e_k to show stability of the closed-loop system of a subsystem. The following proposition gives a stability condition for the overall system taking into account the shared network.

Proposition 1: Let Assumption (A4) hold and the controller and scheduler be given by (6) and (8). If we can guarantee that

$$R > 1 - \frac{1}{\|A\|_2^2}, \tag{11}$$

then the overall system with N identical subsystems sharing a common network is bounded moment stable.

Proof: We use drift criteria to show bounded moment stability [19]. It is straight forward to prove that the underlying Markov chain is ψ-irreducible, aperiodic and the drift of quadratic functions of e_k inside the compact set $\mathcal{M} = \{e_k | \|e_k\|_2 \leq M\}$ is bounded. Thus, it suffices to consider the set of states e_k outside of this compact set [19]. The drift operator is defined as

$$\Delta h(e_k) = \mathsf{E}[h(e_{k+1})|e_k] - h(e_k), \quad e_k \in \mathbb{R}^n.$$

For $e_k \in \mathbb{R}^n \backslash \mathcal{M}$, we have the following difference equation due to assumption (A4)

$$e_{k+1} = (1 - q_k) A e_k + w_k,$$

where q_k is distributed as in (2) and depends on the remaining subsystems. In order to have bounded moment stability, we need to ensure that

$$\Delta h(e_k) \leq -\epsilon \|e_k\|_2^2, \quad e_k \in \mathbb{R}^n \backslash \mathcal{O}, \tag{12}$$

where $\epsilon > 0$ and $\mathcal{O} \supset \mathcal{M}$ is compact. Let us take $h(e_k) = \|e_k\|_2^2$. Due to statistical independence of w_k, q_k and e_k for $e_k \in \mathbb{R}^n \backslash \mathcal{O}$ and the fact that w_k is zero-mean with unit variance, the drift term can be written as

$$\Delta h(e_k) = \mathsf{E}[1 - q_k] \|Ae_k\|_2^2 + 1 - \|e_k\|_2^2.$$

The term $\mathsf{E}[1 - q_k]$ is the average packet drop probability, which is upper bounded by $1 - R$. On the other hand, we have $\|Ae_k\|_2 \leq \|A\|_2 \|e_k\|$. Therefore, the drift is bounded by

$$\Delta h(e_k) \leq ((1 - R)\|A\|_2^2 - 1)\|e_k\|_2^2 + 1.$$

Condition (12) ensures that we can find appropriate ϵ and $\mathcal{O}$, such that the drift criteria given by (12) is satisfied. This completes our proof. ■

Remark 4: In case of heterogeneous subsystems, condition (11) has to be checked for each system matrix A_i.

F. Design approach for heterogeneous subsystems

For identical subsystems, the optimization problem was reduced into a local optimization problem that finds the optimal control and scheduling law in a decentralized way. For solving the optimization problem, the transmission rate R, which is equal to the ratio $\frac{N_{\text{slot}}}{N}$, is assigned apriori to each subsystem. We can proceed in the same way for every subsystem within the heterogeneous system for a particular transmission rate. However, the optimal transmission rates for each subsystem are not given in advance. Fortunately, we have seen in section III-B that the Pareto curve of the cost J and average transmission rate r is convex in the cost region. Therefore, the determination of the optimal transmission rates r_i for subsystems $i \in \{1, \ldots, N\}$ is a resource allocation problem with utility function $-J_i(r_i)$.

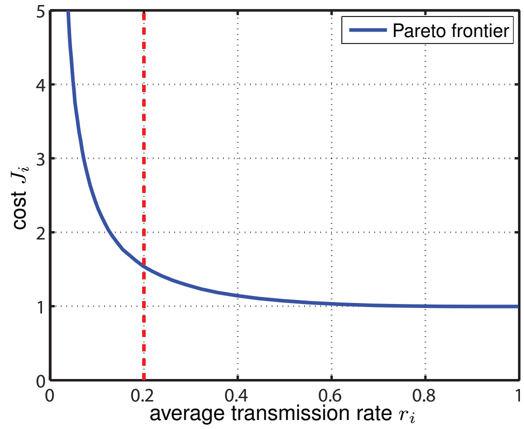

Fig. 2. Pareto frontier of a subsystem and system parameters $\mathcal{K} = (1, 1, 1, 0)$. The vertical line indicates the rate constraint.

This is a well-studied problem in network optimization and admits interpretations like fairness [20]. The resource allocation problem can be written as

$$\min_{\{r_1,\dots,r_N\}} \frac{1}{N}\sum_{i=1}^{N} J_i(r_i), \text{ s.t. } \frac{1}{N}\sum_{i=1}^{N} r_i \le R. \qquad (13)$$

Assumption (A3) is modified by replacing the total transmission rate by the individual rate r_i for each subsystem i.

The design approach for heterogeneous subsystems proceeds splits up into two different optimization stages. Every subsystem solves a local optimization problem by calculating its Pareto frontier of feasible points $[J_i, r_i]$. This is obtained by solving (10) for different $\lambda \in [0, \infty)$. The resulting function $J_i(r_i)$ is used in the global optimization problem allocating transmission rates to each subsystem given by (13).

IV. NUMERICAL VALIDATION

The purpose of this section is twofold. First, the efficacy of the proposed design algorithm is evaluated. This is accomplished by comparing it with optimal TDMA scheduling schemes and the optimal centralized scheme. Second, we illustrate the design approach for the decentralized event-triggered design for a homogeneous and heterogeneous system setup. For sake of illustration, we consider scalar subsystems in the following.

First, suppose we have identical subsystems with parameters $\mathcal{K} = (1, 1, 1, 0)$. The communication network allows a transmission rate $R = 0.2$. The Pareto optimal cost region $[J_i, r_i]$ for a subsystem with parameters $\mathcal{K}$ including the rate constraint is drawn in Fig. 2. We observe that J_i is a decreasing and convex function with respect to r_i.

The optimal cost point is attained at $[J_i^*, R] = [1.54, 0.2]$ by an event-triggered scheduling policy π^* that is given by $\delta_k = \mathbb{1}_{\{|e_k|>1.7\}}$. The optimal control law gain L is given by 1. It should be remarked that the Pareto curve can be obtained individually for every subsystem before runtime without considering the underlying communication system.

Fig. 3 compares the cost of the decentralized event-triggered scheme with the optimal TDMA scheme and the optimal centralized scheduling shows the cost per subsystem for various numbers of identical subsystems N with $R = 0.2$. The resulting costs for $N \in \{5, 25, 100, 250, 500\}$ is determined through Monte Carlo simulations with a time horizon of $T = 10\,000$. The optimal control law for both the optimal TDMA scheme and the optimal centralized scheme are given by $u_k = -L\,\mathsf{E}[x_k|\mathcal{I}_k^{\mathcal{C}}]$ with $L = 1$. In the optimal TDMA scheme, time slots for transmission are assigned successively. Subsystems transmit information periodically with transmission period $\frac{1}{R}$, where we assume that N is a multiple of 5. In the case of identical subsystems, the optimal centralized scheduler selects at each time step k the RN subsystems with maximum magnitude $|e_k|$ whose feedback loop are then closed. It should be noted that this scheduler can be regarded as a lower bound on the performance that can be achieved over the communication networks, but which is not realizable as it needs another communication network gathering the estimation errors of every subsystem.

We observe in Fig. 3 that the cost of the optimal decentralized scheduling algorithm approximates this lower bound very closely and outperforms the optimal TDMA scheme significantly. On the other hand, it can be seen that the costs converge to the asymptotic costs for $N \to \infty$ very rapidly. Already for $N = 100$, the performance gap is less than 1%.

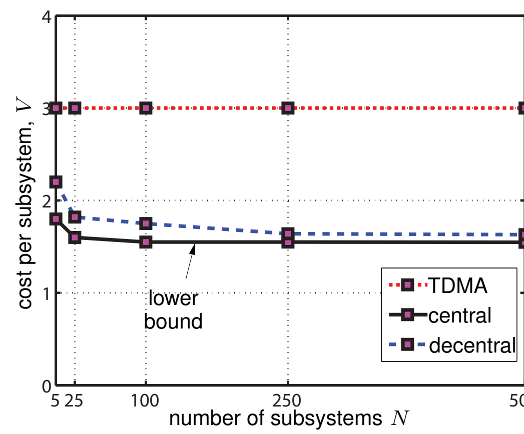

Fig. 3. Numerical validation of the networked control system with homogeneous subsystems and system parameters $\mathcal{K} = (1, 1, 1, 0)$ and $R = 0.2$.

Finally, we consider a heterogeneous system, where we have two different kinds of subsystems occurring at the same amount. The system parameters are $\mathcal{K}_1 = (1.25, 1, 1, 0)$ and $\mathcal{K}_2 = (0.75, 1, 1, 0)$ and the communication network has a transmission rate of $R = 0.5$. We note that the stability condition (11) is satisfied for the underlying subsystems.

Having obtained the Pareto curves for both subsystems sketched in Fig. 4, the resource allocation problem given by (13) determines the optimal rate pair. The dashed line in Fig. 4 depicts the mean cost per subsystem V as a function of r_1 for $N = 2$ without collisions. It can be seen that the total cost V is convex with respect to r_1 and it is minimized at the rate pair $[r_1,\ r_2] = [0.6, 0.4]$ taking a value of 1.07. The optimal control gain is given by $L_i = A_i$ for both

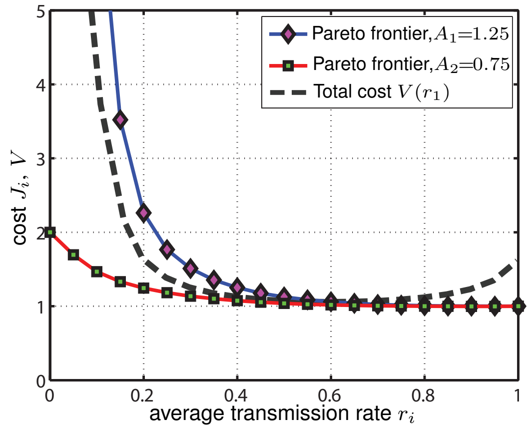

Fig. 4. Solid lines: Pareto frontiers of two different subsystems with system parameters $\mathcal{K}_1 = (1.25, 1, 1, 0)$ and $\mathcal{K}_2 = (0.75, 1, 1, 0)$. Dashed line: Total cost $V(r_1) = \frac{1}{2}(J_1(r_1) + J_2(r_2))$ and constraint $\frac{1}{N}(r_1 + r_2) \leq 0.5$. The optimal rate pair is given at $[r_1,\ r_2] = [0.6, 0.4]$ with total cost $V = 1.07$ for the two subsystems without collisions.

subsystems and the scheduling laws are threshold policies, where $\delta_k^1 = \mathbb{1}_{\{|e_k|>0.5\}}$ for $\mathcal{K}_1$ and $\delta_k^2 = \mathbb{1}_{\{|e_k|>0.95\}}$ for $\mathcal{K}_2$.

Concerning the performance in the presence of the shared network, we consider the mean costs per subsystem depicted in Fig. 5 for $N \in \{2, 10, 50, 100, 250, 500\}$. The optimal TDMA scheme involves a brute-force search over all possible combinations of transmission times. To keep this combinatorial problem numerically tractable, we restricted the admissible transmission scheme to be periodical for subsystems $\mathcal{K}_2$. The optimal periodical transmission scheme is then given by $[\delta_0^1, \delta_1^1, \delta_2^1, \dots] = [1, 1, 0, \dots]$ and $[\delta_0^2, \delta_1^2, \delta_2^2, \dots] = [0, 0, 1, \dots]$ with period 3. A lower bound is given by $V = 1$ assuming no communication constraints on the feedback channels. As can be regarded from Fig. 5, this lower bound is approached with a gap of less than 10% for increasing N and the TDMA scheme is outperformed for every number of subsystems.

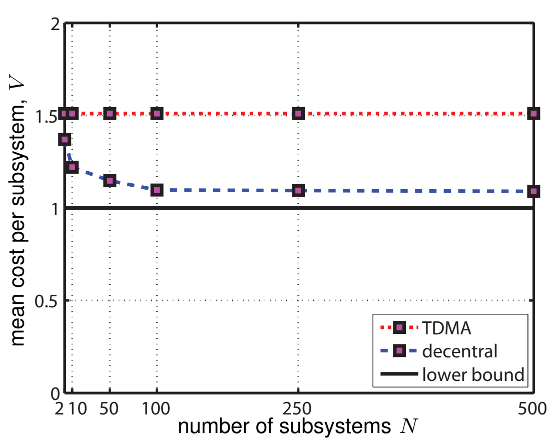

Fig. 5. Numerical validation of the networked control system with heterogeneous subsystems of two classes $\mathcal{K}_1$ and $\mathcal{K}_2$.

V. CONCLUSIONS

This paper shows that decentralized event-triggered scheduling seems to be very promising to achieve a compromise between complexity and performance. The design approach offers a tractable methodology that circumvents the need to take into account the complex behavior of the contention-based network, but guarantees overall stability. The decentralized event-triggered scheme outperforms TDMA scheduling and approaches the optimal centralized scheduling scheme very closely.

Prospective research investigates the online estimation of the network parameters and an adaptation mechanism that adjusts the scheduling law at runtime, as well as the extension to other communication models.

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