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Optimal L^∞ Estimates for the Finite Element Method on Irregular Meshes*

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Abstract. Uniform estimates for the error in the finite element method are derived for a model problem on a general triangular mesh in two dimensions. These are optimal if the degree of the piecewise polynomials is greater than one. Similar estimates of the error are also derived in L^p. As an intermediate step, an L^1 estimate of the gradient of the error in the finite element approximation of the Green's function is proved that is optimal for all degrees.

The finite element method may be briefly described as the Ritz method using a piecewise polynomial trial space: the solution to a differential problem is approximated by minimizing an integral involving the squares of derivatives of the difference between the true solution and piecewise polynomial trial solutions. Thus, there naturally follow estimates for the error in the mean square sense (cf. [1], [23]). It is widely believed that estimates of a similar form should hold in a uniform sense; but until recently, the best general estimates predicted an asymptotically less accurate uniform approximation [15], [8], [12]. It should be noted that optimal uniform estimates were known in one dimension [27], [9], [26] or, in higher dimensions, on a regular mesh [3]–[6], [10], [22]; but those techniques do not generalize to an irregular mesh in higher dimensions. The purpose of this paper is to present a technique for deriving uniform estimates on a general mesh that are optimal in a wide range of cases. We consider a model problem in two dimensions in order to minimize technicalities not relevant to uniform estimates *per se.* Results similar to ours have been obtained independently by Natterer [14] and Nitsche [16].

We now describe our results and the method of proof in some detail. Consider the Neumann problem

$$-\Delta u + u = f \quad \text{in } \Omega \quad \text{and}$$
$$\partial_n u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a convex domain in R^2 with smooth boundary $\partial\Omega$, and let u^* be the finite element approximation to u from the space of C^0 piecewise polynomials of degree $k-1$ on a quasi-uniform mesh of triangles of size h. Theorem 1 shows that as h tends to zero, the L^∞ norm of the error $u - u^*$ tends to zero at the optimal rate h^k if $k \geq 3$ (piecewise quadratics and higher) and at least as fast as $h^2|\log h|$ if $k = 2$ (the piecewise linear case) provided that u has bounded weak derivatives of order k. In Section 6, we derive similar estimates for $u - u^*$, and the derivatives of $u - u^*$, in L^p, $1 \leq p \leq \infty$.

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The proof of the above results is by duality: we derive an L^1 estimate for the gradient of the error $g - g^*$ in the finite element approximation of the Green's function g defined by

$$-\Delta g + g = \delta \quad \text{in } \Omega \quad \text{and}$$
$$\partial_n g = 0 \quad \text{on } \partial\Omega,$$

where δ is the Dirac distribution with singularity at an arbitrary point in Ω. In particular, Theorem 2 shows that the gradient of $g - g^*$ has an L^1 norm of order h if $k \geqslant 3$ and $h|\log h|$ if $k = 2$. This rate of approximation to g is shown to be optimal in the Remark in Section 1. Thus, the duality proof is apparently stuck with the $|\log h|$ in the piecewise linear case. The proof of Theorem 2 contains some new results about $g - g^*$ of independent interest, in particular, Proposition 1 and Lemma 6, as well as the L^1 estimate for the gradient of $g - g^*$ itself.

The idea of the proof of Theorem 2 was inspired by the interior estimates of Nitsche and Schatz [17]. First, the integral of the gradient of $g - g^*$ in a disc of radius $O(h)$ around the singularity of g is shown to be $O(h)$ for all k. Then, in the exterior of this disc, the ideas of [17] are used (with some refinements) to reduce the estimate of the gradient of $g - g^*$ to a global estimate of $g - g^*$ in a negative Sobolev norm, to which [21] applies. (Schatz and Wahlbin [20] have succeeded in modifying our proof to obtain an estimate of $g - g^*$ more directly from interior estimates.) The reader will notice several similarities in some steps of the proof of Theorem 2 to the technique of Natterer [14] and Nitsche [16], although the main thrust of their proof is different. Instead of studying the dual problem $g - g^*$, they derive general estimates for $u - u^*$ in weighted mean square norms. In our proof, weighted norms appear implicitly in the special context of $g - g^*$.

We collect here as a reference some standard notation used throughout the paper. For details, see the book by Stein [24]. For a real valued function u defined on a domain $\Omega \subset R^2$, we use the shorthand notation

$$\int_\Omega u \equiv \int_\Omega u(x)\, dx,$$

where dx denotes Lebesgue measure. As usual, if p is any real number in the range $1 \leqslant p \leqslant \infty$,

$$\|u\|_{L^p(\Omega)} \equiv \left(\int_\Omega |u|^p\right)^{1/p},$$

with the usual modification when $p = \infty$. Given a multi-index $\alpha = (\alpha_1, \alpha_2)$, let $D^\alpha u$ denote the weak derivative of u, that is,

$$\int_\Omega (D^\alpha u)\varphi = (-1)^{|\alpha|} \int_\Omega u \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \varphi$$

for all smooth functions φ having compact support in the interior of Ω, where $|\alpha| = \alpha_1 + \alpha_2$. Define a seminorm by

$$|u|_{W_p^k(\Omega)} \equiv \left(\sum_{|\alpha|=k} \|D^\alpha u\|^p_{L^p(\Omega)}\right)^{1/p}$$

and a norm by

$$\|u\|_{W_p^k(\Omega)} = \sum_{s=0}^{k} |u|_{W_p^s(\Omega)},$$

where k is a nonnegative integer and p is any number that satisfies $1 \leqslant p \leqslant \infty$. The Banach space $W_p^k(\Omega)$ becomes a Hilbert space when $p = 2$, and this is denoted by $H^k(\Omega)$. Its dual space (with the dual norm) is denoted by $H^{-k}(\Omega)$.

1. Presentation of Results. Let $\Omega \subset R^2$ be a bounded convex** domain with smooth boundary, and let u solve

$$\begin{aligned} -\Delta u + u &= f \quad \text{in } \Omega, \\ \partial_n u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

The smoothness of the data f will be specified later implicitly by assumptions on u. Define

$$a(v, w) = \int_\Omega \nabla v \cdot \nabla w + vw. \tag{1.2}$$

We will view this bilinear form as being defined on $W_1^1(\Omega) \times W_\infty^1(\Omega)$. Let $\mathbf{T}$ be a triangulation of Ω having straight interior edges. We associate two parameters with $\mathbf{T}$, namely, for each "triangle" $\tau \in \mathbf{T}$, define $\rho(\tau)$ (resp. $\sigma(\tau)$) to be the diameter of the smallest disc containing τ (resp. largest disc contained in τ), and let

$$h = \max_{\tau \in \mathbf{T}} \rho(\tau), \qquad \gamma = \min_{\tau \in \mathbf{T}} \frac{\sigma(\tau)}{h}. \tag{1.3}$$

(A family of triangulations $\{\mathbf{T}_n\}$ is called *quasi-uniform* if $\gamma \geqslant \gamma_0 > 0$ for all $\mathbf{T}_n$.) Given $\mathbf{T}$ as above, define $\mathbf{S}_k = \mathbf{S}_k(\mathbf{T})$ to be the space of continuous piecewise polynomials of degree $k - 1$, i.e., the subspace of $C^0(\Omega)$ consisting of functions whose restriction to a triangle $\tau \in \mathbf{T}$ is a polynomial of degree $\leqslant k - 1$. Define $u^* \in \mathbf{S}_k$ by

$$a(u^*, v) = (f, v) \quad \text{for all } v \in \mathbf{S}_k. \tag{1.4}$$

The following theorem is the main result of this paper.

THEOREM 1. *Let h and γ be the parameters associated with* $\mathbf{T}$ *in* (1.3). *Then*

$$\sup_\Omega |u - u^*| \leqslant c \begin{Bmatrix} h^2 |\log h| & \text{if } k = 2 \\ h^k & \text{if } k \geqslant 3 \end{Bmatrix} |u|_{W_\infty^k(\Omega)},$$

where c depends only on Ω, γ, and k.

Proof. Let $z_0 \in \Omega$, and consider the Green's function g with singularity at z_0:

$$\begin{aligned} -\Delta g + g &= \delta_{z_0} \quad \text{in } \Omega, \\ \partial_n g &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.5}$$

Thus, we have

$$(u - u^*)(z_0) = a(g, u - u^*). \tag{1.6}$$

Let us introduce the finite element approximation $g^* \in \mathbf{S}_k$ to g:

$$a(g^*, v) = a(g, v) = v(z_0) \quad \text{for all } v \in \mathbf{S}_k. \tag{1.7}$$

Integrating by parts yields the following:

$$a(v, u - u^*) = 0 \quad \text{for all } v \text{ in } \mathbf{S}_k.$$

** The restriction of convexity is not essential.

Thus, we have

$$(1.8)\qquad \begin{aligned}(u-u^*)(z_0) &= a(g, u-u^*) = a(g-g^*, u-u^*)\\ &= a(g-g^*, u-v) \leqslant \|g-g^*\|_{W_1^1(\Omega)}\|u-v\|_{W_\infty^1(\Omega)},\end{aligned}$$

for any $v \in \mathbf{S}_k$. It is well known (cf. [23] or see (1.11) below) that there is a v in $\mathbf{S}_k$ such that

$$(1.9)\qquad \|u-v\|_{W_\infty^1(\Omega)} \leqslant c(\gamma, k)h^{k-1}|u|_{W_\infty^k(\Omega)}.$$

Thus, Theorem 1 will be completed if we prove the following:

THEOREM 2. *Let g and g^* satisfy* (1.5) *and* (1.7), *respectively, for* $z_0 \in \Omega$. *Then*

$$\|g-g^*\|_{W_1^1(\Omega)} \leqslant c\begin{cases} h\,|\log h| & \text{if } k=2 \quad \text{and}\\ h & \text{if } k \geqslant 3,\end{cases}$$

where $c = c(\Omega, \gamma, k)$ is independent of z_0.

Remark. It is at this point that we can explain why the factor of $|\log h|$ appears when $k = 2$. Our proof relies on approximating the singular function g, which just fails to be in $W_1^2(\Omega)$. However, it is in the interpolation space $[W_1^1(\Omega), W_1^3(\Omega)]_{1/2,\infty}$ (see [19]). With piecewise quadratics or better, g may thus be approximated to order h, although this does not say that g^* does. (Our proof that g^* does has the flavor of interpolation in it, but we do not simply reduce to an approximation problem.) However, piecewise linear approximation to g is completely different, as was shown to us by Claes Johnson. The singularity of g is primarily logarithmic, so it suffices to consider $g(z) \equiv \log|z-z_0|$ with z_0 in the interior of Ω. Johnson showed that, on a quasi-uniform mesh,

$$(1.10)\qquad \inf_{v\in \mathbf{S}_2} |g-v|_{W_1^1(\Omega)} \geqslant c(\Omega, z_0, \gamma)h\,|\log h| \quad \text{for } h \leqslant h_0(\Omega, z_0, \gamma).$$

We reproduce his proof here. For $\tau \in \mathbf{T}$, define $r_{\min}(\tau) = \inf_{z\in\tau}|z-z_0|$ and $r_{\max}(\tau) = \sup_{z\in\tau}|z-z_0|$. Let $\tau \in \mathbf{T}$ be such that $r_{\min}(\tau) \geqslant h$. For such τ (see (5.7)), $r_{\max}(\tau) \leqslant 2r_{\min}(\tau)$. Let $\vec{X}^* \in R^2$ minimize $\int_\tau |\nabla g - \vec{X}|$ over $\vec{X} \in R^2$. Then $X_i^* = g_{,i}(z_i)$ for some $z_i \in \tau$, for otherwise we could increase or decrease X_i^* to get a better approximation. We thus have

$$g_{,i}(z) - X_i^* = (z-z_i)\cdot \nabla g_{,i}(z_i) + R_i(z).$$

Using the fact that $|\nabla g_{,i}(z_i)| = |z_i - z_0|^{-2}$ and Taylor's theorem to bound R_i, we find

$$\int_\tau |g_{,i} - X_i^*| \geqslant c_1\frac{h^3}{r_{\max}^2} - c_2\frac{h^4}{r_{\min}^3} \geqslant c_1\frac{h^3}{4r_{\min}^2} - c_2\frac{h^4}{r_{\min}^3},$$

where c_1 depends on γ. Let $A_j = \bigcup\{\tau \in T : jh \leqslant r_{\min}(\tau) < (j+1)h\}$, $j = 1, 2, \ldots$. Then

$$\inf_{v\in\mathbf{S}_2}\int_{A_j} |\nabla g - \nabla v| \geqslant h\sum_{\tau\subset A_j}\left(\frac{c_1}{8j^2} - \frac{c_2}{j^3}\right).$$

Let N be the greatest integer less than $\operatorname{dist}(z_0, \Omega^c)/h$. If $j \leqslant N$, then there are at least πj triangles $\tau \subset A_j$, so we have

$$\inf_{v \in \mathbf{S}_2} \int_{A_j} |\nabla g - \nabla v| \geqslant h\left(\frac{c_3}{j} - \frac{c_4}{j^2}\right).$$

Summing for $j = 1, 2, \ldots, N$, we find

$$\inf_{v \in \mathbf{S}_2} \int_\Omega |\nabla g - \nabla v| \geqslant h(c_5 \log N - c_6) \geqslant c_7 h |\log h| - c_8 h,$$

and this proves (1.10). Note that if we allow mesh refinement near z_0, then a better approximation to g is obtained, so quasi-uniformity is necessary for (1.10) to be valid.

Before beginning the proof of Theorem 2, we collect some well-known facts that will be used throughout the proof. We begin with the concept of the *interpolate* of a continuous function. Let T be a fixed triangle, and choose the following *nodes* for T:

(i) the vertices of T,

(ii) the $k - 2$ points on each edge of T that divide the edge into $k - 1$ equal segments, and

(iii) ½$(k - 3)(k - 2)$ distinct points in the interior of T, chosen so that if a polynomial of degree $k - 4$ vanishes at all of them, it vanishes identically.

Here, (iii) applies only to $k \geqslant 4$ and (ii) applies only to $k \geqslant 3$. Now define nodes for each triangle τ in $\mathbf{T}$ by an affine identification of τ and T. (When τ is a boundary triangle, we identify the two straight sides of τ with two of the sides of T.) Define the interpolate $u_I \in \mathbf{S}_k$ of a continuous function u by the requirement that $u - u_I$ vanish at all the nodes. The following well-known [23] estimate establishes (1.9):

$$(1.11) \quad h^{2/p} \|u - u_I\|_{L^\infty(\Omega)} + \sum_{j=0}^{1} h^j \|u - u_I\|_{W_p^j(\Omega)} \leqslant c(\gamma, k) h^s |u|_{W_p^s(\Omega)},$$

where $1 \leqslant p \leqslant \infty$, $2 \leqslant s \leqslant k$. In addition to this estimate, we will need several estimates used in deriving (1.11), so we recall now its proof. It suffices to prove a local version, namely, for $\tau \in \mathbf{T}$,

$$(1.12) \quad h^{2/p} \|u - u_I\|_{L^\infty(\tau)} + \sum_{j=0}^{1} h^j \|u - u_I\|_{W_p^j(\tau)} \leqslant c(\gamma, k) h^s |u|_{W_p^s(\tau)}.$$

This is proved as follows. Because the nodal basis for $\mathbf{S}_k$ is *uniform* [23, Section 3.1], we have

$$(1.13) \quad \|u_I\|_{W_p^j(\tau)} \leqslant c(\gamma, k) h^{2/p - j} \|u\|_{L^\infty(\tau)}, \qquad j = 0, 1, \ldots, k - 1.$$

By the Bramble-Hilbert lemma, there exists a polynomial P of degree $\leqslant k - 1$ such that

$$(1.14) \quad h^{2/p} \|u - P\|_{L^\infty(\tau)} + \sum_{j=0}^{1} h^j \|u - P\|_{W_p^j(\tau)} \leqslant c h^s |u|_{W_p^s(\tau)}.$$

The proof of this in [2] requires that c depend on τ, but in [11], a proof is given showing that for convex τ, c depends only on γ and k. We may write $u - u_I = u - P - (u - P)_I$ because $P = P_I$. Applying the triangle inequality plus (1.13) and (1.14) yields (1.12).

Finally, we need an inverse relation for $\mathbf{S}_k$. Let ω (resp. ω_0) be the disc of diameter 2 (resp. γ) centered at the origin. Denote by $\mathcal{P}_n$ the space of polynomials of degree n in two variables. Then because of the equivalence of norms on a finite dimensional vector space, we have

$$\|P\|_{W_q^s(\omega)} \leqslant c(\gamma, k)\|P\|_{L^p(\omega_0)} \quad \text{for all } P \in \mathcal{P}_{k-1}.$$

(The constant c may be chosen independent of p and q in view of Hölder's inequality; it may also be assumed to be nonincreasing in γ.) Scaling the variables by a factor h, we find

$$\|P\|_{W_q^s(h\omega)} \leqslant c(\gamma, k)h^{2/q-2/p-s}\|P\|_{L^p(h\omega_0)} \quad \text{for all } P \in \mathcal{P}_{k-1},$$

where $h\omega = \{x \in \mathbb{R}^2 : h^{-1}x \in \omega\}$, etc. Now each $\tau \in \mathbf{T}$ contains a disc of diameter γh (say with center z) and is contained in a disc of diameter $2h$ with center z. Thus (after shifting the origin to z) we have

$$\|v\|_{W_q^s(\tau)} \leqslant c(\gamma, k)h^{2/q-2/p-s}\|v\|_{L^p(\tau)} \quad \text{for } v \in \mathbf{S}_k, \tag{1.15}$$

because $v|\tau \in \mathcal{P}_{k-1}$.

2. Error Estimate in L^2. We begin with the following proposition which extends the results of [21] up to the boundary and has a simplified proof.

PROPOSITION 1. *For* $0 \leqslant s \leqslant k-2$, $\|g - g^*\|_{H^{-s}(\Omega)} \leqslant ch^{1+s}$, *where* $c = c(\Omega, k, \gamma)$ *is independent of* z_0.

Proof. Let τ be a triangle in $\mathbf{T}$ containing z_0, and let Q be the polynomial of degree $k-1$ satisfying $\int_\tau QP = P(z_0)$ for all polynomials P of degree $k-1$. Because τ contains a disc of radius γh, we see that [21]

$$\sup_\tau |Q| \leqslant c(\gamma, k)h^{-2}. \tag{2.1}$$

Define $\widetilde{\delta} \in L^2(\Omega)$ by

$$\widetilde{\delta} = \begin{cases} Q & \text{in } \tau, \\ 0 & \text{in } \Omega - \tau. \end{cases}$$

Then, $\langle \delta - \widetilde{\delta}, v\rangle = 0$ for all $v \in \mathbf{S}_k$ $(\delta = \delta_{z_0})$. Let $\widetilde{g}$ solve

$$-\Delta\widetilde{g} + \widetilde{g} = \widetilde{\delta} \quad \text{in } \Omega,$$
$$\partial_n \widetilde{g} = 0 \quad \text{on } \partial\Omega.$$

Since g^* may be viewed as the finite element approximation to $\widetilde{g}$, we have the well-known estimate (cf. [1, Chapter 6])

$$\|\widetilde{g} - g^*\|_{H^{-s}(\Omega)} \leqslant c(\gamma, \Omega, k)h^{2+s}\|\widetilde{g}\|_{H^2(\Omega)}.$$

From elliptic regularity theory, we have

$$\|\widetilde{g}\|_{H^2(\Omega)} \leqslant c(\Omega)\|\widetilde{\delta}\|_{L^2(\Omega)} \leqslant c(\Omega, \gamma, k)h^{-1}.$$

Therefore,

$$\|\widetilde{g} - g^*\|_{H^{-s}(\Omega)} \leqslant c(\Omega, \gamma, k)h^{1+s}. \tag{2.2}$$

To estimate $g - \widetilde{g}$, we let $\varphi \in H^s(\Omega)$ and solve

$$-\Delta\Phi + \Phi = \varphi \quad \text{in } \Omega,$$
$$\partial_n\Phi = 0 \quad \text{on } \partial\Omega.$$

Integrating by parts (twice), we get

$$(g - \widetilde{g}, \varphi) = a(g - \widetilde{g}, \Phi) = \langle \delta - \widetilde{\delta}, \Phi\rangle.$$

Since $\delta - \widetilde{\delta}$ is orthogonal to $\mathbf{S}_k$, we have

$$(g - \widetilde{g}, \varphi) = \langle \delta - \widetilde{\delta}, \Phi - \Phi_I \rangle. \tag{2.3}$$

In view of (1.11) and (2.1), (2.3) is estimated by

$$\begin{aligned} |(g - \widetilde{g}, \varphi)| &\leqslant c(\gamma, k) h^{1+s} \|\Phi\|_{W_2^{2+s}(\Omega)} \\ &\leqslant c(\Omega, \gamma, k) h^{1+s} \|\varphi\|_{W_2^s(\Omega)}. \end{aligned}$$

Since φ was arbitrary, this means that

$$\|g - \widetilde{g}\|_{H^{-s}(\Omega)} \leqslant c(\Omega, \gamma, k) h^{1+s}.$$

Combined with (2.2), this completes the proposition.

3. Expansion of g to Required Accuracy. The purpose of this section is to show that the only singularities of g that we must contend with are logarithmic. While this is obvious for z_0 in the interior of Ω, it must be demonstrated to hold uniformly as $z_0 \longrightarrow \partial\Omega$. Let $\kappa(z)$ denote the curvature of $\partial\Omega$ at the point $z \in \partial\Omega$, and let $d = \frac{1}{2}(\sup_{z \in \partial\Omega} \kappa(z))^{-1}$. For each $z_0 \in \Omega$ such that $\operatorname{dist}(z_0, \partial\Omega) \leqslant d$, there is a unique $z_1 \in \partial\Omega$ such that $|z_1 - z_0| = \operatorname{dist}(z_0, \partial\Omega)$. For such z_0, let us define

$$\bar{z}_0 = z_1 + (z_1 - z_0)/(1 - \kappa(z_1)|z_1 - z_0|).^{***} \tag{3.1}$$

LEMMA 1. *Let $z_0 \in \Omega$ satisfy* $\operatorname{dist}(z_0, \partial\Omega) \leqslant d$ *and define*

$$G(z) = \log|z - z_0| + \log|z - \bar{z}_0|,$$

where $\bar{z}_0$ is given by (3.1). *Then $g = (1/2\pi)G + W$, where W satisfies $\|W\|_{H^2(\Omega)} \leqslant c(\Omega)$ with $c(\Omega)$ independent of z_0.*

We give the proof of Lemma 1 in the Appendix as it is technical and unrelated to the rest of the paper. To complement the expansion given in Lemma 1, define

$$G(z) = \log|z - z_0| \quad \text{if } \operatorname{dist}(z_0, \partial\Omega) > d.$$

Then, for all z_0 we have $g = (1/2\pi)G + W$, where $\|W\|_{H^2(\Omega)} \leqslant c(\Omega)$, with $c(\Omega)$ independent of $z_0 \in \Omega$. The mapping $u \longrightarrow u^*$ defined by (1.1) and (1.4) is linear, so we may write

$$g - g^* = \frac{1}{2\pi}(G - G^*) + (W - W^*). \tag{3.2}$$

It is well known (cf. [1, Chapter 6]) that for $-1 \leqslant s \leqslant k - 2$,

$$\|W - W^*\|_{H^{-s}(\Omega)} \leqslant c(\Omega, \gamma, k) h^{s+2} \|W\|_{H^2(\Omega)} \leqslant c(\Omega, \gamma, k) h^{s+2}. \tag{3.3}$$

Applying this with $s = -1$ shows that, to complete the proof of Theorem 2, it suffices to verify that

$$\|G - G^*\|_{W_1^1(\Omega)} \leqslant c(\Omega, \gamma, k) \begin{cases} h\,|\log h| & \text{if } k = 2 \quad \text{and} \\ h & \text{if } k \geqslant 3. \end{cases} \tag{3.4}$$

The remainder of the paper is devoted to proving (3.4), and makes use of the simplified form of G. As a first step, we apply (3.3) and Proposition 1 to the decomposition (3.2) to obtain the following:

PROPOSITION 2. *For $0 \leqslant s \leqslant k - 2$, we have*

*** $\bar{z}_0$ is the reflection of z_0 with respect to the osculating circle to $\partial\Omega$ at z_1.

$$\|G - G^*\|_{H^{-s}(\Omega)} \leqslant c(\Omega, \gamma, k)h^{s+1},$$

where the constant does not depend on z_0.

4. Estimates Near z_0. We first prove a result that is analogous to the fact that the logarithm has bounded mean oscillation [13].

LEMMA 2. *Let* z_1 *be any point in* $\mathbb{R}^2$, *and let* $0 < \rho < \infty$. *Then there is a constant* $\bar{G}$ *depending on* z_1 *and* ρ *such that*

$$\int_{\{|z-z_1|\leqslant\rho\}} (G - \bar{G})^2 \leqslant 9\pi\rho^2.$$

Proof. Consider the case $G = \log|z - z_0|$. We first observe that

$$\int_{\{|z-z_0|\leqslant R\}} (\log|z - z_0| - [(\log R) - \tfrac{1}{2}])^2 = \frac{\pi}{4}R^2. \tag{4.1}$$

There are two cases to consider:

(1) $|z_1 - z_0| \leqslant 2\rho$. Apply (4.1) with $R = 3\rho$, using $\bar{G} = (\log 3\rho) - \tfrac{1}{2}$.

(2) $|z_1 - z_0| > 2\rho$. Then $|\nabla G| < 1/\rho$ for $|z - z_1| \leqslant \rho$, and if we take $\bar{G} = G(z_1)$, we have

$$\int_{\{|z-z_1|\leqslant\rho\}} (G - \bar{G})^2 \leqslant \frac{\pi}{2}\rho^2$$

by Taylor's theorem.

Thus, we have shown that we can find $\bar{G}$ such that

$$\int_{\{|z-z_1|\leqslant\rho\}} (G - \bar{G})^2 \leqslant \frac{9}{4}\pi\rho^2.$$

When $G = \log|z - z_0| + \log|z - \bar{z}_0|$, we apply this twice and use the triangle inequality to complete the lemma.

LEMMA 3. *Let* $0 < \rho < c_1 h$ *be given. Then*

$$\int_{\Omega\cap\{|z-z_0|\leqslant\rho\}} |z - z_0|^\beta |\nabla(G - G^*)|^p \leqslant c(\Omega, \gamma, k, \beta, p, c_1)\rho^\beta h^{2-p}$$

for $1 \leqslant p < \beta + 2$.

Proof. We have $|\nabla G| \leqslant r^{-1} + \bar{r}^{-1}$, where $r = |z - z_0|$ and $\bar{r} = |z - \bar{z}_0|$. Since $\bar{r} \geqslant r$ in Ω, we have

$$\int_{\Omega\cap\{r\leqslant\rho\}} r^\beta |\nabla G|^p \leqslant \pi 2^{p+1} \int_0^\rho r^{\beta-p+1}\, dr \leqslant c(\beta, p)\rho^{\beta-p+2}.$$

To estimate ∇G^*, we pick $\tau \in \mathrm{T}$ such that $\tau \cap \{r \leqslant \rho\} \neq \varnothing$. Then choose $\bar{G}$ by Lemma 2 so that

$$\int_\tau (G - \bar{G})^2 \leqslant \frac{9}{4}\pi h^2. \tag{4.2}$$

Now we have

$$\int_\tau r^\beta |\nabla G^*|^p = \int_\tau r^\beta |\nabla(G^* - \bar{G})|^p \leqslant (\rho + h)^\beta h^2 \sup_\tau |\nabla(G^* - \bar{G})|^p. \tag{4.3}$$

Applying (1.15) with $v = G^* - \bar{G}$, we have

$$\sup_T |\nabla(G^* - \bar{G})| \leqslant c(\gamma, k)h^{-2}\Big(\int_T (G^* - \bar{G})^2\Big)^{1/2}.$$

Applying the triangle inequality, (4.2), and Proposition 2, we get

$$\sup_T |\nabla(G^* - \bar{G})| \leqslant c(\Omega, \gamma, k)h^{-1}.$$

Substituting this into (4.3) completes the lemma.

5. Estimates Away From z_0. Lemma 3 shows, in particular, that in a disc of radius $O(h)$ around z_0, the W_1^1 norm of the error $G - G^*$ is $O(h)$, regardless of k. We now concentrate on estimating the error outside such a disc (this is the heart of the proof) using the ideas of interior estimates [17] (here they should be called exterior estimates). Let us write $r = |z - z_0|$, and for "cut-off function" we choose r^α, where $\alpha = 2$ when $k = 2$ and $\alpha = 3$ when $k \geqslant 3$. Let us use the expression E for the error $G - G^*$:

$$E \equiv G - G^*. \tag{5.1}$$

Given any constant $c_1 > 0$, we have

$$\begin{aligned}\int_{\{r \geqslant c_1 h\}\cap\Omega} |\nabla E| &= \int_{\{r \geqslant c_1 h\}\cap\Omega} r^{-\alpha/2} r^{\alpha/2} |\nabla E| \\ &\leqslant \Big(\int_{\{r \geqslant c_1 h\}\cap\Omega} r^{-\alpha}\Big)^{1/2} \Big(\int_{\{r \geqslant c_1 h\}\cap\Omega} r^\alpha |\nabla E|^2\Big)^{1/2} \\ &\leqslant c(c_1, \Omega)\begin{Bmatrix} |\log h|^{1/2} & \text{if } k = 2 \\ h^{-1/2} & \text{if } k \geqslant 3\end{Bmatrix} \Big(\int_\Omega r^\alpha |\nabla E|^2\Big)^{1/2}.\end{aligned} \tag{5.2}$$

We now expand the remaining integral, essentially commuting r^α with ∇:

$$\begin{aligned}\int_\Omega r^\alpha |\nabla E|^2 &= \int_\Omega \nabla E \cdot \nabla(r^\alpha E) - \int_\Omega E(\nabla E \cdot \nabla r^\alpha) \\ &\leqslant \int_\Omega \nabla E \cdot \nabla(r^\alpha E) + \alpha\Big(\int_\Omega r^{\alpha-2}E^2\Big)^{1/2}\Big(\int_\Omega r^\alpha |\nabla E|^2\Big)^{1/2},\end{aligned} \tag{5.3}$$

where we have used the fact that $|\nabla r^\alpha| = \alpha r^{\alpha-1}$. Therefore,

$$\int_\Omega r^\alpha |\nabla E|^2 \leqslant 2\int_\Omega \nabla E \cdot \nabla(r^\alpha E) + \alpha^2 \int_\Omega r^{\alpha-2}E^2. \tag{5.4}$$

LEMMA 4. *Suppose that* $0 < \epsilon \leqslant 1$. *Then*

$$\int_\Omega r^{\alpha-2}E^2 \leqslant \epsilon \int_\Omega r^\alpha |\nabla E|^2 + c(\Omega, \gamma, k)\epsilon^{-1}h^\alpha.$$

Proof. When $k = 2$, this follows from Proposition 2, so assume that $k \geqslant 3$ (and thus $\alpha = 3$). We have (Proposition 2)

$$\begin{aligned}(rE, E) &\leqslant \|rE\|_{H^1(\Omega)}\|E\|_{H^{-1}(\Omega)} \leqslant c(\Omega, \gamma, k)h^2\|rE\|_{H^1(\Omega)} \\ &\leqslant c(\Omega, \gamma, k)h^3\epsilon^{-1} + h\epsilon\|rE\|^2_{H^1(\Omega)}.\end{aligned}$$

Expanding, we have

$$\begin{aligned}\|rE\|^2_{H^1(\Omega)} &\leqslant \int_\Omega (r^2 + 3)E^2 + \int_\Omega 2r^2 |\nabla E|^2 \\ &\leqslant c(\Omega, \gamma, k)h^2 + \int_\Omega 2r^2 |\nabla E|^2.\end{aligned}$$

We have $r^2 \leqslant h^{-1} r^3$ on $\{r \geqslant h\}$, so Lemma 3 implies that

$$\int_\Omega r^2 |\nabla E|^2 = \Big(\int_{\{r \leqslant h\}} + \int_{\{r \geqslant h\} \cap \Omega}\Big) r^2 |\nabla E|^2$$
$$\leqslant c(\Omega, \gamma, k) h^2 + h^{-1} \int_\Omega r^3 |\nabla E|^2.$$

These estimates combine to prove the lemma in view of the fact that $\epsilon \leqslant \epsilon^{-1}$.

Applying Lemma 4 with $\epsilon = 1/18$ to (5.4), we find that

$$\int_\Omega r^\alpha |\nabla E|^2 \leqslant 4 \int_\Omega \nabla E \cdot \nabla(r^\alpha E) + c(\Omega, \gamma, k) h^\alpha.$$

What remains to be shown, to complete Theorem 2, is that

$$\int_\Omega \nabla E \cdot \nabla(r^\alpha E) \leqslant \frac{1}{8} \int_\Omega r^\alpha |\nabla E|^2 + c(\Omega, \gamma, k) h^\alpha |\log h|^{3-\alpha}. \tag{5.5}$$

First, let us define an interpolate of G. Define

$$\Omega_1 = \bigcup \{\tau \in \mathbf{T} : \operatorname{dist}(z_0, \tau) \geqslant h\}.$$

Then $\{z \in \Omega : |z - z_0| \geqslant 2h\} \subset \Omega_1$, and G is a smooth function on Ω_1. Define G_I on Ω_1 by requiring G_I to be the restriction to Ω_1 of a function in $\mathbf{S}_k$ that equals G at all nodes contained in Ω_1.

LEMMA 5. *With G_I defined as above, we have*

$$\int_{\Omega_1} r^{\alpha-2} (G - G_I)^2 \leqslant c(\Omega, \gamma, k) h^\alpha$$

and

$$\int_{\Omega_1} r^{-\alpha} |\nabla(r^\alpha (G - G_I))|^2 \leqslant c(\Omega, \gamma, k) h^\alpha |\log h|^{3-\alpha}.$$

Proof. Let $\tau \in \mathbf{T}$ be contained in Ω_1. Then by (1.12),

$$\|G - G_I\|_{W^s_\infty(\tau)} \leqslant c(\gamma, k) h^{k-s} \|G\|_{W^k_\infty(\tau)} \tag{5.6}$$

for $s = 0, 1$. Let $r_{\max} = \sup_\tau r$ and $r_{\min} = \inf_\tau r$. Then $\|G\|_{W^k_\infty(\tau)} \leqslant c(k)(r_{\min})^{-k}$. Since $r_{\max} - r_{\min} \leqslant h$ and $r_{\min} \geqslant h$, we have

$$r_{\max}/r_{\min} \leqslant 2. \tag{5.7}$$

Thus, we conclude that

$$\int_\tau r^\beta (G - G_I)^2 + h^2 \int_\tau r^\beta |\nabla(G - G_I)|^2 \leqslant c(\gamma, k) \Big(\int_\tau r^{\beta - 2k}\Big) h^{2k}.$$

Summing over all $\tau \subset \Omega_1$, we prove the lemma, since

$$\int_{\Omega_1} r^{\beta - 2k} \leqslant c(\Omega) \begin{cases} |\log h| & \text{if } \beta - 2k + 2 = 0, \\ h^{\beta - 2k + 2}/(-\beta + 2k - 2) & \text{if } \beta - 2k + 2 < 0, \end{cases}$$

and

$$r^{-\alpha} |\nabla r^\alpha (G - G_I)|^2 \leqslant 2\alpha^2 r^{\alpha - 2} (G - G_I)^2 + 2 r^\alpha |\nabla(G - G_I)|^2.$$

LEMMA 6. $\|G_I - G^*\|_{L^\infty(\Omega_1)} \leqslant c(\Omega, \gamma, k)$.

Proof. By (1.15), Proposition 2, and Lemma 5,

$$\begin{aligned}\|G_I - G^*\|_{L^\infty(\Omega_1)} &\leqslant c(\gamma, k)h^{-1}\|G_I - G^*\|_{L^2(\Omega_1)} \\ &\leqslant c(\gamma, k)h^{-1}(\|G - G_I\|_{L^2(\Omega_1)} + \|G - G^*\|_{L^2(\Omega_1)}) \\ &\leqslant c(\gamma, k)h^{-\alpha/2}\|r^{\alpha/2-1}(G - G_I)\|_{L^2(\Omega_1)} + c(\Omega, \gamma, k) \\ &\leqslant c(\Omega, \gamma, k).\end{aligned}$$

LEMMA 7. *Let* $\varphi \in \mathbf{S}_k$. *Then, if* $v = (r^\alpha\varphi)_I$, *we have*

$$\int_{\Omega_1} r^{-\alpha}|\nabla(r^\alpha\varphi - v)|^2 \leqslant c(\gamma, k)\int_{\Omega_1} r^{\alpha-2}\varphi^2.$$

Proof. Let $\tau \in \mathbf{T}$, $\tau \subset \Omega_1$. Then by (1.12),

$$\begin{aligned}|r^\alpha\varphi - v|_{W^1_\infty(\tau)} &\leqslant c(\gamma, k)h^{k-1}|r^\alpha\varphi|_{W^k_\infty(\tau)} \\ &\leqslant c(\gamma, k)h^{k-1}\sum_{j=1}^{k}|r^\alpha|_{W^j_\infty(\tau)}|\varphi|_{W^{k-j}_\infty(\tau)},\end{aligned}\tag{5.8}$$

because $D^\beta(\varphi|\tau) = 0$ for $|\beta| = k$. In view of (5.7) and the fact that $r \geqslant h$ on Ω_1

$$|r^\alpha|_{W^j_\infty(\tau)} \leqslant c(j)\inf_\tau r^{\alpha-j} \leqslant c(j)\left(\inf_\tau r^{\alpha-1}\right)h^{1-j}.$$

Combining (1.15) with (5.7), one obtains

$$\begin{aligned}|\varphi|_{W^{k-j}_\infty(\tau)} &\leqslant c(\gamma, k)h^{j-k-1}\|\varphi\|_{L^2(\tau)} \\ &\leqslant c(\gamma, k)\left(\inf_\tau r^{1-\alpha/2}\right)h^{j-k-1}\|r^{\alpha/2-1}\varphi\|_{L^2(\tau)}.\end{aligned}$$

Applying these estimates in (5.8) yields

$$|r^\alpha\varphi - v|_{W^1_\infty(\tau)} \leqslant c(\gamma, k)\left(\inf_\tau r^{\alpha/2}\right)h^{-1}\|r^{\alpha/2-1}\varphi\|_{L^2(\tau)}.$$

Now Hölder's inequality implies that

$$\int_\tau r^{-\alpha}|\nabla(r^\alpha\varphi - v)|^2 \leqslant c(\gamma, k)\int_\tau r^{\alpha-2}\varphi^2.$$

Summing this over $\tau \subset \Omega_1$ completes the proof.

We return now to the proof of (5.5). Let $v \in \mathbf{S}_k$. Since $a(E, v) = 0$, Lemmas 3 and 4 yield

$$\begin{aligned}\int_\Omega \nabla E\cdot\nabla(r^\alpha E) &= \int_\Omega \nabla E\cdot\nabla(r^\alpha E - v) - \int_\Omega Ev \\ &\leqslant \int_{\Omega_1}\nabla E\cdot\nabla(r^\alpha E - v) + c(\Omega, \gamma, k)(h^\alpha + h|v|_{W^1_\infty(\Omega_1^c)}) \\ &\quad + \frac{1}{4}\int_\Omega r^{\alpha-2}E^2 + \int_\Omega r^{2-\alpha}v^2 \\ &\leqslant \int_{\Omega_1}\nabla E\cdot\nabla(r^\alpha E - v) + c(\Omega, \gamma, k)(h^\alpha + h|v|_{W^1_\infty(\Omega_1^c)}) \\ &\quad + \frac{1}{32}\int_\Omega r^\alpha|\nabla E|^2 + c(\Omega, \gamma, k)h^\alpha + \int_\Omega r^{2-\alpha}v^2.\end{aligned}\tag{5.9}$$

Let $v \in \mathbf{S}_k$ be any function that interpolates $r^\alpha(G_I - G^*)$ in Ω_1. The triangle inequality plus Lemmas 4, 5, and 7 give

$$
\begin{aligned}
\int_{\Omega_1} \nabla E \cdot \nabla(r^\alpha E - v) &\leq \frac{1}{32}\int_\Omega r^\alpha |\nabla E|^2 + 8\int_{\Omega_1} r^{-\alpha}|\nabla(r^\alpha E - v)|^2 \\
&\leq \frac{1}{32}\int_\Omega r^\alpha|\nabla E|^2 + 16\int_{\Omega_1} r^{-\alpha}|\nabla(r^\alpha(G - G_I))|^2 \\
&\quad + 16\int_{\Omega_1} r^{-\alpha}|\nabla(r^\alpha(G_I - G^*) - v)|^2 \\
&\leq \frac{1}{32}\int_\Omega r^\alpha|\nabla E|^2 + c(\Omega,\gamma,k)h^\alpha|\log h|^{3-\alpha} \qquad (5.10)\\
&\quad + c(\gamma,k)\int_{\Omega_1} r^{\alpha-2}(G_I - G^*)^2 \\
&\leq \frac{1}{32}\int_\Omega r^\alpha|\nabla E|^2 + c(\Omega,\gamma,k)h^\alpha|\log h|^{3-\alpha} \\
&\quad + c(\gamma,k)\Big(\int_{\Omega_1} r^{\alpha-2}(G - G_I)^2 + \int_{\Omega_1} r^{\alpha-2}(G - G^*)^2\Big) \\
&\leq \frac{1}{16}\int_\Omega r^\alpha|\nabla E|^2 + c(\Omega,\gamma,k)h^\alpha|\log h|^{3-\alpha}.
\end{aligned}
$$

Applying (5.10) to (5.9), one obtains

$$
\begin{aligned}
\int_\Omega \nabla E\cdot\nabla(r^\alpha E) &\leq \frac{3}{32}\int_\Omega r^\alpha|\nabla E|^2 + c(\Omega,\gamma,k)h^\alpha|\log h|^{3-\alpha} \\
&\quad + \int_\Omega r^{2-\alpha}v^2 + c(\Omega,\gamma,k)h\,|v|_{W^1_\infty(\Omega_1^c)},
\end{aligned}
$$

where $v \in \mathbf{S}_k$ is any function that interpolates $r^\alpha(G_I - G^*)$ in Ω_1. For definiteness, we take v to be zero at the nodes in the interior of Ω_1^c. Thus all that remains is to bound v suitably. From (1.13), (5.7), and (1.15), we find

$$
h|v|_{W^1_\infty(\Omega_1^c)} + h^{\frac{1}{2}\alpha-2}\|r^{1-\alpha/2}v\|_{L^2(\Omega_1^c)} \leq c(\gamma,k)\sup_{\partial\Omega_1-\partial\Omega}|r^\alpha(G_I - G^*)|
$$

and

$$
\|r^{1-\alpha/2}v\|_{L^2(\Omega_1)} \leq c(\gamma,k)\|r^{\frac{1}{2}\alpha+1}(G_I - G^*)\|_{L^2(\Omega_1)}.
$$

The first line is estimated by Lemma 6 and the second is estimated as in (5.10) with the triangle inequality plus Lemmas 4 and 5, proving (5.5). This completes the proof that

$$
|G - G^*|_{W^1_1(\Omega)} \leq c(\Omega,\gamma,k)\begin{cases} h|\log h| & \text{if } k = 2,\\ h & \text{if } k \geq 3.\end{cases}
$$

From Proposition 2, we have $\|G - G^*\|_{L^1(\Omega)} \leq c(\Omega,\gamma,k)h$, so the proof of Theorem 2 is complete.

6. Further Results. We first extend Theorem 1 to give estimates for the error $u - u^*$ in L^p. It is notable that they do not immediately follow by interpolating between the known L^2 estimates and Theorem 1. This would require knowing what are the interpolation spaces between W^k_p and W^k_q for $p = 2$ and $q = \infty$. However, this is known only when $1 < p, q < \infty$ [7], [19].

THEOREM 3. *Let u and u^* be as in Theorem 1. For p a number such that $1 \leq p \leq \infty$ and s an integer such that $1 \leq s \leq k$,*

$$\|u - u^*\|_{L^p(\Omega)} \leqslant c(\Omega, \gamma, k) h^s |\log h|^\beta |u|_{W_p^s(\Omega)},$$

where $\beta = 0$ *if* $k \geqslant 3$ *and* $\beta = |1 - 2/p|$ *if* $k = 2$.

Proof. Following the proof of Theorem 1, let $y \in \Omega$, let g_y be the Green's function with singularity y, and let g_y^* be its projection onto $\mathbf{S}_k$. Define $K(x, y) = g_y(x) - g_y^*(x)$. Then K is a symmetric function and

$$(u - u^*)(x) = \int_\Omega K(x, y) u(y) + \nabla K(x, y) \cdot \nabla u(y)\, dy.$$

The kernel $(K, \nabla K)$ defines a mapping $\mathcal{K}$ for bounded functions $U: \Omega \longrightarrow \mathbf{R}^3$ by

$$\mathcal{K}U(x) = \int K(x, y) U_1(y) + \nabla K(x, y) \cdot (U_2, U_3)(y)\, dy.$$

By Theorem 2 and Theorem 4.1.2 of [18], $\mathcal{K}$ maps $L^p(\Omega)^3 \equiv L^p(\Omega) \times L^p(\Omega) \times L^p(\Omega)$ to $L^p(\Omega)$ and

$$\|\mathcal{K}U\|_{L^p(\Omega)} \leqslant c \begin{Bmatrix} h|\log h| & \text{if } k = 2 \\ h & \text{if } k \geqslant 3 \end{Bmatrix} \|U\|_{L^p(\Omega)^3},$$

where c is the constant in Theorem 2. By (1.7), $\mathcal{K}(v, \nabla v) \equiv 0$ for $v \in \mathbf{S}_k$, so

$$\|u - u^*\|_{L^p(\Omega)} \leqslant c \begin{Bmatrix} h|\log h| & \text{if } k = 2 \\ h & \text{if } k \geqslant 3 \end{Bmatrix} \inf_{v \in \mathbf{S}_k} \|(u, \nabla u) - (v, \nabla v)\|_{L^p(\Omega)^3}.$$

If $s \geqslant 2$, we choose $v = u_I$ to get

$$\|u - u^*\|_{L^p(\Omega)} \leqslant c' \begin{Bmatrix} h^s\,|\log h| & \text{if } k = 2 \\ h^s & \text{if } k \geqslant 3 \end{Bmatrix} |u|_{W_p^s(\Omega)}. \tag{6.1}$$

If $s = 1$, the choice $v = C$ yields the same estimate, where C is the constant such that [2]

$$\|u - C\|_{W_p^1(\Omega)} \leqslant c(\Omega) |u|_{W_p^1(\Omega)}.$$

This proves the theorem if $k \geqslant 3$ or if $k = 2$ and $p = 1$ or ∞.

Now suppose that $k = 2$. It is known [1] that

$$\|u - u^*\|_{L^2(\Omega)} \leqslant c(\Omega, \gamma) h \|u\|_{H^1(\Omega)}. \tag{6.2}$$

As will be shown subsequently, this implies that for all $U \in L^2(\Omega)^3$,

$$\|\mathcal{K}U\|_{L^2(\Omega)} \leqslant c(\Omega, \gamma) h \|U\|_{L^2(\Omega)^3}. \tag{6.3}$$

Assuming this for the moment, the Riesz convexity theorem [25] implies that for all p, $1 \leqslant p \leqslant \infty$,

$$\|\mathcal{K}U\|_{L^p(\Omega)} \leqslant c(\Omega, \gamma) h\, |\log h|^\beta \|U\|_{L^p(\Omega)^3}.$$

The rest of the proof is as before.

Now we prove (6.3). The idea is standard: we simply project a general U onto a function of the form $(u, \nabla u)$ and apply (6.2). So let $U \in L^2(\Omega) \times H^1(\Omega) \times H^1(\Omega)$ be given, and let u solve the boundary value problem

$$-\Delta u + u = U_1 + \nabla \cdot (U_2, U_3) \quad \text{in } \Omega,$$
$$\partial_n u = \vec{n} \cdot (U_2, U_3) \qquad \text{on } \partial\Omega.$$

Integrating by parts shows that $\mathcal{K}U = \mathcal{K}(u, \nabla u)$ because the kernel of $\mathcal{K}$ has the special form $(K, \nabla K)$. So (6.2) yields

$$\|KU\| = \|K(u, \nabla u)\| \leq ch\|(u, \nabla u)\| \leq ch\|U\|,$$

where the norm is the norm in $L^2(\Omega)^3$. This proves (6.3) for $U \in L^2(\Omega) \times H^1(\Omega) \times H^1(\Omega)$, and the general case follows because this space is dense in $L^2(\Omega)^3$.

Up to now, we have estimated only function values. However, as is well known, these estimates suffice to obtain optimal estimates of derivatives also.

THEOREM 4. *Suppose that u and u^* are as in Theorem* 1. *Let s and t be integers with* $1 \leq t \leq s \leq k$ *and let p satisfy* $1 \leq p \leq \infty$. *Then*

$$\left(\sum_{\tau\in\mathbf{T}} \|u - u^*\|^p_{W^t_p(\tau)}\right)^{1/p} \leq c(\Omega, \gamma, k)h^{s-t}|\log h|^\beta |u|_{W^s_p(\Omega)},$$

where $\beta = 0$ *if* $k \geq 3$ *and* $\beta = |1 - 2/p|$ *if* $k = 2$. (*When* $p = \infty$, *the term on the left is replaced by* $\max_{\tau\in\mathbf{T}} \|u - u^*\|_{W^t_\infty(\tau)}$.)

Proof. Write $u - u^* = (u - u_I) + (u_I - u^*)$ and apply the triangle inequality. For $u - u_I$, we apply (1.11). For $u_I - u^*$, apply (1.15) to reduce to norms of function values, rewrite $u_I - u^* = (u - u^*) - (u - u_I)$, and apply Theorem 3 and (1.11) respectively.

Appendix. *Proof of Lemma* 1. Let us focus our attention on z_1 rather than z_0, i.e., let $z_1 \in \partial\Omega$ and consider

$$z_0 = z_1 + t\vec{n}(z_1), \qquad \bar{z}_0 = z_1 - \frac{t}{1 - \kappa(z_1)t}\vec{n}(z_1)$$

for $t \in]0, d]$, where $\vec{n}(z_1)$ is the inward normal to $\partial\Omega$ at z_1. Let $\bar{D}$ be the disc of radius $R = \kappa(z_1)^{-1}$ with center at $\hat{z} \equiv z_1 + R\vec{n}(z_1)$, let D be the domain obtained by deleting the line $\{z = z_1 + t\vec{n}(z_1) : t \in [R, 2R]\}$, and let Γ be ∂D minus this line. Viewing the z's as complex numbers, the function

$$N(z) = \log|\log(z - \hat{z}) - \log(z_0 - \hat{z})| + \log|\log(z - \hat{z}) - \log(\bar{z}_0 - \hat{z})|$$

satisfies $\Delta(G - N) = 0$ in $D_1 \equiv \mathbb{R}^2 - \{z = z_1 + t\vec{n}(z_1) : t \geq R\}$ and $\partial_n N = 0$ on Γ.† To see this, write $z = Re^\zeta + \hat{z}$ and view G and N as functions of ζ for $\zeta \in S \equiv \{\xi + i\eta : \tilde{\eta} < \eta < \tilde{\eta} + 2\pi\}$, where $\tilde{\eta}$ is chosen so that $\vec{n}(z_1) = e^{i\tilde{\eta}}$. We have

$$G(z(\zeta)) - N(z(\zeta)) = \log\left\{R^2 \cdot \frac{|e^\zeta - e^{\zeta_0}|}{|\zeta - \zeta_0|} \cdot \frac{|e^\zeta - e^{\bar{\zeta}_0}|}{|\zeta - \bar{\zeta}_0|}\right\}. \tag{A.1}$$

For any analytic function f, $(f(z) - f(z_0))/(z - z_0)$ is also analytic, so $G - N$ is of the form $\log|\varphi|$, where φ is analytic and $\neq 0$ in S. Thus, $G - N$ is smooth, and since both G and N are harmonic in $D_1 - \{z_0, \bar{z}_0\}$, $G - N$ must be harmonic in all of D_1. (N is obviously harmonic as a function of ζ, and since $z = Re^\zeta + \hat{z}$ is analytic, N is harmonic as a function of z.) As a function of $\zeta = \xi + i\eta$, N is symmetric about the line $\{\xi = 0\}$, hence $N_\xi(0 + i\eta) = 0$. Thus, $\partial_n N$ is zero on the image Γ of $\{\xi = 0, \tilde{\eta} < \eta < \tilde{\eta} + 2\pi\}$ via the conformal map $z = Re^\zeta + \hat{z}$. Let $\Omega_1 = \{z \in \Omega : |z - z_1| < 3d/2\}$. Then

$$\|N - G\|_{H^2(\Omega_1)} \leq c(\Omega)$$

independent of z_0 (i.e., independent of t) in view of (A.1). Now choose a smooth cut-

† When $\kappa(z_1) = 0$, choose $N = G$ and let Γ be the tangent line to $\partial\Omega$ at z_1.

off function χ such that $\chi \equiv 1$ in $\Omega_0 \equiv \{|z - z_1| < 4d/3\}$ and $\chi \equiv 0$ outside Ω_1. We have

$$w = g - \frac{1}{2\pi} G = \left(g - \frac{1}{2\pi} \chi N\right) + \frac{1}{2\pi} \chi(N - G) + \frac{1}{2\pi} (\chi - 1)G.$$

The last two terms are bounded in $H^2(\Omega)$ independently of t, so it remains to prove the same of $w_1 \equiv g - \chi N/2\pi$, which satisfies the equations

$$\begin{aligned} -\Delta w_1 + w_1 &= \frac{1}{2\pi} (N\Delta\chi + \nabla N \cdot \nabla\chi - N\chi) && \text{in } \Omega, \\ \partial_n w_1 &= -\frac{1}{2\pi} (N\partial_n \chi + \chi\partial_n N) && \text{on } \partial\Omega. \end{aligned}$$

The function $N\Delta\chi + \nabla N \cdot \nabla\chi - N\chi$ is in $L^2(\Omega)$, boundedly in t, and the function $N\partial_n\chi$ is in $H^1(\partial\Omega)$, boundedly in t. Hence, it remains to show that

$$\|\chi\partial_n N\|_{H^1(\partial\Omega)} \leqslant c(\Omega) \tag{A.2}$$

independently of t. (Elliptic regularity then implies that $\|w_1\|_{H^2(\Omega)} \leqslant c(\Omega)$, completing the proof.) The reason (A.2) is valid is that $\partial_n N = 0$ on Γ, and Γ is a third order approximation to $\partial\Omega$ at z_1. Let us define a mapping from $\partial\Omega$ to Γ (near z_1) by choosing $r = r(s)$ so that

$$\tilde{z}(s) \equiv z(s) + r\vec{n}(z(s)) \in \Gamma,$$

where $z(s)$ parametrizes $\partial\Omega$ by arc length with $z_1 = z(0)$. The function $\tilde{z}(s)$ is smooth for s near 0, and

$$\frac{d^i}{ds^i} (z - \tilde{z})(0) = 0, \qquad i = 0, 1, 2.$$

Let $\vec{\nu}(s)$ be the normal vector to Γ at $\tilde{z}(s)$. Then $\vec{\nu}$ is smooth for s near 0, and

$$\frac{d^i}{ds^i} (\vec{\nu} - \vec{n})(0) = 0, \qquad i = 0, 1,$$

where $\vec{n}(s) = \vec{n}(z(s))$. We have

$$\begin{aligned} \partial_n N(z(s)) &= \vec{n}(s) \cdot \nabla N(z(s)) \\ &= \vec{n}(s) \cdot [\nabla N(z(s)) - \nabla N(\tilde{z}(s))] + [\vec{n}(s) - \vec{\nu}(s)] \cdot \nabla N(\tilde{z}(s)) \end{aligned}$$

because $\vec{\nu}(s) \cdot \nabla N(\tilde{z}(s)) \equiv 0$. Viewing N as a function also of t, we have

$$\partial_n N(z(s)) = \varphi(s, t),$$

where

$$|\varphi(s, t)| \leqslant \frac{c|s|^3}{s^2 + t^2} + \frac{cs^2}{(s^2 + t^2)^{1/2}}$$

and

$$\left|\frac{\partial}{\partial s} \varphi(s, t)\right| \leqslant \frac{c|s|^3}{(s^2 + t^2)^{3/2}} + \frac{c|s|^3}{(s^2 + t^2)} + \frac{c|s|}{(s^2 + t^2)^{1/2}},$$

where we have made use of the observation that $|D_z^\alpha N| \leqslant c(s^2 + t^2)^{-|\alpha|/2}$. These terms are all bounded as $t \longrightarrow 0$, so we actually have $\|\partial_n N\|_{W^1_\infty(\omega)} \leqslant c(\Omega)$ for some neighborhood ω of z_1 in $\partial\Omega$. Outside of any such neighborhood, $\partial_n N$ is clearly bounded, so we are done.

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