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Optimal Discrete-Time Control for Nonlinear Cascade Systems

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Abstract

In this paper we develop an optimality-based framework for designing controllers for discrete-time nonlinear cascade systems. Specifically, using a nonlinear-nonquadratic optimal control framework we develop a family of globally stabilizing backstepping-type controllers parameterized by the cost functional that is minimized. Furthermore, it is shown that the control Lyapunov function guaranteeing closed-loop stability is a solution to the steady-state Bellman equation for the controlled system and thus guarantees both optimality and stability.

1. Introduction

Since most physical processes evolve naturally in continuous time, it is not surprising that the bulk of nonlinear control theory has been developed for continuous-time systems. Nevertheless, it is the overwhelming trend to implement controllers digitally. Despite this fact the development of nonlinear control theory for discrete-time systems has lagged its continuous-time counterpart. This is in part due to the fact that concepts such as zero dynamics, normal forms, and minimum phase are much more intricate for discrete-time systems. For example, in contrast to the continuous-time case, technicalities involving passivity analysis tools needed to prove global stability via smooth feedback controllers [1] as well as system relative degree requirements [2] are more involved in the discrete-time case.

Recent work involving differential geometric methods [3] employing concepts of zero dynamics and feedback linearization have been applied to discrete-time systems. In particular, these results parallel continuous-time results on linearization of nonlinear systems via state and output feedback. However, as in the continuous-time case, these techniques cancel out system nonlinearities and may therefore lead to inefficient designs since the resulting feedback linearizing controller may generate large control effort to cancel beneficial nonlinearities.

Backstepping control for continuous-time systems has recently received a great deal of attention in the nonlinear control literature [4]. The popularity of this control methodology can be explained in a large part due to the fact that it provides a framework for designing stabilizing nonlinear controllers for a large class of nonlinear cascade systems. Even though discrete-time recursive backstepping techniques have not been developed, the closest discrete-time analog to backstepping is given in [2,5]. Specifically, in [2,5] discrete-time passivity analysis tools are used to construct control Lyapunov functions guaranteeing global asymptotic stability for block cascade discrete-time systems.

In this paper we develop an optimality-based control design theory for nonlinear discrete-time cascade systems. The key motivation for developing an optimal nonlinear control theory framework for discrete-time cascade systems is that it provides a family of candidate controllers parameterized by the cost functional that is minimized. In order to address the optimality-based nonlinear control problem we use the nonlinear-nonquadratic optimal control framework developed in [6]. The basic underlying ideas of the results in [6] rely on the fact that the steady-state solution of the discrete-time Bellman equation is a control Lyapunov function for the nonlinear controlled system thus guaranteeing both optimality and stability. Finally, we use the following standard notation. Let $\mathbb{N}^{n\times n}$ (resp., $\mathbb{P}^{n\times n}$) denote the set of $n \times n$ nonnegative (resp., positive) definite matrices and let $\mathcal{N}$ denote the set of nonnegative integers.

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2. Optimal Control for Nonlinear Systems

In this section we consider the nonlinear system

$$x(k+1) = f(x(k)) + g(x(k))u(k),\ x(0) = x_0,\ k \in \mathcal{N}, \quad (1)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(0) = 0$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, with performance criterion

$$J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} [L_1(x(k)) + L_2(x(k))u(k) + u^{\mathrm{T}}(k)R_2u(k)]. \quad (2)$$

where $L_1 : \mathbb{R}^n \to \mathbb{R}, L_2 : \mathbb{R}^n \to \mathbb{R}^{1\times m}$, and $R_2 \in \mathbb{P}^{m\times m}$. Furthermore, define the set of asymptotically stabilizing controllers for the nonlinear system (1) by

$$\mathcal{S}(x_0) \triangleq \{u : \mathbb{R}^m \times \mathcal{N} \to \mathbb{R}^m : x(\cdot) \text{ given by (1) satisfies } x(k) \to 0 \text{ as } k \to \infty\}.$$

Theorem 2.1. Consider the controlled system (1) with performance functional (2). Assume there exist functions $V : \mathbb{R}^n \to \mathbb{R}$, $L_2 : \mathbb{R}^n \to \mathbb{R}^{1\times m}$, $P_{12} : \mathbb{R}^n \to \mathbb{R}^{1\times m}$, and a nonnegative-definite function $P_2 : \mathbb{R}^n \to \mathbb{N}^{m\times m}$ such that

$$L_2(0) = 0,\ P_{12}(0) = 0,\ V(0) = 0, \quad (3)$$

$$V(x) > 0, \quad x \in \mathbb{R}^n,\ x \neq 0, \quad (4)$$

$$V\big[f(x) - \tfrac{1}{2}g(x)(R_2 + P_2(x))^{-1} \cdot L_2^{\mathrm{T}}(x)\big] - V(x) < 0, \quad x \in \mathbb{R}^n,\ x \neq 0, \quad (5)$$

$$V(f(x) + g(x)u) = V(f(x)) + P_{12}(x)u + u^{\mathrm{T}}P_2(x)u, \quad (6)$$

where $V(x) \to \infty$ as $\|x\| \to \infty$. Then the solution $x(k) = 0$, $k \in \mathcal{N}$, of the closed-loop system

$$x(k+1) = f(x(k)) + g(x(k))\phi(x(k)),\ x(0) = x_0,\ k \in \mathcal{N}, \quad (7)$$

is globally asymptotically stable with the feedback control law $\phi(x) = -\frac{1}{2}(R_2 + P_2(x))^{-1}[L_2(x) + P_{12}(x)]^{\mathrm{T}}$, and the performance functional (2), with

$$L_1(x) = \phi^{\mathrm{T}}(x)(R_2 + P_2(x))\phi(x) - V(f(x)) + V(x), \quad (8)$$

is minimized in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)), \ x_0 \in \mathbb{R}^n. \quad (9)$$

Finally, $J(x_0, \phi(x(\cdot))) = V(x_0)$, for all $x_0 \in \mathbb{R}^n$.

3. Optimal Block Backstepping Controllers

In this section we consider nonlinear cascade systems with nonlinear input subsystems of the form

$$\begin{aligned} x(k+1) &= f(x(k)) + g(x(k))y(k), \ x(0) = x_0, & (10)\\ \hat{x}(k+1) &= \hat{f}(\hat{x}(k)) + \hat{g}(\hat{x}(k))u(k), \ \hat{x}(0) = \hat{x}_0, & (11)\\ y(k) &= h(\hat{x}(k)) + J(\hat{x}(k))u(k), & (12)\end{aligned}$$

where $k \in \mathcal{N}$, $\hat{x} \in \mathbb{R}^q$, $u, y \in \mathbb{R}^m$, $\hat{f} : \mathbb{R}^q \to \mathbb{R}^q$, $\hat{g} : \mathbb{R}^q \to \mathbb{R}^{q\times m}$, $h : \mathbb{R}^q \to \mathbb{R}^m$, and $J : \mathbb{R}^q \to \mathbb{R}^{m\times m}$ such that $\hat{f}(0) = 0$ and $h(0) = 0$. Here, we consider the case in which the nonlinear input subsystem (11), (12) is feedback strictly passive with positive definite storage function $V_{\rm s} : \mathbb{R}^q \to \mathbb{R}$ such that $V_{\rm s}(0) = 0$ and

$$V_{\rm s}(\hat{f}(\hat{x}) + \hat{g}(\hat{x})u) = V_{\rm s}(\hat{f}(\hat{x})) + \hat{P}_{12}(\hat{x})u + u^{\rm T}\hat{P}_2(\hat{x})u, \quad (13)$$

where $\hat{P}_{12} : \mathbb{R}^q \to \mathbb{R}^{1\times m}$ and $\hat{P}_2 : \mathbb{R}^q \to \mathbb{P}^{m\times m}$. Specifically, we assume there exist functions $k : \mathbb{R}^q \to \mathbb{R}^m$, $l : \mathbb{R}^q \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^q \to \mathbb{R}^{p\times m}$ such that $l(0) = 0$, $k(0) = 0$,

$$0 > V_{\rm s}(\hat{f}(\hat{x}) + \hat{g}(\hat{x})k(\hat{x})) - V_{\rm s}(\hat{x}) + l^{\rm T}(\hat{x})l(\hat{x}), \ \hat{x} \neq 0, \quad (14)$$

$$0 = \tfrac{1}{2}\hat{P}_{12}^{\rm T}(\hat{x}) + \mathcal{W}^{\rm T}(\hat{x})l(\hat{x}) - (h(\hat{x}) + J(\hat{x})k(\hat{x})) + \hat{P}_2(\hat{x})k(\hat{x}), \quad (15)$$

$$0 = \hat{P}_2(\hat{x}) + \mathcal{W}^{\rm T}(\hat{x})\mathcal{W}(\hat{x}) - (J(\hat{x}) + J^{\rm T}(\hat{x})). \quad (16)$$

Theorem 3.1. Consider the cascade system (10)–(12) with performance functional

$$\tilde{J}(x_0, \hat{x}_0, u(\cdot)) \triangleq \sum_{k=0}^{\infty} \big[\tilde{L}_1(x(k), \hat{x}(k)) + \tilde{L}_2(x(k), \hat{x}(k))u(k) + u^{\rm T}(k)R_2u(k)\big], \quad (17)$$

where $(x(k), \hat{x}(k))$, $k \in \mathcal{N}$, solves (10), (11). Assume that the input subsystem (11), (12) is feedback strictly passive and the subsystem (10) has a globally stable equilibrium at $x(k) = 0$, $k \in \mathcal{N}$, and Lyapunov function $V_{\rm sub}(x)$ so that $V_{\rm sub}(f(x)) < V_{\rm sub}(x)$, for all $x \in \mathbb{R}^n$ such that $x \neq 0$. Furthermore, assume there exist functions $\tilde{L}_2 : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^{1\times m}$, $\tilde{P}_{12} : \mathbb{R}^n \to \mathbb{R}^{1\times m}$, and $\tilde{P}_2 : \mathbb{R}^n \to \mathbb{N}^{m\times m}$ such that

$$\tilde{L}_2(0,0) = 0, \ \tilde{P}_{12}(0) = 0, \quad (18)$$

$$V_{\rm sub}(f(x) + g(x)y) = V_{\rm sub}(f(x)) + \tilde{P}_{12}(x)y + y^{\rm T}\tilde{P}_2(x)y, \quad (19)$$

$$y^{\rm T}\Big\{\tilde{P}_{12}^{\rm T}(x) + \tilde{P}_2(x)h(\hat{x}) - 2k(\hat{x}) - (I_m + \tfrac{1}{2}\tilde{P}_2(x)J(\hat{x})) \cdot \tilde{R}_{2a}^{-1}(x,\hat{x})[\hat{P}_{12}^{\rm T}(\hat{x}) + 2J^{\rm T}(\hat{x})\tilde{P}_2(x)h(\hat{x}) + \tilde{L}_2^{\rm T}(x,\hat{x}) + J^{\rm T}(\hat{x})\tilde{P}_{12}^{\rm T}(x)]\Big\} \leq 0, \quad (x,\hat{x}) \in \mathbb{R}^n \times \mathbb{R}^q, \quad (20)$$

where $\tilde{R}_{2a}(x,\hat{x}) \triangleq R_2 + \hat{P}_2(\hat{x}) + J^{\rm T}(\hat{x})\tilde{P}_2(x)J(\hat{x})$ and $k(\hat{x})$ satisfies (14)–(16). Then the solution $(x(k), \hat{x}(k)) = (0,0)$, $k \in \mathcal{N}$, of the cascade system (10), (11) is globally asymptotically stable with the feedback control law $u = \tilde{\phi}(x, \hat{x})$, where

$$\tilde{\phi}(x,\hat{x}) = -\tfrac{1}{2}\tilde{R}_{2a}^{-1}(x,\hat{x})[\hat{P}_{12}^{\rm T}(\hat{x}) + 2J^{\rm T}(\hat{x})\tilde{P}_2(x)h(\hat{x}) + \tilde{L}_2^{\rm T}(x,\hat{x}) + J^{\rm T}(\hat{x})\tilde{P}_{12}^{\rm T}(x)]. \quad (21)$$

Furthermore, for $(x_0, \hat{x}_0) \in \mathbb{R}^n \times \mathbb{R}^m$

$$\tilde{J}(x_0, \hat{x}_0, \tilde{\phi}(x(\cdot), \hat{x}(\cdot))) = V(x_0, \hat{x}_0), \quad (22)$$

where $V(x, \hat{x}) = V_{\rm sub}(x) + V_{\rm s}(\hat{x})$, and the performance functional (17), with

$$\tilde{L}_1(x,\hat{x}) = \tilde{\phi}^{\rm T}(x,\hat{x})\tilde{R}_{2a}(x,\hat{x})\tilde{\phi}(x,\hat{x}) + V_{\rm sub}(x) - V_{\rm sub}(f(x) + g(x)h(\hat{x})) + V_{\rm s}(\hat{x}) - V_{\rm s}(\hat{f}(\hat{x})), \quad (23)$$

is minimized in the sense that

$$\tilde{J}(x_0, \hat{x}_0, \tilde{\phi}(x(\cdot), \hat{x}(\cdot))) = \min_{u\in\mathcal{S}(x_0,\hat{x}_0)} \tilde{J}(x_0, \hat{x}_0, u(\cdot)). \quad (24)$$

Remark 3.1. Assuming $\det(I_m + \frac{1}{2}\tilde{P}_2(x)J(\hat{x})) \neq 0$ for all $(x,\hat{x}) \in \mathbb{R}^n \times \mathbb{R}^q$, a particular choice of $\tilde{L}_2(x,\hat{x})$ satisfying conditions (18) and (20) is given by

$$\tilde{L}_2(x,\hat{x}) = [\tilde{P}_{12}^{\rm T}(x) + \tilde{P}_2(x)h(\hat{x}) - 2k(\hat{x})]^{\rm T} \cdot (I_m + \tfrac{1}{2}\tilde{P}_2(x)J(\hat{x}))^{-\rm T}\tilde{R}_{2a}(x,\hat{x}) - \tilde{P}_{12}(x)J(\hat{x}) - \hat{P}_{12}(\hat{x}) - 2h^{\rm T}(\hat{x})\tilde{P}_2(x)J(\hat{x}). \quad (25)$$

In this case,

$$\tilde{\phi}(x,\hat{x}) = k(\hat{x}) - \tfrac{1}{2}(I_m + \tfrac{1}{2}\tilde{P}_2(x)J(\hat{x}))^{-1}[\tilde{P}_{12}^{\rm T}(x) + \tilde{P}_2(x)\,(h(\hat{x}) + J(\hat{x})k(\hat{x}))\,]. \quad (26)$$

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