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OPTIMAL DISTRIBUTED CONTROL OF A NONLOCAL CONVECTIVE CAHN–HILLIARD EQUATION BY THE VELOCITY IN THREE DIMENSIONS*

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Abstract. In this paper we study a distributed optimal control problem for a nonlocal convective Cahn–Hilliard equation with degenerate mobility and singular potential in three dimensions of space. While the cost functional is of standard tracking type, the control problem under investigation cannot easily be treated via standard techniques for two reasons: the state system is a highly nonlinear system of PDEs containing singular and degenerating terms, and the control variable, which is given by the velocity of the motion occurring in the convective term, is nonlinearly coupled to the state variable. The latter fact makes it necessary to state rather special regularity assumptions for the admissible controls, which, while looking a bit nonstandard, are, however, quite natural in the corresponding analytical framework. In fact, they are indispensable prerequisites to guarantee the well-posedness of the associated state system. In this paper, we employ recently proved existence, uniqueness, and regularity results for the solution to the associated state system in order to establish the existence of optimal controls and appropriate first-order necessary optimality conditions for the optimal control problem.

Key words. distributed optimal control, first-order necessary optimality conditions, nonlocal models, integrodifferential equations, convective Cahn–Hilliard equation, phase separation

AMS subject classifications. 49J20, 49J50, 35R09, 45K05, 74N99

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1. Introduction.

This paper is concerned with the study of a distributed control problem for a Cahn–Hilliard-type PDE system that may be considered as a model for an isothermal phase separation of two constituents taking place in a fluid flow whose velocity is given. More precisely, we investigate the case of a *nonlocal* Cahn–Hilliard equation with a convective term, degenerate mobility, and singular potential. In fact, while the standard Cahn–Hilliard equation (cf., e.g., [3, 4, 5]) is widely used, it seems that a more realistic version of the Cahn–Hilliard equation can be characterized by a (spatially) nonlocal free energy. Although the physical relevance of nonlocal interactions was already pointed out in the pioneering paper [31] (see also [12, Chap. 4.2] and the references therein), the isothermal and nonisothermal models containing nonlocal terms have only recently been studied from the analytical viewpoint (cf., e.g., [1, 9, 15, 17, 18, 23] and the references given there). We also remark that recently increasing attention has been paid to nonlocal models also from the viewpoint of numerics (cf., e.g., [20, 19]).

The main difference between local and nonlocal models is given by the choice of the interaction potential. Typically, the nonlocal contribution to the free energy

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has the form $\int_\Omega k(x,y)\,|\varphi(x)-\varphi(y)|^2\,\mathrm{d}y$, with a given symmetric kernel k defined on $\Omega\times\Omega$, where Ω denotes a (sufficiently regular and bounded) domain in $\mathbb{R}^3$ in which the phase separation takes place; its local Ginzburg–Landau counterpart is given by $(\sigma/2)|\nabla\varphi(x)|^2$, where the positive parameter σ is a measure for the thickness of the interface. Here, φ represents the local concentration of one of the two phases, which typically attains values in a bounded interval, say, in $[0,1]$. The local potential can be obtained as a formal limit as $m\to\infty$ from the nonlocal one with the choice $k(x,y)=m^5k(|m(x-y)|^2)$, where k is a nonnegative function with support in $[0,1]$. This follows from the formula (which was formally deduced in [22])

$$\int_\Omega m^5k(|m(x-y)|^2)\,|\varphi(x)-\varphi(y)|^2\,\mathrm{d}y=\int_{\Omega_m(x)}k(|z|^2)\left|\frac{\varphi\left(x+\frac{z}{m}\right)-\varphi(x)}{\frac{1}{m}}\right|^2\mathrm{d}z$$
$$\overset{m\to\infty}{\longrightarrow}\int_{\mathbb{R}^3}k(|z|^2)\left\langle\nabla\varphi(x),z\right\rangle^2\,\mathrm{d}z=\frac{\sigma}{2}|\nabla\varphi(x)|^2$$

for a sufficiently regular φ, where $\sigma=2/3\int_{\mathbb{R}^3}k(|z|^2)|z|^2\,\mathrm{d}z$ and $\Omega_m(x)=m(\Omega-x)$. Here we have used that $\int_{\mathbb{R}^3}k(|z|^2)\left\langle e,z\right\rangle^2\,\mathrm{d}z=1/3\int_{\mathbb{R}^3}k(|z|^2)|z|^2\,\mathrm{d}z$ for every unit vector $e\in\mathbb{R}^3$. As a consequence, the local Cahn–Hilliard equation can be viewed as an approximation of the nonlocal one and vice versa. We remark at this point that typical integral kernels, which arise in applications and meet the regularity assumptions stated below in section 2, are given by the classical Newton potential

$$k(x)=\kappa\,|x|^{-1},\quad x\neq 0,\quad \text{where } \kappa>0 \text{ is a constant},$$

by the usual mollifiers, and by the Gaussian kernels

$$k(x)=\kappa_2\,\exp\left(-|x|^2/\kappa_3\right),\quad x\in\mathbb{R}^3,\quad \text{where } \kappa_2>0 \text{ and } \kappa_3 \text{ are constants.}$$

In the seminal paper [11], the authors established the existence of a weak solution to the local Cahn–Hilliard equation with degenerate mobility and singular potentials endowed with no-flux boundary conditions. However, in the local case no uniqueness proof is known in the case of degenerate mobility and singular potential. This is one of the main advantages of considering the nonlocal potential: for the nonlocal Cahn–Hillard system, indeed, in the case of periodic boundary conditions, an existence and uniqueness result was proved in [18]. Later, a more general case was considered in [15]. More recently, the convergence to single equilibria was studied in [26, 27] (cf. also [16] for further results), and in [14] the existence of a global attractor for a convective nonlocal Cahn–Hilliard equation with degenerate mobility and singular potential was proved in the three-dimensional case. Moreover, for the two-dimensional case, also the long-time dynamics of its coupling with the Navier–Stokes equation (the nonlocal version of the so-called *H-model*) was analyzed in [14]. For this model, uniqueness of weak solutions and existence of the global attractor in two dimensions have been proved recently in [13].

Concerning the problem of deriving first-order necessary optimality conditions for optimal control problems involving *local* Cahn–Hilliard equations, we can quote the following references: in [34], the authors studied the case of a polynomially growing potential f (in (1.3)) with constant mobility m in (1.2), while more recently in [21] the case of the double obstacle potential $f=I_{[0,1]}$ in (1.3) with constant mobility m in (1.2) was investigated; first-order necessary optimality conditions were obtained by means of a regularization procedure. Moreover, the convective one-dimensional case has been dealt with in [35], and the recent paper [36] discusses the two-dimensional case, where the boundary conditions $\varphi = \Delta\varphi = 0$ were prescribed in place of the usual no-flux conditions for φ and the chemical potential. Notice that in all of the abovementioned contributions a distributed control was assumed which was not related to the fluid velocity. Let us finally recall the papers [8, 7], where the authors studied the optimal control problem associated with a nonstandard phase field model of Cahn–Hilliard type, and [2], respectively, where optimization techniques were used in order to solve variational inequalities related to Allen–Cahn and Cahn–Hilliard equations.

While optimal control problems for certain classes of PDEs coupled with nonlocal *boundary conditions* have already been studied in the literature (cf., e.g., [10, 28, 29, 30]), to our best knowledge no analytical contribution exists in the literature to the study of optimal control problems for nonlocal phase field models of convective Cahn-Hilliard type and, more generally, for nonlocal PDEs where the nonlocal operator appears in the PDEs and not on the boundary.

Another novelty of this paper is the use of the fluid velocity field as the control parameter. This entails that through the convective term there arises a nonlinear coupling between control and state in product form that renders the analysis difficult. Practical applications of this concept arise (at least indirectly) in the growth of bulk semiconductor crystals. A typical case is the block solidification of large silicon crystals for photovoltaic applications: in this industrial process a mixture of several species of atoms (impurities) dissolved in the silicon melt has to be moved by the flow (i.e., by the velocity field $\mathbf{v}$) to the boundary of the solidifying silicon in order to maximize the purified high quality part of the resulting silicon ingot. In other words, the flow pattern acts as a control to optimize the final distribution of the impurities. Notice that in this application the control through the velocity $\mathbf{v}$ is only *indirect* since the flow pattern itself is controlled via magnetic fields that induce a Lorentz force in the electrically conducting silicon melt. For a description of such a block solidification process, we refer the reader to, e.g., [24].

Throughout this paper, we will generally assume that $\Omega \subset \mathbb{R}^3$ is a bounded and connected domain with smooth boundary $\partial\Omega$ and outward unit normal $\mathbf{n}$, and we denote $Q := \Omega \times (0,T)$ and $\Sigma := \partial\Omega \times (0,T)$, where $T > 0$ is a prescribed final time. We then consider the following control problem:

(CP) Minimize the cost functional

(1.1)
$$J(\varphi,\mathbf{v}) = \frac{\beta_1}{2}\int_0^T\!\!\int_\Omega |\varphi-\varphi_Q|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\beta_2}{2}\int_\Omega |\varphi(T)-\varphi_\Omega|^2\,\mathrm{d}x + \frac{\beta_3}{2}\int_0^T\!\!\int_\Omega |\mathbf{v}|^2\,\mathrm{d}x\,\mathrm{d}t\,,$$

subject to the initial-boundary value problem (the *state system*)

$$\varphi_t - \operatorname{div}(m(\varphi)\nabla\mu) = -\mathbf{v}\cdot\nabla\varphi \quad \text{in } Q\,, \tag{1.2}$$

$$\mu = f'(\varphi) + w \quad \text{in } Q\,, \tag{1.3}$$

$$w(x,t) = \int_\Omega k(|x-y|)(1-2\varphi(y,t))\,\mathrm{d}y \quad \text{in } Q\,, \tag{1.4}$$

$$m(\varphi)\nabla\mu\cdot\mathbf{n} = 0 \quad \text{on } \Sigma\,, \tag{1.5}$$

$$\varphi(0) = \varphi_0 \quad \text{in } \Omega\,, \tag{1.6}$$

and to the constraint that the velocity $\mathbf{v}$, which plays the role of the control, has to belong to a suitable closed, bounded, and convex subset (to be specified later) of the space

$$\mathcal{V} := \{\mathbf{v} \in L^2(0,T; H^1_{div}(\Omega)) \cap L^\infty(Q)^3 : \ \exists\, \mathbf{v}_t \in L^2(0,T; L^3(\Omega)^3)\}, \tag{1.7}$$

where

$$H^1_{div}(\Omega) := \{\mathbf{v} \in H^1_0(\Omega)^3 : \mathrm{div}(\mathbf{v}) = 0\}\,. \tag{1.8}$$

Notice that the velocity is assumed divergence-free, and we recall that through the convective term $-\mathbf{v}\cdot\nabla\varphi$ the coupling between control and state is nonlinear. This nonlinear coupling between control and state is the reason for the strong and a bit nonstandard regularity assumption for the time derivative of the control $\mathbf{v}$. We also remark that both $H^1_{div}(\Omega)$ and $\mathcal{V}$ are Banach spaces when equipped with their natural norms and that the embedding $\mathcal{V} \subset C^0([0,T]; L^3(\Omega)^3)$ is continuous.

The singular potential f will be taken in the typical logarithmic form (cf. the original paper [4])

$$f(\varphi) = \varphi \log(\varphi) + (1-\varphi)\log(1-\varphi),$$

and the mobility m, which degenerates at the pure phases $\varphi = 0$ and $\varphi = 1$, has to satisfy the compatibility condition (cf. [11, 15, 27])

$$m(\varphi) = \frac{c_0}{f''(\varphi)} = c_0\varphi(1-\varphi) \quad \text{with some constant } c_0 > 0,$$

which entails that we have the relations

$$m(\varphi) f''(\varphi) \equiv c_0\,, \quad m(\varphi)\nabla\mu = c_0\,\nabla\varphi + m(\varphi)\,\nabla w. \tag{1.9}$$

Moreover, throughout this paper we assume that the given constants β_1, β_2, β_3 in (1.1) are nonnegative, while $\varphi_Q \in L^2(Q)$ and $\varphi_\Omega \in L^2(\Omega)$ represent prescribed target functions of the cost functional J. We could generalize both the expressions of J and of the potential f, but we restrict ourselves to the above situation for the sake of a simpler exposition. In particular, we could consider the case when

$$f \in C^4(0,1) \text{ is strictly convex in } (0,1), \quad \mathrm{Im}(f')^{-1} = [0,1],$$
$$\frac{1}{f''} \text{ is strictly concave in } (0,1),$$

and, for example,

$$m \in C^2([0,1]) \quad \text{satisfies } \ m(\varphi)f''(\varphi) \geq c_0 > 0 \ \text{ for every } \ \varphi \in [0,1].$$

Other interesting problems would be related not only to the case of more general potentials and mobilities but also to the optimal control problem related to the coupling of (1.2)–(1.6) with a Navier–Stokes system governing the evolution of the velocity $\mathbf{v}$. The existence of weak solutions to such coupled systems and their long-time behavior have recently been studied in [14] in the two- and three-dimensional cases. The analysis of an associated control problem in the two-dimensional case will be the subject of a forthcoming paper.

Plan of the paper. The paper is organized as follows: in section 2, we recall known results regarding the well-posedness of the PDE system (1.2)–(1.6) as well as the related separation property. We also prove a continuous dependence result (Proposition 2.2) which is needed for the analysis of the control problem. In section 3, we prove the main results of this paper concerning existence and first-order necessary optimality conditions for the optimal control problem **(CP)**.

Throughout this paper we will denote the norm of a Banach space E by $\|\cdot\|_E$. In the following, we will make repeated use of Young's inequality

$$a\,b \le \delta\, a^2 + \frac{1}{4\delta} b^2 \quad \forall\, a, b \in \mathbb{R} \quad \text{and } \ \delta > 0, \tag{1.10}$$

as well as of the fact that for three dimensions of space the embeddings $H^1(\Omega) \subset L^p(\Omega)$, $1 \le p \le 6$, and $H^2(\Omega) \subset C^0(\overline{\Omega})$ are continuous and (in the first case only for $1 \le p < 6$) compact. Moreover, we recall that for smooth and bounded three-dimensional domains there hold the special Gagliardo–Nirenberg inequalities

$$\|v\|_{L^3(\Omega)} \le \widehat{K}_1 \left(\|v\|^{1/2}_{L^2(\Omega)} \|v\|^{1/2}_{H^1(\Omega)} + \|v\|_{L^2(\Omega)} \right) \quad \forall\, v \in H^1(\Omega)\,, \tag{1.11}$$

$$\|v\|_{L^4(\Omega)} \le \widehat{K}_2 \left(\|v\|^{1/4}_{L^2(\Omega)} \|v\|^{3/4}_{H^1(\Omega)} + \|v\|_{L^2(\Omega)} \right) \quad \forall\, v \in H^1(\Omega)\,, \tag{1.12}$$

where the constants $\widehat{K}_1 > 0$ and $\widehat{K}_2 > 0$ depend only on Ω; observe that (1.10) and the continuity of the embedding $W^{1,4}(\Omega) \subset L^\infty(\Omega)$ imply that for every $\delta > 0$ it holds that

$$\|v\|^2_{L^3(\Omega)} \le \delta\, \|v\|^2_{H^1(\Omega)} + \frac{\widehat{K}_3}{\delta}\, \|v\|^2_{L^2(\Omega)} \quad \forall\, v \in H^1(\Omega)\,, \tag{1.13}$$

$$\|v\|^2_{L^\infty(\Omega)} \le \delta\, \|v\|^2_{H^2(\Omega)} + \frac{\widehat{K}_4}{\delta}\, \|v\|^2_{H^1(\Omega)} \quad \forall\, v \in H^2(\Omega)\,, \tag{1.14}$$

where also $\widehat{K}_3 > 0$ and $\widehat{K}_4 > 0$ depend only on Ω. We also recall the well-known fact that the trace operator $\varphi \mapsto \varphi_{|\partial\Omega}$ is a continuous mapping from $H^1(\Omega) \cap L^\infty(\Omega)$ into $H^{1/2}(\partial\Omega) \cap L^\infty(\partial\Omega)$; moreover, it follows from the form of the intrinsic norm of $H^{1/2}(\partial\Omega)$ that we have for products the implications

$$u, v \in H^{1/2}(\partial\Omega) \cap L^\infty(\partial\Omega) \quad \Longrightarrow \quad u\, v \in H^{1/2}(\partial\Omega) \cap L^\infty(\partial\Omega), \tag{1.15}$$

$$u, v \in L^2(0,T; H^{1/2}(\partial\Omega)) \cap L^\infty(\Sigma) \quad \Longrightarrow \quad u\, v \in L^2(0,T; H^{1/2}(\partial\Omega) \cap L^\infty(\partial\Omega))\,. \tag{1.16}$$

Finally, for the sake of a shorter exposition, we denote by $\mathcal{K}$ the integral operator that assigns to φ the function w through (1.4); that is, we put

$$\mathcal{K}(\varphi)(x,t) := \int_\Omega k(|x-y|)(1 - 2\varphi(y,t))\, \mathrm{d}y\,. \tag{1.17}$$

2. Well-posedness of the state system. In the following, we study the state system (1.2)–(1.6). To fix things, we assume for the set of admissible controls the following:

(H1) $\quad \mathcal{V}_{\rm ad} := \{ \mathbf{v} = (v_1, v_2, v_3) \in \mathcal{V} : \ \widetilde{v}_{1_i} \le v_i \le \widetilde{v}_{2_i}$ a.e. in Q, $i = 1, 2, 3$,
$\quad\quad \|\mathbf{v}\|_{L^2(0,T;H^1(\Omega)^3)} + \|\mathbf{v}_t\|_{L^2(0,T;L^3(\Omega)^3)} \le V \}$,

where $V > 0$ is a given constant and $\widetilde{v}_{1_i}, \widetilde{v}_{2_i} \in L^\infty(Q)$, $i = 1, 2, 3$, are given threshold functions; we generally assume that $\mathcal{V}_{\rm ad} \neq \emptyset$.

Observe that $\mathcal{V}_{\rm ad}$ is a bounded, closed, and convex subset of $\mathcal{V}$, which is certainly contained in some bounded open subset of $\mathcal{V}$. For convenience, we fix such a set once and for all, noting that any other such set could be used instead:

(H2) $\mathcal{V}_R \subset \mathcal{V}$ is an open set satisfying $\mathcal{V}_{\rm ad} \subset \mathcal{V}_R$ such that, for all $\mathbf{v} \in \mathcal{V}_R$,

$$\|\mathbf{v}\|_{L^2(0,T;H^1(\Omega)^3)} + \|\mathbf{v}\|_{L^\infty(Q)^3} + \|\mathbf{v}_t\|_{L^2(0,T;L^3(\Omega)^3)} \leq R\,. \tag{2.1}$$

Before stating some results on the well-posedness of the state system (1.2)–(1.6), we now formulate the general assumptions for the problem data. We remark at this place that not all of these assumptions are needed to ensure the respective results concerning existence, separation, uniqueness, and regularity; however, they are indispensable prerequisites for the continuous dependence result of Proposition 2.2 below, which will be needed for the derivation of necessary optimality conditions for the control problem. Since we focus on optimal control here, we have decided to impose the corresponding (stronger) conditions from the very beginning in order to avoid any confusion. We make the following assumptions:

(H3) $\varphi_0 \in H^2(\Omega)$, there is some $\kappa_0 > 0$ such that $0 < \kappa_0 \leq \varphi_0 \leq 1 - \kappa_0 < 1$ a.e. in Ω, and it holds a.e. in Ω that

$$\begin{aligned} 0 &= \left(c_0\, \nabla\varphi_0 + m(\varphi_0)\, \nabla \int_\Omega k(|x-y|)(1 - 2\,\varphi_0(y))\,{\rm d}y \right) \cdot \mathbf{n} \\ &= m(\varphi_0)\, \nabla\mu(\cdot, 0) \cdot \mathbf{n}. \end{aligned}$$

(H4) $f(\varphi) = \varphi\,\log(\varphi) + (1-\varphi)\,\log(1-\varphi)$ for $0 < \varphi < 1$, $f(0) = f(1) = 0$, $f(\varphi_0) \in L^1(\Omega)$.

(H5) $m(\varphi) = \frac{c_0}{f''(\varphi)}$ for $0 < \varphi < 1$, with some $c_0 > 0$.

(H6) $\int_\Omega \int_\Omega k(|x-y|)\,{\rm d}x\,{\rm d}y =: k_0 < +\infty$, $\sup_{x\in\Omega} \int_\Omega |k(|x-y|)|\,{\rm d}y =: \bar{k} < +\infty\,.$

(H7) $\forall\, p \in [1, +\infty]\ \exists\, k_p > 0 : \left\| -2 \int_\Omega k(|x-y|)\, z(y)\,{\rm d}y \right\|_{W^{1,p}(\Omega)} \leq k_p\, \|z\|_{L^p(\Omega)}$ for all $z \in W^{1,p}(\Omega)$.

(H8) For $p \in \{2, 3\}$ there is some $s_p > 0$ such that for all $z \in W^{1,p}(\Omega)$ it holds that

$$\left\| -2 \int_\Omega k(|x-y|)\, z(y)\,{\rm d}y \right\|_{W^{2,p}(\Omega)} \leq s_p\, \|z\|_{W^{1,p}(\Omega)}\,.$$

We now establish some results for the state system. The following result was essentially shown in [27, Thm. 2.2] for the case $\mathbf{v} = 0$.

PROPOSITION 2.1. *The system* (1.2)–(1.6) *admits under the hypotheses* **(H1)**–**(H8)** *for any* $\mathbf{v} \in \mathcal{V}_R$ *a unique solution triple* (φ, w, μ) *such that*

$$\varphi \in C^1([0,T]; L^2(\Omega)) \cap H^1(0,T; H^1(\Omega)) \cap L^\infty(0,T; H^2(\Omega)) \cap C^0(\overline{Q}). \tag{2.2}$$

Moreover, there is some $\kappa \in (0,1)$*, which does not depend on the choice of* $\mathbf{v} \in \mathcal{V}_R$*, such that*

$$0 < \kappa \leq \varphi \leq 1 - \kappa < 1 \quad \textit{a.e. in } Q\,. \tag{2.3}$$

Proof. At first, adapting the proof by Gajewski and Zacharias (see [15, Thm. 3.5] and also [14, Thm. 4 and Prop. 4]) to the case $\mathbf{v} \neq 0$, one can establish the existence of a unique weak solution (φ, w, μ) to (1.2)–(1.6) such that

$$\varphi \in H^1(0,T;H^1(\Omega)^*) \cap L^2(0,T;H^1(\Omega)) \cap C^0([0,T];L^2(\Omega)), \tag{2.4}$$

$$\int_\Omega \varphi(x,t)\,\mathrm{d}x = \int_\Omega \varphi_0(x)\,\mathrm{d}x \quad \forall\, t \in [0,T], \quad 0 < \varphi < 1 \quad \text{a.e. in } Q\,, \tag{2.5}$$

$$w \in L^\infty(0,T;W^{1,\infty}(\Omega))\,, \tag{2.6}$$

$$\int_0^T \int_\Omega m(\varphi)\,|\nabla\mu|^2\,\mathrm{d}x\,\mathrm{d}t \,<\, +\infty\,. \tag{2.7}$$

Next, it is not difficult to see that the additional convective term $-\mathbf{v}\cdot\nabla\varphi$ on the right-hand side of (1.2) does not create major problems in modifying the proof of [27, Prop. 3.1] to the convective case, provided the velocity is (as in our case) bounded; in fact, just as there it turns out that the expressions $\|\ln(\varphi(t)\|_{L^r(\Omega)}$ and $\|\ln(1-\varphi(t))\|_{L^r(\Omega)}$ are bounded by a constant that depends neither on $r \in [1,+\infty)$ nor on $t \in [0,T]$, whence it can be concluded that there is a constant $\kappa \in (0,1)$, which is independent of the choice of $\mathbf{v} \in \mathcal{V}_R$, such that the weak solution satisfies the separation property (2.3).

In order to prove the regularity property (2.2), we can follow along the lines of the proof of [27, Thm. 2.2], in which the asserted regularity was shown for the case without convection. We provide here the details of the argument since they differ from those given there. To this end, we will first show that (cf. eq. (4.1) in [27])

$$\varphi_t \in L^\infty(0,T;H^1(\Omega)^*) \cap L^2(Q)\,. \tag{2.8}$$

The derivation of (2.8) requires the introduction of a functional analytic tool which is standard in the framework of Cahn–Hilliard equations. To this end, we denote by $\langle\,\cdot\,,\,\cdot\,\rangle$ the dual pairing between $H^1(\Omega)^*$ and $H^1(\Omega)$, and denoting by $|\Omega|$ the Lebesgue measure of Ω, we introduce for functions $\psi \in H^1(\Omega)^*$ and $\varphi \in L^1(0,T;H^1(\Omega)^*)$ the generalized mean values

$$\psi^\Omega := \frac{1}{|\Omega|}\langle\psi,\mathbf{1}\rangle \quad \text{and} \quad \varphi^\Omega(t) := (\varphi(t))^\Omega \quad \text{for a.e. } t \in (0,T). \tag{2.9}$$

We then introduce the operator $\mathcal{N}$ as the inverse of the Laplacian with zero Neumann boundary condition as follows: we define

$$\mathrm{dom}\,\mathcal{N} := \left\{\psi_* \in H^1(\Omega)^* : \psi_*^\Omega = 0\right\} \quad \text{and } \mathcal{N} : \mathrm{dom}\,\mathcal{N} \to \left\{\psi \in H^1(\Omega) : \psi^\Omega = 0\right\}$$

by setting

$$\mathcal{N}\psi_* \in H^1(\Omega), \quad (\mathcal{N}\psi_*)^\Omega = 0, \quad \text{and} \quad \int_\Omega \nabla\mathcal{N}\psi_*\cdot\nabla z\,\mathrm{d}x = \langle\psi_*, z\rangle \quad \forall\, z \in H^1(\Omega)\,.$$

In other words, $\psi = \mathcal{N}\psi_*$ is the unique solution to the generalized Neumann problem $-\Delta\psi = \psi_*$ in Ω, $\partial\psi/\partial\mathbf{n} = 0$ on $\partial\Omega$, which has zero mean value. It is a well-known fact that through the formula

$$\|\psi_*\|_*^2 \,:=\, \left\|\nabla\mathcal{N}\left(\psi_* - \psi_*^\Omega\right)\right\|_{L^2(\Omega)}^2 \,+\, \left|\psi_*^\Omega\right|^2 \quad \forall\,\psi_* \in H^1(\Omega)^* \tag{2.10}$$

a norm is defined on $H^1(\Omega)^*$, which is equivalent to the standard norm of $H^1(\Omega)^*$ and has the following properties:

$$\langle \psi_*, \mathcal{N}\varphi_* \rangle = \langle \varphi_*, \mathcal{N}\psi_* \rangle = \int_\Omega (\nabla \mathcal{N}\psi_*) \cdot (\nabla \mathcal{N}\varphi_*)\, \mathrm{d}x \quad \forall\, \varphi_*, \psi_* \in \operatorname{dom} \mathcal{N}, \tag{2.11}$$

$$\langle \psi_*, \mathcal{N}\psi_* \rangle = \|\psi_*\|_*^2 = \int_\Omega |\nabla \mathcal{N}\psi_*|^2\, \mathrm{d}x \quad \forall\, \psi_* \in \operatorname{dom} \mathcal{N}, \tag{2.12}$$

$$2\, \langle \partial_t \psi_*(t), \mathcal{N}\psi_*(t) \rangle = \frac{d}{\mathrm{d}t} \int_\Omega |\nabla \mathcal{N}\psi_*(t)|^2\, \mathrm{d}x = \frac{d}{\mathrm{d}t} \|\psi(t)\|_*^2 \quad \text{a.e. in } (0,T) \tag{2.13}$$

for any $\psi_* \in H^1(0,T; H^1(\Omega)^*)$ satisfying $\psi_*^\Omega(t) = 0$ for a.e. $t \in (0,T)$.

We are now in the position to prove (2.8). In the remainder of the proof, we will denote by C generic positive constants that depend only on the data of the system and may change within formulas and/or even within lines. Moreover, we will argue formally, noting that all of the following arguments can be made rigorous by using difference quotients with respect to time.

Now recall that $\mathbf{v}(t)$ is divergence-free and vanishes on $\partial\Omega$ for almost all $t \in (0,T)$, whence it follows that $\mathbf{v}(t) \cdot \nabla\varphi(t) \in L^2(\Omega)$ has zero mean value. It is thus an easy consequence of (1.2) and (1.5) that $\varphi_t(t)$ belongs to $\operatorname{dom}\mathcal{N}$ for almost every $t \in (0,T)$. We may therefore (formally) differentiate the variational formulation of the state system (1.2)–(1.6) with respect to t and insert $\mathcal{N}\varphi_t(t)$ as test function. As in the proof of [27, Thm. 2.2], this leads for almost every $t \in (0,T)$ to an estimate of the form

$$\|\varphi_t(t)\|_*^2 + \int_0^t \int_\Omega \varphi_t^2\, \mathrm{d}x\, \mathrm{d}s \le \|\varphi_t(0)\|_*^2 + C \left(\int_0^t \|\varphi_t(s)\|_*^2\, \mathrm{d}s + I_1(t) + I_2(t) \right), \tag{2.14}$$

where the terms $I_1(t)$ and $I_2(t)$ originate from the convective term and will be estimated below.

Notice that $\|\varphi_t(0)\|_*$ is bounded since this is true for $\|\varphi_t(0)\|_{L^2(\Omega)}$; indeed, the assumption $\varphi_0 \in H^2(\Omega)$, in combination with (1.9) and **(H7)**, yields that

$$\operatorname{div}(m(\varphi_0)\nabla\mu(0)) \in L^2(\Omega),$$

and since $\mathbf{v} \in C^0([0,T]; L^3(\Omega)^3)$ and $\nabla\varphi_0 \in L^6(\Omega)$, it is easily seen that also $\mathbf{v}(0) \cdot \nabla\varphi_0 \in L^2(\Omega)$.

Next, we have, using the fact that $\mathbf{v}(t) \in H^1_{div}(\Omega) \cap L^\infty(\Omega)^3$ for almost every $t \in (0,T)$ and invoking (2.12),

$$\begin{aligned} I_1(t) &= \left| \int_0^t \int_\Omega (\mathbf{v} \cdot \nabla\varphi_t)\, \mathcal{N}\varphi_t\, \mathrm{d}x\, \mathrm{d}s \right| = \left| \int_0^t \int_\Omega \varphi_t\, (\mathbf{v} \cdot \nabla \mathcal{N}\varphi_t)\, \mathrm{d}x\, \mathrm{d}s \right| \\ &\le \int_0^t \|\mathbf{v}(s)\|_{L^\infty(\Omega)^3}\, \|\varphi_t(s)\|_{L^2(\Omega)}\, \|\nabla\mathcal{N}\varphi_t(s)\|_{L^2(\Omega)}\, \mathrm{d}s \\ &\le \frac{1}{2} \int_0^t \int_\Omega \varphi_t^2\, \mathrm{d}x\, \mathrm{d}s + C \int_0^t \|\varphi_t(s)\|_*^2\, \mathrm{d}s\,. \end{aligned} \tag{2.15}$$

Also,

$$
\begin{aligned}
I_2(t) \;&= \left| \int_0^t \int_\Omega (\mathbf{v}_t \cdot \nabla\varphi)\, \mathcal{N}\varphi_t \,\mathrm{d}x\,\mathrm{d}s \right| \\
&\le \int_0^t \|\mathbf{v}_t(s)\|_{L^3(\Omega)^3}\, \|\nabla\varphi(s)\|_{L^2(\Omega)^3}\, \|\mathcal{N}\varphi_t(s)\|_{L^6(\Omega)}\,\mathrm{d}s \\
&\le C + \int_0^t \|\nabla\varphi(s)\|^2_{L^2(\Omega)^3}\, \|\varphi_t(s)\|^2_*\,\mathrm{d}s\,,
\end{aligned}
\tag{2.16}
$$

where we have used that

$$
\|\mathcal{N}\varphi_t(s)\|_{L^6(\Omega)} \le C\,\|\mathcal{N}\varphi_t(s)\|_{H^1(\Omega)} \le C\,\|\nabla\mathcal{N}\varphi_t(s)\|_{L^2(\Omega)^3} \le C\,\|\varphi_t(s)\|_*\,.
$$

Combining (2.14)–(2.16), and noting that the function $s \mapsto \|\nabla\varphi(s)\|^2_{L^2(\Omega)^3}$ is known to belong to $L^1(0,T)$, we can finally verify the claim (2.8) using Gronwall's lemma.

Next, we can infer from (1.9), (2.8), and from the fact that $-\mathbf{v}\cdot\nabla\varphi \in L^2(Q)$ that

$$
\mathrm{div}(m(\varphi)\,\nabla\mu) = c_0\,\Delta\varphi + m'(\varphi)\,\nabla\varphi\cdot\nabla w + m(\varphi)\,\Delta w
$$

belongs to $L^2(Q)$. But then it follows from (2.4), (2.6), and **(H8)** that also

$$
\Delta\varphi \in L^2(Q)\,. \tag{2.17}
$$

Moreover, we know already from (2.4), (2.6), and **(H8)** that $\nabla w \in L^2(0,T;H^1(\Omega)^3) \cap L^\infty(Q)^3$, so that $\partial w/\partial\mathbf{n} = (\nabla w)_{|\partial\Omega}\cdot\mathbf{n}$ belongs to $L^2(0,T;H^{1/2}(\partial\Omega)) \cap L^\infty(\Sigma)$. Since obviously $m(\varphi)_{|\partial\Omega}$ belongs to the same space, it follows from the boundary condition (1.5) and the product rule (1.16) that the same is true for $\partial\varphi/\partial\mathbf{n}$. Hence we can infer from standard elliptic estimates that

$$
\varphi \in L^2(0,T;H^2(\Omega))\,. \tag{2.18}
$$

It then follows from the continuity of the embedding $H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^2(\Omega)) \subset C^0([0,T];H^1(\Omega))$ and from **(H7)** that also

$$
\varphi \in C^0([0,T];H^1(\Omega)), \qquad w \in C^0([0,T];H^2(\Omega)), \tag{2.19}
$$

and analogous reasoning as above shows that we also have

$$
\frac{\partial\varphi}{\partial\mathbf{n}} \in L^\infty(0,T;H^{1/2}(\partial\Omega))\,. \tag{2.20}
$$

In the next step we show that it holds (cf. eq. (4.3) in [27]) that

$$
\varphi_t \in C^0([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))\,. \tag{2.21}
$$

To this end, we differentiate the variational formulation of problem (1.2)–(1.6) with respect to time again and test by φ_t. As in [27], we obtain for every $t \in (0,T]$ an inequality of the form

(2.22)

$$
\|\varphi_t(t)\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega |\nabla\varphi_t|^2\,\mathrm{d}x\,\mathrm{d}s \le \|\varphi_t(0)\|^2_{L^2(\Omega)} + C\int_0^t\int_\Omega \varphi_t^2\,\mathrm{d}x\,\mathrm{d}s + C\,I(t),
$$

where we have $\varphi_t(0) \in L^2(\Omega)$ and where the expression $I(t)$ originating from the convective term has to be estimated. Employing (2.1), (2.8), and (2.18), and invoking Hölder's and Young's inequalities as well as the continuity of the embedding $H^1(\Omega) \subset L^6(\Omega)$, we have, for any $\delta > 0$,

$$
\begin{aligned}
I(t) &= \left| \int_0^t \int_\Omega (\mathbf{v} \cdot \nabla\varphi_t + \mathbf{v}_t \cdot \nabla\varphi)\, \varphi_t \,\mathrm{d}x\,\mathrm{d}s \right| \\
&\le \int_0^t \|\mathbf{v}(s)\|_{L^\infty(\Omega)^3} \, \|\nabla\varphi_t(s)\|_{L^2(\Omega)^3} \, \|\varphi_t(s)\|_{L^2(\Omega)} \,\mathrm{d}s \\
&\quad + \int_0^t \|\mathbf{v}_t(s)\|_{L^3(\Omega)^3} \, \|\nabla\varphi(s)\|_{L^6(\Omega)^3} \, \|\varphi_t(s)\|_{L^2(\Omega)} \,\mathrm{d}s \\
&\le \frac{C}{\delta} \left(1 + \int_0^t \left(1 + \|\mathbf{v}_t(s)\|^2_{L^3(\Omega)^3} \right) \|\varphi_t(s)\|^2_{L^2(\Omega)} \,\mathrm{d}s \right) \\
&\quad + \delta \int_0^t \int_\Omega |\nabla\varphi_t|^2 \,\mathrm{d}x\,\mathrm{d}s\,.
\end{aligned} \tag{2.23}
$$

Observing that the function $s \mapsto \|\mathbf{v}_t(s)\|^2_{L^3(\Omega)^3}$ belongs to $L^1(0,T)$, and adjusting $\delta > 0$ appropriately small, we obtain (2.21) by an application of Gronwall's lemma.

Next, we observe that we have a.e. in Q that

$$
c_0\, \Delta\varphi = \varphi_t + \mathbf{v} \cdot \nabla\varphi - m'(\varphi)\, \nabla\varphi \cdot \nabla w - m(\varphi)\, \Delta w\,, \tag{2.24}
$$

and since all terms on the right-hand side are known to belong to $L^\infty(0,T;L^2(\Omega))$, the same holds for $\Delta\varphi$. Invoking (2.19) and (2.20), we therefore obtain from standard elliptic estimates that

$$
\varphi \in L^\infty(0,T;H^2(\Omega))\,. \tag{2.25}
$$

Finally, we conclude from the continuity of the embedding $H^1(0,T;H^1(\Omega)) \cap L^\infty(0,T;H^2(\Omega)) \subset C^0([0,T];H^s(\Omega))$ for every $s \in [0,2)$ that also $\varphi \in C^0(\overline{Q})$, which concludes the proof of the assertion. ∎

Remark 1. A closer inspection of the above proof reveals that there is a constant $\widehat{C}_1 > 0$, which depends only on the data of the system and on the constant R, such that we have

$$
\|\varphi\|_{C^1([0,T];L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)) \cap L^\infty(0,T;H^2(\Omega)) \cap C^0(\overline{Q})} \le \widehat{C}_1 \tag{2.26}
$$

whenever φ is the first component of a solution (φ, w, μ) associated with some $\mathbf{v} \in \mathcal{V}_R$. But then it follows from hypotheses **(H7)** and **(H8)** that, in particular,

$$
\|w_t\|_{L^2(0,T;H^2(\Omega)) \cap C^0([0,T];H^1(\Omega))} + \|w\|_{C^0([0,T];H^2(\Omega)) \cap L^\infty(0,T;W^{1,\infty}(\Omega))} \le \widehat{C}_2, \tag{2.27}
$$

where also $\widehat{C}_2$ depends only on the data and R. Moreover, the separation property (2.3) holds even pointwise for every $(x,t) \in \overline{Q}$, whence it follows that

$$
\max_{1 \le i \le 4} \|f^{(i)}(\varphi)\|_{C^0(\overline{Q})} \le \widehat{C}_3, \tag{2.28}
$$

where, again, $\widehat{C}_3$ depends only on the data and R. Therefore, we can conclude from (1.3) that

$$
\|\mu_t\|_{C^0([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))} + \|\mu\|_{L^\infty(0,T;H^2(\Omega))} \le \widehat{C}_4, \tag{2.29}
$$

where also $\widehat{C}_4$ depends only on the data and R. In the remainder of this paper, we denote $K_1^* := \max_{1\le i\le 4} \widehat{C}_i$.

Remark 2. The separation property (2.3) and hypotheses **(H4)** and **(H5)** also entail the estimate

$$\frac{c_0}{4} \ge m(\varphi(x,t)) \ge c_0\,\kappa(1-\kappa) > 0 \quad \text{for every } (x,t)\in\overline{Q}. \tag{2.30}$$

This means that under the given hypotheses neither the possible degeneracy of m nor the possible singularity of f' can become active. Also, we may without loss of generality assume (by possibly choosing a larger K_1^*) that

$$\|m(\varphi)\|_{C^0(\overline{Q})} + \|m'(\varphi)\|_{C^0(\overline{Q})} + \|m''(\varphi)\|_{C^0(\overline{Q})} \le K_1^*\,. \tag{2.31}$$

We will now show a global stability estimate. We have the following result.

PROPOSITION 2.2. *Let hypotheses* **(H1)**–**(H8)** *be satisfied. Then there exists a constant* $K_2^* > 0$*, which depends only on the data of the state system and on* R*, such that the following holds: whenever* $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}_R$ *are given and* $\varphi_1, \varphi_2 \in C^1([0,T];L^2(\Omega))\cap H^1(0,T;H^1(\Omega))\cap L^\infty(0,T;H^2(\Omega))$ *denote the associated solutions to the state system* (1.2)–(1.6) *and* w_1, w_2 *the corresponding nonlocal operators according to* (1.4)*, then we have for* $\varphi := \varphi_1 - \varphi_2$ *and* $\mathbf{v} := \mathbf{v}_1 - \mathbf{v}_2$ *that for all* $t\in[0,T]$ *it holds that*

$$\int_0^t \|\varphi_t(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s + \max_{0\le s\le t} \|\varphi(s)\|^2_{H^1(\Omega)} \le K_2^* \int_0^t \|\mathbf{v}(s)\|^2_{L^3(\Omega)^3}\,\mathrm{d}s\,. \tag{2.32}$$

Proof. In the following, the symbol C will denote positive constants, which possibly differ from line to line or even within lines. They may depend only on the problem data and R. To begin with, we put

$$w := w_1 - w_2, \quad \mu_i := f'(\varphi_i) + w_i, \ \ i=1,2, \quad \text{and } \ \mu := \mu_1 - \mu_2$$

and observe that (φ, w, μ) satisfies

$$\varphi_t - \operatorname{div}(m(\varphi_1)\nabla\mu_1 - m(\varphi_2)\nabla\mu_2) = -\mathbf{v}\cdot\nabla\varphi_1 - \mathbf{v}_2\cdot\nabla\varphi \quad \text{a.e. in } Q\,, \tag{2.33}$$

$$\mu = f'(\varphi_1) - f'(\varphi_2) + w \quad \text{a.e. in } Q\,, \tag{2.34}$$

$$w(x,t) = -2\int_\Omega k(|x-y|)\varphi(y,t)\,\mathrm{d}y \quad \text{a.e. in } Q\,, \tag{2.35}$$

$$\varphi(0) = w(0) = \mu(0) = 0 \quad \text{a.e. in } \Omega\,. \tag{2.36}$$

We also notice that, owing to (1.3), **(H5)**, and (1.9), we have

$$\begin{aligned}(m(\varphi_1)\nabla\mu_1 - m(\varphi_2)\nabla\mu_2) &= c_0\,\nabla\varphi + (m(\varphi_1)\nabla w_1 - m(\varphi_2)\nabla w_2)\\ &= c_0\,\nabla\varphi + (m(\varphi_1)-m(\varphi_2))\nabla w_1 + m(\varphi_2)\nabla w\,.\end{aligned} \tag{2.37}$$

Hence, testing (2.33) by φ, we have, for every $t>0$,

$$\begin{aligned}\frac12\|\varphi(t)\|^2_{L^2(\Omega)} + c_0\int_0^t\!\!\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x\,\mathrm{d}s &\le \int_0^t\!\!\int_\Omega |\mathbf{v}_2||\varphi||\nabla\varphi|\,\mathrm{d}x\,\mathrm{d}s\\ &+ \int_0^t\!\!\int_\Omega |m(\varphi_1)-m(\varphi_2)||\nabla w_1||\nabla\varphi|\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\!\!\int_\Omega |m(\varphi_2)||\nabla w||\nabla\varphi|\,\mathrm{d}x\,\mathrm{d}s\\ &+ \int_0^t\!\!\int_\Omega |\mathbf{v}||\nabla\varphi_1||\varphi|\,\mathrm{d}x\,\mathrm{d}s\,.\end{aligned} \tag{2.38}$$

We denote the four integrals on the right-hand side by $I_j(t)$, $1 \le j \le 4$, in that order and estimate them individually. At first, it follows from (2.1) and Young's inequality that

$$I_1(t) \,\le\, \frac{c_0}{8} \int_0^t\!\!\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x\,\mathrm{d}s \,+\, C \int_0^t\!\!\int_\Omega |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}s\,. \tag{2.39}$$

Next, from the mean value theorem, (2.31), (2.27), and Young's inequality, we infer that

$$\begin{aligned} I_2(t) &\le \frac{c_0}{8} \int_0^t\!\!\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x\,\mathrm{d}s \,+\, C \int_0^t \|\nabla w_1(s)\|^2_{L^\infty(Q)^3}\, \|\varphi(s)\|^2_{L^2(\Omega)} \,\mathrm{d}s \\ &\le \frac{c_0}{8} \int_0^t\!\!\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x\,\mathrm{d}s \,+\, C \int_0^t\!\!\int_\Omega |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}s\,. \end{aligned} \tag{2.40}$$

Moreover, (2.30) and Young's inequality imply that

$$\begin{aligned} I_3(t) &\le \frac{c_0}{8} \int_0^t\!\!\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x\,\mathrm{d}s \,+\, C \int_0^t\!\!\int_\Omega |\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}s \\ &\le \frac{c_0}{8} \int_0^t\!\!\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x\,\mathrm{d}s \,+\, C \int_0^t\!\!\int_\Omega |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}s\,, \end{aligned} \tag{2.41}$$

where the last inequality follows from **(H7)**. Finally, we employ (2.26), Hölder's and Young's inequalities, and the continuity of the embedding $H^1(\Omega) \subset L^4(\Omega)$ to conclude that

$$\begin{aligned} I_4(t) &\le C \int_0^t \|\mathbf{v}(s)\|_{L^2(\Omega)^3}\, \|\nabla\varphi_1(s)\|_{L^4(\Omega)^3}\, \|\varphi(s)\|_{L^4(\Omega)} \,\mathrm{d}s \\ &\le \frac{c_0}{8} \int_0^t \|\varphi(s)\|^2_{H^1(\Omega)} \,\mathrm{d}s \,+\, C \int_0^t \|\varphi_1(s)\|^2_{H^2(\Omega)}\, \|\mathbf{v}(s)\|^2_{L^2(\Omega)^3} \,\mathrm{d}s \\ &\le \frac{c_0}{8} \int_0^t\!\!\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x\,\mathrm{d}s \,+\, C \int_0^t \Big(\|\varphi(s)\|^2_{L^2(\Omega)} \,+\, \|\mathbf{v}(s)\|^2_{L^2(\Omega)^3} \Big) \,\mathrm{d}s\,. \end{aligned} \tag{2.42}$$

Combining estimates (2.38)–(2.42) and invoking Gronwall's lemma, we have thus shown that for any $t \in [0, T]$ we have

$$\max_{0 \le s \le t} \|\varphi(s)\|^2_{L^2(\Omega)} \,+\, \int_0^t \|\varphi(s)\|^2_{H^1(\Omega)} \,\mathrm{d}s \,\le\, C \int_0^t \|\mathbf{v}(s)\|^2_{L^2(\Omega)^3} \,\mathrm{d}s\,, \tag{2.43}$$

where the constant C depends only on the data of the system and R.

Having established the stability estimate (2.43), we can now proceed to prove the stronger estimate (2.32). To this end, we multiply (2.33) by φ_t and integrate over $\Omega \times [0, t]$, where $t > 0$. Integration by parts and (2.37) yield that

$$\int_0^t\!\!\int_\Omega |\varphi_t|^2 \,\mathrm{d}x\,\mathrm{d}s \,+\, \frac{c_0}{2}\, \|\nabla\varphi(t)\|^2_{L^2(\Omega)^3} \,\le\, \sum_{i=1}^{3} I_i(t)\,, \tag{2.44}$$

where

$$I_1(t) := -\int_0^t\!\!\int_\Omega (m(\varphi_1)\nabla w_1 - m(\varphi_2)\nabla w_2)\cdot\nabla\varphi_t\,\mathrm{d}x\,\mathrm{d}s\,,$$

$$I_2(t) := -\int_0^t\!\!\int_\Omega (\mathbf{v}\cdot\nabla\varphi_1)\,\varphi_t\,\mathrm{d}x\,\mathrm{d}s\,,$$

$$I_3(t) := -\int_0^t\!\!\int_\Omega (\mathbf{v}_2\cdot\nabla\varphi)\,\varphi_t\,\mathrm{d}x\,\mathrm{d}s\,.$$

We estimate these expressions individually. The last two terms are easily handled. Indeed, owing to (2.26), Hölder's and Young's inequalities, and the continuity of the embedding $H^1(\Omega)\subset L^6(\Omega)$, we have, for any $\gamma>0$ (to be specified later),

$$\begin{aligned}
|I_2(t)| &\le C\int_0^t \|\mathbf{v}(s)\|_{L^3(\Omega)^3}\,\|\nabla\varphi_1(s)\|_{L^6(\Omega)^3}\,\|\varphi_t(s)\|_{L^2(\Omega)}\,\mathrm{d}s\\
&\le \gamma\int_0^t\!\!\int_\Omega|\varphi_t|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{C}{\gamma}\int_0^t\|\varphi_1(s)\|^2_{H^2(\Omega)}\,\|\mathbf{v}(s)\|^2_{L^3(\Omega)^3}\,\mathrm{d}s\\
&\le \gamma\int_0^t\!\!\int_\Omega|\varphi_t|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{C}{\gamma}\int_0^t\|\mathbf{v}(s)\|^2_{L^3(\Omega)^3}\,\mathrm{d}s\,.
\end{aligned}\tag{2.45}$$

Similarly, by also using (2.1) and (2.43), we obtain that

$$\begin{aligned}
|I_3(t)| &\le \int_0^t \|\mathbf{v}_2(s)\|_{L^\infty(\Omega)^3}\,\|\nabla\varphi(s)\|_{L^2(\Omega)^3}\,\|\varphi_t(s)\|_{L^2(\Omega)}\,\mathrm{d}x\,\mathrm{d}s\\
&\le \gamma\int_0^t\!\!\int_\Omega|\varphi_t|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{C}{\gamma}\int_0^t\|\varphi(s)\|^2_{H^1(\Omega)}\,\mathrm{d}s\\
&\le \gamma\int_0^t\!\!\int_\Omega|\varphi_t|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{C}{\gamma}\int_0^t\|\mathbf{v}(s)\|^2_{L^2(\Omega)^3}\,\mathrm{d}s\,.
\end{aligned}\tag{2.46}$$

It remains to estimate the first integral. First notice that integration by parts with respect to time, together with (2.36), yields

$$\begin{aligned}
I_1(t) = &-\int_\Omega ((m(\varphi_1)\nabla w_1 - m(\varphi_2)\nabla w_2)\cdot\nabla\varphi)\,(t)\,\mathrm{d}x\\
&+\int_0^t\!\!\int_\Omega (m(\varphi_1)\nabla w_1 - m(\varphi_2)\nabla w_2)_t\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}s =: I_{11}(t)+I_{12}(t)\,.
\end{aligned}$$

Using the mean value theorem, (2.27), (2.31), **(H7)**, and Young's inequality, we obtain

$$\begin{aligned}
|I_{11}(t)| \le &\int_\Omega |m(\varphi_1(t)) - m(\varphi_2(t))||\nabla w_1(t)||\nabla\varphi(t)|\,\mathrm{d}x\\
&+\int_\Omega |m(\varphi_2(t))||\nabla w(t)||\nabla\varphi(t)|\,\mathrm{d}x\\
\le\,& C\left(\|\varphi(t)\|_{L^2(\Omega)}\,\|\nabla w_1(t)\|_{L^\infty(\Omega)^3} + \|\nabla w(t)\|_{L^2(\Omega)^3}\right)\|\nabla\varphi(t)\|_{L^2(\Omega)^3}\\
\le\,& \gamma\,\|\nabla\varphi(t)\|^2_{L^2(\Omega)^3} + \frac{C}{\gamma}\|\varphi(t)\|^2_{L^2(\Omega)}\\
\le\,& \gamma\,\|\nabla\varphi(t)\|^2_{L^2(\Omega)^3} + \frac{C}{\gamma}\int_0^t\|\mathbf{v}(s)\|^2_{L^2(\Omega)^3}\,\mathrm{d}s\,,
\end{aligned}\tag{2.47}$$

where the last inequality follows from (2.43).

Finally, we estimate $I_{12}(t)$. We have

$$
\begin{aligned}
|I_{12}(t)| \;\leq\; & \int_0^t\!\!\int_\Omega |m'(\varphi_1)-m'(\varphi_2)||\varphi_{1,t}||\nabla w_1||\nabla\varphi|\,\mathrm{d}x\,\mathrm{d}s \\
&+ \int_0^t\!\!\int_\Omega |m'(\varphi_2)||\varphi_t||\nabla w_1||\nabla\varphi|\,\mathrm{d}x\,\mathrm{d}s \\
&+ \int_0^t\!\!\int_\Omega |m'(\varphi_2)||\varphi_{2,t}||\nabla w||\nabla\varphi|\,\mathrm{d}x\,\mathrm{d}s \\
&+ \int_0^t\!\!\int_\Omega |m(\varphi_1)-m(\varphi_2)||\nabla w_{1,t}||\nabla\varphi|\,\mathrm{d}x\,\mathrm{d}s \\
&+ \int_0^t\!\!\int_\Omega |m(\varphi_2)||\nabla w_t||\nabla\varphi|\,\mathrm{d}x\,\mathrm{d}s \\
=:\; & J_1(t)+J_2(t)+J_3(t)+J_4(t)+J_5(t)\,.
\end{aligned} \tag{2.48}
$$

We estimate the terms on the right-hand side individually. First, invoking the mean value theorem, Hölder's inequality, and the continuity of the embedding $H^1(\Omega)\subset L^4(\Omega)$, as well as (2.26), (2.27), (2.31), and (2.43), we find that for every $\gamma>0$ it holds that

$$
\begin{aligned}
|J_1(t)| &\leq C\int_0^t \|\varphi(s)\|_{L^4(\Omega)}\,\|\varphi_{1,t}(s)\|_{L^4(\Omega)}\,\|\nabla w_1(s)\|_{L^\infty(\Omega)^3}\,\|\nabla\varphi(s)\|_{L^2(\Omega)^3}\,\mathrm{d}s \\
&\leq C\max_{0\leq s\leq t}\|\varphi(s)\|_{H^1(\Omega)}\left(\int_0^t \|\nabla\varphi(s)\|^2_{L^2(\Omega)^3}\,\mathrm{d}s\right)^{1/2}\|\varphi_{1,t}\|_{L^2(0,t;H^1(\Omega))} \\
&\leq \gamma\max_{0\leq s\leq t}\|\varphi(s)\|^2_{H^1(\Omega)} + \frac{C}{\gamma}\int_0^t \|\mathbf{v}(s)\|^2_{L^2(\Omega)^3}\,\mathrm{d}s.
\end{aligned} \tag{2.49}
$$

Moreover, using (2.27), (2.31), (2.43), and Young's inequality, we have

$$
\begin{aligned}
|J_2(t)| &\leq C\int_0^t \|m'(\varphi_2(s))\|_{L^\infty(\Omega)}\,\|\varphi_t(s)\|_{L^2(\Omega)}\,\|\nabla w_1(s)\|_{L^\infty(\Omega)^3}\,\|\nabla\varphi(s)\|_{L^2(\Omega)^3}\,\mathrm{d}s \\
&\leq \gamma\int_0^t\!\!\int_\Omega |\varphi_t|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{C}{\gamma}\int_0^t \|\mathbf{v}(s)\|^2_{L^2(\Omega)^3}\,\mathrm{d}s\,.
\end{aligned} \tag{2.50}
$$

Also, invoking (2.26), (2.31), **(H7)**, the continuity of the embedding $H^1(\Omega)\subset L^4(\Omega)$, and (2.43), we find the estimate

$$
\begin{aligned}
|J_3(t)| &\leq \int_0^t \|m'(\varphi_2(s)\|_{L^\infty(\Omega)}\,\|\varphi_{2,t}(s)\|_{L^4(\Omega)}\,\|\nabla w(s)\|_{L^4(\Omega)^3}\,\|\nabla\varphi(s)\|_{L^2(\Omega)^3}\,\mathrm{d}s \\
&\leq C\int_0^t \|\varphi_{2,t}(s)\|_{H^1(\Omega)}\,\|\varphi(s)\|_{L^4(\Omega)}\,\|\nabla\varphi(s)\|_{L^2(\Omega)^3}\,\mathrm{d}s \\
&\leq C\max_{0\leq s\leq t}\|\varphi(s)\|_{H^1(\Omega)}\,\|\varphi\|_{L^2(0,t;H^1(\Omega))}\,\|\varphi_{2,t}\|_{L^2(0,t;H^1(\Omega))} \\
&\leq \gamma\max_{0\leq s\leq t}\|\varphi(s)\|^2_{H^1(\Omega)} + \frac{C}{\gamma}\int_0^t \|\mathbf{v}(s)\|^2_{L^2(\Omega)^3}\,\mathrm{d}s\,.
\end{aligned} \tag{2.51}
$$

Similarly, again using (2.26), (2.31), (2.43), together with Hölder's inequality and **(H7)**, we obtain that

$$
\begin{aligned}
(2.52)\qquad |J_4(t)| &\le C\int_0^t \|\varphi(s)\|_{L^4(\Omega)}\|\nabla w_{1,t}(s)\|_{L^4(\Omega)^3}\,\|\nabla\varphi(s)\|_{L^2(\Omega)^3}\,\mathrm{d}s\\
&\le C\int_0^t \|\varphi(s)\|_{H^1(\Omega)}\,\|\varphi_{1,t}(s)\|_{L^4(\Omega)}\,\|\nabla\varphi(s)\|_{L^2(\Omega)^3}\,\mathrm{d}s\\
&\le \gamma \max_{0\le s\le t}\,\|\varphi(s)\|^2_{H^1(\Omega)} + \frac{C}{\gamma}\int_0^t \|\mathbf{v}(s)\|^2_{L^2(\Omega)^3}\,\mathrm{d}s\,.
\end{aligned}
$$

Finally, we conclude from (2.31), **(H7)**, and (2.43), as well as Hölder's and Young's inequalities, that

$$
\begin{aligned}
(2.53)\qquad |J_5(t)| &\le C\int_0^t \|\nabla w_t(s)\|_{L^2(\Omega)^3}\,\|\nabla\varphi(s)\|_{L^2(\Omega)^3}\,\mathrm{d}s\\
&\le C\int_0^t \|\varphi_t(s)\|_{L^2(\Omega)}\,\|\varphi(s)\|_{H^1(\Omega)}\,\mathrm{d}s\\
&\le \gamma\int_0^t\!\!\int_\Omega |\varphi_t|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{C}{\gamma}\int_0^t \|\mathbf{v}(s)\|^2_{L^2(\Omega)^3}\,\mathrm{d}s\,.
\end{aligned}
$$

Combining estimates (2.44)–(2.53) and observing the continuity of the embedding $L^3(\Omega)\subset L^2(\Omega)$, we have thus shown an estimate of the form

$$
\begin{aligned}
(2.54)\qquad &(1-4\,\gamma)\int_0^t\!\!\int_\Omega |\varphi_t|^2\,\mathrm{d}x\,\mathrm{d}s + \left(\frac{c_0}{2}-\gamma\right)\|\nabla\varphi(t)\|^2_{L^2(\Omega)^3}\\
&\quad\le 3\,\gamma \max_{0\le s\le t}\,\|\varphi(s)\|^2_{H^1(\Omega)} + \frac{C}{\gamma}\int_0^t \|\mathbf{v}(s)\|^2_{L^3(\Omega)^3}\,\mathrm{d}s\,.
\end{aligned}
$$

From this, invoking (2.43), and adjusting $\gamma>0$ appropriately small, it is easily seen that (2.32) is satisfied. ∎

Remark 3. By virtue of **(H7)** and **(H8)**, the stability estimates (2.32) and (2.43) entail corresponding estimates for w and μ. In particular, we may without loss of generality assume (by possibly choosing an appropriately larger $K_2^*>0$) that for all $t\in[0,T]$ we have

$$
(2.55)\qquad \int_0^t \|\nabla w_t(s)\|^2_{L^2(\Omega)^3}\,\mathrm{d}s + \|w\|^2_{L^\infty(0,t;H^2(\Omega))} \le K_2^*\int_0^t \|\mathbf{v}(s)\|^2_{L^3(\Omega)^3}\,\mathrm{d}s\,,
$$

$$
(2.56)\qquad \int_0^t \|\nabla \mu_t(s)\|^2_{L^2(\Omega)^3}\,\mathrm{d}s + \|\mu\|^2_{L^\infty(0,t;H^2(\Omega))} \le K_2^*\int_0^t \|\mathbf{v}(s)\|^2_{L^3(\Omega)^3}\,\mathrm{d}s\,.
$$

3. Optimal control.

In this section, we study the optimal control problem **(CP)** with $\mathcal{V}_{\mathrm{ad}}$ defined as in (**H1**), and we assume that the general assumptions **(H2)**–**(H8)** are satisfied. Notice that, owing to Propositions 2.1 and 2.2, the *control-to-state operator*

$$
\mathcal{S}:\mathcal{V}_R\to C^1([0,T];L^2(\Omega))\cap H^1(0,T;H^1(\Omega))\cap L^\infty(0,T;H^2(\Omega));\ \ \mathbf{v}\mapsto\varphi
$$

is well defined and Lipschitz continuous as a mapping from $\mathcal{V}_R$ (viewed as a subset of $L^2(0,T;L^3(\Omega)^3)$) into $H^1(0,T;L^2(\Omega))\cap C^0([0,T];H^1(\Omega))$. Moreover, all of the global bounds (2.26)–(2.31), as well as all of the stability estimates (2.32), (2.43), (2.55) and (2.56), are satisfied.

We are now ready to prove existence for the control problem **(CP)**.

THEOREM 3.1. *Suppose that hypotheses* **(H1)**–**(H8)** *are fulfilled. Then problem* **(CP)** *admits a solution* $\bar{\mathbf{v}} \in \mathcal{V}_{ad}$.

Proof. Let $\{\mathbf{v}_n\} \subset \mathcal{V}_{ad}$ be a minimizing sequence for **(CP)**, and let $\varphi_n = \mathcal{S}(\mathbf{v}_n)$, $n \in \mathbb{N}$. Then it follows from **(H1)** and (2.26) that there exist $(\bar{\mathbf{v}}, \bar{\varphi})$ such that, possibly for a subsequence which is again indexed by n, we have

$$\begin{aligned}
\mathbf{v}_n \to \bar{\mathbf{v}} \quad & \text{weakly in } L^2(0,T;H^1_{div}(\Omega)) \cap H^1(0,T;L^3(\Omega)^3) \\
& \text{and weakly-star in } L^\infty(Q), \\
\varphi_n \to \bar{\varphi} \quad & \text{weakly in } H^1(0,T;H^1(\Omega)) \\
& \text{and weakly-star in } L^\infty(0,T;H^2(\Omega)), \\
\partial_t \varphi_n \to \partial_t \bar{\varphi} \quad & \text{weakly-star in } L^\infty(0,T;L^2(\Omega)).
\end{aligned}$$

Clearly, $\bar{\mathbf{v}} \in \mathcal{V}_{\rm ad}$. In addition, by virtue of standard compactness lemmas (cf. [25, Thm. 5.1, p. 58] and [32, sect. 8, Cor. 4]), we have the strong convergences

$$\begin{aligned}
\mathbf{v}_n \to \bar{\mathbf{v}} \quad & \text{strongly in } C^0([0,T];L^2(\Omega)^3), \\
\varphi_n \to \bar{\varphi} \quad & \text{strongly in } C^0([0,T];H^s(\Omega)) \quad \forall s \in [0,2),
\end{aligned}$$

which implies, in particular, that

$$\varphi_n \to \bar{\varphi} \quad \text{strongly in } C^0(\overline{Q}),$$

as well as

$$\mathbf{v}_n \cdot \nabla \varphi_n \to \bar{\mathbf{v}} \cdot \nabla \bar{\varphi} \quad \text{strongly in } L^1(Q).$$

Owing to the separation property (2.3) and the assumptions on f and m, we also have

$$f'(\varphi_n) \to f'(\bar{\varphi}) \quad \text{and} \quad m(\varphi_n) \to m(\bar{\varphi}), \quad \text{both strongly in } C^0(\overline{Q}).$$

Finally, it is easily deduced from **(H6)** that $\{w_n := \mathcal{K}(\varphi_n)\}$ converges strongly in $C^0(\overline{Q})$ to $\bar{w} := \mathcal{K}(\bar{\varphi})$ (recall (1.4) and (1.17)). In summary, we can pass to the limit as $n \to \infty$ in (1.2)–(1.6), written for $(\mathbf{v}_n, \varphi_n)$, finding that $\bar{\varphi} = \mathcal{S}(\bar{\mathbf{v}})$; i.e., the pair $(\bar{\mathbf{v}}, \bar{\varphi})$ is admissible for **(CP)**. It then follows from the weak sequential lower semicontinuity properties of J that $\bar{\mathbf{v}}$, together with the associated state $\bar{\varphi} = \mathcal{S}(\bar{\mathbf{v}})$, is a solution to **(CP)**. ∎

We now turn our interest to the derivation of necessary first-order optimality conditions for problem **(CP)**. Referring the reader to [33] for a detailed discussion and description of the various techniques related to optimality conditions, we proceed as follows: we first prove a suitable differentiability property for the control-to-state operator $\mathcal{S}$, using the *linearized* system, and then we establish the necessary optimality conditions in terms of a *variational inequality* and the associated *adjoint state equation*. In the following, we will always (unless it is explicitly stated otherwise) assume that $\bar{\mathbf{v}} \in \mathcal{V}_R$ is fixed and that $(\bar{\varphi}, \bar{w}, \bar{\mu})$ is the associated triple solving the state system, i.e.,

$$\bar{\varphi} = \mathcal{S}(\bar{\mathbf{v}}), \quad \bar{w} = \mathcal{K}(\bar{\varphi}), \quad \bar{\mu} = f'(\bar{\varphi}) + \bar{w}.$$

The linearized system. Suppose that an arbitrary $\mathbf{h} \in \mathcal{V}$ is given. As a preparation for the proof of differentiability, we consider the following system, which is obtained by linearizing the state system (1.2)–(1.6) at $\bar\varphi = S(\bar{\mathbf{v}})$:

$$\xi_t - c_0\,\Delta\xi - \operatorname{div}\left(m'(\bar\varphi)\,\xi\,\nabla\bar w - 2\,m(\bar\varphi)\,\nabla\left(\int_\Omega k(|x-y|)\,\xi(y,\,\cdot\,)\,\mathrm{d}y\right)\right) \tag{3.1}$$
$$= -\,\mathbf{h}\cdot\nabla\bar\varphi - \bar{\mathbf{v}}\cdot\nabla\xi \quad \text{a.e. in } Q\,,$$

$$\bar w(x,t) = \int_\Omega k(|x-y|)(1-2\bar\varphi(y,t))\,\mathrm{d}y \quad \text{a.e. in } Q\,, \tag{3.2}$$

$$\left(c_0\,\nabla\xi + m'(\bar\varphi)\,\xi\,\nabla\bar w - 2\,m(\bar\varphi)\,\nabla\left(\int_\Omega k(|x-y|)\,\xi(y,\,\cdot\,)\,dy\right)\right)\cdot\mathbf{n} = 0 \tag{3.3}$$
$$\text{a.e. on } \Sigma\,,$$

$$\xi(0) = 0 \quad \text{a.e. in } \Omega\,. \tag{3.4}$$

After proving that (3.1)–(3.4) has a unique solution ξ, we expect that $\xi = D\mathcal{S}(\bar{\mathbf{v}})\mathbf{h}$, where $D\mathcal{S}(\bar{\mathbf{v}})$ denotes the Fréchet derivative of $\mathcal{S}$ at $\bar{\mathbf{v}}$. Recalling the global bounds (2.26)–(2.31), we can expect the regularity

$$\xi \in H^1(0,T;L^2(\Omega)) \cap C^0([0,T];H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))\,. \tag{3.5}$$

We have the following result.

Proposition 3.2. *Suppose that hypotheses* **(H1)**–**(H8)** *are fulfilled. Then the problem* (3.1)–(3.4) *has a unique solution satisfying* (3.5)*.*

Proof. The proof is performed via a Faedo–Galerkin scheme inspired by [6, sect. 4]. To this end, we choose $\{\psi_j\}_{j\in\mathbb{N}}$ to be the family of (appropriately orthonormalized and ordered) eigenfunctions to the eigenvalue problem

$$-\Delta\psi + \psi = \lambda\,\psi \quad \text{in } \Omega\,, \qquad \frac{\partial\psi}{\partial\mathbf{n}} = 0 \quad \text{on } \partial\Omega$$

as a Galerkin basis in $H^1(\Omega)$. Putting $\xi_n(x,t) := \sum_{k=1}^n a_k(t)\psi_k(x)$, we then look for a solution to the approximating problem

$$\int_\Omega \xi_n'(t)\,\psi\,\mathrm{d}x + \int_\Omega c_0\nabla\xi_n(t)\cdot\nabla\psi\,\mathrm{d}x + \int_\Omega\Bigg(m'(\bar\varphi(t))\,\xi_n(t)\,\nabla\bar w(t) \tag{3.6}$$
$$-\,2\,m(\bar\varphi(t))\,\nabla\left(\int_\Omega k(|x-y|)\xi_n(y,t)\,\mathrm{d}y\right)\Bigg)\cdot\nabla\psi\,\mathrm{d}x$$
$$= -\int_\Omega(\mathbf{h}(t)\cdot\nabla\bar\varphi(t))\,\psi\,\mathrm{d}x - \int_\Omega(\bar{\mathbf{v}}(t)\cdot\nabla\xi_n(t))\,\psi\,\mathrm{d}x \quad \text{for } t\in(0,T],$$

$$\xi_n(0) = 0 \quad \text{a.e. in } \Omega \tag{3.7}$$

for every $\psi \in \Psi_n := \operatorname{span}\{\psi_1,\ldots,\psi_n\}$. Apparently, this is nothing but an initial value problem for a system of linear ordinary differential equations for the unknown functions $a_1,\ldots,a_n$, where, owing to the global bounds (2.26)–(2.31), all occurring coefficient functions are continuous on $[0,T]$. It is therefore a standard matter to show that there exists some $T_n \in (0,T]$ such that the ODE system has a maximal solution $\mathbf{a} := (a_1,\dots,a_n) \in C^1([0,T_n);\mathbb{R}^n)$ that specifies a solution $\xi_n \in C^1([0,T_n);H^3(\Omega))$. Observe that

$$\frac{\partial \xi_n}{\partial \mathbf{n}} = \frac{\partial \Delta\xi_n}{\partial \mathbf{n}} = 0 \quad \text{on } \Sigma \quad \forall n \in \mathbb{N}. \tag{3.8}$$

We now aim to prove a (uniform in $n \in \mathbb{N}$) estimate for ξ_n in $C^0([0,T];H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$. Once this is shown, it is a standard matter to show that $T_n = T$ and to pass to the limit as $n \to \infty$ to recover a solution with the asserted regularity to the linearized problem (3.1)–(3.4). Due to regularity of the coefficients and of the known functions in the system, we can also prove that the solution is unique, simply by testing the difference between two equations (3.1), written for two possible different solutions ξ_1 and ξ_2, by $\xi_1 - \xi_2$ and then exploiting the linearity of the problem. Since these arguments are straightforward, we can allow ourselves to be brief here and to restrict ourselves to the derivation of the asserted global bounds.

To this end, let $t \in (0,T_n)$ be arbitrary. In what follows, C_i, $i \in \mathbb{N}$, will denote positive constants that may depend on the data of the system but not on $n \in \mathbb{N}$, and we will make repeated use of the global bounds (2.26)–(2.31) and of (2.1). First observe that the global bounds and hypothesis **(H7)** imply that, for any $s \in [0,t]$,

$$\left\| m'(\bar\varphi(s))\,\xi_n(s)\,\nabla\bar w(s) \;-\; 2\,m(\bar\varphi(s))\,\nabla\left(\int_\Omega k(|x-y|)\xi_n(y,s)\,\mathrm{d}y\right)\right\|_{L^2(\Omega)} \le C_1\,\|\xi_n(s)\|_{L^2(\Omega)}. \tag{3.9}$$

Now we insert $\psi = \xi_n(t)$ in (3.6) and integrate over $[0,t]$ to find that

$$\frac12\,\|\xi_n(t)\|^2_{L^2(\Omega)} \;+\; c_0\int_0^t\!\!\int_\Omega |\nabla\xi_n|^2\,\mathrm{d}x\,\mathrm{d}s \;\le\; \sum_{j=1}^3 I_j(t), \tag{3.10}$$

with expressions $I_j(t)$, $1 \le j \le 3$, that will be specified and estimated below.

Let $\gamma > 0$ be arbitrary (to be specified later). We have, using (3.9) and Young's inequality,

$$\begin{aligned}|I_1(t)| &= \left|\int_0^t\!\!\int_\Omega\left[m'(\bar\varphi)\,\xi_n\,\nabla\bar w - 2\,m(\bar\varphi)\,\nabla\left(\int_\Omega k(|x-y|)\xi_n(y,s)\,\mathrm{d}y\right)\right]\cdot\nabla\xi_n\,\mathrm{d}x\,\mathrm{d}s\right| \\ &\le \gamma\int_0^t\!\!\int_\Omega|\nabla\xi_n|^2\,\mathrm{d}x\,\mathrm{d}s \;+\; \frac{C_2}{\gamma}\int_0^t\!\!\int_\Omega|\xi_n|^2\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{3.11}$$

Moreover,

$$\begin{aligned}|I_2(t)| \;&=\; \left|\int_0^t\!\!\int_\Omega(\mathbf{h}\cdot\nabla\bar\varphi)\,\xi_n\,\mathrm{d}x\,\mathrm{d}s\right| \;\le\; \int_0^t\|\mathbf{h}(s)\|_{L^3(\Omega)^3}\,\|\nabla\bar\varphi(s)\|_{L^6(\Omega)^3}\,\|\xi_n(s)\|_{L^2(\Omega)}\,\mathrm{d}s \\ &\le C_3\left(\int_0^t\|\mathbf{h}(s)\|^2_{L^3(\Omega)^3}\,\mathrm{d}s \;+\; \int_0^t\!\!\int_\Omega|\xi_n|^2\,\mathrm{d}x\,\mathrm{d}s\right),\end{aligned} \tag{3.12}$$

as well as

$$
\begin{aligned}
(3.13)\qquad |I_3(t)| \,&=\, \left| \int_0^t\!\!\int_\Omega (\bar{\mathbf{v}}\cdot\nabla\xi_n)\,\xi_n\,\mathrm{d}x\,\mathrm{d}s \right| \\
&\le\, C_4 \int_0^t \|\bar{\mathbf{v}}(s)\|_{L^\infty(\Omega)^3}\,\|\nabla\xi_n(s)\|_{L^2(\Omega)^3}\,\|\xi_n(s)\|_{L^2(\Omega)}\,\mathrm{d}s \\
&\le\, \gamma \int_0^t\!\!\int_\Omega |\nabla\xi_n|^2\,\mathrm{d}x\,\mathrm{d}s \,+\, \frac{C_5}{\gamma}\int_0^t\!\!\int_\Omega |\xi_n|^2\,\mathrm{d}x\,\mathrm{d}s\,.
\end{aligned}
$$

Combining the estimates (3.10)–(3.13), choosing $\gamma > 0$ small enough, and applying Gronwall's lemma, we have thus shown the estimate

$$
(3.14)\qquad \max_{0\le s\le t}\,\|\xi_n(s)\|^2_{L^2(\Omega)} \,+\int_0^t \|\xi_n(s)\|^2_{H^1(\Omega)}\,\mathrm{d}s \,\le\, C_6\int_0^t \|\mathbf{h}(s)\|^2_{L^3(\Omega)^3}\,\mathrm{d}s \,\le\, C_7\,.
$$

Next, we insert $\psi = -\Delta\xi_n(t)$ in (3.6), integrate by parts using the boundary condition (3.8), and then integrate over $[0,t]$. We then obtain

$$
\begin{aligned}
(3.15)\qquad \frac{1}{2}\|\nabla\xi_n(t)\|^2_{L^2(\Omega)^3} \,+\, c_0\int_0^t\!\!\int_\Omega |\Delta\xi_n|^2\,\mathrm{d}x\,\mathrm{d}s \,\le\, \int_0^t\!\!\int_\Omega (|g_1| \,+\, |g_2|)\,|\Delta\xi_n|\,\mathrm{d}x\,\mathrm{d}s \\
+\int_0^t\!\!\int_\Omega |\mathbf{h}||\nabla\bar\varphi||\Delta\xi_n|\,\mathrm{d}x\,\mathrm{d}s \,+\int_0^t\!\!\int_\Omega |\bar{\mathbf{v}}||\nabla\xi_n||\Delta\xi_n|\,\mathrm{d}x\,\mathrm{d}s\,,
\end{aligned}
$$

where the functions g_1, g_2 will be specified and estimated below. Now let $\gamma > 0$ be arbitrary (to be specified later). The last two integrals on the right-hand side of (3.15) are easily estimated. In fact, using the general bounds (2.26), as well as Hölder's and Young's inequalities and (2.1), we have

(3.16)

$$
\begin{aligned}
\int_0^t\!\!\int_\Omega |\mathbf{h}||\nabla\bar\varphi||\Delta\xi_n|\,\mathrm{d}x\,\mathrm{d}s \,&\le\, C_8\int_0^t \|\mathbf{h}(s)\|_{L^3(\Omega)^3}\,\|\nabla\bar\varphi(s)\|_{L^6(\Omega)^3}\,\|\Delta\xi_n(s)\|_{L^2(\Omega)}\,\mathrm{d}s \\
&\le\, \gamma\int_0^t\!\!\int_\Omega |\Delta\xi_n|^2\,\mathrm{d}x\,\mathrm{d}s \,+\, \frac{C_9}{\gamma}\int_0^t \|\mathbf{h}(s)\|^2_{L^3(\Omega)^3}\,\mathrm{d}s\,,
\end{aligned}
$$

as well as

(3.17)

$$
\begin{aligned}
\int_0^t\!\!\int_\Omega |\bar{\mathbf{v}}||\nabla\xi_n||\Delta\xi_n|\,\mathrm{d}x\,\mathrm{d}s \,&\le\, \int_0^t \|\bar{\mathbf{v}}(s)\|_{L^\infty(\Omega)^3}\,\|\nabla\xi_n(s)\|_{L^2(\Omega)^3}\,\|\Delta\xi_n(s)\|_{L^2(\Omega)}\,\mathrm{d}s \\
&\le\, \gamma\int_0^t\!\!\int_\Omega |\Delta\xi_n|^2\,\mathrm{d}x\,\mathrm{d}s \,+\, \frac{C_{10}}{\gamma}\int_0^t \|\xi_n(s)\|^2_{H^1(\Omega)}\,\mathrm{d}s\,.
\end{aligned}
$$

It remains to estimate the first integral on the right-hand side of (3.15). To this end, we first infer from the global bounds (2.26)–(2.31) that a.e. on Q it holds that

$$
|g_1|:=|\,\mathrm{div}\,[m'(\bar\varphi)\,\xi_n\,\nabla\bar w]\,| \,\le\, C_{11}\Big(|\xi_n|\,(|\nabla\bar\varphi| \,+\, |\Delta\bar w|) \,+\, |\nabla\xi_n||\nabla\bar\varphi|\Big)\,,
$$

where it is easily verified that the expression in the inner bracket, which we denote by z, is bounded in $C^0([0,T];L^2(\Omega))$. We thus have, invoking (1.10) and (1.14),

$$\begin{aligned}(3.18)\quad \int_0^t\int_\Omega |\xi_n||z||\Delta\xi_n|\,\mathrm{d}x\,\mathrm{d}s \,&\le\, \int_0^t \|\xi_n(s)\|_{L^\infty(\Omega)}\,\|z(s)\|_{L^2(\Omega)}\,\|\Delta\xi_n(s)\|_{L^2(\Omega)}\,\mathrm{d}s\\ &\le\, \gamma\int_0^t\int_\Omega |\Delta\xi_n|^2\,\mathrm{d}x\,\mathrm{d}s \,+\, \frac{C_{12}}{\gamma}\int_0^t \|\xi_n(s)\|^2_{L^\infty(\Omega)}\,\mathrm{d}s\\ &\le\, 2\,\gamma\int_0^t \|\xi_n(s)\|^2_{H^2(\Omega)}\,\mathrm{d}s \,+\, C_{13}\,(\gamma^{-1}+\gamma^{-3})\int_0^t \|\xi_n(s)\|^2_{H^1(\Omega)}\,\mathrm{d}s\,.\end{aligned}$$

Moreover, by (2.26), (1.13), and Hölder's and Young's inequalities, it holds that

(3.19)

$$\begin{aligned}\int_0^t\int_\Omega |\nabla\bar\varphi||\nabla\xi_n||\Delta\xi_n|\,\mathrm{d}x\,\mathrm{d}s \,&\le\, C_{14}\int_0^t \|\nabla\bar\varphi(s)\|_{L^6(\Omega)^3}\,\|\nabla\xi_n(s)\|_{L^3(\Omega)^3}\,\|\Delta\xi_n(s)\|_{L^2(\Omega)}\,\mathrm{d}s\\ &\le\, \gamma\int_0^t\int_\Omega |\Delta\xi_n|^2\,\mathrm{d}x\,\mathrm{d}s \,+\, \frac{C_{15}}{\gamma}\int_0^t \|\nabla\xi_n(s)\|^2_{L^3(\Omega)^3}\,\mathrm{d}s\\ &\le\, 2\,\gamma\int_0^t \|\xi_n(s)\|^2_{H^2(\Omega)}\,\mathrm{d}s \,+\, C_{16}\,(\gamma^{-1}+\gamma^{-3})\int_0^t \|\xi_n(s)\|^2_{H^1(\Omega)}\,\mathrm{d}s\,.\end{aligned}$$

Finally, notice that for a.e. $(x,t)\in Q$ we have

$$|g_2(x,t)| \,:=\, \left|\operatorname{div}\left[2\,m(\bar\varphi(x,t))\,\nabla\biggl(\int_\Omega k(|x-y|)\,\xi_n(y,t)\,\mathrm{d}y\biggr)\right]\right|\,,$$

and it easily follows from (2.26), (2.31), and hypotheses **(H7)** and **(H8)** that

$$\int_0^t \|g_2(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s \,\le\, C_{17}\int_0^t \|\xi_n(s)\|^2_{H^1(\Omega)}\,\mathrm{d}s\,,$$

whence we obtain that

$$(3.20)\qquad \int_0^t\int_\Omega |g_2||\Delta\xi_n|\,\mathrm{d}x\,\mathrm{d}s \,\le\, \gamma\int_0^t\int_\Omega |\Delta\xi_n|^2\,\mathrm{d}x\,\mathrm{d}s \,+\, \frac{C_{18}}{\gamma}\int_0^t \|\xi_n(s)\|^2_{H^1(\Omega)}\,\mathrm{d}s\,.$$

Now observe that $\partial\xi_n/\partial\mathbf{n}=0$, so that standard elliptic estimates imply that

$$\|\xi_n(s)\|_{H^2(\Omega)} \,\le\, C_{19}\left(\|\Delta\xi_n(s)\|_{L^2(\Omega)} \,+\, \|\xi_n(s)\|_{H^1(\Omega)}\right)\,,$$

where $C_{19}>0$ depends only on Ω. Therefore, choosing $\gamma>0$ appropriately small and invoking (3.14), we can infer from the estimates (3.15)–(3.20) that

$$(3.21)\quad \max_{0\le s\le t}\|\xi_n(s)\|^2_{H^1(\Omega)} \,+\, \int_0^t \|\xi_n(s)\|^2_{H^2(\Omega)}\,\mathrm{d}s \,\le\, C_{20}\int_0^t \|\mathbf{h}(s)\|^2_{L^3(\Omega)^3}\,\mathrm{d}s \,\le\, C_{21}\,.$$

This concludes the proof of the assertion. ∎

Remark 4. From (3.21) it follows, in particular, that the linear mapping $\mathbf{h}\mapsto\xi=:\xi^{\mathbf{h}}$ is continuous as a mapping from $\mathcal{V}$ into the space $C^0([0,T];H^1(\Omega))\cap L^2(0,T;H^2(\Omega))$.

Differentiability of the control-to-state mapping. In this section we are going to prove the following result.

PROPOSITION 3.3. *Let hypotheses* **(H1)**–**(H8)** *be satisfied. Then the control-to-state operator*

$$\mathcal{S}\,:\,\mathcal{V}_R \to C^1([0,T];L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)) \cap L^\infty(0,T;H^2(\Omega)), \quad \mathbf{v} \mapsto \varphi\,,$$

is Fréchet differentiable in $\mathcal{V}_R$ *as a mapping from* $\mathcal{V}$ *into* $\mathcal{Y} := C^0([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$*, and for every* $\bar{\mathbf{v}} \in \mathcal{V}_R$ *the Fréchet derivative* $D\mathcal{S}(\bar{\mathbf{v}}) \in \mathcal{L}(\mathcal{V},\mathcal{Y})$ *is defined as follows: for every* $\mathbf{h} \in \mathcal{V}$ *we have*

$$D\mathcal{S}(\bar{\mathbf{v}})\mathbf{h} = \xi^{\mathbf{h}}\,, \tag{3.22}$$

where $\xi^{\mathbf{h}}$ *is the unique solution to the linearized system* (3.1)–(3.4) *with* $\bar{\varphi} = \mathcal{S}(\bar{\mathbf{v}})$.

Proof. Let $\bar{\mathbf{v}} \in \mathcal{V}_R$ be fixed, and let $\bar{\varphi} = \mathcal{S}(\bar{\mathbf{v}})$. Since $\mathcal{V}_R$ is open, there is some $\Lambda > 0$ such that $\bar{\mathbf{v}} + \mathbf{h} \in \mathcal{V}_R$ whenever $\|\mathbf{h}\|_{\mathcal{V}} \le \Lambda$. In the following, we consider only such perturbations $\mathbf{h}$ and set

$$\mathbf{v}^{\mathbf{h}} = \bar{\mathbf{v}} + \mathbf{h}, \quad \varphi^{\mathbf{h}} = \mathcal{S}(\mathbf{v}^{\mathbf{h}}), \quad y^{\mathbf{h}} = \varphi^{\mathbf{h}} - \bar{\varphi} - \xi^{\mathbf{h}}\,.$$

Since the linear mapping $\mathbf{h} \mapsto \xi^{\mathbf{h}}$ is by Remark 4 continuous as a mapping from $\mathcal{V}$ into $\mathcal{Y}$, it suffices to show that there exists an increasing mapping $Z\,:\,(0,+\infty) \to (0,+\infty)$ such that $\lim_{\lambda\searrow 0} Z(\lambda)/\lambda^2 = 0$ and

$$\|y^{\mathbf{h}}\|^2_{C^0([0,T];L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \le Z(\|\mathbf{h}\|_{\mathcal{V}})\,. \tag{3.23}$$

In the following, we will denote by C_i, $i \in \mathbb{N}$, positive constants that may depend on the data of the system and on R but not on the special choice of $\mathbf{h} \in \mathcal{V}$ with $\|\mathbf{h}\|_{\mathcal{V}} \le \Lambda$. For a shorter exposition, we also often omit the arguments of the involved functions if no confusion may arise. Notice that the global bounds (2.26), (2.28), and (2.31) are satisfied by $\varphi^{\mathbf{h}}$ for any perturbation $\mathbf{h}$ with $\|\mathbf{h}\|_{\mathcal{V}} \le \Lambda$, and, owing to the weak sequential lower semicontinuity of norms, it follows from (3.21) that for all such perturbations we have

$$\|\xi^{\mathbf{h}}\|_{C^0([0,T];H^1(\Omega))\cap L^2(0,T;H^2(\Omega))} \,\le\, C_1\,. \tag{3.24}$$

First of all, it is easily verified that $y^{\mathbf{h}}$ is a strong solution to the following system:

$$\begin{aligned} y^{\mathbf{h}}_t - c_0\,\Delta y^{\mathbf{h}} - \operatorname{div}\Bigg[& m(\varphi^{\mathbf{h}})\nabla\left(\int_\Omega k(|x-y|)(1-2\varphi^{\mathbf{h}})\,\mathrm{d}y\right) \\ & - m(\bar{\varphi})\nabla\left(\int_\Omega k(|x-y|)(1-2\bar{\varphi})\,\mathrm{d}y\right) \\ & - m'(\bar{\varphi})\,\xi^{\mathbf{h}}\,\nabla\left(\int_\Omega k(|x-y|)(1-2\bar{\varphi})\,\mathrm{d}y\right) \\ & + 2\,m(\bar{\varphi})\nabla\left(\int_\Omega k(|x-y|)\,\xi^{\mathbf{h}}(y)\,\mathrm{d}y\right)\Bigg] \\ + \mathbf{v}\cdot\nabla y^{\mathbf{h}} \, + \, & \mathbf{h}\cdot(\nabla\varphi^{\mathbf{h}} - \nabla\bar{\varphi}) = 0 \quad \text{a.e. in } Q\,, \end{aligned} \tag{3.25}$$

$$\begin{aligned}
(3.26)\qquad &\Bigg[c_0\,\nabla y^{\mathbf{h}} + m(\varphi^{\mathbf{h}})\nabla\bigg(\int_\Omega k(|x-y|)(1-2\varphi^{\mathbf{h}})\,\mathrm{d}y\bigg)\\
&-m(\bar\varphi)\nabla\bigg(\int_\Omega k(|x-y|)(1-2\bar\varphi)\,\mathrm{d}y\bigg)\\
&-m'(\bar\varphi)\,\xi^{\mathbf{h}}\,\nabla\bigg(\int_\Omega k(|x-y|)(1-2\bar\varphi)\,\mathrm{d}y\bigg)\\
&+2\,m(\bar\varphi)\nabla\bigg(\int_\Omega k(|x-y|)\xi^{\mathbf{h}}(y)\,\mathrm{d}y\bigg)\Bigg]\cdot\mathbf{n}=0\quad\text{a.e. on }\Sigma,
\end{aligned}$$

$$(3.27)\qquad y^{\mathbf{h}}(0)=0\quad\text{a.e. in }\Omega.$$

We now test (3.25) by $y^{\mathbf{h}}$, integrate over $(0,t)$, where $t\in(0,T]$, and use (3.26) and (3.27) to get

$$\begin{aligned}
(3.28)\qquad &\frac12\|y^{\mathbf{h}}(t)\|^2_{L^2(\Omega)} + \int_0^t\!\!\int_\Omega|\nabla y^{\mathbf{h}}|^2\,\mathrm{d}x\,\mathrm{d}s\\
&+\int_0^t\!\!\int_\Omega\nabla y^{\mathbf{h}}\cdot\Bigg\{m(\varphi^{\mathbf{h}})\nabla\bigg(\int_\Omega k(|x-y|)(1-2\varphi^{\mathbf{h}})\,\mathrm{d}y\bigg)\\
&\qquad-m(\bar\varphi)\nabla\bigg(\int_\Omega k(|x-y|)(1-2\bar\varphi)\,\mathrm{d}y\bigg)\\
&\qquad-m'(\bar\varphi)\,\xi^{\mathbf{h}}\,\nabla\bigg(\int_\Omega k(|x-y|)(1-2\bar\varphi)\,\mathrm{d}y\bigg)\\
&\qquad+2\,m(\bar\varphi)\,\nabla\bigg(\int_\Omega k(|x-y|)\xi^{\mathbf{h}}(y)\,\mathrm{d}y\bigg)\Bigg\}\,\mathrm{d}x\,\mathrm{d}s\\
&+\int_0^t\!\!\int_\Omega y^{\mathbf{h}}(\bar{\mathbf{v}}\cdot\nabla y^{\mathbf{h}})\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\!\!\int_\Omega\mathbf{h}\cdot(\nabla\varphi^{\mathbf{h}}-\nabla\bar\varphi)\,y^{\mathbf{h}}\,\mathrm{d}x\,\mathrm{d}s = 0.
\end{aligned}$$

We have $\int_0^t\!\int_\Omega y^{\mathbf{h}}\,(\bar{\mathbf{v}}\cdot\nabla y^{\mathbf{h}})\,\mathrm{d}x\,\mathrm{d}s=0$ since $\bar{\mathbf{v}}$ vanishes on $\partial\Omega$ and is divergence-free. Moreover, using Hölder's and Young's inequalities, as well as the stability estimate (2.32) and the continuity of the embedding $H^1(\Omega)\subset L^6(\Omega)$, we obtain that

$$\begin{aligned}
&(3.29)\\
&\left|\int_0^t\!\!\int_\Omega\mathbf{h}\cdot(\nabla\varphi^{\mathbf{h}}-\nabla\bar\varphi)y^{\mathbf{h}}\,\mathrm{d}x\,\mathrm{d}s\right|\le\int_0^t\|\nabla(\varphi^{\mathbf{h}}-\bar\varphi)(s)\|_{L^2(\Omega)^3}\|y^{\mathbf{h}}(s)\|_{L^6(\Omega)}\|\mathbf{h}(s)\|_{L^3(\Omega)^3}\,\mathrm{d}s\\
&\quad\le\gamma\int_0^t\|y^{\mathbf{h}}(s)\|^2_{H^1(\Omega)}\,\mathrm{d}s+\frac{C_2}{\gamma}\int_0^t\|\mathbf{h}(s)\|^2_{L^3(\Omega)}\,\|\nabla(\varphi^{\mathbf{h}}-\bar\varphi)(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s\\
&\quad\le\gamma\int_0^t\|y^{\mathbf{h}}(s)\|^2_{H^1(\Omega)}\,\mathrm{d}s+\frac{C_3}{\gamma}\bigg(\int_0^t\|\mathbf{h}(s)\|^2_{L^3(\Omega)}\,\mathrm{d}s\bigg)^2
\end{aligned}$$

for every positive γ (to be chosen later).

It remains to estimate the third summand in (3.28). To this end, we observe that the expression in the curly bracket in (3.28) equals the sum of the following three expressions:

$$A_1(x,s) := \big(m(\varphi^{\mathbf{h}}) - m(\bar\varphi) - m'(\bar\varphi)\,\xi^{\mathbf{h}}\big)(x,s)\,\nabla\int_\Omega k(|x-y|)\,(1-2\,\bar\varphi(y,s))\,\mathrm{d}y,$$

$$A_2(x,s) := -2\,(m(\varphi^{\mathbf{h}}) - m(\bar\varphi))(x,s)\,\nabla\int_\Omega k(|x-y|)(\varphi^h(y,s)-\bar\varphi(y,s))\,\mathrm{d}y,$$

$$A_3(x,s) := -2\,m(\bar\varphi(x,s))\,\nabla\int_\Omega k(|x-y|)\,y^{\mathbf{h}}(y,s)\,\mathrm{d}y\,.$$

Moreover, Taylor's theorem, using also the separation property (2.3) and the global bounds (2.31), yields that a.e. in Q it holds that

$$(3.30)\quad |m(\varphi^{\mathbf{h}}) - m(\bar\varphi) - m'(\bar\varphi)\xi^{\mathbf{h}}| \le C_4\,|y^{\mathbf{h}}| + \frac{1}{2}\max_{\kappa\le\sigma\le 1-\kappa}|m''(\sigma)||\varphi^{\mathbf{h}}-\bar\varphi|^2 \le C_4\,|y^{\mathbf{h}}| + C_5\,|\varphi^{\mathbf{h}}-\bar\varphi|^2\,.$$

Now, by virtue of hypothesis **(H7)** and (2.27), and by invoking Hölder's and Young's inequalities, we have

$$(3.31)\int_0^t\!\!\int_\Omega |A_1||\nabla y^{\mathbf{h}}|\,\mathrm{d}x\,\mathrm{d}s \le C_6\int_0^t\!\!\int_\Omega \big(|y^{\mathbf{h}}| + |\varphi^{\mathbf{h}}-\bar\varphi|^2\big)\,|\nabla y^{\mathbf{h}}|\,\mathrm{d}x\,\mathrm{d}s$$
$$\le \int_0^t\!\!\int_\Omega\left(\gamma|\nabla y^{\mathbf{h}}|^2 + \frac{C_7}{\gamma}\,|y^{\mathbf{h}}|^2\right)\mathrm{d}x\,\mathrm{d}s + \int_0^t\Big(\|\varphi^{\mathbf{h}}-\bar\varphi\|^2_{L^4(\Omega)}\|\nabla y^{\mathbf{h}}\|_{L^2(\Omega)^3}\Big)(s)\,\mathrm{d}s$$
$$\le 2\,\gamma\int_0^t\!\!\int_\Omega|\nabla y^{\mathbf{h}}|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{C_8}{\gamma}\left(\int_0^t\!\!\int_\Omega|y^{\mathbf{h}}|^2\,\mathrm{d}x\,\mathrm{d}s + \left(\int_0^t\|\mathbf{h}(s)\|^2_{L^3(\Omega)^3}\,\mathrm{d}s\right)^2\right),$$

where again (2.32) was employed. Similarly,

$$(3.32)\quad \int_0^t\!\!\int_\Omega |A_2||\nabla y^{\mathbf{h}}|\,\mathrm{d}x\,\mathrm{d}s \le C_9\int_0^t\|\nabla y^{\mathbf{h}}(s)\|_{L^2(\Omega)^3}\,\|\varphi^{\mathbf{h}}(s)-\bar\varphi(s)\|^2_{L^4(\Omega)}\,\mathrm{d}s$$
$$\le \gamma\int_0^t\!\!\int_\Omega|\nabla y^{\mathbf{h}}|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{C_{10}}{\gamma}\int_0^t\|\varphi^{\mathbf{h}}(s)-\bar\varphi(s)\|^4_{L^4(\Omega)}\,\mathrm{d}s\,,$$

as well as, using hypothesis **(H7)** once more,

(3.33)

$$\int_0^t\!\!\int_\Omega |A_3||\nabla y^{\mathbf{h}}|\,\mathrm{d}x\,\mathrm{d}s \le C_{11}\int_0^t\|\nabla y^{\mathbf{h}}(s)\|_{L^2(\Omega)^3}\left\|\nabla\int_\Omega k(|x-\eta|)y^{\mathbf{h}}(\eta,s)\,\mathrm{d}\eta\right\|_{L^2(\Omega)^3}\mathrm{d}s$$
$$\le \gamma\int_0^t\!\!\int_\Omega|\nabla y^{\mathbf{h}}|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{C_{12}}{\gamma}\int_0^t\!\!\int_\Omega|y^{\mathbf{h}}|^2\,\mathrm{d}x\,\mathrm{d}s\,.$$

Collecting the estimates (3.28), (3.29), and (3.31)–(3.33), choosing $\gamma > 0$ small enough, and invoking the stability estimate (2.32), we can finally conclude from Gronwall's lemma that

$$(3.34)\quad \|y^{\mathbf{h}}\|^2_{C^0([0,T];L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \le C_{13}\left(\int_0^T\|\mathbf{h}(t)\|^2_{L^3(\Omega)^3}\,\mathrm{d}t\right)^2 \le C_{14}\,\|\mathbf{h}\|^4_{\mathcal{V}}\,.$$

This concludes the proof of Proposition 3.3. ∎

Using the convexity of $\mathcal{V}_{\mathrm{ad}}$, we immediately conclude from Proposition 3.3 the following result.

COROLLARY 3.4. *Assume that hypotheses* **(H1)**–**(H8)** *are fulfilled, and let* $\bar{\mathbf{v}} \in \mathcal{V}_{\mathrm{ad}}$ *be an optimal control for problem* **(CP)** *with associated state* $\bar{\varphi} = \mathcal{S}(\bar{\mathbf{v}})$. *Then we have for every* $\mathbf{v} \in \mathcal{V}_{\mathrm{ad}}$ *the inequality*

$$\beta_1 \int_0^T\!\!\int_\Omega (\bar{\varphi} - \varphi_Q)\, \xi^{\mathbf{h}} \,\mathrm{d}x\,\mathrm{d}s \,+\, \beta_2 \int_\Omega (\bar{\varphi}(T) - \varphi_\Omega)\, \xi^{\mathbf{h}}(T)\,\mathrm{d}x \\ +\, \beta_3 \int_0^T\!\!\int_\Omega \bar{\mathbf{v}} \cdot (\mathbf{v} - \bar{\mathbf{v}})\,\mathrm{d}x\,\mathrm{d}s \,\geq\, 0\,, \tag{3.35}$$

where $\xi^{\mathbf{h}}$ *is the unique solution to the linearized system* (3.1)–(3.4) *associated with* $\mathbf{h} = \mathbf{v} - \bar{\mathbf{v}}$.

The adjoint system and first-order necessary optimality conditions. In order to establish the necessary first-order optimality conditions for **(CP)**, we need to eliminate $\xi^{\mathbf{h}}$ from inequality (3.35). To this end, we introduce the *adjoint system*, which formally reads as follows:

$$-\,p_t - c_0\,\Delta p - \nabla p \cdot \left[\bar{\mathbf{v}} + m'(\bar{\varphi})\nabla\left(\int_\Omega k(|x-y|)(1-2\bar{\varphi}(y,t))\,\mathrm{d}y\right)\right] \\ -\,2\int_\Omega \nabla p(y,t)\, m(\bar{\varphi}(y,t)) \cdot \nabla k(|x-y|)\,\mathrm{d}y = \beta_1(\bar{\varphi} - \varphi_Q) \quad \text{in } Q\,, \tag{3.36}$$

$$\frac{\partial p}{\partial \mathbf{n}} = 0 \quad \text{on } \Sigma\,, \tag{3.37}$$

$$p(T) = \beta_2(\bar{\varphi}(T) - \varphi_\Omega) \quad \text{a.e. in } \Omega\,. \tag{3.38}$$

Since the final value $p(T)$ belongs only to $L^2(\Omega)$, we can at best expect the regularity

$$p \in H^1(0,T;H^1(\Omega)^*) \cap C^0([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))\,,$$

which entails that (3.36)–(3.37) must be understood in the weak variational sense. To this end, we rewrite (3.36)–(3.37) in the form

$$\langle p_t(t), \eta\rangle \,+\, c_0 \int_\Omega \nabla p(t) \cdot \nabla \eta\,\mathrm{d}x \\ -\int_\Omega \eta\,\nabla p(t) \cdot \left[\bar{\mathbf{v}}(t) \,+\, m'(\bar{\varphi}(t))\nabla\left(\int_\Omega k(|x-y|)(1-2\bar{\varphi}(y,t))\,\mathrm{d}y\right)\right]\mathrm{d}x \\ -2\int_\Omega \eta \int_\Omega \nabla p(y,t)\, m(\bar{\varphi}(y,t)) \cdot \nabla k(|x-y|)\,\mathrm{d}y\,\mathrm{d}x \,=\, \int_\Omega \eta\,\beta_1\,(\bar{\varphi}(t) - \varphi_Q(t))\,\mathrm{d}x \tag{3.39}$$

for every $\eta \in H^1(\Omega)$ and almost every $t \in (0,T)$.

We have the following existence and uniqueness result.

PROPOSITION 3.5. *The adjoint system* (3.36)–(3.38)*, written in the weak form* (3.39)*, has a unique solution*

$$p \in H^1(0,T;H^1(\Omega)^*) \cap C^0([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))\,.$$

Proof. The proof is analogous to the first part of the proof of Proposition 3.2. In fact, one can again devise a Faedo–Galerkin approximation scheme, for which estimates similar to those leading to (3.14) can be performed. An estimate resembling (3.21) cannot be derived since $\beta_1\,(\bar\varphi(T)-\varphi_\Omega)$ does not necessarily belong to $H^1(\Omega)$. One then obtains a weak solution that enjoys the asserted regularity and turns out to be unique. Since these arguments are rather standard and straightforward, we can allow ourselves to omit the details here. ∎

We are now in the position to eliminate $\xi^{\mathbf{h}}$ from (3.35). We have the following result.

THEOREM 3.6. *Assume that hypotheses* **(H1)**–**(H8)** *are fulfilled, and let* $\bar{\mathbf{v}}\in\mathcal{V}_{\rm ad}$ *be an optimal control for problem* **(CP)** *with associated state* $\bar\varphi=\mathcal{S}(\bar{\mathbf{v}})$ *and adjoint state* p. *Then we have for every* $\mathbf{v}\in\mathcal{V}_{\rm ad}$ *the inequality*

$$\beta_3\int_0^T\!\!\int_\Omega \bar{\mathbf{v}}\cdot(\mathbf{v}-\bar{\mathbf{v}})\,{\rm d}x\,{\rm d}t\,+\,\int_0^T\!\!\int_\Omega p(\mathbf{v}-\bar{\mathbf{v}})\cdot\nabla\bar\varphi\,{\rm d}x\,{\rm d}t\,\ge\,0\,. \tag{3.40}$$

Proof. This is a standard calculation that can be left to the reader. We note only that we have

$$\begin{aligned}
&\beta_1\int_0^T\!\!\int_\Omega(\bar\varphi-\varphi_Q)\,\xi^{\mathbf{h}}\,{\rm d}x\,{\rm d}t\,+\,\beta_2\int_\Omega(\bar\varphi(T)-\varphi_\Omega)\,\xi^{\mathbf{h}}(T)\,{\rm d}x\\
&=\beta_1\int_0^T\!\!\int_\Omega(\bar\varphi-\varphi_Q)\,\xi^{\mathbf{h}}\,{\rm d}x\,{\rm d}t\,+\,\int_0^T\big(\langle p_t(t),\xi^{\mathbf{h}}(t)\rangle\,+\,\langle\xi^{\mathbf{h}}_t(t),p(t)\rangle\big)\,{\rm d}t\\
&=\int_0^T\!\!\int_\Omega p\,(\mathbf{v}-\bar{\mathbf{v}})\cdot\nabla\bar\varphi\,{\rm d}x\,{\rm d}t\,,
\end{aligned}\tag{3.41}$$

where the last equality easily follows from expressing $p_t(t)$ and $\xi^{\mathbf{h}}_t(t)$ via the adjoint equation (3.39) and the linearized system (3.1)–(3.4) and then integrating by parts. ∎

Remark 5. The state system (1.2)–(1.6), written for $\varphi=\bar\varphi$, the adjoint system, and the variational inequality (3.40) together form the first-order necessary optimality conditions. Observe that we have $p\in L^2(0,T;H^1(\Omega))\cap C^0([0,T];L^2(\Omega))$ and $\bar\varphi\in L^\infty(0,T;H^2(\Omega))$, whence it follows that $p\,\nabla\bar\varphi\in L^2(Q)^3\cap L^\infty(0,T;L^{3/2}(\Omega)^3)$, so that the variational inequality (3.40) is meaningful. Moreover, since $\mathcal{V}_{\rm ad}$ is a nonempty, closed, and convex subset of $L^2(Q)^3$, we can infer from (3.40) that for $\beta_3>0$ the optimal control $\bar{\mathbf{v}}$ is the $L^2(Q)^3$-orthogonal projection of $-\beta_3^{-1}p\,\nabla\bar\varphi$ onto $\mathcal{V}_{\rm ad}$. In particular, if the function $\widetilde{\mathbf{v}}=(\widetilde v_1,\widetilde v_2,\widetilde v_3)\in L^2(Q)^3$, which is given by

$$\widetilde v_i(x,t):=\max\,\big\{\widetilde v_{1_i}(x,t),\,\min\,\big\{\widetilde v_{2_i}(x,t),\,-\beta_3^{-1}\,p(x,t)\,\partial_i\bar\varphi(x,t)\big\}\big\} \tag{3.42}$$

for $i=1,2,3$ and almost every $(x,t)\in Q$, belongs to $\mathcal{V}_{\rm ad}$, then $\widetilde{\mathbf{v}}=\bar{\mathbf{v}}$, and the optimal control $\bar{\mathbf{v}}$ turns out to be a pointwise projection. Notice, however, that the requirement $\widetilde{\mathbf{v}}\in\mathcal{V}_{\rm ad}$ implies that we should have $\widetilde{\mathbf{v}}_t\in L^2(0,T;L^3(\Omega)^3)$, which in general cannot be expected since we can guarantee only the regularity $p_t\in L^2(0,T;H^1(\Omega)^*)$. Therefore, the information about the optimal control that can be recovered from the projection property may be rather weak, in general. This is in contrast to the nonconvective local case (see, e.g., [21, Thm. 3.16]) and to the convective local two-dimensional case (see [36], where different boundary conditions are considered); it is in fact the price to be paid for considering the three-dimensional case with the flow velocity as the control parameter.

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