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Optimal evaluation of a Toader-type mean by power mean

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Abstract

In this paper, we present the best possible parameters $p, q \in \mathbb{R}$ such that the double inequality $M_p(a,b) < T[A(a,b), Q(a,b)] < M_q(a,b)$ holds for all $a, b > 0$ with $a \neq b$, and we get sharp bounds for the complete elliptic integral $\mathcal{E}(t) = \int_0^{\pi/2}(1 - t^2\sin^2\theta)^{1/2}\,d\theta$ of the second kind on the interval $(0, \sqrt{2}/2)$, where $T(a,b) = \frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,d\theta$, $A(a,b) = (a+b)/2$, $Q(a,b) = \sqrt{(a^2+b^2)/2}$, $M_r(a,b) = [(a^r + b^r)/2]^{1/r}$ $(r \neq 0)$, and $M_0(a,b) = \sqrt{ab}$ are the Toader, arithmetic, quadratic, and rth power means of a and b, respectively.

MSC: 33E05; 33C05; 26E60

Keywords: arithmetic mean; Toader mean; quadratic mean

1 Introduction

For $r \in \mathbb{R}$ and $a, b > 0$, the Toader mean $T(a,b)$ (see [1]) and rth power mean $M_r(a,b)$ are defined by

$$T(a,b) = \frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,d\theta \tag{1.1}$$

and

$$M_r(a,b) = \begin{cases} \left(\frac{a^r+b^r}{2}\right)^{1/r}, & r \neq 0, \\ \sqrt{ab}, & r = 0, \end{cases} \tag{1.2}$$

respectively.

It is well known that $M_r(a,b)$ is continuous and strictly increasing with respect to $r \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$. Many classical bivariate means are a special case of the power mean, for example, $H(a,b) = 2ab/(a+b) = M_{-1}(a,b)$ is the harmonic mean, $G(a,b) = \sqrt{ab} = M_0(a,b)$ is the geometric mean,

$$A(a,b) = (a+b)/2 = M_1(a,b) \tag{1.3}$$

is the arithmetic mean, and

$$Q(a,b) = \sqrt{\left(a^2+b^2\right)/2} = M_2(a,b) \tag{1.4}$$

is the quadratic mean. The main properties of the power mean are given in [2]. The Toader mean $T(a,b)$ has been well known in the mathematical literature for many years, it satisfies

$$T(a,b) = R_E\left(a^2,b^2\right),$$

where

$$R_E(a,b) = \frac{1}{\pi}\int_0^\infty \frac{[a(t+b)+b(t+a)]t}{(t+a)^{3/2}(t+b)^{3/2}}\,dt$$

stands for the symmetric complete elliptic integral of the second kind (see [3–5]), therefore it cannot be expressed in terms of the elementary transcendental functions.

Let $r \in (0,1)$, $\mathcal{K}(r) = \int_0^{\pi/2}(1-r^2\sin^2\theta)^{-1/2}\,d\theta$, and $\mathcal{E}(r) = \int_0^{\pi/2}(1-r^2\sin^2\theta)^{1/2}\,d\theta$ be, respectively, the complete elliptic integrals of the first and second kind. Then $\mathcal{K}(0^+) = \mathcal{E}(0^+) = \pi/2$, the Toader mean $T(a,b)$ given in (1.1) can be expressed as

$$T(a,b) = \begin{cases} \frac{2a}{\pi}\mathcal{E}(\sqrt{1-(\frac{b}{a})^2}), & a > b, \\ \frac{2b}{\pi}\mathcal{E}(\sqrt{1-(\frac{a}{b})^2}), & a < b, \end{cases} \tag{1.5}$$

and $\mathcal{K}(r)$ and $\mathcal{E}(r)$ satisfy the derivatives formulas (see [6], Appendix E, p. 474-475)

$$\frac{d\mathcal{K}(r)}{dr} = \frac{\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{r(1-r^2)}, \qquad \frac{d\mathcal{E}(r)}{dr} = \frac{\mathcal{E}(r)-\mathcal{K}(r)}{r}, \qquad \frac{d(\mathcal{K}(r)-\mathcal{E}(r))}{dr} = \frac{r\mathcal{E}(r)}{1-r^2}.$$

Numerical computations show that

$$\mathcal{E}\left(\frac{\sqrt{2}}{2}\right) = 1.3506\ldots, \qquad \mathcal{K}\left(\frac{3}{5}\right) = 1.7507\ldots, \qquad \mathcal{E}\left(\frac{3}{5}\right) = 1.4180\ldots,$$
$$\mathcal{K}\left(\frac{17}{25}\right) = 1.8234\ldots, \qquad \mathcal{E}\left(\frac{17}{25}\right) = 1.3693\ldots.$$

Recently, the power mean $M_r(a,b)$ and Toader mean $T(a,b)$ have been the subject of intensive research. In particular, many remarkable inequalities for both means can be found in the literature [7–18].

Vuorinen [19] conjectured that the inequality

$$M_{3/2}(a,b) < T(a,b)$$

holds for all $a,b > 0$ with $a \neq b$. This conjecture was proved by Qiu and Shen [20], and Barnard *et al.* [21], respectively.

Alzer and Qiu [22] presented a best possible upper power mean bound for the Toader mean as follows:

$$T(a,b) < M_{\log 2/(\log\pi - \log 2)}(a,b)$$

for all $a,b > 0$ with $a \neq b$.

Neuman [3], and Kazi and Neuman [4] proved that the inequalities

$$\frac{(a+b)\sqrt{ab}-ab}{AGM(a,b)} < T(a,b) < \frac{4(a+b)\sqrt{ab}+(a-b)^2}{8AGM(a,b)},$$
$$T(a,b) < \frac{1}{4}\left(\sqrt{(2+\sqrt{2})a^2+(2-\sqrt{2})b^2}+\sqrt{(2+\sqrt{2})b^2+(2-\sqrt{2})a^2}\right)$$

hold for all $a,b>0$ with $a\neq b$, where $AGM(a,b)$ is the arithmetic-geometric mean of a and b.

Let $\lambda,\mu,\alpha,\beta\in(1/2,1)$. Then Chu *et al.* [23], and Hua and Qi [24] proved that the double inequalities

$$C\bigl[\lambda a+(1-\lambda)b,\lambda b+(1-\lambda)a\bigr] < T(a,b) < C\bigl[\mu a+(1-\mu)b,\mu b+(1-\mu)a\bigr],$$
$$\overline{C}\bigl[\alpha a+(1-\alpha)b,\alpha b+(1-\alpha)a\bigr] < T(a,b) < \overline{C}\bigl[\beta a+(1-\beta)b,\beta b+(1-\beta)a\bigr]$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\lambda\leq 3/4$, $\mu\geq 1/2+\sqrt{\pi(4-\pi)}/(2\pi)$, $\alpha\leq 1/2+\sqrt{3}/4$, and $\beta\geq 1/2+\sqrt{12/\pi-3}/2$, where $C(a,b)=(a^2+b^2)/(a+b)$ and $\overline{C}(a,b)=2(a^2+ab+b^2)/[3(a+b)]$ are, respectively, the contraharmonic and centroidal means of a and b.

In [25–29], the authors proved that the double inequalities

$$\alpha_1 Q(a,b)+(1-\alpha_1)A(a,b) < T(a,b) < \beta_1 Q(a,b)+(1-\beta_1)A(a,b),$$
$$Q^{\alpha_2}(a,b)A^{(1-\alpha_2)}(a,b) < T(a,b) < Q^{\beta_2}(a,b)A^{(1-\beta_2)}(a,b),$$
$$\alpha_3 C(a,b)+(1-\alpha_3)A(a,b) < T(a,b) < \beta_3 C(a,b)+(1-\beta_3)A(a,b),$$
$$\frac{\alpha_4}{A(a,b)}+\frac{1-\alpha_4}{C(a,b)} < \frac{1}{T(a,b)} < \frac{\beta_4}{A(a,b)}+\frac{1-\beta_4}{C(a,b)},$$
$$\alpha_5 C(a,b)+(1-\alpha_5)H(a,b) < T(a,b) < \beta_5 C(a,b)+(1-\beta_5)H(a,b),$$
$$\alpha_6\bigl[C(a,b)-H(a,b)\bigr]+A(a,b) < T(a,b) < \beta_6\bigl[C(a,b)-H(a,b)\bigr]+A(a,b),$$
$$\alpha_7\overline{C}(a,b)+(1-\alpha_7)A(a,b) < T(a,b) < \beta_7\overline{C}(a,b)+(1-\beta_7)A(a,b),$$
$$\frac{\alpha_8}{A(a,b)}+\frac{1-\alpha_8}{\overline{C}(a,b)} < \frac{1}{T(a,b)} < \frac{\beta_8}{A(a,b)}+\frac{1-\beta_8}{\overline{C}(a,b)},$$
$$\alpha_9 Q(a,b)+(1-\alpha_9)H(a,b) < T(a,b) < \beta_9 Q(a,b)+(1-\beta_9)H(a,b),$$
$$\frac{\alpha_{10}}{H(a,b)}+\frac{1-\alpha_{10}}{Q(a,b)} < \frac{1}{T(a,b)} < \frac{\beta_{10}}{H(a,b)}+\frac{1-\beta_{10}}{Q(a,b)}$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1\leq 1/2$, $\beta_1\geq(4-\pi)/[(\sqrt{2}-1)\pi]$, $\alpha_2\leq 1/2$, $\beta_2\geq 4-2\log\pi/\log 2$, $\alpha_3\leq 1/4$, $\beta_3\geq 4/\pi-1$, $\alpha_4\leq\pi/2-1$, $\beta_4\geq 3/4$, $\alpha_5\leq 5/8$, $\beta_5\geq 2/\pi$, $\alpha_6\leq 1/8$, $\beta_6\geq 2/\pi-1/2$, $\alpha_7\leq 3/4$, $\beta_7\geq 12/\pi-3$, $\alpha_8\leq\pi-3$, $\beta_8\geq 1/4$, $\alpha_9\leq 5/6$, $\beta_9\geq 2\sqrt{2}/\pi$, $\alpha_{10}\leq 0$, and $\beta_{10}\geq 1/6$.

The main purpose of this paper is to present the best possible parameters $p,q\in\mathbb{R}$ such that the double inequality

$$M_p(a,b) < T\bigl[A(a,b),Q(a,b)\bigr] < M_q(a,b)$$

holds for all $a,b>0$ with $a\neq b$.

2 Lemmas

In order to prove our main results, we need several lemmas which we present in this section.

Lemma 2.1 (See [30], Theorem 1.1) *The inequality $\mathcal{E}[M_p(x,y)] > M_q[\mathcal{E}(x),\mathcal{E}(y)]$ holds for all $x,y \in (0,1)$ if and only if*

$$p \leq C(q) := \inf_{r\in(0,1)}\left\{\frac{r^2\mathcal{E}(r)}{(1-r^2)[\mathcal{K}(r)-\mathcal{E}(r)]} + \frac{(1-q)[\mathcal{K}(r)-\mathcal{E}(r)]}{\mathcal{E}(r)}\right\},$$

where $q \to C(q)$ is a continuous function which satisfies $C(q) = 2$ for all $q \leq 5/2$ and $C(q) < 2$ for all $q > 5/2$.

Lemma 2.2 *The double inequality*

$$\frac{(1-t^2)^{5/8}+1}{(1-t^2)^{1/8}+1} < 1 - \frac{t^2}{4} < \left[\frac{(\sqrt{1-t^2}+t)^{3/2} + (\sqrt{1-t^2}-t)^{3/2}}{2}\right]^{2/3} \tag{2.1}$$

holds for all $t \in (0, \sqrt{2}/2)$.

Proof Let $u = (1-t^2)^{1/8}$. Then $u \in (1/\sqrt[8]{2}, 1)$, $t^2 = 1 - u^8$, and the first inequality of (2.1) is equivalent to

$$\frac{u^5+1}{u+1} < \frac{u^8+3}{4} \tag{2.2}$$

for all $u \in (1/\sqrt[8]{2}, 1)$.

We clearly see that (2.2) follows from

$$(u+1)\left(u^8+3\right) - 4\left(u^5+1\right) = (u+1)\left(u^2+1\right)(1-u)^2\left[(2u-1)+2u^2+2u^3+u^4\right] > 0$$

for all $u \in (1/\sqrt[8]{2}, 1)$.

For the second inequality of (2.1), let $v = \sqrt{1-t^2} \in (\sqrt{2}/2, 1)$, then it suffices to prove that

$$\begin{aligned}\rho(v) &:= \frac{[(v+\sqrt{1-v^2})^{3/2} + (v-\sqrt{1-v^2})^{3/2}]^2}{4} - \frac{(v^2+3)^3}{64} \\ &= \frac{1}{2}\left[3v - 2v^3 + \left(2v^2-1\right)^{3/2} - \frac{(v^2+3)^3}{32}\right] > 0\end{aligned} \tag{2.3}$$

for all $v \in (\sqrt{2}/2, 1)$.

We claim that

$$\left(2v^2-1\right)^{3/2} > 2 - 6v + 3v^2 + 2v^3 \tag{2.4}$$

for all $v \in (\sqrt{2}/2, 1)$.

Indeed, if $v \in (\sqrt{2}/2, (\sqrt{6}-1)/2]$, then we clearly see that the function $2-6v+3v^2+2v^3$ is strictly increasing on $(\sqrt{2}/2, (\sqrt{6}-1)/2]$, and (2.4) follows from

$$\begin{aligned}2-6v+3v^2+2v^3 &\leq 2-6\times\frac{\sqrt{6}-1}{2}+3\times\left(\frac{\sqrt{6}-1}{2}\right)^2+2\times\left(\frac{\sqrt{6}-1}{2}\right)^3 \\ &= \frac{22-9\sqrt{6}}{4}<0.\end{aligned}$$

If $v \in ((\sqrt{6}-1)/2, 1)$, then (2.4) follows easily from

$$\begin{aligned}\left(2v^2-1\right)^3-\left(2-6v+3v^2+2v^3\right)^2 &= \left(1-v^4\right)\left(-5+4v+4v^2\right) \\ &> \left(1-v^4\right)\left[-5+4\times\frac{\sqrt{6}-1}{2}+4\left(\frac{\sqrt{6}-1}{2}\right)^2\right]=0.\end{aligned}$$

Therefore, inequality (2.3) follows from (2.4) and

$$\begin{aligned}3v-2v^3+\left(2v^2-1\right)^{3/2}-\frac{(v^2+3)^3}{32} &> 3v-2v^3+\left(2-6v+3v^2+2v^3\right)-\frac{(v^2+3)^3}{32} \\ &= \frac{(1-v^3)(37+15v+3v^2+v^3)}{32}>0\end{aligned}$$

for all $v \in (\sqrt{2}/2, 1)$. □

Lemma 2.3 *The inequality*

$$\mathcal{E}(t) > \frac{\pi}{2}\left(1-\frac{5t^2}{18}\right)$$

holds for all $t \in (0, 3/5)$.

Proof Let

$$f(t) = \mathcal{E}(t) - \frac{\pi}{2}\left(1-\frac{5t^2}{18}\right). \tag{2.5}$$

Then simple computations lead to

$$f\left(0^+\right)=0, \qquad f\left(\frac{3}{5}\right)=0.00436\ldots>0, \tag{2.6}$$

$$f'(t) = tf_1(t), \tag{2.7}$$

where

$$f_1(t) = \frac{\mathcal{E}(t)-\mathcal{K}(t)}{t^2}+\frac{5\pi}{18},$$

$$f_1\left(0^+\right)=\frac{\pi}{36}>0, \qquad f_1\left(\frac{3}{5}\right)=-0.0514\ldots<0, \tag{2.8}$$

$$f_1'(t) = \frac{f_2(t)}{t^3(1-t^2)}, \tag{2.9}$$

where

$$f_2(t) = 2(1-t^2)\mathcal{K}(t) - (2-t^2)\mathcal{E}(t),$$
$$f_2(0^+) = 0, \tag{2.10}$$
$$f_2'(t) = -3t[\mathcal{K}(t) - \mathcal{E}(t)] < 0 \tag{2.11}$$

for $t \in (0, 3/5)$.

From (2.9)-(2.11) we clearly see that $f_1(t)$ is strictly decreasing on $(0, 3/5)$. Then (2.7) and (2.8) lead to the conclusion that there exists $t_0 \in (0, 3/5)$ such that $f(t)$ is strictly increasing on $(0, t_0]$ and strictly decreasing on $[t_0, 3/5)$.

Therefore, Lemma 2.3 follows easily from (2.5) and (2.6) together with the piecewise monotonicity of $f(t)$. □

Lemma 2.4 *The inequality*

$$\left(\frac{18+13t^2}{18\sqrt{1+t^2}}\right)^{7/5} > 1 + \frac{14t^2}{45}$$

holds for all $t \in (0, 3/4)$.

Proof It suffices to prove that the inequalities

$$\frac{18+13t^2}{18\sqrt{1+t^2}} > 1 + \frac{2t^2}{9} \tag{2.12}$$

and

$$\left(1 + \frac{2t^2}{9}\right)^{7/5} > 1 + \frac{14t^2}{45} \tag{2.13}$$

hold for all $t \in (0, 3/4)$.

Indeed, inequalities (2.12) and (2.13) follow easily from the identities

$$(18+13t^2)^2 - 4(1+t^2)(9+2t^2)^2 = t^4(3-4t)(3+4t)$$

and

$$\left(1+\frac{2t^2}{9}\right)^7 - \left(1+\frac{14t^2}{45}\right)^5$$
$$= 4t^2\left(\frac{7}{405} + \frac{14t^2}{675} + \frac{2{,}632t^4}{273{,}375} + \frac{390{,}544t^6}{184{,}528{,}125} + \frac{112t^8}{531{,}441} + \frac{32t^{10}}{4{,}782{,}969}\right).$$ □

Lemma 2.5 *Let* $\lambda = 2\log 2/[2\log\pi - \log 2 - 2\log\mathcal{E}(\sqrt{2}/2)] = 1.3930\ldots$ *and*

$$g(t) = \frac{2}{\pi}\sqrt{1+t^2}\,\mathcal{E}\left(\frac{t}{\sqrt{1+t^2}}\right) - \left[\frac{(1+t)^\lambda + (1-t)^\lambda}{2}\right]^{1/\lambda}.$$

Then $g(t) > 0$ *for all* $t \in (0, 3/4)$.

Proof It follows from $t/\sqrt{1+t^2} \in (0,3/5)$, $\lambda < 7/5$, Lemma 2.3, Lemma 2.4 and the monotonicity of $M_r(1+t,1-t)$ with respect to $r \in \mathbb{R}$ that

$$\begin{aligned} g(t) &> \frac{2}{\pi}\sqrt{1+t^2} \times \frac{\pi}{2}\left[1 - \frac{5t^2}{18(1+t^2)}\right] - \left[\frac{(1+t)^{7/5}+(1-t)^{7/5}}{2}\right]^{5/7} \\ &= \frac{18+13t^2}{18\sqrt{1+t^2}} - \left[\frac{(1+t)^{7/5}+(1-t)^{7/5}}{2}\right]^{5/7} \\ &> \left(1+\frac{14t^2}{45}\right)^{5/7} - \left[\frac{(1+t)^{7/5}+(1-t)^{7/5}}{2}\right]^{5/7} \end{aligned} \tag{2.14}$$

for $t \in (0,3/4)$. Let

$$g_1(t) = 2\left(1+\frac{14t^2}{45}\right) - \left[(1+t)^{7/5}+(1-t)^{7/5}\right]. \tag{2.15}$$

Then simple computations lead to

$$g_1(0) = 0, \qquad g_1\left(\frac{3}{4}\right) = 0.0173\ldots > 0, \tag{2.16}$$

$$g_1'(t) = \frac{7}{45}\left[8t - 9(1+t)^{2/5} + 9(1-t)^{2/5}\right],$$

$$g_1'(0) = 0, \qquad g_1'\left(\frac{3}{4}\right) = -0.0138\ldots < 0, \tag{2.17}$$

$$g_1''(t) = \frac{14}{225}\left[20 - \frac{9}{(1+t)^{3/5}} - \frac{9}{(1-t)^{3/5}}\right],$$

$$g_1''(0) = \frac{28}{225} > 0, \qquad g_1''\left(\frac{3}{4}\right) = -0.4423\ldots < 0, \tag{2.18}$$

$$g_1'''(t) = \frac{42}{125}\left[\frac{1}{(1+t)^{8/5}} - \frac{1}{(1-t)^{8/5}}\right] < 0 \tag{2.19}$$

for $t \in (0,3/4)$.

From (2.18) and (2.19) we know that there exists $t_1 \in (0,3/4)$ such that $g_1'(t)$ is strictly increasing on $(0,t_1]$ and strictly decreasing on $[t_1,3/4)$. Then (2.17) leads to the conclusion that there exists $t_2 \in (0,3/4)$ such that $g_1(t)$ is strictly increasing on $(0,t_2]$ and strictly decreasing on $[t_2,3/4)$.

Therefore, Lemma 2.5 follows from (2.14)-(2.16) and the piecewise monotonicity of $g_1(t)$. □

Lemma 2.6 *Let* $\lambda = 2\log 2/[2\log\pi - \log 2 - 2\log\mathcal{E}(\sqrt{2}/2)] = 1.3930\ldots$. *Then the function* $t^{-1}\mathcal{E}^{\lambda-1}(t)[\mathcal{E}(t)-\mathcal{K}(t)]$ *is strictly decreasing on* $(0,1)$.

Proof From Lemma 2.1 we clearly see that the inequality

$$\mathcal{E}\big(M_\lambda(x,y)\big) > M_\lambda\big(\mathcal{E}(x),\mathcal{E}(y)\big) = \left(\frac{\mathcal{E}^\lambda(x)+\mathcal{E}^\lambda(y)}{2}\right)^{1/\lambda} \tag{2.20}$$

holds for all $x, y \in (0,1)$ with $x \neq y$.

It follows from the monotonicity of the function $\mathcal{E}(t)$ and the power mean $M_p(x,y)$ with respect to $p \in \mathbb{R}$ together with $\lambda > 1$ that

$$\mathcal{E}\left(\frac{x+y}{2}\right) = \mathcal{E}\big(M_1(x,y)\big) > \mathcal{E}\big(M_\lambda(x,y)\big) \tag{2.21}$$

for all $x, y \in (0,1)$ with $x \neq y$.

Inequalities (2.20) and (2.21) lead to

$$\mathcal{E}^\lambda\left(\frac{x+y}{2}\right) > \frac{\mathcal{E}^\lambda(x) + \mathcal{E}^\lambda(y)}{2}$$

for all $x, y \in (0,1)$ with $x \neq y$, which implies that the function $\mathcal{E}^\lambda(t)$ is strictly concave on $(0,1)$.

Note that

$$t^{-1}\mathcal{E}^{\lambda-1}(t)\big[\mathcal{E}(t) - \mathcal{K}(t)\big] = \frac{1}{\lambda}\frac{d\mathcal{E}^\lambda(t)}{dt}. \tag{2.22}$$

□

Therefore, Lemma 2.6 follows easily from (2.22) and the concavity of $\mathcal{E}^\lambda(t)$ on $(0,1)$.

Lemma 2.7 *Let* $\lambda = 2\log 2/[2\log\pi - \log 2 - 2\log\mathcal{E}(\sqrt{2}/2)] = 1.3930\ldots$

$$h_1(t) = \frac{2^{1+\lambda}}{\pi^\lambda}\mathcal{E}^\lambda(t) - \frac{(5+t)^\lambda + (5-7t)^\lambda}{4^\lambda}$$

and

$$h_2(t) = \frac{2^{1+\lambda}}{\pi^\lambda}\mathcal{E}^\lambda(t) - (\sqrt{2} - 2t)^\lambda - 2^{\lambda/2}.$$

Then $h_1(t) > 0$ *for* $t \in [3/5, 17/25)$ *and* $h_2(t) > 0$ *for* $t \in [17/25, \sqrt{2}/2)$.

Proof Simple computations lead to

$$h_1\left(\frac{17}{25}\right) = 0.0022\ldots > 0, \qquad h_2\left(\frac{\sqrt{2}}{2}\right) = 0, \tag{2.23}$$

$$h_1'(t) = \frac{\lambda}{4^\lambda}\left[\frac{2^{3\lambda+1}}{\pi^\lambda}t^{-1}\mathcal{E}^{\lambda-1}(t)\big(\mathcal{E}(t) - \mathcal{K}(t)\big) + 7(5-7t)^{\lambda-1} - (5+t)^{\lambda-1}\right], \tag{2.24}$$

$$h_2'(t) = 2\lambda\left[\left(\frac{2}{\pi}\right)^\lambda t^{-1}\mathcal{E}^{\lambda-1}(t)\big(\mathcal{E}(t) - \mathcal{K}(t)\big) + (\sqrt{2} - 2t)^{\lambda-1}\right], \tag{2.25}$$

$$h_1'\left(\frac{3}{5}\right) = -0.0471\ldots < 0, \qquad h_2'\left(\frac{17}{25}\right) = -0.236\ldots < 0. \tag{2.26}$$

From (2.24) and (2.25) together with Lemma 2.6 we clearly see that both $h_1'(t)$ and $h_2'(t)$ are strictly decreasing on $(0, \sqrt{2}/2)$. Then (2.26) leads to the conclusion that $h_1(t)$ is strictly decreasing on $[3/5, 17/25]$ and $h_2(t)$ is strictly decreasing on $[17/25, \sqrt{2}/2)$.

Therefore, Lemma 2.7 follows from (2.23) and the monotonicity of $h_1(t)$ on $[3/5, 17/25]$ and $h_2(t)$ on $[17/25, \sqrt{2}/2)$. □

Lemma 2.8 (See [18], Corollary 3.2) *The inequality*

$$\frac{2}{\pi}\mathcal{E}(t) < \frac{(1-t^2)^{5/8}+1}{(1-t^2)^{1/8}+1} \tag{2.27}$$

holds for all $t \in (0,1)$.

3 Main results

Theorem 3.1 *Let* $\lambda = 2\log 2/[2\log\pi - \log 2 - 2\log\mathcal{E}(\sqrt{2}/2)] = 1.3930\ldots$. *Then the double inequality*

$$M_p(a,b) < T\big[A(a,b),Q(a,b)\big] < M_q(a,b)$$

holds for all $a,b>0$ *with* $a \neq b$ *if and only if* $p \leq \lambda$ *and* $q \geq 3/2$.

Proof Since the arithmetic mean $A(a,b)$, quadratic mean $Q(a,b)$, Toader mean $T(a,b)$, and rth power mean $M_r(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a > b$. Let $t = (a-b)/\sqrt{2(a^2+b^2)}$. Then $t \in (0, \sqrt{2}/2)$ and equations (1.2)-(1.5) lead to

$$M_r(a,b) = \frac{A(a,b)}{\sqrt{1-t^2}}\left[\frac{(\sqrt{1-t^2}+t)^r + (\sqrt{1-t^2}-t)^r}{2}\right]^{1/r}, \tag{3.1}$$

$$T\big[A(a,b),Q(a,b)\big] = \frac{2A(a,b)\mathcal{E}(t)}{\pi\sqrt{1-t^2}}. \tag{3.2}$$

We divide the proof into three cases.

Case 1 $r \geq 3/2$. Then it follows from (3.1) and (3.2) together with the monotonicity of $M_r(a,b)$ with respect to r that

$$\begin{aligned}
&T\big[A(a,b),Q(a,b)\big] - M_r(a,b) \\
&\quad\leq T\big[A(a,b),Q(a,b)\big] - M_{3/2}(a,b) \\
&\quad= \frac{A(a,b)}{\sqrt{1-t^2}}\left[\frac{2}{\pi}\mathcal{E}(t) - \frac{(1-t^2)^{5/8}+1}{(1-t^2)^{1/8}+1}\right] \\
&\qquad+ \frac{A(a,b)}{\sqrt{1-t^2}}\left[\frac{(1-t^2)^{5/8}+1}{(1-t^2)^{1/8}+1} - \left(\frac{(\sqrt{1-t^2}+t)^{3/2}+(\sqrt{1-t^2}-t)^{3/2}}{2}\right)^{2/3}\right].
\end{aligned} \tag{3.3}$$

Therefore,

$$T\big[A(a,b),Q(a,b)\big] < M_r(a,b)$$

for all $a,b>0$ with $a \neq b$ follows from Lemmas 2.2 and 2.8 together with (3.3).

Case 2 $r \leq \lambda$. Then equations (3.1) and (3.2) together with the monotonicity of $M_r(a,b)$ with respect to r lead to

$$\begin{aligned} &T\big[A(a,b),Q(a,b)\big]-M_r(a,b) \\ &\quad\geq T\big[A(a,b),Q(a,b)\big]-M_\lambda(a,b) \\ &\quad=\frac{A(a,b)}{\sqrt{1-t^2}}\left[\frac{2}{\pi}\mathcal{E}(t)-\left(\frac{(\sqrt{1-t^2}+t)^\lambda+(\sqrt{1-t^2}-t)^\lambda}{2}\right)^{1/\lambda}\right]. \end{aligned} \tag{3.4}$$

We divide the proof into two subcases.

Subcase 2.1 $t \in (0,3/5)$. Let $u = t/\sqrt{1-t^2}$. Then $u \in (0,3/4)$ and (3.4) leads to

$$\begin{aligned} &T\big[A(a,b),Q(a,b)\big]-M_r(a,b) \\ &\quad> A(a,b)\left[\frac{2}{\pi}\sqrt{1+u^2}\mathcal{E}\left(\frac{u}{\sqrt{1+u^2}}\right)-\left(\frac{(1+u)^\lambda+(1-u)^\lambda}{2}\right)^{1/\lambda}\right]. \end{aligned} \tag{3.5}$$

Therefore,

$$T\big[A(a,b),Q(a,b)\big]>M_r(a,b)$$

for $0<|a-b|/\sqrt{2(a^2+b^2)}<3/5$ with $a \neq b$ follows from Lemma 2.5 and (3.5).

Subcase 2.2 $t \in [3/5,\sqrt{2}/2)$. Let

$$h(t)=\frac{2^{1+\lambda}}{\pi^\lambda}\mathcal{E}^\lambda(t)-\big(\sqrt{1-t^2}+t\big)^\lambda-\big(\sqrt{1-t^2}-t\big)^\lambda. \tag{3.6}$$

It is easy to verify that

$$\sqrt{1-t^2}\leq\frac{5-3t}{4}\quad\text{and}\quad\sqrt{1-t^2}<\sqrt{2}-t \tag{3.7}$$

for all $t \in (0,\sqrt{2}/2)$.

Equation (3.6) and inequality (3.7) lead to

$$h(t)>\frac{2^{1+\lambda}}{\pi^\lambda}\mathcal{E}^\lambda(t)-\frac{(5+t)^\lambda+(5-7t)^\lambda}{4^\lambda} \tag{3.8}$$

and

$$h(t)>\frac{2^{1+\lambda}}{\pi^\lambda}\mathcal{E}^\lambda(t)-(\sqrt{2}-2t)^\lambda-2^{\lambda/2}. \tag{3.9}$$

Therefore,

$$T\big[A(a,b),Q(a,b)\big]>M_r(a,b)$$

for $3/5\leq|a-b|/\sqrt{2(a^2+b^2)}$ with $a \neq b$ follows from Lemma 2.7, (3.4), (3.6), (3.8), and (3.9).

Case 3 $\lambda < r < 3/2$. On the one hand, equations (1.2) and (1.5) lead to

$$\begin{aligned}
&\lim_{x\to 0^+}\bigl[\log T\bigl[A(1,x),Q(1,x)\bigr]-\log M_r(1,x)\bigr] \\
&\quad=\log\biggl[\frac{\sqrt{2}\mathcal{E}(\frac{\sqrt{2}}{2})}{\pi}\biggr]+\frac{\log 2}{r} \\
&\quad=-\frac{(r-\lambda)\log 2}{\lambda r}<0. \qquad (3.10)
\end{aligned}$$

Inequality (3.10) implies that there exists $\delta_1 > 0$ such that

$$T\bigl[A(a,b),Q(a,b)\bigr] < M_r(a,b)$$

for all $a, b > 0$ with $a/b \in (0, \delta_1)$.

On the other hand, by the Taylor expansion and let $x > 0$ and $x \to 0$, then equations (1.2) and (1.5) lead to

$$\begin{aligned}
&T\bigl[A(1,1-x),Q(1,1-x)\bigr]-M_r(1,1-x) \\
&\quad=\frac{2}{\pi}\sqrt{1-x+\frac{x^2}{2}}\mathcal{E}\biggl(\frac{x}{2\sqrt{1-x+\frac{x^2}{2}}}\biggr)-\biggl[\frac{1+(1-x)^r}{2}\biggr]^{1/r} \\
&\quad=1-\frac{x}{2}+\frac{x^2}{16}-\biggl[1-\frac{x}{2}+\biggl(\frac{1}{16}-\frac{3-2r}{16}\biggr)x^2\biggr]+o\bigl(x^2\bigr) \\
&\quad=\frac{3-2r}{16}x^2+o\bigl(x^2\bigr). \qquad (3.11)
\end{aligned}$$

Equation (3.11) implies there exists $\delta_2 \in (0, 1)$ such that

$$T\bigl[A(a,b),Q(a,b)\bigr] > M_r(a,b)$$

for all $a, b > 0$ with $a/b \in (1 - \delta_2, 1)$. □

From Theorem 3.1 we get Corollary 3.2 immediately.

Corollary 3.2 *Let* $\lambda = 2\log 2/[2\log\pi - \log 2 - 2\log\mathcal{E}(\sqrt{2}/2)] = 1.3930\ldots$. *Then the double inequality*

$$\frac{\pi}{2}\biggl[\frac{(\sqrt{1-t^2}+t)^p+(\sqrt{1-t^2}-t)^p}{2}\biggr]^{1/p}<\mathcal{E}(t)<\frac{\pi}{2}\biggl[\frac{(\sqrt{1-t^2}+t)^q+(\sqrt{1-t^2}-t)^q}{2}\biggr]^{1/q}$$

holds for all $t \in (0, \sqrt{2}/2)$ *if and only if* $p \le \lambda$ *and* $q \ge 3/2$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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