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Optimal Domains for Kernel Operators via Interpolation

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Abstract. The problem of finding optimal lattice domains for kernel operators with values in rearrangement invariant spaces on the interval [0,1] is considered. The techniques used are based on interpolation theory and integration with respect to $C([0,1])$–valued measures.

1. Introduction

Given a continuous linear operator T between spaces E and X, there arise situations when T has a natural extension (still with values in X) to a larger space F into which E is continuously embedded (e. g. as in the Riesz representation theorem for $C(K)$–spaces). A basic problem in this regard is to identify the optimal F within a particular class of spaces. Recently Edmunds, Kerman and Pick have studied this problem for the Sobolev embedding; see [12]. Our aim here is to study this "optimal extension procedure" for particular kinds of operators T and spaces E and X.

More precisely, let E be a Banach function space on [0,1] and X be a Banach space. We will denote by $[T, X]$ the *optimal lattice domain* for T, that is, the maximal Banach function space (containing E) to which T can be extended as a continuous linear operator, in the sense that any other Banach function space F to which T can be extended, is continuously embedded in $[T, X]$. Under certain conditions we associate to the operator T (linear, but not necessarily continuous) a vector measure ν with values in X such that E is continuously embedded into the space $L^1(\nu)$ of ν–integrable functions (see Section 2 for the definition) and that the integration operator $f \mapsto \int f\, d\nu$ extends T from E to the bigger domain space $L^1(\nu)$. In this case, $L^1(\nu)$ is the optimal lattice domain for T within the class of Banach function spaces with absolutely continuous norm. We will identify the situations in which $[T, X] = L^1(\nu)$, with the advantage, in this case, that the role of the properties of ν (which come

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from T) and X in determining the space $L^1(\nu)$, hence also the space $[T, X]$, are well understood; see Section 3.

Our attention will be focused on the case when T takes its values in a function space X on [0,1] and is generated by a kernel. That is, suppose there is a measurable function $K : [0,1] \times [0,1] \to \mathbf{R}$ such that $Tf(x) = \int_0^1 f(y)K(x,y)\,dy$, for each function f in the domain of T; denote T by T_K in this case. The measure ν associated to the operator T_K is given by $\nu(A)(x) = \int_A K(x,y)\,dy$, for each Borel set A. In Section 4 we determine properties of K which ensure that ν takes its values in $C([0,1])$. Then ν is also σ–additive as an X–valued measure for any Banach function space X on [0,1]; denote ν by ν_X in this case.

Many interesting kernels of the above kind occur. In the theory of semigroups of operators, for instance, the Gauss–Weierstrass kernel (on the circle) [9, §I.10], and the kernel $K(x,y) = \exp(-\lambda(y-x))\chi_{[x,1]}(y)$, where λ is a real constant, arising from the nilpotent left translation semigroup [14], are well known. Other important examples are the kernel of the Riemann–Liouville fractional integral $K(x,y) = |x-y|^{\alpha-1}$, for $0 < \alpha < 1$, and the Volterra kernel $K(x,y) = \chi_\Delta(x,y)$, where $\Delta = \{(x,y) \in [0,1]^2 : 0 \le y \le x\}$.

Our main interest is in identifying the extended spaces $\left[T_K, X\right]$ for large classes of kernels K and spaces X. This is the purpose of Section 5, where the methods of interpolation theory and vector integration play a crucial role. Attention is focused on kernels K which are non–negative and monotone. In these cases it is possible to identify precisely the spaces $\left[T_K, X\right]$ when either $X = C([0,1])$ or $X = L^1([0,1])$. If K additionally satisfies a certain constraint, then a precise description of $\left[T_K, X\right]$ is also possible for an extensive class of spaces X. For instance, suppose that K is non–negative and monotone and $X = \left(L^1, L^\infty\right)_\rho$ is any rearrangement invariant space on [0,1]. Then, with equivalence of norms, $\left[T_K, X\right]$ is the interpolation space $\left(L^1_\omega, L^1_\xi\right)_\rho$ (for the definition of these spaces, see Section 2), where the weights are given by $\xi(y) = K(1,y)$ and $\omega(y) = \int_0^1 K(x,y)\,dx$. Moreover, if, in addition, X has absolutely continuous norm, then $\left[T_K, X\right] = L^1\left(\nu_X\right)$. We note that the required constraint holds for the Volterra kernel and the kernel generated by the nilpotent left translation semigroup, but not for all non–negative, monotone kernels; see Section 5. The types of conditions imposed on the kernel K and the space X are essentially optimal. Counterexamples are exhibited when K fails to satisfy the constraint condition mentioned above, or the norm in X is not absolutely continuous. We end by considering the classical fractional integral operator, which has no monotonicity properties.

2. Preliminaries

Throughout the paper, $\mathcal{B}$ is the σ–algebra of all Borel subsets of [0,1]. The Banach space $C([0,1])$ consists of all continuous, $\mathbf{R}$–valued functions on [0,1] equipped with the supremum norm. All functions are $\mathbf{R}$–valued and all Banach spaces are over $\mathbf{R}$.

A Banach function space X on [0,1] is a Banach space of integrable functions on [0,1], which contains the simple functions and satisfies $g \in X$ with $\|g\| \le \|f\|$ whenever $f \in X$ and $|g| \le |f|$. So $L^\infty([0,1]) \subset X \subset L^1([0,1])$ continuously; [17, 1.b.17]. It is rearrangement invariant if it satisfies the Fatou property and $f \in X$ implies that $g \in X$ with $\|g\| = \|f\|$ whenever g and f are equimeasurable; [2, II.4.1]. In particular, if f^* is the decreasing rearrangement of $f \in X$, then $f^* \in X$ with $\|f^*\| = \|f\|$. An important property of Banach function spaces is absolute continuity of the norm: order bounded, increasing sequences are norm convergent; [2, I.3.1].

We recall the K–method of Peetre. If (X_0, X_1) is a compatible pair of Banach spaces, then the K–functional of $f \in X_0 + X_1$ is, for $t > 0$,

$$K(t, f; X_0, X_1) \;=\; \inf\{\|f_0\| + t\,\|f_1\| : f = f_0 + f_1;\; f_0 \in X_0,\; f_1 \in X_1\}\,.$$

From a rearrangement invariant norm ρ on [0,1] (see [2, I.1.1 and II.4.1]) we can generate interpolation spaces $(X_0, X_1)_\rho$; see [2, V.1.18 and V.1.13]. These spaces have the *monotonicity property.* Indeed, let (Y_0, Y_1) be another pair of Banach spaces. If $f \in (X_0, X_1)_\rho$ and for every $t > 0$ we have $K(t, g; Y_0, Y_1) \le K(t, f; X_0, X_1)$ then, $g \in (Y_0, Y_1)_\rho$ and $\|g\|_{(Y_0,Y_1)_\rho} \;\le\; \|f\|_{(X_0,X_1)_\rho}$; see [2, V.(1.47) and V.1.19]. Every rearrangement invariant space X on [0,1] arises as $X = \big(L^1, L^\infty\big)_\rho$ for a suitable ρ; see [2, V.1.17].

We briefly recall the theory of integration of real functions with respect to a vector measure due to Bartle, Dunford and Schwartz [1]. Let (Ω, Σ) be a measurable space, X a Banach space and $\nu : \Sigma \to X$ a countably additive vector measure. A measurable set A is ν–null if $\nu(B) = 0$ for every measurable set $B \subset A$. The integral of a simple function $f = \sum_1^n a_i \chi_{A_i}$ over a set A is defined by $\int_A f\, d\nu = \sum_1^n a_i \nu(A \cap A_i)$. A measurable function $f : \Omega \to \mathbf{R}$ is *integrable* with respect to ν if there exists a sequence $\{f_n\}$ of simple functions converging ν–a. e. to f, for which the sequence $\big\{\int_A f_n\, d\nu\big\}$ converges in the norm of X for each $A \in \Sigma$. Let X^* be the dual space of X and, for each $x^* \in X^*$, denote the $\mathbf{R}$–valued measure $A \mapsto \langle x^*, \nu(A)\rangle$ by $x^*\nu$ and its variation measure by $|x^*\nu|$. The ν–integrable functions form a linear space in which

$$\|f\|_\nu \;=\; \sup\left\{\int |f|\, d|x^*\nu| : x^* \in X^*,\; \|x^*\| \le 1\right\}$$

is a seminorm. Identifying functions which differ in a ν–null set, we obtain a Banach space (of classes) of ν–integrable functions, denoted by $L^1(\nu)$. It is a Banach function space for the ν–a. e. order and has absolutely continuous norm. Simple functions are dense in $L^1(\nu)$ and the ν–essentially bounded functions are contained in $L^1(\nu)$. The integration operator from $L^1(\nu)$ to X is defined by $f \mapsto \int f\, d\nu$. It is a continuous linear operator of norm at most one. Observe that no assumptions have been made on the variation of the measure ν in the definition of $L^1(\nu)$. For more details concerning $L^1(\nu)$ see [5], [6], [18]. In particular, we point out that such spaces $L^1(\nu)$ can, in general, be quite different to the classical L^1–spaces of scalar measures and may be difficult to identify explicitly, [7]. Indeed, every Banach lattice with absolutely continuous norm and having a weak unit $\big($e. g. $L^2([0,1])\big)$ is the L^1–space of some vector measure, [5, Theorem 8].

In general we will follow the notation in [2], [11] and [17].

3. Optimal lattice domains for operators

Let T be a linear operator defined on a Banach function space E and taking values in a Banach space X. We will denote by $[T, X]$ the maximal Banach function space (containing E) to which T can be extended as a continuous linear operator, still with values in X. This maximality is to be understood in the following sense. There is a continuous linear extension of T (which we still denote by T) $T : [T, X] \to X$, and if T has a continuous, linear extension $\widetilde{T} : F \to X$, where F is a Banach function space, then F is continuously embedded in $[T, X]$. The space $[T, X]$ is then the *optimal lattice domain* for T. If we consider the class of Banach function spaces with absolutely continuous norm, then we have the space $[T, X]_a$, which is the *absolutely continuous optimal lattice domain* for T.

In the next result, under certain conditions we associate to the operator T a vector measure ν with values in X. The space $L^1(\nu)$ is then the absolutely continuous optimal lattice domain for T.

Theorem 3.1. *Let E be a Banach function space over a finite measure space $(\Omega, \Sigma, \lambda)$, X be a Banach space and $T : E \to X$ be a linear operator with the property that $Tf_n \to Tf$ weakly in X whenever $\{f_n\} \subset E$ is a positive sequence increasing λ–a.e. to $f \in E$. Then there exists a countably additive vector measure $\nu : \Sigma \to X$ such that E is continuously embedded in $L^1(\nu)$ and the integration operator from $L^1(\nu)$ to X extends T.*

Proof. Define $\nu : \Sigma \to X$ by $\nu(A) = T\big(\chi_A\big)$. It is additive since T is linear. Let (A_i) be disjoint measurable sets. Then $0 \le \chi_{\cup_1^n A_i}$, for $n \in \mathbf{N}$, increases to $\chi_{\cup A_i} \in E$. By hypothesis $T(\chi_{\cup_1^n A_i}) = \sum_1^n \nu(A_i)$ converges weakly to $T(\chi_{\cup A_i}) = \nu(\cup A_i)$ in X. So, ν is weakly countably additive. Hence, ν is countably additive by the Orlicz–Pettis theorem; see [11, I.4.4].

We will use an equivalent definition of integrability with respect to a vector measure, given by Lewis in [15]. Namely, a function f is ν–integrable if and only if $\int |f|\, d|x^*\nu| < \infty$, for every $x^* \in X^*$, and for each $A \in \Sigma$ there exists an element of X, denoted by $\int_A f\, d\nu$, such that $\big\langle x^*, \int_A f\, d\nu\big\rangle = \int_A f\, dx^*\nu$, for every $x^* \in X^*$.

Let f be in E. Since E is a lattice we can assume that $f \ge 0$. Let $\{f_n\}$ be a sequence of non–negative simple functions increasing to f. Let A be a measurable set. Since $f_n\chi_A$ increases to $f\chi_A$, the sequence $T\big(f_n\chi_A\big)$ converges weakly to $T\big(f\chi_A\big)$ in X. For any simple function $S = \sum_1^n a_i\chi_{B_i}$ we see that

$$T\big(S\chi_A\big) \;=\; \sum_1^n a_i T\big(\chi_{A\cap B_i}\big) \;=\; \sum_1^n a_i\nu(A\cap B_i) \;=\; \int S\chi_A\, d\nu \;=\; \int_A S\, d\nu\,.$$

Fix $x^* \in X^*$. Then $\big\langle x^*, T\big(f_n\chi_A\big)\big\rangle$ converges to $\big\langle x^*, T\big(f\chi_A\big)\big\rangle$ and hence,

$$\big\langle x^*, T\big(f_n\chi_A\big)\big\rangle \;=\; \int_A f_n\, d(x^*\nu) \;\longrightarrow\; \big\langle x^*, T\big(f\chi_A\big)\big\rangle\,.$$

Let B_0, B_1 be the Hahn decomposition of the real measure $x^*\nu$. Replacing A with $A \cap B_j$, $j = 0, 1$, we see that $\int_{A\cap B_j} f_n\, d(x^*\nu) \to \big\langle x^*, T\big(f\chi_{A\cap B_j}\big)\big\rangle$. Accordingly,

$\{\int_A f_n\, d|x^*\nu|\}$ is convergent. Since $\{f_n\}$ increases to f it follows that f is integrable with respect to the measure $|x^*\nu|$ and

$$\langle x^*, T(f\chi_A)\rangle \;=\; \lim_n \int_A f_n\, d(x^*\nu) \;=\; \int_A f\, d(x^*\nu)\,.$$

Hence, f is in $L^1(\nu)$ and $Tf = \int f\, d\nu$. The embedding of E into $L^1(\nu)$ is continuous since it is positive and linear between Banach lattices; see [17, p. 2]. □

Remark 3.2. The assumptions of the theorem hold if E has absolutely continuous norm and T is continuous and linear. They also hold for $E = L^\infty(\lambda)$ and T weak*–to–weak continuous. Continuity of T alone does not suffice in general; consider the identity operator on $L^\infty([0,1])$. The result can be extended to σ–finite measure spaces, or even general measure spaces provided E contains a weak unit (see [17, p. 9]), that is, a function $\varphi > 0$, λ–a. e. In this case the measure ν is defined by $\nu(A) = T(\varphi\chi_A)$ and the embedding from E into $L^1(\nu)$ is $f \mapsto f/\varphi$.

Corollary 3.3. *Given the conditions of the above theorem, the space $L^1(\nu)$ is the absolutely continuous optimal lattice domain for T.*

Proof. Suppose that T has a continuous, linear extension $\widetilde{T} : F \to X$, where F is a Banach function space having absolutely continuous norm. By Remark 3.2 there is a vector measure $\tilde{\nu}$ such that $\widetilde{T}$ can be extended to the space $L^1(\tilde{\nu})$, in which F is continuously embedded. But, the measures $\tilde{\nu}$ and ν coincide. So, F is continuously embedded in $L^1(\nu)$. Since $L^1(\nu)$ has absolutely continuous norm ([5, Theorem 1]) it follows that $[T, X]_a = L^1(\nu)$. □

Remark 3.4. The interest in the identification of the space $[T, X]_a$ as $L^1(\nu)$, stems from the fact that the interplay between the properties of a vector measure, the Banach space where it takes its values and the resulting space $L^1(\nu)$, are well understood; see [6].

Remark 3.5. Theorem 3.1 provides an integral representation for certain operators defined on a function space via integration with respect to a vector measure. This representation even applies in cases where the Bochner or Pettis integrals do not exist.

For instance, consider the Volterra operator $T : L^\infty([0,1]) \to C([0,1])$, namely $Tf(x) = \int_0^x f(y)\, dy$. Theorem 3.1 shows that T has the integral representation $Tf = \int f\, d\nu$ for the measure ν given by $\nu(A)(x) = \int_0^x \chi_A(y)\, dy$; see Example 4.3 below and also [20, Proposition 3.1]. However, there is no Bochner or Pettis integrable function $G : [0,1] \to C([0,1])$ such that $Tf = \int_{[0,1]} f(t)G(t)\, dt$; see [11, p. 73].

Another such example is given by the fractional integral. For $0 < \alpha < 1$, following the definition of Riemann and Liouville, the fractional integral of order α of a function f at a point $x \in [0,1]$ is given by

$$I_\alpha f(x) \;=\; \int_0^1 \frac{f(t)}{|x-t|^{1-\alpha}}\, dt$$

$\big($up to a constant $\Gamma(\alpha)^{-1}$ which we will omit$\big)$ whenever it is defined. We can consider I_α as an operator $I_\alpha : L^\infty([0,1]) \to C([0,1])$. Applying Theorem 3.1 we obtain the measure ν given by $\nu(A)(x) = \int_A |x-y|^{\alpha-1}\,dy$, see Example 4.6 below. We can also consider the fractional integral as an operator $I_\alpha : L^\infty([0,1]) \to L^p([0,1])$. Again by Theorem 3.1, $I_\alpha(f) = \int f\,d\nu_{_{L^p}}$, where $\nu_{_{L^p}}$ denotes ν considered as taking its values in $L^p([0,1])$. This can be done for any $1 \le p \le \infty$. However, in general there is no Bochner or Pettis integrable function $G : [0,1] \to L^p([0,1])$ such that $I_\alpha(f) = \int_{[0,1]} f(t)G(t)\,dt$ since, in this case, $G(t)$ must be the t–translate of the function $|x|^{\alpha-1}$. This requires the restriction $(1-\alpha)p < 1$.

4. Kernel measures taking values in $C([0,1])$

Let $K : [0,1]\times[0,1] \to \mathbf{R}$ be a measurable function such that, for every $x \in [0,1]$, the function $K_x : y \mapsto K(x,y)$, $y \in [0,1]$, is integrable with respect to Lebesgue measure m in [0,1]. Then, for every $A \in \mathcal{B}$, there is an $\mathbf{R}$–valued function, that we denote by $\nu(A)$, defined by

$$\nu(A) \ : \ x \ \longmapsto \ \int_A K_x(y)\,dy\,, \quad x \in [0,1]\,.$$

Due to the additivity of the integral, we have an additive set function ν on $\mathcal{B}$ associated with the kernel K.

The following proposition identifies the connection between properties of the kernel K and those of its associated measure ν. This result will be applied to some relevant kernels, which we discuss throughout the paper.

Proposition 4.1. *Let K be a kernel with the above properties and ν be its associated additive set function.*

(a) *ν takes its values in $C([0,1])$ if and only if $K_{x_n} \to K_{x_0}$ weakly in $L^1([0,1])$ whenever $x_0 \in [0,1]$ and $x_n \to x_0$. In this case ν is necessarily countably additive.*

(b) *ν is countably additive, takes its values in $C([0,1])$ and has relatively compact range if and only if $K_{x_n} \to K_{x_0}$ in the norm of $L^1([0,1])$ whenever $x_0 \in [0,1]$ and $x_n \to x_0$.*

(c) *Suppose that ν is $C([0,1])$–valued. Then ν has bounded variation if and only if the set $\{K_x : x \in [0,1]\}$ is order bounded in $L^1([0,1])$.*

Proof. (a) The function $\nu(A)$ is continuous at $x_0 \in [0,1]$ if and only if $\int_A K_{x_n}(y)\,dy \to \int_A K_{x_0}(y)\,dy$ whenever $x_n \to x_0$. For fixed $x_n \to x_0$ this condition holds for every set A precisely when K_{x_n} converges weakly to K_{x_0} in $L^1([0,1])$; see [10, Corollary p. 91].

Consider the map $x \mapsto K_x$ from [0,1] to $L^1([0,1])$. Since it is continuous for the weak topology, the set $\{K_x : x \in [0,1]\}$ is weakly compact in $L^1([0,1])$. Hence, by the Dunford–Pettis theorem (see [10, p. 93]), it is uniformly integrable: given any $\varepsilon > 0$ there exists $\delta > 0$ such that if $m(A) < \delta$, then $\int_A |K_x(y)|\,dy < \varepsilon$ for every $x \in [0,1]$. It follows that $\|\nu(A)\|_\infty = \sup_x \left|\int_A K_x(y)\,dy\right| \le \varepsilon$, whenever $m(A) < \delta$. Accordingly, ν is additive and absolutely continuous with respect to Lebesgue measure. Hence ν is countably additive.

(b) Since norm convergence implies weak convergence we have (a). So we have only to consider relative compactness of the range of ν. Consider the inequality

$$\frac{1}{2}\left\|K_{x_1}-K_{x_2}\right\|_1 \;\le\; \sup_A\left|\int_A \big(K_{x_1}(y)-K_{x_2}(y)\big)\,dy\right| \;\le\; \left\|K_{x_1}-K_{x_2}\right\|_1\,,$$

and the Ascoli–Arzelà theorem. If the set $\{\nu(A) : A \in \mathcal{B}\}$ is relatively compact in $C([0,1])$, then it is equicontinuous. It follows that the map $x \mapsto K_x$ from [0,1] to $L^1([0,1])$ is norm continuous. Conversely, the countable additivity of ν implies that its range is a bounded subset of $C([0,1])$. From the previous inequality, if the map $x \mapsto K_x$ is norm continuous, then the set $\{\nu(A) : A \in \mathcal{B}\}$ is equicontinuous. Hence, it is relatively compact in $C([0,1])$.

(c) Let $f \in L^1([0,1])$ be such that, for every $x \in [0,1]$, we have $|K_x| \le f$ a.e.. Then ν has bounded variation since

$$\|\nu(A)\|_\infty \;\le\; \sup_{0\le x\le 1}\int_A |K_x(y)|\,dy \;\le\; \int_A f(y)\,dy\,.$$

Conversely, suppose ν has bounded variation. Let $|\nu|$ denote the variation of ν. Since $|\nu|$ is countably additive and $m(A) = 0$ implies $|\nu|(A) = 0$, it follows that $|\nu|$ is absolutely continuous with respect to m. Let $g \in L^1([0,1])$ be the Radon–Nikodym derivative of $|\nu|$ with respect to m. Fix $x \in [0,1]$. For a measurable set A let $B = \{y \in A : K_x(y) \ge 0\}$ and $C = A \setminus B$. Then

$$\begin{aligned}\int_A |K_x(y)|\,dy &= \int_B K_x(y)\,dy - \int_C K_x(y)\,dy\\ &\le \|\nu(B)\|_\infty + \|\nu(C)\|_\infty\\ &\le \int_A g(y)\,dy\,.\end{aligned}$$

This holds for every set A. Hence $|K_x| \le g$ a.e., for every $x \in [0,1]$. □

Remark 4.2. A sufficient condition for guaranteeing that ν is $C([0,1])$–valued, countably additive and has relatively compact range, is that $\{K_x : x \in [0,1]\}$ is uniformly integrable and K_{x_n} converges to K_{x_0} a.e. whenever $x_0 \in [0,1]$ and $x_n \to x_0$. A consequence of a theorem of De la Vallée–Poussin is useful in this regard. Namely, a subset $\mathcal{H}$ of $L^1([0,1])$ is uniformly integrable if and only if there exists an Orlicz function space Y on [0,1] such that $\mathcal{H}$ is bounded in Y and $\|\chi_A\|_{Y^*} \to 0$ as $m(A) \to 0$; see [19, I.2.2].

Example 4.3. The classical Volterra operator in $C([0,1])$ is given by the kernel $K(x,y) = \chi_\Delta(x,y)$ where $\Delta = \big\{(x,y) \in [0,1]^2 : 0 \le y \le x\big\}$. The associated measure ν is countably additive, has relatively compact range and bounded variation in $C([0,1])$.

Example 4.4. Arising from nilpotent left translation semigroups (see [14, §19.4]) we have the kernel $K(x,y) = \exp(-\lambda(y-x))\chi_{[x,1]}(y)$, where λ is a real constant. The associated measure ν is countably additive, has relatively compact range and bounded variation in $C([0,1])$.

Example 4.5. The kernel $K(x,y) = \arctan(y/x)$ for $x \neq 0$ and $K(0,y) = \pi/2$, arises (when restricted to $[0,1] \times [0,1]$) from the Poisson semigroup, [14, p. 579]. The associated measure ν is countably additive, has relatively compact range and bounded variation in $C([0,1])$.

Example 4.6. The fractional integral, for $0 < \alpha < 1$, is generated by the kernel $K(x,y) = |x-y|^{\alpha-1}$, $y \neq x$. The associated measure ν is countably additive with values in $C([0,1])$ and has relatively compact range. This follows from Remark 4.2 since $\sup_x \|K_x\|_{L^p} < \infty$ if $(1-\alpha)p < 1$. Choosing $p > 1$, we have $\left\|\chi_A\right\|_{L^{p'}} \to 0$ as $m(A) \to 0$. We note that ν does not have bounded variation since $\sup_x |K(x,y)| = \infty$, for every $y \in [0,1]$.

5. Non–negative and monotone kernels

Throughout this section K will be a kernel on $[0,1] \times [0,1]$ satisfying the conditions of the previous section, that is, all functions $K_x = K(x,\cdot)$ are Lebesgue integrable on [0,1]. We denote by T_K the operator associated to the kernel K and defined by

$$T_K f(x) \;=\; \int_0^1 f(y)K(x,y)\,dy\,, \quad x \in [0,1]\,,$$

for any function f for which it is meaningful to do so. We will suppose that the measure ν associated to K takes its values in $C([0,1])$ and, hence is countably additive; see Proposition 4.1 and Remark 4.2.

The kernel K is called non–decreasing if the family $\{K_x : x \in [0,1]\}$ is non–decreasing, in the sense that $K_{x_1} \leq K_{x_2}$ a. e. on [0,1] whenever $0 \leq x_1 \leq x_2 \leq 1$. Similarly, we have non–increasing kernels. Kernels with these properties abound. The Volterra kernel of Example 4.3 is non–decreasing, as are the Volterra convolution kernels of the form $K(x,y) = \phi(x-y)\chi_{[0,x]}(y)$, considered in [8]. The kernels of Example 4.4 (for $\lambda < 0$) and Example 4.5 are non–increasing.

Proposition 5.1. *Let K be a kernel and ν its associated $C([0,1])$–valued measure.*
(a) *$L^1(\nu) \subset \left[T_K, C([0,1])\right]$ and if $f \in L^1(\nu)$, then $\int f\,d\nu = T_K f \in C([0,1])$.*
(b) *If K is non–negative, then $\left[T_K, C([0,1])\right] = L^1(\nu)$.*
(c) *If, in addition, K is non–decreasing, then $\left[T_K, C([0,1])\right] = L^1_\xi$ where the weight $\xi(y) = K(1,y)$.*

Proof. (a) Observe that no assumptions are made concerning the sign of K. We first check that T_K is well defined as a linear operator from $L^\infty([0,1])$ into $C([0,1])$. Let $f \geq 0$ belong to $L^\infty([0,1])$ and $\{f_n\}$ be a sequence of simple functions increasing to f. Then f is also ν–essentially bounded, and hence $f \in L^1(\nu)$. By the dominated convergence theorem for vector measures [1, Theorem 2.8], $\left\{\int f_n\,d\nu\right\}$ converges to $\int f\,d\nu$ in $C([0,1])$. For a fixed $x \in [0,1]$, by considering the sets $\{y : K(x,y) \geq 0\}$ and $\{y : K(x,y) \leq 0\}$, it follows that $T_K f_n(x)$ converges to $T_K f(x)$. Since f_n are simple $\int f_n\,d\nu = T_K f_n$. We deduce that $T_K f = \int f\,d\nu \in C([0,1])$.

Arguing as before, we see that if now $\{f_n\}$ is a positive sequence increasing to $f \in L^\infty([0,1])$, then $T_K f_n$ converges to $T_K f$ weakly in $C([0,1])$ (in fact in the norm). We can then apply Theorem 3.1, choosing $E = L^\infty([0,1])$, $X = C([0,1])$, and $T = T_K$, and deduce that the integration operator extends T_K to the space $L^1(\nu)$.

(b) Let $f \in \big[T_K, C([0,1])\big]$. Then $T_K(|f|\chi_A) \in C([0,1])$ for every measurable set A. Let $\{f_n\}$ be a sequence of non–negative simple functions increasing to $|f|$. Since f_n are simple functions, $T_K\big(f_n\chi_A\big) = \int_A f_n \, d\nu \in C([0,1])$. Since K is non–negative, by the monotone convergence theorem $\big\{T_K\big(f_n\chi_A\big)\big\}$ is a sequence of continuous functions that increases pointwise to the function $T_K\big(|f|\,\chi_A\big)$. This last function is continuous by assumption, so by Dini's theorem the convergence is uniform. Hence, for every A, the sequence $\big\{\int_A f_n \, d\nu\big\}$ is convergent in $C([0,1])$, so $f \in L^1(\nu)$.

(c) In view of (b), we will check that $L^1(\nu) = L^1_\xi$. Let $f \in L^1(\nu)$. Then $\int |f| \, d\nu = T_K\,|f| \in C([0,1])$. Since K is non–negative, the measure ν is non–negative in the sense that $\nu(A)$ is a non–negative function in $C([0,1])$, for every set A. Hence, since $C([0,1])^*$ is a lattice, we have

$$\|f\|_{L^1(\nu)} = \sup_{\|x^*\|\le 1} \int |f| \, d|x^*\nu| = \sup_{0\le x^*,\, \|x^*\|\le 1} \int |f| \, d(x^*\nu) = \Big\| \int |f| \, d\nu \Big\|_\infty = \big\|T_K\,|f|\,\big\|_\infty .$$

Since K is non–negative and non–decreasing, we have

$$\|T_K|f|\,\|_\infty = \sup_{0\le x\le 1} \int |f(y)|\, K(x,y)\, dy = \int |f(y)|\, K(1,y)\, dy = \|f\|_{L^1_\xi} . \tag{5.1}$$

Hence $f \in L^1_\xi$. Now suppose $f \in L^1_\xi$. Let $\{f_n\}$ be a sequence of simple functions increasing to $|f|$. For a set A, by applying (5.1) to $(f_m - f_n)\chi_A$, with $m > n$, we deduce that the sequence $\big\{T_K\big(f_n\chi_A\big)\big\}$ is convergent in $C([0,1])$. Since $\int_A f_n \, d\nu = T_K\big(f_n\chi_A\big)$ we have, $|f| \in L^1(\nu)$ and so also $f \in L^1(\nu)$. □

Let X be a Banach function space on [0,1]. Then $C([0,1])$ is continuously embedded in X. The measure ν is then also X–valued and countably additive in X; it is denoted by ν_X in this case. Observe that for a Banach function space X, absolute continuity of the norm together with the Fatou property are equivalent to the condition that norm bounded, increasing sequences are convergent. For further equivalences see [17, 1.c.4].

Proposition 5.2. *Let K be a kernel and X be a Banach function space on $[0,1]$.*
(a) *$L^1(\nu_X) \subset \big[T_K, X\big]$ and if $f \in L^1\big(\nu_X\big)$, then $\int f \, d\nu_X = T_K f \in X$.*
(b) *If K is non–negative, then $\big[T_K, X\big] = \big\{f : T_K\,|f| \in X\big\}$.*
(c) *If, in addition, X has absolutely continuous norm, then $\big[T_K, X\big] = L^1\big(\nu_X\big)$.*

Proof. (a) The result follows from Proposition 5.1(a) since $C([0,1])$ is continuously embedded in X.

(b) If $f \in \big[T_K, X\big]$, then $|f| \in \big[T_K, X\big]$ and hence, $T_K|f| \in X$. Conversely, let $T_K|f| \in X$ and $|g| \le |f|$. Since K is non–negative $T_K|g| \le T_K|f|$, and X being a Banach function space, $T_K|g| \in X$. So $\big\{f : T_K|f| \in X\big\}$ is a lattice ideal, where $\|f\| := \|T_K|f|\|_X$ is a complete norm, which makes T_K continuous. Hence $\big\{f : T_K|f| \in X\big\} \subset [T_K, X]$.

(c) Assume $T_K|f| \in X$. Let $\{f_n\}$ be a sequence of non–negative simple functions increasing to $|f|$, and A be a measurable set. Since K is non–negative, by the monotone convergence theorem the sequence $\{T_K(f_n\chi_A)\}$ increases pointwise to the function $T_K(|f|\,\chi_A)$. Since $T_K(f_n\chi_A) \le T_K(|f|\,\chi_A) \le T_K|f| \in X$, from the absolute continuity of the norm of X we deduce that $\{T_K(f_n\chi_A)\}$ converges in X. Since $T_K(f_n\chi_A) = \int_A f_n\, d\nu_X \in X$, it follows that $f \in L^1(\nu_X)$. □

The part of the proof in Proposition 5.1(c) establishing $\|f\|_{L^1(\nu)} = \|T_K\,|f|\,\|$ is general. That is, a similar argument shows that if K is any non–negative kernel and X is any Banach function space on [0,1], in which case X^* is a lattice, then $\|f\|_{L^1(\nu_X)} = \|T_K\,|f|\,\|_X$, for $f \in L^1(\nu_X)$.

Remark 5.3. We have seen that $L^1(\nu_X) \subset [T_K, X]$ is always valid, with equality if X has absolutely continuous norm. Without this condition on X equality may fail to hold. Let X be the Zygmund space $L_{\exp}$, where $\|f\| = \sup_{0<t<1} f^{**}(t)\,(1-\log t)^{-1}$; see [2, IV.6.1]. It does not have absolutely continuous norm. Consider the measure ν associated to the Volterra kernel K of Example 4.3. The corresponding operator is the antiderivative $T_K f(x) = \int_0^x f$. Then, for $f(x) = (1-x)^{-1}$ we have $T_K f(x) = -\log(1-x)$ which belongs to X. However $f \notin L^1(\nu_X)$. Indeed, the bounded functions $f_n = \min\{f, n\}$ belong to $L^1(\nu_X)$. Since $\int f_n\, d\nu_X = T_K f_n$ and the sequence $\{T_K f_n\}$ is not convergent in X, we deduce that $f \notin L^1(\nu_X)$.

We now consider the measure ν as being $L^1([0,1])$–valued. We will denote ν_{L^1} simply by ν_1.

Proposition 5.4. *Let K be a non–negative kernel. Then* $[T_K, L^1([0,1])] = L^1(\nu_1) = L^1_\omega$, *where the weight* $\omega(y) = \int_0^1 K(x,y)\,dx$.

P r o o f. The first equality follows from Proposition 5.2(c) since $L^1([0,1])$ has absolutely continuous norm. Let $f \in L^1(\nu_1)$. Then

$$\|f\|_{L^1(\nu_1)} = \|T_K\,|f|\,\|_1 = \int_0^1\int_0^1 |f(y)|\,K(x,y)\,dy\,dx = \int_0^1 |f(y)|\,\omega(y)\,dy = \|f\|_{L^1_\omega}.$$

Hence, $f \in L^1_\omega$. Assume now that $f \in L^1_\omega$. Arguing as in the proof of Proposition 5.1(c) it follows that $f \in L^1(\nu_1)$. □

Proposition 5.5. *Let K be a non–negative kernel and X be a rearrangement invariant space on* $[0,1]$, *say* $X = (L^1, L^\infty)_\rho$. *Then* $(L^1_\omega, L^1(\nu))_\rho$ *is continuously embedded in* $[T_K, X]$.

P r o o f. Let f be in $(L^1_\omega, L^1(\nu))_\rho$. Proposition 5.4 and the interpolation property of the spaces $(L^1_\omega, L^1(\nu))_\rho$ and $(L^1, L^\infty)_\rho$ with respect to the couples $(L^1_\omega, L^1(\nu))$ and (L^1, L^∞) for the operator T_K (see [2, V.1.19]), imply that $T_K\,|f| \in X = (L^1, L^\infty)_\rho$ and $\|T_K\,|f|\,\|_X \le \|f\|_{(L^1_\omega, L^1(\nu))_\rho}$. □

In general, the inclusion $(L^1_\omega, L^1(\nu))_\rho \subset [T_K, X]$ as in Proposition 5.5 is proper; see Remark 5.7 and Example 5.15 below. However, under suitable monotonicity conditions on K equality does occur, as given in Theorem 5.11 and Theorem 5.12 below.

Proposition 5.6. *Let K be a non–negative, non–decreasing kernel with the property that there exists a constant $\beta > 0$ such that, for every $t > 0$ and every $y \in [0,1]$*

$$(*) \qquad \int_{\max\{0,1-t\}}^{1} K(x,y)\,dx \;\geq\; \beta \cdot \min\left\{ \int_0^1 K(x,y)\,dx;\; t \cdot K(1,y) \right\}.$$

Let $X = \big(L^1, L^\infty\big)_\rho$ be a rearrangement invariant space on $[0,1]$. Then $\big[T_K, X\big]$ is continuously embedded in $\big(L^1_\omega, L^1_\xi\big)_\rho$.

Proof. We calculate K–functionals with respect to the pair $\big(L^1([0,1]), C([0,1])\big)$. A result of Gagliardo shows that it is the same as calculating them with respect to the pair $\big(L^1([0,1]), L^\infty([0,1])\big)$; see [2, V.1. (1.32)].

Let $f \geq 0$. Then $T_K f$ is a non–decreasing function. Hence, its decreasing rearrangement $\big(T_K f\big)^*(x) = T_K f(1-x)$ for $x \in [0,1]$ and $\big(T_K f\big)^* = 0$ on $(1,\infty)$. Accordingly, we have by [2, II.6.2] that

$$\begin{aligned} K\big(t, T_K f; L^1, L^\infty\big) &= \int_0^{\min\{1,t\}} T_K f(1-x)\,dx \\ &= \int_0^{\min\{1,t\}} \int_0^1 f(y) K(1-x,y)\,dy\,dx \\ &= \int_0^1 f(y) \int_0^{\min\{1,t\}} K(1-x,y)\,dx\,dy \\ &= \int_0^1 f(y) \int_{\max\{0,1-t\}}^1 K(x,y)\,dx\,dy\,. \end{aligned}$$

The K–functional of f with respect to $\big(L^1_\omega, L^1_\xi\big)$ is given by (see [4, (3.1.39) p. 307])

$$\begin{aligned} K\big(t, f; L^1_\omega, L^1_\xi\big) &= \int_0^1 f(y) \min\{\omega(y), t\cdot\xi(y)\}\,dy \\ &= \int_0^1 f(y) \min\left\{ \int_0^1 K(x,y)dx;\; t\cdot K(1,y) \right\} dy\,. \end{aligned}$$

From condition $(*)$, there exists $\beta > 0$ such that for every $t > 0$

$$(5.2) \qquad \beta \cdot K\big(t, f; L^1_\omega, L^1_\xi\big) \;\leq\; K\big(t, T_K f; L^1, L^\infty\big)\,.$$

Let $f \in \big[T_K, X\big]$, with $f \geq 0$. Then $T_K f \in X = \big(L^1, L^\infty\big)_\rho$. From (5.2) and the monotonicity of the norm ρ, we see that $f \in \big(L^1_\omega, L^1_\xi\big)_\rho$ and

$$\beta \cdot \|f\|_{(L^1_\omega, L^1_\xi)_\rho} \;\leq\; \big\|T_K f\big\|_X\,. \qquad \square$$

Remark 5.7. The Volterra kernel of Example 4.3 is non–negative, non–decreasing and satisfies $(*)$ in Proposition 5.6, where $\beta = 1$ works. For Volterra convolution kernels of the form $K(x,y) = \phi(x-y)\chi_{[0,x]}(y)$, condition $(*)$ has been studied in terms of the function ϕ; see [8].

For non–negative, non–increasing kernels K a corresponding result holds.

Proposition 5.8. *Let K be a non–negative and non–increasing kernel. Let* $X = \big(L^1, L^\infty\big)_\rho$ *be a rearrangement invariant space on* $[0,1]$.

(a) $\big[T_K, C([0,1])\big] = L^1(\nu) = L^1_\eta$ *where the weight* $\eta(y) = K(0,y)$.

(b) *Suppose that K has the property that there exists a constant $\beta > 0$ such that, for every $t > 0$ and every $y \in [0,1]$*

$$(**)\qquad \int_0^{\min\{1,t\}} K(x,y)\,dx \;\ge\; \beta \cdot \min\left\{ \int_0^1 K(x,y)dx;\; t\cdot K(0,y)\right\}.$$

Then $[T_K, X]$ *is continuously embedded in* $\big(L^1_\omega, L^1_\eta\big)_\rho$.

Remark 5.9. Using the monotonicity properties of the kernel K, it can be seen that conditions $(*)$ and $(**)$ have the following equivalent expressions, which are more convenient for computational purposes;

$$(*)\qquad \inf\left\{ \frac{\int_{1-t}^1 K(x,y)\,dx}{\int_0^1 K(x,y)\,dx} : y\in[0,1],\; t = \frac{\int_0^1 K(x,y)\,dx}{K(1,y)} \right\} \;>\; 0\,,$$

$$(**)\qquad \inf\left\{ \frac{\int_0^t K(x,y)\,dx}{\int_0^1 K(x,y)\,dx} : y\in[0,1],\; t = \frac{\int_0^1 K(x,y)\,dx}{K(0,y)} \right\} \;>\; 0\,.$$

Remark 5.10. The following example shows that, for a non–negative monotone kernel, when condition $(*)$ or $(**)$ does not hold, then the conclusion of Proposition 5.6 or Proposition 5.8(b) may fail to hold. The kernel $K(x,y) = \arctan(y/x)$ of Example 4.5 is non–negative and non–increasing but it fails to satisfy $(**)$. Let ν be the measure associated to K and X be the Lorentz space $L^{p,1}([0,1])$, with $1 < p < \infty$. The measure ν_X has a Bochner integrable derivative $y \in [0,1] \mapsto G(y) \in L^{p,1}([0,1])$, given by $G(y)(x) = \arctan(y/x)$, for $x \in (0,1]$. The variation measure $|\nu_X|$ of ν_X is the measure with density $\|G(y)\|_{p,1} = \int_0^1 x^{1/p-1}\arctan(y/x)\,dx$, with respect to Lebesgue measure. Consider the function $f(y) = (y\log(1/y))^{-1-1/p}$ near zero. Since $\|G(y)\|_{p,1} \sim y^{1/p}$, for $y\in[0,1]$, and $\int_0^1 f(y)y^{1/p}\,dy < \infty$, it follows that f belongs to the space $L^1(|\nu_X|)$. A result of Lewis gives that $L^1(|\nu_X|) \subset L^1\big(\nu_X\big)$ [16, Theorem 4.1]. Since $X = L^{p,1}([0,1])$ has absolutely continuous norm, from Proposition 5.2, $L^1(\nu_X) = [T_K, X]$. Hence $f \in [T_K, X]$.

From Proposition 5.1 and Proposition 5.4 we have that $L^1(\nu) = L^1([0,1])$, with equivalence of norms, and $L^1\big(\nu_1\big) = L^1_\omega$ where $\omega(y) = \int_0^1 \arctan(y/x)\,dx$. Recall, in the usual notation of the K–method, that $L^{p,1}([0,1]) = \big(L^1, L^\infty\big)_{1-1/p,1}$. From an interpolation result of Stein and Weiss (see [3, 5.4.1]), we have that $\big(L^1_\omega, L^1([0,1])\big)_{1-1/p,1} = L^1\big(\omega^{1/p}\big)$. Since $\int_0^1 \arctan(y/x)\,dx \sim y\log(1/y)$, we have

$$\int_0^1 (y\log(1/y))^{-1-1/p}\Big(\int_0^1 \arctan(y/x)\,dx\Big)^{1/p} dy \;=\; \infty\,,$$

and deduce that the function f is not in $\big(L^1_\omega, L^1([0,1])\big)_{1-1/p,1}$.

Combining Proposition 5.5 and Proposition 5.6 we are now able to give a precise description of the spaces $\big[T_{K}, X\big]$ and $L^1\big(\nu_X\big)$ for certain kernels K and spaces X.

Theorem 5.11. *Let K be a kernel whose associated measure ν takes its values in $C([0,1])$. Assume K is non–negative, non–decreasing and satisfies property $(*)$. Let $X = \big(L^1, L^\infty\big)_\rho$ be a rearrangement invariant space on $[0,1]$. Then, with equivalence of norms,*

$$\big[T_{K}, X\big] \;=\; \big(L^1_\omega, L^1_\xi\big)_\rho\,,$$

where the weights are given by $\xi(y) = K(1,y)$ and $\omega(y) = \int_0^1 K(x,y)\,dx$. If, in addition, X has absolutely continuous norm, then $[T_{K}, X] = L^1\big(\nu_X\big)$.

Combining Proposition 5.5 and Proposition 5.8 we obtain the corresponding result for non–negative and non–increasing kernels.

Theorem 5.12. *Let K be a kernel whose associated measure ν takes its values in $C([0,1])$. Assume K is non–negative, non–increasing and satisfies property $(**)$. Let $X = \big(L^1, L^\infty\big)_\rho$ be a rearrangement invariant space on $[0,1]$. Then, with equivalence of norms,*

$$\big[T_{K}, X\big] \;=\; \big(L^1_\omega, L^1_\eta\big)_\rho\,,$$

where the weights are given by $\eta(y) = K(0,y)$ and $\omega(y) = \int_0^1 K(x,y)\,dx$. If, in addition, X has absolutely continuous norm, then $[T_{K}, X] = L^1\big(\nu_X\big)$.

Remark 5.13. The sharpness of condition $(*)$ follows from the fact:

If K is a non–negative, non–decreasing kernel such that for all rearrangement invariant spaces $X = \big(L^1, L^\infty\big)_\rho$ the spaces $\big[T_{K}, X\big]$ and $\big(L^1_\omega, L^1_\xi\big)_\rho$ are uniformly isomorphic, then K satisfies condition $(*)$.

The proof of this fact follows the lines of [8, Theorem 2]. An analogous result for non–increasing kernels and condition $(**)$ also holds.

Remark 5.14. The procedure for applying Theorem 5.11 or Theorem 5.12 starts by identifying the spaces L^1_ω and L^1_ξ, and then checking that either the condition $(*)$ or $(**)$ holds, depending on the monotonicity properties of K. Next the K–functional with respect to the pair $\big(L^1_\omega, L^1_\xi\big)$ has to be computed. This can be done via the known formula which we have already used; see [4, (3.1.39) p. 307]. Finally the corresponding rearrangement invariant norm determines the space $\big[T_{K}, X\big]$.

For example, the kernel $K(x,y) = \exp(-\lambda(y-x))\chi_{[x,1]}(y)$ of Example 4.4 is non–negative, non–increasing for $\lambda < 0$ and satisfies $(**)$, so 5.12 applies. Since $\omega(y) = \big(1 - e^{-\lambda y}\big)/\lambda$ and $\eta(y) = e^{-\lambda y}$, on [0,1], we have that $L^1_\omega = L^1(ydy)$ and $L^1_\eta = L^1([0,1])$, with equivalence of norms. The K–functional is then $K\big(t, f; L^1_\omega, L^1_\eta\big) = \int_0^1 f(y)\min\{y,t\}\,dy$. Consider, for instance, the case when X is a Lorentz space

$L^{p,q}([0,1])$, with $1 < p < \infty$ and $1 \le q < \infty$. Then the corresponding space $\left[T_K, X\right] = L^1\left(\nu_X\right)$ is precisely the space of functions f satisfying

$$\int_0^\infty \left(t^{1/p-1} \int_0^1 |f(y)| \min\{y,t\}\, dy \right)^q \frac{dt}{t} \; < \; \infty\,;$$

see [2, V.1.7]. This implies that the norm of $\left[T_K, X\right]$ is equivalent to $\int_0^1 |f(y)|\, y^{1/p}\, dy$.

Example 5.15. The fractional integral operator I_α exhibits interesting behavior with respect to the properties considered above.

(a) When no monotonicity conditions are assumed on the kernel, then the conclusion of Proposition 5.6 may fail. The kernel of the fractional integral is non–negative, but has no monotonicity properties. Let ν be the associated measure; see Example 4.6. From the results above we have that $[I_\alpha, C([0,1])] = L^1(\nu)$, $\left[I_\alpha, L^1([0,1])\right] = L^1([0,1])$, with equivalent norms, and $[I_\alpha, X] = \left\{ f : I_\alpha |f| \in X \right\}$. It will be shown that for $f_s = \chi_{[0,s]}$ and a suitable choice of $X = \left(L^1, L^\infty\right)_\rho$, we have

$$\lim_{s\to 0} \frac{\|f_s\|_{(L^1_\omega, L^1(\nu))_\rho}}{\|f_s\|_{[I_\alpha, X]}} \;=\; \infty\,. \tag{5.3}$$

Direct computation shows that $I_\alpha f_s(x) \le (2s/\alpha) x^{\alpha-1}$. Hence, for $X = L^p([0,1])$ with $1 \le p < (1-\alpha)^{-1}$ we have, for a constant $C > 0$ depending on α, that

$$\|f_s\|_{[I_\alpha, X]} \;=\; \|I_\alpha f_s\|_X \;\le\; Cs\,. \tag{5.4}$$

Since the spaces $L^1([0,1])$ and $L^1(\nu)$ are lattices, it follows from [4, (3.9.10) p. 467] that

$$\begin{aligned}
& K\big(t, f_s; L^1([0,1]), L^1(\nu)\big) \\
=\; & \inf_{A\subset[0,1]} \left\{ \left\|f_s \chi_{A^c}\right\|_1 + t\cdot \left\|f_s\chi_A\right\|_{L^1(\nu)} \right\} \\
=\; & \inf_{A\subset[0,1]} \left\{ m([0,s]\cap A^c) + t\cdot \sup_{0\le x\le 1} \int_{[0,s]\cap A} |x-y|^{\alpha-1}\, dy \right\} \\
\ge\; & \inf_{A\subset[0,1]} \left\{ m([0,s]\cap A^c) + t\cdot \sup_{0\le x\le s} \int_{[0,s]\cap A} |x-y|^{\alpha-1}\, dy \right\} \\
\ge\; & \inf_{A\subset[0,1]} \left\{ m([0,s]\cap A^c) + t\cdot s^{\alpha-1} m([0,s]\cap A) \right\}.
\end{aligned}$$

Either $m([0,s]\cap A^c) \ge s/2$ or $m([0,s]\cap A) \ge s/2$. Hence, we deduce for every $t > 0$, that

$$K\big(t, f_s; L^1([0,1]), L^1(\nu)\big) \;\ge\; \frac{1}{2} \min\{s, ts^\alpha\} \;=\; \int_0^t g_s^* \;=\; K\big(t, g_s; L^1, L^\infty\big)\,,$$

where $g_s = \frac{1}{2}\, s^\alpha \chi_{[0,s^{1-\alpha}]}$. For $X = L^p([0,1])$ with $1 \le p < \infty$, the monotonicity property of the interpolation functor implies that

$$\|f_s\|_{(L^1_\omega, L^1(\nu))_\rho} \;\ge\; \|g_s\|_{L^p} \;=\; \frac{1}{2}\, s^\alpha s^{(1-\alpha)/p}\,. \tag{5.5}$$

From (5.4) and (5.5), if $1 < p < (1-\alpha)^{-1}$, we deduce (5.3). Observe that, since $X = L^p([0,1])$ for $1 < p < (1-\alpha)^{-1}$ has absolutely continuous norm, then $[I_\alpha, X] = L^1(\nu_X)$, so this example shows also that $L^1(\nu_X)$ is not continuously embedded in $\big(L^1_\omega, L^1(\nu)\big)_\rho$.

(b) The fractional integral operator I_α is interesting for another reason. In general, the L^1–space of a vector measure is not rearrangement invariant. One such non–trivial example is given by the space of functions on [0,1] whose Rademacher–Fourier coefficients, on every measurable set, belong to ℓ^2, [7]. Another such example is the space $[I_\alpha, C([0,1])] = L^1(\nu)$, for ν the measure associated to I_α. To see that $L^1(\nu)$ is not rearrangement invariant we proceed as follows.

For $n \geq 1$ consider the function $S(t) = \sum_1^{2^n} \chi_{I_i}$, for the intervals $I_i = \big[\frac{i-1}{2^n}, \frac{i-1}{2^n} + \frac{1}{4^n}\big]$. We want to estimate

$$\|S\|_{L^1(\nu)} \;=\; \sup_{0\leq x\leq 1} \int_0^1 \frac{S(t)}{|x-t|^{1-\alpha}}\,dt\,.$$

For $0 \leq x \leq 1$ we have

$$\int_0^1 \frac{S(t)}{|x-t|^{1-\alpha}}\,dt \;=\; \int_0^x \frac{S(x-u)}{u^{1-\alpha}}\,du + \int_0^{1-x} \frac{S(x+u)}{u^{1-\alpha}}\,du \;\leq\; 2\int_0^1 \frac{S(t)}{t^{1-\alpha}}\,dt\,.$$

Hence,

$$\sup_{0\leq x\leq 1} \int_0^1 \frac{S(t)}{|x-t|^{1-\alpha}}\,dt \;\sim\; \int_0^1 \frac{S(t)}{t^{1-\alpha}}\,dt\,.$$

We now estimate

$$\int_0^1 \frac{S(t)}{t^{1-\alpha}}\,dt \;=\; \sum_{i=1}^{2^n} \int_{(i-1)2^{-n}}^{(i-1)2^{-n}+4^{-n}} \frac{dt}{t^{1-\alpha}} \;=\; \frac{1}{2^{n\alpha}}\left(\frac{1}{\alpha 2^{n\alpha}} + \Delta_n\right),$$

where

$$\Delta_n \;=\; \sum_{i=2}^{2^n} \int_{i-1}^{(i-1)+2^{-n}} \frac{dt}{t^{1-\alpha}} \;\sim\; \frac{1}{\alpha 2^{n(1-\alpha)}}\,.$$

Since $S^*(t) = \chi_{[0,2^{-n}]}$, we have $\|S^*\|_{L^1(\nu)} = \int_0^1 S^*(t)t^{\alpha-1}\,dt = 1/(\alpha 2^{\alpha n})$. Hence, we conclude that

$$\frac{\|S\|_{L^1(\nu)}}{\|S^*\|_{L^1(\nu)}} \;\sim\; \frac{1}{2^{n\alpha}} + \frac{1}{2^{n(1-\alpha)}}\,.$$

For every $n \in \mathbf{N}$ the functions S and S^* are in $L^1(\nu)$. But, as $n \to \infty$, the ratio of their norms in $L^1(\nu)$ goes to zero. Hence, the norm in the space $L^1(\nu)$ is not equivalent to a rearrangement invariant norm.

(c) The problem of optimal *rearrangement invariant* domains has been considered by Edmunds, Kerman and Pick for the Sobolev embedding; see [12]. In our setting, consider the fractional integral with values in $C([0,1])$. If $p > 1/\alpha$ and $f \in L^p([0,1])$ then $I_\alpha f \in C([0,1])$. Hardy and Littlewood proved that for $1 < p < 1/\alpha$, if $f \in L^p([0,1])$ then $I_\alpha f \in L^q([0,1])$, for $q = \frac{p}{1-p\alpha}$ [13, Theorem 4]. In the limiting case, if we want $I_\alpha f$ to be continuous, then necessarily $q = \infty$, that is, $p = 1/\alpha$.

But, this is not true as Hardy and Littlewood observed [13, 3.5.(iv)]. Hence, all spaces $L^p([0,1])$ for $p > 1/\alpha$ are in $[I_\alpha, C([0,1])] = L^1(\nu)$, but $L^{1/\alpha}([0,1])$ is not. Even though $L^1(\nu)$ is not rearrangement invariant, it is possible to identify the largest rearrangement invariant space included in $L^1(\nu)$, namely the Lorentz space $L^{1/\alpha,1}([0,1])$.

To see this, let f be in $L^{1/\alpha,1}([0,1])$. Assume that $f \geq 0$ and let $\{f_n\}$ be a sequence of non–negative simple functions increasing to f. Let $g(t) = |t|^{\alpha-1}$ and $g_x(t) = |x-t|^{\alpha-1}$ be its translate by x. Consider these functions on the interval $[0,1]$ and for $x \in [0,1]$. They belong to the space $L^{1/(1-\alpha),\infty}([0,1])$, where they are bounded, and $\|g_x\|_{1/(1-\alpha),\infty} \leq 2^{1-\alpha}$ for every $x \in [0,1]$ (observe that translation is not continuous in this space). Hence, by Hölder's inequality, for $m > n$ we have

$$\left\| \int_A (f_m - f_n)\, d\nu \right\|_\infty = \sup_{0\leq x\leq 1} \int_A \frac{f_m(t) - f_n(t)}{|x-t|^{1-\alpha}}\, dt \leq 2^{1-\alpha} \|f_m - f_n\|_{1/\alpha,1} .$$

This last expression tends to zero, since $L^{1/\alpha,1}([0,1])$ has absolutely continuous norm. So, for every set A the integrals $\{\int_A f_n\, d\nu\}$ converge in $C([0,1])$. Hence, $f \in L^1(\nu)$. Moreover, the embedding is continuous since $\left(|x-t|^{\alpha-1}\right)^* \leq 2^{1-\alpha} t^{\alpha-1}$, and hence

$$\|f\|_{L^1(\nu)} \leq \sup_{0\leq x\leq 1} \int_0^1 f^*(t) \left(\frac{1}{|x-t|^{1-\alpha}} \right)^* dt = 2^{1-\alpha} \|f\|_{1/\alpha,1} .$$

Let $f \in L^1(\nu)$ be such that its decreasing rearrangement $f^* \in L^1(\nu)$. Then

$$\|f^*\|_{L^1(\nu)} = \sup_{0\leq x\leq 1} \int_0^1 \frac{f^*(t)}{|x-t|^{1-\alpha}}\, dt = \int_0^1 \frac{f^*(t)}{t^{1-\alpha}}\, dt = \|f\|_{1/\alpha,1} ,$$

i.e. $f \in L^{1/\alpha,1}([0,1])$. So, $L^{1/\alpha,1}([0,1])$ is the largest rearrangement invariant space in $L^1(\nu)$.

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