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Optimal L_∞ Error Estimates for Galerkin Approximations to Solutions of Two-Point Boundary Value Problems*

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Abstract. A priori error estimates in the maximum norm are derived for Galerkin approximations to solutions of two-point boundary value problems. The class of Galerkin spaces considered includes almost all (quasiuniform) piecewise-polynomial spaces that are used in practice. The estimates are optimal in the sense that no better rate of approximation is possible in general in the spaces employed.

1. Introduction. Consider the two-point boundary value problem

$$-(a(x)y')' + b(x)y' + d(x)y = f(x), \qquad x \in I = (0, 1), y(0) = y(1) = 0,$$

or, in weak form, the problem of finding $y \in \overset{\circ}{H}{}^1$ such that

$$(ay', v') + (by', v) + (dy, v) = (f, v), \qquad v \in \overset{\circ}{H}{}^1. \tag{1.1}$$

To seek an approximate solution to the problem (1.1), consider a piecewise-polynomial spline space M_k^r, $-1 \leqslant k < r$, defined as

$$M_k^r = \{v \in C^k(I) : v|_{I_i} \in \Pi_r(I_i),\ i = 1, \cdots, N\}.$$

Here, $I_i = (x_{i-1}, x_i)$, $0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1$, and $\Pi_r(I_i)$ denotes the set of polynomials on I_i of degree not greater than r. It is assumed that, as the meshes vary, they are quasiuniform; i.e., with $h_i = x_i - x_{i-1}$, there exists a constant c_0 such that

$$\max_{i,j} h_i h_j^{-1} \leqslant c_0. \tag{1.2}$$

Let $h = \max_i h_i$.

The approximate solution Y to (1.1) is sought in the space

$$M = \overset{\circ}{M}{}_k^r = M_k^r \cap \{v : v(0) = v(1) = 0\}$$

according to the rule

$$(aY', V') + (bY', V) + (dY, V) = (f, V), \qquad V \in M. \tag{1.3}$$

Here, it is assumed that $0 \leqslant k < r$ so that $M \subset \overset{\circ}{H}{}^1$.

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Assume throughout the remainder of the paper that

(i) $a(x) \geqslant c_1 > 0, \ x \in I$,

(ii) $a, a', b, b', d \in L_\infty$,

(iii) for all $f \in L_2$, there exists a unique $y \in \overset{\circ}{H}{}^1$ satisfying (1.1).

From these assumptions it follows that (1.3) has a unique solution $Y \in \mathcal{M}$ for h sufficiently small (Schatz [5]; see also [3] for a proof). For h sufficiently small it is known (Nitsche [4]) that

$$\|y - Y\|_{L_2} + h\|y - Y\|_{H^1} \leqslant c_2 h^{r+1}\|y\|_{H^{r+1}}, \tag{1.4}$$

where c_2 depends on the L_∞-norms of the functions specified in assumption (ii). For simplicity, we shall also assume that the particular y of (1.1) that we shall approximate is an element of $W_\infty^{r+1} \cap \overset{\circ}{H}{}^1$.

Under the assumptions above, our result for the error in the maximum norm is:

THEOREM 1.1. *There exists a constant c*

$$c = c(c_0, c_1, c_2, \|a\|_{W_\infty^1}, \|b\|_{H^1}, \|d\|_{L_2})$$

such that

$$\|y - Y\|_{L_\infty} \leqslant ch^{r+1}\|y\|_{W_\infty^{r+1}}.$$

Theorem 1.1 was proved for $k = 0$, i.e., continuous piecewise-polynomial splines in [6] and, in that case, without the assumption of quasiuniformity.

Outline of the Paper. In Section 2, the notation used is defined, and a basic extension lemma is proved. In Section 3, the problem is reduced to the special case when $a(x) \equiv 1$, $b(x) \equiv d(x) \equiv 0$. In Section 4, it is first noted that, in this case, the derivative of the elliptic projection W of y into $\overset{\circ}{\mathcal{M}}{}_k^r$ is the L_2-projection of y' into $\mathcal{M}_{k-1}^{r-1}$. An estimate for the error in the L_2-projection in the maximum norm is derived, giving an estimate for $y' - W'$. The proof of this estimate uses the extension lemma to prove that the L_2-projection of a function of small support decreases rapidly outside that support. The estimate for $y' - W'$ then gives an estimate for $y - W$ via a duality argument.

Remark 1.1. The result (4.4) below (stability in the maximum norm of the L_2-projection) also gives estimates for the error in the maximum norm for smooth spline interpolation, see [1, Lemma 2.1].

2. Notation and an Extension Lemma.

For an open interval J, let $H^s(J)$ and $W_p^s(J)$ denote the closure of $C^\infty(\bar{J})$ in the norms

$$\|v\|_{H^s(J)} = \left(\sum_{i=0}^{s} \|v^{(i)}\|_{L_2(J)}^2\right)^{1/2} \quad \text{and} \quad \|v\|_{W_p^s(J)} = \sum_{i=0}^{s} \|v^{(i)}\|_{L_p(J)},$$

respectively. When $J = I = (0, 1)$, we drop the dependence on the interval in the notation.

We note that

$$\|v\|_{L_\infty} \leqslant 2^{1/2}\|v\|_{H^1}. \tag{2.1}$$

Let $\mathring{H}^s(J)$ denote the closure of $C_0^\infty(J)$ in the norm $\|\cdot\|_{H^s(J)}$; then for $v \in \mathring{H}^1$,

$$\|v\|_{H^1} \leqslant 2^{1/2}\|v'\|_{L_2}. \tag{2.2}$$

Let (v, w) denote $\int_0^1 v(x)w(x)dx$, and for $i, l, m \in \mathbf{Z}$, let

$$I(i, l, m) = (I_{i-l} \cup I_{i-l+1} \cup \cdots \cup I_{i+m}) \cap I,$$

where $\cdots, I_{-2}, I_{-1}, I_0, I_{N+1}, \cdots$ are arbitrarily defined.

The letters c and C will denote constants, not necessarily the same at each occurrence unless indexed.

The rest of this section is devoted to the proof of the following lemma, which allows us to construct appropriate piecewise-polynomial extensions.

LEMMA 2.1. *Given r and k, $-1 \leqslant k < r$, and $\mathcal{M}_k^r$ subject to* (1.2), *there exist constants $n = n(k, r) \in \mathbf{Z}$ and $c = c(c_0, k, r)$ such that, given $V \in \Pi_r(I_i)$, there exists a function $f_i \in \mathcal{M}_k^r$ such that*

$$f_i = V \text{ on } I_i; \quad \operatorname{supp} f_i \subset I(i, n, n); \quad \|f_i\|_{L_2} \leqslant c\|V\|_{L_2(I_i)}.$$

Proof. The case $k = -1$ is trivial, since we can set $f_i = 0$ outside I_i. Assume $0 \leqslant k < r$. We consider the problem of extending V to the right of I_i to fulfill the conditions of the lemma. Let $(k + 1)/(r - k) = n - \sigma$, where n is an integer and $0 \leqslant \sigma < 1$. Put $s = \sigma(r - k)$. Assume for simplicity that $i + n < N$.

Define $f \in \mathcal{M}_k^r(x_i, 1)$ (in obvious notation) by the requirements

$$f \equiv 0 \quad \text{outside } I_{i+n}, \tag{2.3}$$

$$f^{(l)}(x_i) = V^{(l)}(x_i), \qquad l = 0, \cdots, k, \tag{2.4}$$

$$f^{(l)}(x_{i+n}) = 0, \qquad l = 0, \cdots, k + s. \tag{2.5}$$

We must show that these requirements determine f on $(x_i, 1)$. Let

$$f|_{I_m} = \sum_{j=0}^{r} f_{j,m}(x - x_{m-1})^j, \qquad m = i + 1, \cdots, i + n.$$

We have $n(r + 1)$ coefficients to determine and the requirements

$$l!f_{l,i+1} = V^{(l)}(x_i), \qquad l = 0, \cdots, k, \tag{2.6}$$

$$\begin{gathered}\sum_{j=0}^{r} j(j-1)\cdots(j-l+1)f_{j,m}(x_m - x_{m-1})^{j-l} - l!f_{l,m+1} = 0,\\ l = 0, \cdots, k; \; m = i + 1, \cdots, i + n - 1,\end{gathered} \tag{2.7}$$

$$\begin{gathered}\sum_{j=0}^{r} j(j-1)\cdots(j-l+1)f_{j,i+n}(x_{i+n} - x_{i+n-1})^{j-l} = 0,\\ l = 0, \cdots, k + s,\end{gathered} \tag{2.8}$$

to fulfill. These requirements total

$$k + 1 + (n - 1)(k + 1) + k + s + 1 = n(k + 1) + (k + s + 1)$$

$$= n(r + 1) + n(k - r) + k + s + 1 = n(r + 1),$$

since $s = \sigma(r - k) = n(r - k) - (k + 1)$.

Hence, it suffices to show that if $V^{(l)}(x_i) = 0$, $l = 0, \cdots, k$, then $f \equiv 0$. For this, consider the continuous function $f^{(k)}$: This function is a piecewise polynomial of degree not greater than $r - k$. On each of $\bar{I}_{i+2}, \cdots, \bar{I}_{i+n-1}$ where $f^{(k)} \not\equiv 0$, it has at most $r - k$ roots. Similarly, if $f^{(k)} \not\equiv 0$ on the open interval I_{i+1}, it has at most $r - k - 1$ roots there, and on I_{i+n}, it has at most $r - k - s - 1$ roots. Altogether, on subintervals where $f^{(k)} \not\equiv 0$, it has at most

$$(n - 2)(r - k) + r - k - 1 + r - k - s - 1 = n(r - k) - 2 - s$$

$$= (n - \sigma)(r - k) - 2 = k - 1$$

roots not coinciding with x_i or x_{i+n}. Hence, we can find a polynomial $p(x)$ of degree $k - 1$ such that

$$f^{(k)}(x)p(x) \geqslant 0, \quad x_i \leqslant x \leqslant x_{i+n}, \quad \text{and} \quad f^{(k)}(x)p(x) > 0 \quad \text{if } f^{(k)}(x) \neq 0.$$

However, by repeated partial integration, we find that, since $f^{(j)}(x_i) = f^{(j)}(x_{i+n}) = 0$, $j \leqslant k$,

$$\int_{x_i}^{x_{i+n}} f^{(k)} p \, dx = 0.$$

Thus, $f^{(k)} \equiv 0$, and $f \equiv 0$. Hence, (2.3)–(2.5) determine $f \in M_k^r(x_i, 1)$.

To establish the norm inequality of the lemma, multiply (2.6)–(2.8) by h_{m+1}^l ($h_{m+1} = x_{m+1} - x_m$). The corresponding linear system of equations for the quantities $g_{j,m} = f_{j,m} h_m^j$ is:

$$(2.6)' \qquad l!\, g_{l,i+1} = V^{(l)}(x_i) h_{i+1}^l, \qquad l = 0, \cdots, k,$$

$$(2.7)' \qquad \sum_{j=0}^{r} j(j - 1) \cdots (j - l + 1) \left(\frac{h_{m+1}}{h_m}\right)^l g_{j,m} - l!\, g_{l,m+1} = 0,$$

$$l = 0, \ldots, k; \quad m = i + 1, \ldots, i + n - 1,$$

$$(2.8)' \qquad \sum_{j=0}^{r} j(j - 1) \cdots (j - l + 1) g_{j,i+n} = 0, \qquad l = 0, \cdots, k + s.$$

Since the determinant of this system is never zero, and since, by (1.2), h_{m+1}/h_m varies over a compact interval, it follows that there exists a constant $c = (c_0)$ such that

$$\max_{m,j} |f_{j,m} h_m^j| \leqslant c \max_{l} |V^{(l)}(x_i) h_{i+1}^l|.$$

Since

$$\|f\|_{L_2((x_i,\, x_{i+n}))} \leqslant c h^{1/2} \max_{m,j} |f_{j,m} h_m^j|$$

and

$$\max_l |V^{(l)}(x_i)h_i^l| \leqslant ch_i^{-1/2} \|V\|_{L_2(I_i)},$$

it follows that

$$\|f\|_{L_2(x_i,1)} \leqslant c\|V\|_{L_2(I_i)}. \tag{2.9}$$

Apply the analogous construction leftwards; this concludes the proof.

3. Comparison of Different Elliptic Projections. We shall consider three different elliptic projections, Y, Z, and W, of the solution y of (1.1) into $\mathcal{M} = \overset{\circ}{\mathcal{M}}{}^r_k$. Here, Y is given by (1.3) or, equivalently, by

$$(a(y' - Y'), V') + (b(y' - Y'), V) + (d(y - Y), V) = 0, \qquad V \in \mathcal{M}, \tag{3.1}$$

and Z and W are given by

$$(a(y' - Z'), V') = 0, \quad V \in \mathcal{M}, \tag{3.2}$$

$$(y' - W', V') = 0, \quad V \in \mathcal{M}. \tag{3.3}$$

Since the bilinear forms corresponding to (3.2) and (3.3) are positive definite on $\mathcal{M}$, Z and W are defined. We shall prove that the three elliptic projections defined above differ in H^1 by $O(h^{r+1})$.

LEMMA 3.1. *There exists a constant c,*

$$c = c(c_1, c_2, \|b\|_{H^1}, \|d\|_{L_2}),$$

such that $\|Y - Z\|_{H^1} \leqslant c\|y - Y\|_{L_2}$.

Proof (cf. [6]). We have

$$\begin{aligned} 0 &= (a(y' - Y'), V') + (b(y' - Y'), V) + (d(y - Y), V) \\ &= (a(Z' - Y'), V') + ((y - Y), dV - (bV)'). \end{aligned}$$

Choosing $V = Z - Y$, we obtain

$$\begin{aligned} c_1\|Z' - Y'\|_{L_2}^2 &\leqslant \|y - Y\|_{L_2}(\|d\|_{L_2}\|Z - Y\|_{L_\infty} + \|b\|_{L_\infty}\|Z' - Y'\|_{L_2} + \|b'\|_{L_2}\|Z - Y\|_{L_\infty}) \\ &\leqslant (\|d\|_{L_2} + \|b\|_{L_\infty} + \|b'\|_{L_2})\|y - Y\|_{L_2}\|Z' - Y'\|_{L_2}. \end{aligned}$$

LEMMA 3.2. *There exists a constant c, $c = c(c_0, c_1, \|a\|_{W^1_\infty})$, such that* $\|Z - W\|_{H^1} \leqslant ch\|y - W\|_{H^1}$.

Proof. Let $\vartheta = Z - W$. From the definitions of Z and W, we see that for $\chi \in \mathcal{M}$,

$$c_1\|\vartheta'\|_{L_2}^2 \leqslant (a\vartheta', \vartheta') = (a(y - W)', \vartheta') = (y' - W', a\vartheta' - \chi').$$

Since $y' - W'$ has zero average value, we can use instead of χ' any $\nu \in \mathcal{M}^* = \mathcal{M}^{r-1}_{k-1}$. Thus

$$\|\vartheta'\|_{L_2}^2 \leqslant c\|y - W\|_{H^1} \inf_{\nu \in \mathcal{M}^*} \|a\vartheta' - \nu\|_{L_2}. \tag{3.4}$$

In order to prove the result, it suffices to show that for $V \in \mathcal{M}^*$

$$\inf_{v \in \mathcal{M}^*} \|aV - v\|_{L_2} \leqslant ch \|V\|_{L_2}. \tag{3.5}$$

In order to establish (3.5), we first remark that there is a constant c such that, if $W \in C^{k-1}(I)$ and $W|_{I_i} \in H^r(I_i)$, $i = 1, \cdots, N$, then

$$\inf_{v \in \mathcal{M}^*} \|W - v\|_{L_2} \leqslant ch^r \|W^{(r)}\|_{L_2} := \left(\sum_{i=1}^{N} \|W^{(r)}\|^2_{L_2(I_i)} \right)^{1/2} \tag{3.6}$$

This is easily seen by adding a function $v_1 \in \mathcal{M}^*$ to W so that $W + v_1 \in H^r$ and then noting that

$$\begin{aligned} \inf_{v \in \mathcal{M}^*} \|W - v\|_{L_2} &= \inf_{v \in \mathcal{M}^*} \|W + v_1 - v\|_{L_2} \\ &\leqslant ch^r \|(W + v_1)^{(r)}\|_{L_2} = ch^r \|W^{(r)}\|_{L_2}. \end{aligned}$$

Next, note that there exists a function $\psi \in \mathcal{M}^*$ such that

$$\|a - \psi\|_{L_\infty} \leqslant c \|a'\|_{L_\infty} h \tag{3.7}$$

and

$$\|\psi^{(l)}\|_{L_\infty} \leqslant c \|a'\|_{L_\infty} h^{1-l}, \qquad l = 1, 2, \cdots, r-1; \tag{3.8}$$

this is easily seen by modifying a and applying an estimate like (4.1) of the next section.

Thus, from (3.6), (3.7) and (3.8), we see that for $V \in \mathcal{M}^*$

$$\begin{aligned} \inf_{\chi \in \mathcal{M}} \|aV - \chi\|_{L_2} &\leqslant \|(a - \psi)V\|_{L_2} + \inf_{\chi \in \mathcal{M}^*} \|\psi V - \chi\|_{L_2} \\ &\leqslant ch \|V\|_{L_2} + ch^r \sum_{l=1}^{r-1} \|\psi^{(l)}\|_{L_\infty} \|V^{(r-l)}\|_{L_2} \\ &\leqslant ch \|V\|_{L_2}, \end{aligned}$$

where we used the quasiuniformity of the mesh to estimate the terms $\|V^{(r-l)}\|_{L_2}$.

4. Proof of Theorem 1.1. It is sufficient, as a consequence of the reduction of the last section, to prove Theorem 1.1 in the case $a \equiv 1$, $b \equiv d \equiv 0$. We begin by summarizing the approximation-theoretic properties of the space $\mathcal{M}_l^s$, $-1 \leqslant l < s$, that we need.

LEMMA 4.1 (DE BOOR [2]). *There exists a constant c such that, if $u \in W_\infty^{s+1}$ and $v \in \overset{\circ}{W}{}_1^2 \cap \overset{\circ}{H}{}^1$, there exists $\chi \in \mathcal{M}_l^s$ and $\psi \in \overset{\circ}{\mathcal{M}}{}_l^s$ such that*

$$\|u - \chi\|_{L_\infty} \leqslant ch^{s+1} \|u\|_{W_\infty^{s+1}}, \tag{4.1}$$

$$\|v - \psi\|_{W_1^1} \leqslant ch \|v\|_{W_1^2}. \tag{4.2}$$

Let Pu denote the L_2-projection of a function $u \in L_2$ into $\mathcal{M}_l^s$, $-1 \leqslant l < s$, defined by

$$(Pu - u, V) = 0, \qquad V \in \mathcal{M}_l^s.$$

LEMMA 4.2. *There exists a constant* $c = c(c_0)$ *such that, given* $u \in W_\infty^{s+1}$,

$$\|Pu - u\|_{L_\infty} \leqslant ch^{s+1}\|u\|_{W_\infty^{s+1}}.$$

The proof of this lemma is postponed until the end of this section.

Remark 4.1. It is easily seen by duality using Lemma 4.2 that P gives optimal approximation in the L_1-norm. It then follows from interpolation that P gives optimal approximation in any L_p-norm, $1 \leqslant p \leqslant \infty$.

LEMMA 4.3. *There exists a constant* $c = c(c_0)$ *such that, given* $y \in W_\infty^{r+1} \cap \overset{\circ}{H}{}^1$, *and with* W *defined by* (3.3),

$$\|y - W\|_{L_\infty} \leqslant ch^{r+1}\|y\|_{W_\infty^{r+1}}.$$

Proof. Since $(y' - W', 1) = 0$, (3.3) implies that W' is the L_2-projection of y' into $\mathcal{M}_{k-1}^{r-1}$. By Lemma 4.2, it follows that

$$\|y' - W'\|_{L_\infty} \leqslant ch^r\|y'\|_{W_\infty^r}. \tag{4.3}$$

We now apply a duality argument [4]. Given $g \in L_1$, let G be such that $G'' = -g$, $G(0) = G(1) = 0$. Then

$$(y - W, g) = (y' - W', G') = (y' - W', G' - \chi'), \qquad \chi \in \mathcal{M}.$$

By Lemma 4.1, χ can be chosen so that

$$|(y - W, g)| \leqslant ch\|y' - W'\|_{L_\infty}\|g\|_{L_1},$$

and it follows from (4.3) that

$$\|y - W\|_{L_\infty} = \sup_{\|g\|_{L_1} = 1} |(y - W, g)| \leqslant ch^{r+1}\|y\|_{W_\infty^{r+1}}.$$

Proof of Theorem 1.1. We have (cf. (3.2), (3.3))

$$\|y - Y\|_{L_\infty} \leqslant \|y - W\|_{L_\infty} + \|Y - Z\|_{L_\infty} + \|Z - W\|_{L_\infty}.$$

Using (2.1), Lemmas 3.1, 3.2, and 4.3, and (1.4) and its counterpart for $y - W$, the theorem follows.

It remains to prove Lemma 4.2.

Proof of Lemma 4.2. Let χ be as in (4.1). Since $P\chi = \chi$, we have

$$\|Pu - u\|_{L_\infty} \leqslant \|u - \chi\|_{L_\infty} + \|P(u - \chi)\|_{L_\infty},$$

and hence it suffices to show that there exists a constant c such that

$$\|Pu\|_{L_\infty} \leqslant c\|u\|_{L_\infty}, \qquad u \in L_\infty. \tag{4.4}$$

Let $u = \sum_{i=1}^N u_i$, where

$$u_i(x) = \begin{cases} u(x), & x \in I_i, \\ 0, & x \notin I_i. \end{cases}$$

Consider Pu_i on I_m, $m < i$. By Lemma 2.1, there exists $f_{i,m} \in M_l^s$ which agrees with Pu_i on I_m, and satisfies

$$\|f_{i,m}\|_{L_2} \leqslant c\|Pu_i\|_{L_2(I_m)}.$$

Since the L_2-projection Pu_i minimizes the L_2-norm of the difference $V - u_i$ for V in M_l^s, and since $u_i = 0$ outside I_i, it follows that Pu_i minimizes the $L_2((0, x_{m-1}))$-norm of elements of M_l^s agreeing with Pu_i on I_m. Thus,

$$\|Pu_i\|^2_{L_2((0,x_{m-1}))} \leqslant \|f_{i,m}\|^2_{L_2} \leqslant c_4\|Pu_i\|^2_{L_2(I_m)}, \qquad m \leqslant i.$$

Hence, with $p_{i,m} = \|Pu_i\|^2_{L_2(I_m)}$, we have

$$\sum_{\alpha<m} p_{i,\alpha} \leqslant c_4 p_{i,m}, \qquad m \leqslant i. \tag{4.5}$$

From this it follows that

$$p_{i,m} \geqslant c_4^{-1}(1 + c_4^{-1})^{m-q-1}p_{i,q}, \qquad 0 \leqslant q < m \leqslant i, \tag{4.6}$$

which we proceed to show by induction. Assume that (4.6) holds for q, m such that $0 \leqslant q < m \leqslant L$. (Note that for $L = 1$, i.e., $m = 1$ and $q = 0$, (4.6) is immediate from (4.5).) For any $q < L + 1$, we then obtain by (4.5) and the induction hypothesis

$$\begin{aligned} p_{i,L+1} &\geqslant c_4^{-1}\sum_{\alpha\leqslant L} p_{i,\alpha} \geqslant c_4^{-1}\Big(\sum_{q<\alpha\leqslant L} p_{i,\alpha} + p_{i,q}\Big) \\ &\geqslant c_4^{-1}p_{i,q}\Big(\sum_{q<\alpha\leqslant L} c_4^{-1}(1 + c_4^{-1})^{\alpha-q-1} + 1\Big) \\ &= c_4^{-1}p_{i,q}(1 + c_4^{-1})^{(L+1)-q-1}. \end{aligned}$$

This establishes (4.6).

A similar result holds for intervals to the right of I_i, and, taking $m = i$ in (4.6), we find that there exist positive constants c and C, depending only on c_0, such that

$$\|Pu_i\|_{L_2(I_q)} \leqslant Ce^{-c|i-q|}\|Pu_i\|_{L_2(I_i)}. \tag{4.7}$$

Since Pu is a polynomial of fixed degree on I_q, we have

$$\|Pu\|_{L_\infty(I_q)} \leqslant ch_q^{-1/2}\|Pu\|_{L_2(I_q)}.$$

Using this, (1.2), (4.7), and the fact that $\|Pu\|_{L_2} \leqslant \|u\|_{L_2}$, we have, for any q,

$$\begin{aligned} \|Pu\|_{L_\infty(I_q)} &\leqslant ch_q^{-1/2}\|Pu\|_{L_2(I_q)} \leqslant ch_q^{-1/2}\sum_i \|Pu_i\|_{L_2(I_q)} \\ &\leqslant Ch_q^{-1/2}\sum_i e^{-c|i-q|}\|Pu_i\|_{L_2(I_i)} \leqslant Ch_q^{-1/2}\sum_i e^{-|i-q|}\|u_i\|_{L_2} \\ &\leqslant Ch_q^{-1/2}\sum_i e^{-c|i-q|}h_i^{1/2}\|u_i\|_{L_\infty} \leqslant c\|u\|_{L_\infty}. \end{aligned}$$

This proves (4.4) and establishes the lemma.

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1. CARL DE BOOR, "On the convergence of odd-degree spline interpolation," *J. Approximation Theory*, v. 1, 1968, pp. 452–463. MR **38** #6273.

2. CARL DE BOOR, "On local spline approximation by moments," *J. Math. Mech.*, v. 17, 1967/68, pp. 729–735. MR **36** #6850.

3. J. DOUGLAS, JR. & T. DUPONT, "Galerkin approximations for the two point boundary problem using continuous, piecewise polynomial spaces," *Numer. Math.*, v. 22, 1974, pp. 99–109.

4. JOACHIM A. NITSCHE, "Ein Kriterium für die Quasi-Optimalität des Ritzschen Verfahrens," *Numer. Math.*, v. 11, 1968, pp. 346–348. MR **38** #1823.

5. A. SCHATZ, "An observation concerning Ritz-Galerkin methods with indefinite bilinear forms," *Math. Comp.*, v. 28, 1974, pp. 959–962.

6. M. F. WHEELER, "An optimal L_∞ error estimate for Galerkin approximations to solutions of two point boundary value problems," *SIAM J. Numer. Anal.*, v. 10, 1973, pp. 914–917.