

Optimal Execution in a General One-Sided Limit-Order Book

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Abstract. We construct an optimal execution strategy for the purchase of a large number of shares of a financial asset over a fixed interval of time. Purchases of the asset have a nonlinear impact on price, and this is moderated over time by resilience in the limit-order book that determines the price. The limit-order book is permitted to have arbitrary shape. The form of the optimal execution strategy is to make an initial lump purchase and then purchase continuously for some period of time during which the rate of purchase is set to match the order book resiliency. At the end of this period, another lump purchase is made, and following that there is again a period of purchasing continuously at a rate set to match the order book resiliency. At the end of this second period, there is a final lump purchase. Any of the lump purchases could be of size zero. A simple condition is provided that guarantees that the intermediate lump purchase is of size zero.

Key words. Optimal execution, limit-order book, price impact.

AMS subject classifications. 91B26, 91G80, 49K45, 90C25

1. Introduction. We consider optimal execution over a fixed time interval of a large asset purchase in the face of a one-sided limit-order book. We assume that the ask price for the underlying asset is a continuous martingale that undergoes two adjustments during the period of purchase. The first adjustment is that orders consume a part of the limit-order book, and this increases the ask price for subsequent orders. The second adjustment is that resilience in the limit-order book causes the effect of these prior orders to decay over time.

We assume that there is a fixed shadow limit-order book shape toward which resilience returns the limit-order book. At any time, the actual limit-order book relative to the martingale component of the ask price has this shape, but with some left-hand part missing due to prior purchases. An investor is given a period of time and a target amount of asset to be purchased within that period. His goal is to distribute his purchasing over the period in order to minimize the expected cost of purchasing the target. We permit purchases to occur in lumps or to be spread continuously over time. We show that the optimal execution strategy consists of three lump purchases, one or more of which may be of size zero, i.e., does not occur. One of these lump purchases is made at the initial time, one at an intermediate time, and one at the final time. Between these lump purchases, the optimal strategy purchases at a constant rate matched to the limit-order book recovery rate so that the ask price remains constant. We provide a simple condition under which the intermediate lump purchase is of size zero (see Theorem 4.2 and Remark 4.3 below).

Dynamic models for optimal execution were developed by Bertsimas and Lo [3]

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and Almgren and Chriss [2]. Trading in [3] is on a discrete-time grid, and the price impact of a trade is linear in the size of the trade and is permanent. In [3], the expected-cost-minimizing liquidation strategy for an order is to divide the order into equal pieces, one for each trading date. Trading in [2] is also on a discrete-time grid, and there are linear permanent and temporary price impacts. In [2] variance of the cost of execution is taken into account. This leads to the construction of an efficient frontier of trading strategies.

The present paper is inspired by Obizhaeva and Wang [4], who explicitly model the one-sided limit-order book as a means to capture the price impact of order execution. Empirical evidence for the model of [4] and its generalization by Alfonsi, Almgren and Schulz [1] are reported in [4] and [1]. In [4], the limit-order book has a block shape, and in this case the price impact of a purchase is linear, the same as in [3] and [2]. However, the change of mind set is important. This change of mind set was exploited by [1], who permit more general limit-order book shapes, subject to the condition discussed in Remark 4.3 below. In both [4] and [1], trading is discrete and it is shown that for an optimal purchasing strategy all purchases except the first and last are of the same size. Furthermore, the size of the intermediate purchases is chosen so that the price impact of each purchase is exactly offset by the order book resiliency before the next purchase.

In contrast to [4] and [1], we permit the order book shape to be completely general. However, in our model all price impact is transient; [4] and [1] also include the possibility of a permanent linear price impact. In contrast to [1], we do not assume that the limit order book has a positive density. It can be discrete or continuous and can have gaps. In contrast to both [4] and [1], we permit the resilience in the order book to be a function of the adjustments to the martingale component of the ask price.

Finally, we set up our model so as to allow for both discrete-time and continuous-time trading, whereas [4] and [1] begin with discrete-time trading and then study the limit of their optimal strategies as trading frequency approaches infinity. The simplicity afforded by a fully continuous model is evident in the analysis below. In particular, we provide constructive proofs of Theorems 4.2 and 4.4 that describe the form of the optimal purchasing strategies.

Section 2 of this paper presents our model. It contains the definition of the cost of purchasing in our more general framework, and that is preceded by a justification of the definition. Section 3 shows that randomness can be removed from the optimal purchasing problem and reformulates the cost function into a convenient form. In Section 4, we solve the problem, first in the case that is analogous to the one solved by [1], and then in full generality. Sections 4.1 and 4.3 contain examples.

2. The model. Let T be a positive constant. We assume that the ask price of some asset, in the absence of the large investor modeled by this paper, is a continuous nonnegative martingale $A(t)$, $0 \leq t \leq T$, relative to some filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ satisfying the usual conditions. We show below that for the optimal execution problem of this paper, one can assume without loss of generality that this martingale is identically zero. We make this assumption beginning in Section 3 in order to simplify the presentation.

For some extended positive real number M , let μ be an infinite measure on $[0, M)$

that is finite on each compact subset of $[0, M)$. Denote the associated left-continuous cumulative distribution function by

$$F(x) \triangleq \mu([0, x]), \quad x \geq 0.$$

This is the *shadow limit-order book*, in the sense described below. We assume $F(x) > 0$ for every $x > 0$. If B is a measurable subset of $[0, M)$, then in the absence of the large investor modeled in this paper, at time $t \geq 0$ the number of limit orders with prices in $B + A(t) \triangleq \{b + A(t); b \in B\}$ is $\mu(B)$.

There is a strictly positive constant \bar{X} such that our large investor must purchase \bar{X} shares over the time interval $[0, T]$. His *purchasing strategy* is a non-decreasing right-continuous adapted process X with $X_T = \bar{X}$. We interpret X_t to be the cumulative amount of purchasing done by time t . We adopt the convention $X_{0-} = 0$, so that $X_0 = \Delta X_0$ is the number of shares purchased at time zero. Here and elsewhere, we use the notation ΔX_t to denote the jump $X_t - X_{t-}$ in X at time t .

The effect the purchasing strategy X on the limit-order book is determined by a *resilience function* h , a strictly increasing, locally Lipschitz function defined on $[0, \infty)$ and satisfying $h(0) = 0$ and $\lim_{x \rightarrow \infty} h(x) > \frac{\bar{X}}{T}$. This last condition ensures that h^{-1} is defined on $[0, \bar{Y}]$, where \bar{Y} is defined by (4.2) below. The function h together with X determine the *residual effect process* E satisfying

$$E_t = X_t - \int_0^t h(E_s) ds, \quad 0 \leq t \leq T. \quad (2.1)$$

It is shown in Appendix A that there is a unique nonnegative right-continuous finite-variation adapted process E satisfying (2.1). As with X , we adopt the convention $E_{0-} = 0$. We note that $\Delta X_t = \Delta E_t$ for $0 \leq t \leq T$.

Let B be a measurable subset of $[0, M)$. The interpretation of E is that in the presence of the large investor using strategy X , at time $t \geq 0$ the number of limit orders with prices in $B + A(t)$ is $\mu_t(B)$, where μ_t is the σ -finite infinite measure on $[0, M)$ with left-continuous cumulative distribution function $(F(x) - E_t)^+$, $x \geq 0$. In other words, E_t units of mass have been removed from the shadow limit-order book μ . In any interval in which no purchases are made, (2.1) implies $\frac{d}{dt} E_t = -h(E_t)$. Hence, in the absence of purchases, the residual effect process decays toward zero and the limit-order book tends toward the shadow limit-order book μ , displaced by the ask price A .

To calculate the cost to the investor of using the strategy X , we introduce the following notation. We first define the left-continuous inverse of F ,

$$\psi(y) \triangleq \sup\{x \geq 0 | F(x) < y\}, \quad y > 0.$$

We set $\psi(0) = \psi(0+) = 0$, where the second equality follows from the assumption that $F(x) > 0$ for every $x > 0$. The ask price in the presence of the large investor is $A_t + D_t$, where

$$D_t \triangleq \psi(E_t), \quad 0 \leq t \leq T. \quad (2.2)$$

This is the price after any lump purchases by the investor at time t (see Fig. 2.1).

EXAMPLE 2.1 (Block order book). Let q be a fixed positive number. If q is the quantity of shares available at each price, then for each $x \geq 0$, the quantity available

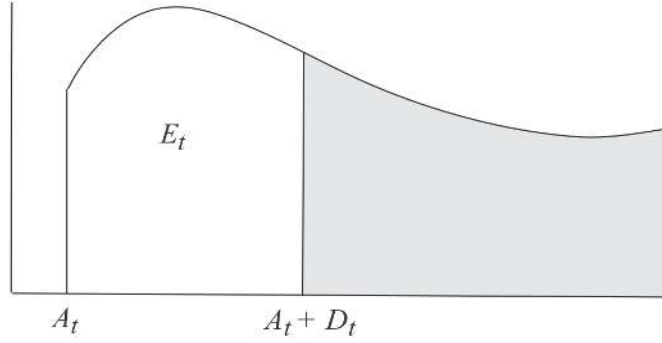


FIG. 2.1. *Limit order book at time t . The shadow area corresponds to the remaining shares. The white area equals E_t and corresponds to the amount of shares, missing from the order book at time t . The current ask price is $A_t + D_t$.*

at prices in $[0, x]$ is $F(x) = qx$. This is the *block order book* considered by [4]. In this case, $\psi(y) = y/q$ and $F(\psi(y)) = y$ for all $y \geq 0$. \square

EXAMPLE 2.2 (Modified block order book). Let $0 < a < b < \infty$ be given, and suppose

$$F(x) = \begin{cases} x, & 0 \leq x \leq a, \\ a, & a \leq x \leq b, \\ x - (b - a), & b \leq x < \infty. \end{cases} \quad (2.3)$$

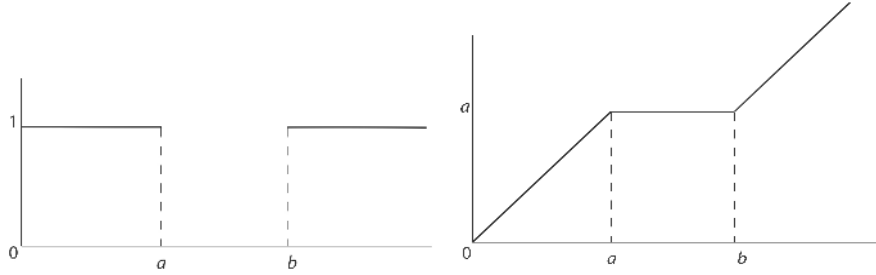


FIG. 2.2. *Density and cumulative distribution of the modified block order book*

This is a block order book, except that the orders with prices between a and b are not present (see Fig 2.2). In this case,

$$\psi(y) = \begin{cases} y, & 0 \leq y \leq a, \\ y + b - a, & a < y < \infty. \end{cases} \quad (2.4)$$

We have $F(\psi(y)) = y$ for all $y \geq 0$. \square

EXAMPLE 2.3 (Discrete order book). Suppose that

$$F(x) = \sum_{i=0}^{\infty} \mathbb{I}_{(i, \infty)}(x), \quad x \geq 0, \quad (2.5)$$

which corresponds to an order of size one at each of the nonnegative integers (see Fig.

3). Then

$$\psi(y) = \sum_{i=1}^{\infty} \mathbb{I}_{(i, \infty)}(y), \quad y \geq 0. \quad (2.6)$$

For every nonnegative integer j , we have $F(j) = j$, $F(j+) = j + 1$, $\psi(j + 1) = j$, $\psi(j+) = j$, $F(\psi(j) +) = j$ and $\psi(F(j) +) = j$ (see Fig. 2.3). \square

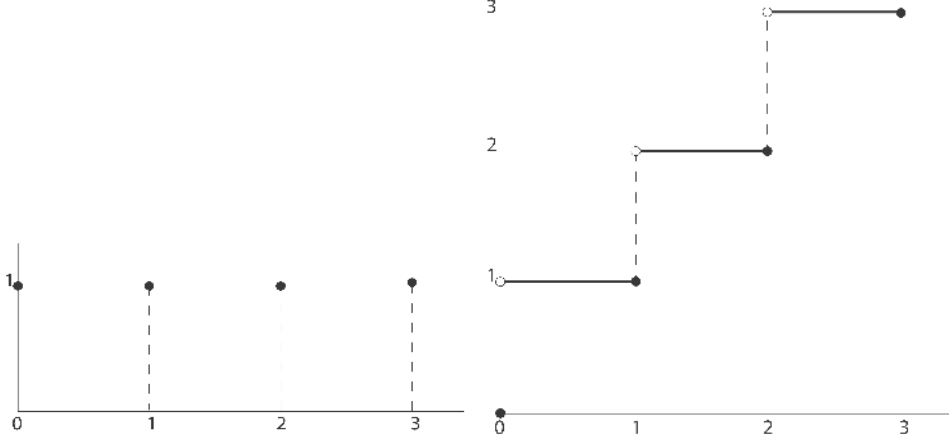


FIG. 2.3. Measure and cumulative distribution function of the discrete order book

Suppose, as in Example 2.2, F is constant on an interval $[\alpha, \beta]$, but strictly increasing to the left of α and to the right of β . Let $y = F(x)$ for $\alpha \leq x \leq \beta$. Then $\psi(y) = \alpha$ and $\psi(y+) = \beta$. Suppose at time t , we have $E_t = y$. Then $D_t = \alpha$, but the measure μ_t assigns mass zero to $[0, \beta]$. The ask price is $A_t + D_t$, but there are no shares for sale at this price, nor in an interval to the right of this price. Nonetheless, it is reasonable to call $A_t + D_t$ the ask price for an infinitesimal purchase because if the agent will wait an infinitesimal amount of time before making this purchase, shares will appear at the price $A_t + D_t$ due to resilience. We make this argument more precise.

Suppose the agent wishes to purchase a small number $\epsilon > 0$ shares at time t at the ask price $A_t + D_t$. This purchase can be approximated by first purchasing zero shares in the time interval $[t, t + \delta]$, where δ is chosen so that $\int_t^{t+\delta} h(E_s) ds = \epsilon$ and

$$E_s = X_t - \int_0^s h(E_u) du, \quad t \leq s < t + \delta.$$

In other words, E_s for $t \leq s < t + \delta$ is given by (2.1) with X held constant (no purchases) over this interval. With δ chosen this way, $E_{(t+\delta)-} = E_t - \epsilon$. Resilience in the order book has created ϵ shares. Suppose the investor purchases these shares at time $t + \delta$, which means that $\Delta X_{t+\delta} = \Delta E_{t+\delta} = \epsilon$ and $E_{t+\delta} = E_t$. Immediately before the purchase, the ask price is $A_{t+\delta} + \psi(E_t - \epsilon)$; immediately after the purchase, the ask price is $A_{t+\delta} + \psi(E_t) = A_{t+\delta} + \alpha$. The cost of purchasing these shares is

$$\epsilon A_{t+\delta} + \int_{[\psi(E_t - \epsilon), \alpha]} \xi d(F(\xi) - E_t + \epsilon)^+, \quad (2.7)$$

Because $\int_{[\psi(E_t - \epsilon), \alpha]} d(F(\xi) - E_t + \epsilon)^+ = \epsilon$, the integral in (2.7) is bounded below by $\epsilon\psi(E_t - \epsilon)$ and bounded above by $\epsilon\alpha$. But $\alpha = \psi(E_t) = D_t$ and ψ is left continuous, so the cost per share obtained by dividing (2.7) by ϵ converges to $A_t + \alpha = A_t + D_t$ as ϵ (and hence δ) converge down to zero.

On the other hand, an impatient agent who does not wait before purchasing shares could choose a different method of approximating an infinitesimal purchase at time t that leads to a limiting cost per share $A_t + \beta$. In particular, it is not the case that our definition of ask price is consistent with all limits of discrete-time purchasing strategies. Our definition is designed to capture the limit of discrete-time purchasing strategies that seek to minimize cost.

To simplify calculations of the type just presented, we define the functions

$$\varphi(x) = \int_{[0, x]} \xi dF(\xi), \quad x \geq 0, \quad (2.8)$$

$$\Phi(y) = \varphi(\psi(y)) + [y - F(\psi(y))]\psi(y), \quad y \geq 0. \quad (2.9)$$

We note that $\Phi(0) = 0$, and we extend Φ to be zero on the negative half-line. In the absence of the large investor, the cost one would pay to purchase all the shares available at prices in the interval $[A(t), A(t) + x]$ at time t would be $A(t) + \varphi(x)$. The function $\Phi(y)$ captures the cost, in excess of A_t , of purchasing y shares in the absence of the large investor. The first term on the right-hand side of (2.9) is the cost less A_t of purchasing all the shares with prices in the interval $[A_t, A_t + \psi(y)]$. If F has a jump at $\psi(y)$, this might be fewer than y shares. The difference, $y - F(\psi(y))$ shares, can be purchased at price $A_t + \psi(y)$, and this explains the second term on the right-hand side of (2.9). We present these functions in the three examples considered earlier.

Example 2.1 (Block order book, continued) We have simply $\varphi(x) = q \int_0^x \xi d\xi = \frac{q}{2}x^2$ for all $x \geq 0$, and $\Phi(y) = \frac{q}{2}\psi^2(y) = \frac{1}{2q}y^2$ for all $y \geq 0$. Note that Φ is convex and $\Phi'(y) = \psi(y)$ for all $y \geq 0$, including at $y = 0$ because we define Φ to be identically zero on the negative half-line. \square

Example 2.2 (Modified block order book, continued) With F and ψ given by (2.3) and (2.4), we have

$$\varphi(x) = \begin{cases} \frac{1}{2}x^2, & 0 \leq x \leq a, \\ \frac{1}{2}a^2, & a \leq x \leq b, \\ \frac{1}{2}(x^2 + a^2 - b^2), & b \leq x < \infty, \end{cases} \quad (2.10)$$

and

$$\Phi(y) = \begin{cases} \frac{1}{2}y^2, & 0 \leq y \leq a, \\ \frac{1}{2}((y + b - a)^2 + a^2 - b^2), & a \leq y < \infty. \end{cases} \quad (2.11)$$

Note that Φ is convex with subdifferential

$$\partial\Phi(y) = \begin{cases} \{y\}, & 0 \leq y < a, \\ [a, b], & y = a, \\ \{y + b - a\}, & a < y < \infty. \end{cases} \quad (2.12)$$

In particular, $\partial\Phi(y) = [\psi(y), \psi(y+)]$ for all $y \geq 0$ (see Fig. 2.4). \square

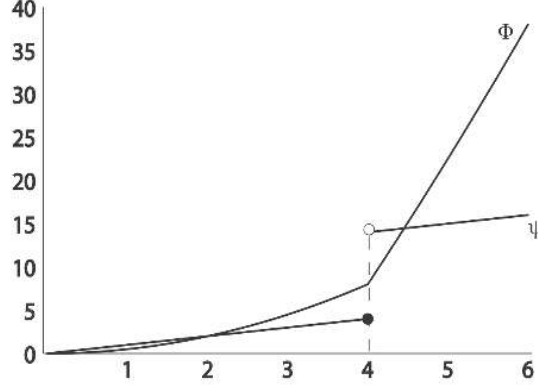


FIG. 2.4. Functions Φ and ψ for the modified block order book with parameters $a = 4$ and $b = 14$

Example 2.3 (Discrete order book, continued) With F given by (2.5), we have $\varphi(x) = \sum_{i=0}^{\infty} i \mathbb{I}_{(i, \infty)}(x)$. In particular, $\varphi(0) = 0$ and for integers $k \geq 1$ and $k < y \leq k + 1$,

$$\varphi(y) = \sum_{i=0}^{k-1} i = \frac{k(k-1)}{2}. \quad (2.13)$$

For $0 \leq y \leq 1$, $\psi(y) = 0$ and hence $\varphi(\psi(y)) = 0$, $[y - F(\psi(y))]\psi(y) = 0$, and $\Phi(y) = 0$. For integers $k \geq 1$ and $k < y \leq k + 1$, (2.6) gives $\psi(y) = k$, and hence $\varphi(\psi(y)) = \frac{k(k-1)}{2}$. Finally, for y in this range, $[y - F(\psi(y))]\psi(y) = k(y - k)$. We conclude that

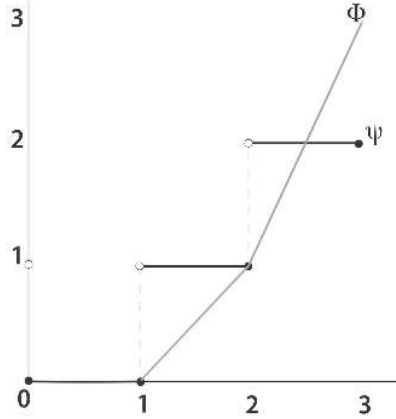
$$\Phi(y) = \sum_{k=1}^{\infty} k \left(y - \frac{1}{2}k - \frac{1}{2} \right) \mathbb{I}_{(k, k+1]}(y). \quad (2.14)$$

For each positive integer k , $\Phi(k-) = \Phi(k+) = \frac{1}{2}k(k-1)$, so Φ is continuous. Furthermore, $\partial\Phi(k) = [k-1, k] = [\psi(k), \psi(k+)]$. For nonnegative integers k and $k < y < k + 1$, $\Phi'(y)$ is defined and is equal to $\psi(y) = k$. Furthermore $\Phi'(0) = \psi(0) = 0$. Once again we have $\partial\Phi(y) = [\psi(y), \psi(y+)]$ for all $y \geq 0$, and because ψ is nondecreasing, Φ is convex (see Fig. 2.5). \square

We decompose the purchasing strategy X into its continuous and pure jump parts

$$X_t = X_t^c + \sum_{0 \leq s \leq t} \Delta X_s.$$

The investor pays price $A_t + D_t$ for infinitesimal purchases at time t , and hence the total cost of these purchases is $\int_0^T (A_t + D_t) dX_t^c$. On the other hand, if $\Delta X_t > 0$, the investor makes a lump purchase of size $\Delta X_t = \Delta E_t$ at time t . Because mass E_{t-} is missing in the shadow order book immediately prior to time t , the cost of this purchase is the difference between purchasing E_t and purchasing E_{t-} from the

FIG. 2.5. Functions Φ and ψ for the discrete order book

shadow order book, i.e., the difference in what the costs of these purchases would be in the absence of the large investor. Therefore, the cost of the purchase ΔX_t at time t is $A_t \Delta X_t + \Phi(E_t) - \Phi(E_{t-})$. These considerations lead to the following definition.

DEFINITION 2.4. *The total cost incurred by the investor using purchasing strategy X over the interval $[0, T]$ is*

$$\begin{aligned} C(X) &\triangleq \sum_{0 \leq t \leq T} [A_t \Delta X_t + \Phi(E_t) - \Phi(E_{t-})] + \int_0^T (A_t + D_t) dX_t^c \\ &= \sum_{0 \leq t \leq T} [\Phi(E_t) - \Phi(E_{t-})] + \int_0^T D_t dX_t^c + \int_{[0, T]} A_t dX_t. \end{aligned} \quad (2.15)$$

Our goal is to determine the purchasing strategy X that minimizes $\mathbb{E}C(X)$.

3. Problem simplifications. To compute the expectation of $C(X)$ defined by (2.15), we invoke the integration by parts formula

$$\int_{[0, T]} A_t dX_t = A_T X_T - A_0 X_{0-} - \int_0^T X_t dA_t$$

for the bounded variation process X and the continuous martingale A . Our investor's strategies must satisfy $0 = X_{0-} \leq X_t \leq X_T = \bar{X}$, $0 \leq t \leq T$, and hence

$$\mathbb{E} \int_0^T A_t dX_t = \bar{X} \mathbb{E} A_T = \bar{X} A_0.$$

It follows that

$$\mathbb{E}C(X) = \mathbb{E} \sum_{0 \leq t \leq T} [\Phi(E_t) - \Phi(E_{t-})] + \mathbb{E} \int_0^T D_t dX_t^c + \bar{X} A_0. \quad (3.1)$$

Since the third term on the right-hand side of (3.1) does not depend on X , minimization of $\mathbb{E}C(X)$ is equivalent to minimization of the first two terms. But the first two

terms do not depend on A , and hence we may assume without loss of generality that A is identically zero. Under this assumption, the cost of using strategy X is

$$C(X) = \sum_{0 \leq t \leq T} [\Phi(E_t) - \Phi(E_{t-})] + \int_0^T D_t dX_t^c. \quad (3.2)$$

But with $A \equiv 0$, there is no longer a source of randomness in the problem. Consequently, without loss of generality we may restrict the search for an optimal strategy to nonrandom functions of time. Once we find a nonrandom purchasing strategy minimizing (3.3) below, then even if A is a continuous non-zero nonnegative martingale, we have found a purchasing strategy that minimizes the expected value of (2.15) over all (possibly random) purchasing strategies.

THEOREM 3.1. *Under the assumption (made without loss of generality) that A is identically zero, the cost (3.2) associated with a nonrandom nondecreasing right continuous function X_t , $0 \leq t \leq T$, satisfying $X_{0-} = 0$ and $X_T = \bar{X}$ is equal to*

$$C(X) = \Phi(E_T) + \int_0^T D_t h(E_t) dt. \quad (3.3)$$

PROOF: The proof proceeds in two steps. In Step 1 we show that, as we have seen in the examples, Φ is a convex function with subdifferential

$$\partial\Phi(y) = [\psi(y), \psi(y+)], \quad y \geq 0. \quad (3.4)$$

In Step 2 we justify the integration formula

$$\Phi(E_T) = \int_0^T D^- \Phi(E_t) dE_t^c + \sum_{0 \leq t \leq T} [\Phi(E_t) - \Phi(E_{t-})], \quad (3.5)$$

where $D^- \Phi(E_t)$ denotes the left-hand derivative $\psi(E_t) = D_t$ of Φ at E_t , and E^c is the continuous part of E : $E_t^c = E_t - \sum_{0 \leq s \leq t} \Delta E_s$. From (2.1) and (3.5) we have immediately that

$$\Phi(E_T) = \int_{[0, T]} D_t dX_t^c - \int_0^T D_t h(E_t) dt + \sum_{0 \leq t \leq T} [\Phi(E_t) - \Phi(E_{t-})],$$

and (3.3) follows from (3.2).

Step 1. Using the integration by parts formula $xF(x) = \int_{[0, x]} \xi dF(\xi) + \int_0^x F(\xi) d\xi$, we write

$$\begin{aligned} \Phi(y) &= \int_{[0, \psi(y))} \xi dF(\xi) + [y - F(\psi(y))]\psi(y) \\ &= \int_0^{\psi(y)} (y - F(\xi)) d\xi \\ &= \int_0^{\psi(y)} \int_{F(\xi)}^y d\eta d\xi \\ &= \int_0^y \int_0^{\psi(\eta)} d\xi d\eta, \end{aligned}$$

where the last step follows from the fact that the symmetric difference of the sets $\{(\eta, \xi) | \xi \in [0, \psi(y)], \eta \in [F(\xi), y]\}$ and $\{(\eta, \xi) | \eta \in [0, y], \xi \in [0, \psi(\eta)]\}$ is at most a countable union of line segments and thus has two-dimensional Lebesgue measure 0. Therefore,

$$\Phi(y) = \int_0^y \psi(\eta) d\eta, \quad (3.6)$$

and by Problem 3.6.20, p. 213 of [6], with ψ and Φ extended to be 0 for the negative reals, we conclude that Φ is convex and that $\partial\Phi(y) = [\psi(y), \psi(y+)]$, as desired.

Step 2. We mollify ψ , taking ρ to be a C^∞ function with support on $[-1, 0]$ and integral 1, defining $\rho_n(\eta) = n\rho(n\eta)$, and defining

$$\psi_n(y) = \int_{\mathbb{R}} \psi(y + \eta)\rho_n(\eta) d\eta = \int_{\mathbb{R}} \psi(\zeta)\rho_n(\zeta - y) d\zeta.$$

Then each ψ_n is a C^∞ function satisfying $0 \leq \psi_n(y) \leq \psi(y)$ for all $y \geq 0$. Furthermore, $\psi(y) = \lim_{n \rightarrow \infty} \psi_n(y)$ for every $y \in \mathbb{R}$. We set $\Phi_n(y) = \int_0^y \psi_n(\eta) d\eta$, so that each Φ_n is also a C^∞ function and $\lim_{n \rightarrow \infty} \Phi'_n(y) = D^-\Phi(y)$.

Because $\Phi_n(E_0) = \Phi(0) = 0$, we have

$$\Phi_n(E_T) = \int_0^T \Phi'_n(E_t) dE_t^c + \sum_{0 \leq t \leq T} [\Phi_n(E_t) - \Phi_n(E_{t-})]; \quad (3.7)$$

see, e.g., [5], p. 78. The function E_t , $0 \leq t \leq T$, is bounded. Letting $n \rightarrow \infty$ in (3.7) and using the bounded convergence theorem, we obtain

$$\Phi(E_T) = \int_0^T D^-\Phi(E_t) dE_t^c + \lim_{n \rightarrow \infty} \sum_{0 \leq t \leq T} [\Phi_n(E_t) - \Phi_n(E_{t-})]. \quad (3.8)$$

To conclude the proof of (3.5), we divide the sum in (3.8) into two parts. Given $\delta > 0$, we define $S_\delta = \{t \in [0, T] : 0 < \Delta E_t \leq \delta\}$ and $S'_\delta = \{t \in [0, T] : \Delta E_t > \delta\}$. The sum in (3.8) is over $t \in S_\delta \cup S'_\delta$, and because E has finite variation, $\sum_{t \in S_\delta \cup S'_\delta} \Delta E_t < \infty$. Let $\epsilon > 0$ be given. We choose $\delta > 0$ so small that $\sum_{t \in S_\delta} \Delta E_t \leq \epsilon$. Because ψ and hence each ψ_n is bounded on $[0, E_T]$, the function Φ and each Φ_n is Lipschitz continuous on $[0, E_T]$ with the same Lipschitz constant $L = \psi(E_T)$. It follows that

$$\begin{aligned} \sum_{t \in S_\delta} [\Phi(E_t) - \Phi(E_{t-})] &\leq L \sum_{t \in S_\delta} \Delta E_t \leq L\epsilon, \\ \sum_{t \in S_\delta} [\Phi_n(E_t) - \Phi_n(E_{t-})] &\leq L \sum_{t \in S_\delta} \Delta E_t \leq L\epsilon, \quad n = 1, 2, \dots \end{aligned}$$

Hence the difference between $\sum_{t \in S_\delta} [\Phi(E_t) - \Phi(E_{t-})]$ and any limit point of $\sum_{t \in S_\delta} [\Phi_n(E_t) - \Phi_n(E_{t-})]$ is at most $2L\epsilon$. On the other hand, the set S'_δ contains only finitely many elements, and thus

$$\lim_{n \rightarrow \infty} \sum_{t \in S'_\delta} [\Phi_n(E_t) - \Phi_n(E_{t-})] = \sum_{t \in S'_\delta} [\Phi(E_t) - \Phi(E_{t-})].$$

Since $\epsilon > 0$ is arbitrary, (3.8) reduces to (3.5). \square

4. Solution of the Optimization Problem. In view of Theorem 3.1, we want to minimize $\Phi(E_T) + \int_0^T D_t h(E_t) dt$ over the set of deterministic purchasing strategies. The main result of this paper is that there exists an optimal strategy X under which the trader buys a lump quantity $X_0 = E_0$ of shares at time 0, then buys at a constant rate $dX_t = h(E_0) dt$ up to time t_0 (so as to keep $E_t = E_0$ for $t \in [0, t_0)$), then buys another lump quantity of shares at time t_0 , subsequently trades again at a constant rate $dX_t = h(E_{t_0}) dt$ until time T (so as to keep $E_t = E_{t_0}$ for $t \in [t_0, T)$), and finally buys the remaining shares at time T . We shall call this strategy a *Type B strategy*. We further show that if the nonnegative function

$$g(y) \triangleq y\psi(h^{-1}(y)) \quad (4.1)$$

is convex, then the purchase at time t_0 consists of 0 shares (so X has only jumps at times 0 and T). We call such a strategy a *Type A strategy*. Clearly the latter is a special case of the former.

Although g is naturally defined on $[0, \infty)$ by (4.1), we will want it to be defined only on a compact subset of $[0, \infty)$. We set

$$\bar{Y} = \max \left\{ h(\bar{X}), \frac{\bar{X}}{T} \right\}. \quad (4.2)$$

The function g we consider has domain $[0, \bar{Y}]$. For future reference, we make three observations about the function g . First,

$$\lim_{y \downarrow 0} g(y) = g(0) = 0. \quad (4.3)$$

Secondly, using the definition (2.2) of D_t , we can rewrite the cost function formula (3.3) as

$$C(X) = \Phi(E_T) + \int_0^T g(h(E_t)) dt. \quad (4.4)$$

Lemma A.1(iv) in the appendix shows that $0 \leq E_t \leq \bar{X}$, so the domain $[0, \bar{Y}]$ of g is large enough in order for (4.4) to make sense. Because h^{-1} is strictly increasing and continuous and ψ is nondecreasing and left continuous, the function g is nondecreasing and left continuous, hence lower semicontinuous. In particular,

$$g(\bar{Y}) = \lim_{y \uparrow \bar{Y}} g(y). \quad (4.5)$$

4.1. Convexity and Type A Strategies.

REMARK 4.1. Let X^A be a Type A strategy and let E^A be related to X^A via (2.1), so that $E_t^A = X_0^A$ for $0 \leq t < T$. Then

$$X_{T-}^A = E_{T-}^A + \int_0^T h(E_t^A) dt = X_0^A + h(X_0^A)T, \quad (4.6)$$

$$\Delta X_T^A = \bar{X} - X_{T-}^A = \bar{X} - X_0^A - h(X_0^A)T, \quad (4.7)$$

$$E_T^A = E_{T-}^A + \Delta X_T^A = \bar{X} - h(X_0^A)T. \quad (4.8)$$

According to (4.4) and (4.8), the cost of this strategy is

$$C(X^A) = \Phi(E_T^A) + Tg(h(X_0^A)) = \Phi(E_T^A) + Tg\left(\frac{\bar{X} - E_T^A}{T}\right). \quad (4.9)$$

A Type A strategy is fully determined by its initial condition X_0^A , or equivalently, by the terminal value E_T^A of the corresponding E^A (see (4.8)). We must choose X_0^A so that $X_0^A \geq 0$ and X_{T-}^A given by (4.6) does not exceed \bar{X} . The function $k(x) \triangleq x + h(x)T$ is strictly increasing and continuous on $[0, \infty)$, and $k(\bar{X}) > \bar{X}$. Therefore, there exists a unique $\bar{e} \in (0, \bar{X})$ such that $k(\bar{e}) = \bar{X}$. i.e.,

$$\bar{e} + h(\bar{e})T = \bar{X}. \quad (4.10)$$

The constraint on the initial condition of Type A strategies that guarantees that the strategy is feasible is

$$0 \leq X_0^A \leq \bar{e}. \quad (4.11)$$

From (4.8) and (4.10) we see that the corresponding feasibility condition on E_T^A for Type A strategies is

$$\bar{e} \leq E_T^A \leq \bar{X}. \quad (4.12)$$

□

THEOREM 4.2. *If g given by (4.1) is convex on $[0, \bar{Y}]$, then there exists a Type A purchasing strategy that minimizes $C(X)$ over all purchasing strategies X .*

PROOF: Assume that g is convex and let X be a purchasing strategy. Jensen's inequality applied to (4.4) yields the lower bound

$$C(X) = \Phi(E_T) + T \int_0^T g(h(E_t)) \frac{dt}{T} \geq \Phi(E_T) + Tg\left(\int_0^T h(E_t) \frac{dt}{T}\right).$$

From (2.1) we further have $\int_0^T h(E_t) dt = \bar{X} - E_T$, and thus the lower bound can be rewritten as

$$C(X) \geq \Phi(E_T) + Tg\left(\frac{\bar{X} - E_T}{T}\right). \quad (4.13)$$

Recall that $0 \leq E_T \leq \bar{X}$, so the argument of g in (4.13) is in $[0, \bar{Y}]$.

This leads us to consider minimization of the function

$$G(e) \triangleq \Phi(e) + Tg\left(\frac{\bar{X} - e}{T}\right)$$

over $e \in [0, \bar{X}]$. By assumption, the function g is convex on $[0, \bar{Y}]$ and hence continuous on $(0, \bar{Y})$. Equations (4.3) and (4.5) show that g is also continuous at the endpoints of its domain. Because Φ has the integral representation (3.6), it also is convex and continuous on $[0, \bar{X}]$. Therefore, G is a convex continuous function on $[0, \bar{X}]$, and hence the minimum is attained.

We show next that the minimum of G over $[0, \bar{X}]$ is attained in $[\bar{e}, \bar{X}]$. For this, we first observe that because g is convex,

$$D^+g(y) \geq \frac{g(y) - g(0)}{y} = \psi(h^{-1}(y)), \quad 0 < y \leq \bar{Y}. \quad (4.14)$$

This inequality together with (3.4) and (4.10) implies

$$D^-G(\bar{e}) = \psi(\bar{e}) - D^+g(y) \Big|_{y=\frac{\bar{X}-\bar{e}}{T}} \leq \psi(\bar{e}) - \psi\left(h^{-1}\left(\frac{\bar{X}-\bar{e}}{T}\right)\right) = 0.$$

Therefore, the minimum of the convex function G over $[0, \bar{X}]$ is obtained in $[\bar{e}, \bar{X}]$.

Let $e^* \in [\bar{e}, \bar{X}]$ attain the minimum of G over $[0, \bar{X}]$. The Type A strategy X^A with initial condition $X_0^A = h^{-1}\left(\frac{\bar{X}-e^*}{T}\right)$ satisfies $E_T^A = e^*$ (see (4.8)), and hence the strategy is feasible (see (4.12)). The cost associated with this strategy is less than or equal to the right-hand side of (4.13) (see (4.9)). This strategy is therefore optimal. \square

REMARK 4.3. Alfonsi, Schied and Schulz [1] consider the case that the measure μ has a strictly positive density f . In this case, the function $F(x) = \int_0^x f(\xi) d\xi$ is strictly increasing and continuous with derivative $F'(x) = f(x)$, and its inverse ψ is likewise strictly increasing and continuous with derivative $\psi'(y) = 1/f(\psi(y))$. Furthermore, in [1] the resilience function is $h(x) = \rho x$, where ρ is a positive constant. In this case,

$$g'(y) = \psi(y/\rho) + \frac{y/\rho}{f(\psi(y/\rho))},$$

and Theorem 4.2 guarantees the existence of a Type A strategy under the assumption that g' is nondecreasing. This is equivalent to the condition that

$$\psi(y) + \frac{y}{f(\psi(y))}$$

is nondecreasing.

Alfonsi, Schied and Schulz [1] obtain a discrete-time version of a Type A strategy under the assumption that

$$h_1(y) \triangleq \psi(y) - e^{-\rho\tau}\psi(e^{-\rho\tau}y) \quad (4.15)$$

is strictly increasing, where τ is the time between trading dates. In order to study the limit of their model as $\tau \downarrow 0$, they observe that

$$\lim_{\tau \downarrow 0} h_1(y)/(1 - e^{-\rho\tau}) = \psi(y) + \frac{y}{f(\psi(y))},$$

which is thus nondecreasing. Thus g given by (4.1) is convex in their model.

To find a simpler formulation of the hypothesis of Theorem 4.2 under the assumption that μ has a strictly positive density f and $h(x) = \rho x$ for a positive constant ρ , we compute

$$\frac{d}{dy} \left(\psi(y) + \frac{y}{f(\psi(y))} \right) = \frac{2}{f(\psi(y))} - \frac{yf'(\psi(y))}{f^3(\psi(y))}.$$

This is nonnegative if and only if $2f^2(\psi(y)) \geq yf'(\psi(y))$. Replacing y by $F(x)$, we obtain the condition

$$2f^2(x) \geq F(x)f'(x), \quad x \geq 0. \quad (4.16)$$

This is clearly satisfied under the assumption of [4] that f is a positive constant. \square

Example 2.1 (Block order book, continued) In the case of the block order book with $h(x) = \rho x$, where ρ is a strictly positive constant,

$$g(y) = \frac{yh^{-1}(y)}{q} = \frac{y^2}{\rho q},$$

which is convex. Theorem 4.2 implies that there is an optimal strategy of Type A. Because $\Phi(e) = \frac{1}{2q}e^2$, we have

$$G(e) = \frac{e^2}{2q} + \frac{(\bar{X} - e)^2}{\rho q T}.$$

The minimizer is $e^* = \frac{2\bar{X}}{2+\rho T}$, which lies between $\bar{e} = \frac{\bar{X}}{1+\rho T}$ and \bar{X} , as expected. According to Remark 4.1, the optimal strategy of Type A is to make an initial purchase of size

$$X_0^A = h^{-1}\left(\frac{\bar{X} - e^*}{T}\right) = \frac{\bar{X}}{2 + \rho T},$$

then purchase continuously at rate $dX_t^A = h(X_0^A) dt = \frac{\rho\bar{X}}{2+\rho T} dt$ over the time interval $[0, T]$, and conclude with a lump purchase

$$e^* - X_0^A = \frac{\bar{X}}{2 + \rho T}$$

at the final time T . In particular, the initial and final lump purchases are the same size, and there is no intermediate lump purchase.

4.2. Convex Hulls and Type B Strategies.

THEOREM 4.4. *In the absence of the assumption that g given by (4.1) is convex, there exists a Type B purchasing strategy that minimizes $C(X)$ over all purchasing strategies X .*

PROOF: We construct the *convex hull* of g , defined for $0 \leq y \leq \bar{Y}$ by

$$\widehat{g}(y) \triangleq \sup \{ \ell(y) : \ell \text{ is an affine function and } \ell(\eta) \leq g(\eta) \forall \eta \in [0, \bar{Y}] \}. \quad (4.17)$$

For each $0 \leq y < \bar{Y}$, the supremum in (4.17) is obtained by the support line of \widehat{g} at y . For $y = 0$ the zero function is such a support line, and hence $0 \leq \widehat{g}(0) \leq g(0) = 0$ (recall (4.3)). At $y = \bar{Y}$ the only support line might be vertical, in which case the supremum in (4.17) is not attained. The function \widehat{g} is the largest convex function defined on $[0, \bar{Y}]$ that is dominated by g there. Because $\widehat{g}(0) = 0$, \widehat{g} is nonnegative, and \widehat{g} is convex, we know that \widehat{g} is also nondecreasing. Being convex, \widehat{g} is continuous on $(0, \bar{Y})$, upper semi-continuous on $[0, \bar{Y}]$, and we have continuity at 0 because of (4.3).

We next argue that $\widehat{g}(\overline{Y}) = g(\overline{Y})$. Suppose, on the contrary, we had $\widehat{g}(\overline{Y}) < g(\overline{Y})$. The function g is continuous at \overline{Y} (see (4.5)) and \widehat{g} is upper semicontinuous. Therefore, there is a one-sided neighborhood $[\gamma, \overline{Y}]$ of \overline{Y} (with $\gamma < \overline{Y}$) on which $g - \widehat{g}$ is bounded away from zero by a positive number ϵ . The function

$$\widehat{g}(y) + \frac{\epsilon(y - \gamma)}{\overline{Y} - \gamma}, \quad 0 \leq y \leq \overline{Y},$$

is convex, lies strictly above \widehat{g} at \overline{Y} , and lies below g everywhere. This contradicts the fact that \widehat{g} is the largest convex function dominated by g . We must therefore have $\widehat{g}(\overline{Y}) = g(\overline{Y})$.

Finally, we describe the situation when for some $y^* \in [0, \overline{Y}]$, we have $\widehat{g}(y^*) < g(y^*)$. We have shown that this can happen only if $0 < y^* < \overline{Y}$. Let ℓ be an affine function that attains the maximum in (4.17) at the point y^* (which is attained because $y^* < \overline{Y}$), i.e.,

$$\ell(y^*) = \widehat{g}(y^*). \quad (4.18)$$

Define

$$\alpha = \sup\{\eta \in [0, y^*] : g(\eta) - \ell(\eta) = 0\}, \quad (4.19)$$

$$\beta = \inf\{\eta \in [y^*, \overline{Y}] : g(\eta) - \ell(\eta) = 0\}. \quad (4.20)$$

Because g is continuous, the minimum of $g - \ell$ over $[0, \overline{Y}]$ is attained. This minimum cannot be a positive number ϵ , for then $\ell + \epsilon$ would be an affine function lying below g . Therefore, either the supremum in (4.19) or the infimum in (4.20) is taken over a nonempty set. In the former case, we must have $g(\alpha) = \ell(\alpha)$, whereas in the latter case $g(\beta) = \ell(\beta)$.

Let us consider first the case that $g(\alpha) = \ell(\alpha)$. Define $\gamma = \frac{1}{2}(\alpha + y^*)$. Like α , γ is strictly less than y^* . The function $g - \ell$ attains its minimum over $[\gamma, \overline{Y}]$. If this minimum were a positive number ϵ , then the affine function

$$\ell(y) + \frac{\epsilon(y - \gamma)}{\overline{Y} - \gamma}, \quad 0 \leq y \leq \overline{Y},$$

would lie below g but have a larger value at y^* than ℓ , violating the choice of ℓ . It follows that $g - \ell$ attains the minimum value zero on $[\gamma, \overline{Y}]$, and since this function is strictly positive on $[\gamma, y^*]$, the minimum is attained to the right of y^* . This implies that $g(\beta) = \ell(\beta)$. Similarly, if we begin with the assumption that $g(\beta) = \ell(\beta)$, we can argue that $g(\alpha) = \ell(\alpha)$.

In conclusion, α and β defined by (4.19) and (4.20) satisfy

$$0 \leq \alpha < y^* < \beta \leq \overline{Y}, \quad (4.21)$$

$$g(\alpha) = \ell(\alpha) = \widehat{g}(\alpha), \quad g(\beta) = \ell(\beta) = \widehat{g}(\beta). \quad (4.22)$$

Finally, (4.22) shows that ℓ restricted to $[\alpha, \beta]$ is the largest affine function lying below g on this interval, and hence

$$\ell(y) = \widehat{g}(y) < g(y), \quad \alpha < y < \beta. \quad (4.23)$$

Using \widehat{g} in place of g in (4.4), we define the *modified cost function*

$$\widehat{C}(X) \triangleq \Phi(E_T) + \int_0^T \widehat{g}(h(E_t)) dt. \quad (4.24)$$

For any purchasing strategy X , we obviously have $\widehat{C}(X) \leq C(X)$. Analogously to (4.13), for any purchasing strategy X the lower bound

$$\widehat{C}(X) \geq \Phi(E_T) + T\widehat{g}\left(\frac{\overline{X} - E_T}{T}\right) \quad (4.25)$$

holds. This leads us to consider minimization of the function

$$\widehat{G}(e) \triangleq \Phi(e) + T\widehat{g}\left(\frac{\overline{X} - e}{T}\right) \quad (4.26)$$

over $e \in [0, \overline{X}]$. As in the proof of Theorem 4.2, this function attains its minimum at some $e^* \in [0, \overline{X}]$.

However, we do not have an inequality analogous to (4.14) for \widehat{g} , and so we proceed somewhat differently from the proof of Theorem 4.2. We assume without loss of generality that e^* is the largest number in $[0, \overline{X}]$ that minimizes \widehat{G} . We then construct below a Type B strategy X^B (which is actually a simpler Type A strategy in three of the four cases into which the proof is divided) that satisfies $E_T^B = e^*$ and

$$C(X^B) = \widehat{G}(e^*). \quad (4.27)$$

This construction will prove the theorem.

For the remainder of the proof, we use the notation

$$y^* = \frac{\overline{X} - e^*}{T}, \quad x^* = h^{-1}(y^*), \quad (4.28)$$

where e^* is the largest minimizer of \widehat{G} in $[0, \overline{X}]$. There are four cases.

Case I. $x^* = 0$.

In this case, $y^* = 0$, $e^* = \overline{X}$, and $\widehat{G}(e^*) = G(e^*)$. The Type A strategy that waits until the final time T and then purchases \overline{X} is optimal. In particular, this strategy satisfies the initial condition $X_0^A = x^*$.

Case II. $\widehat{g}(y^*) = g(y^*)$ and $0 < x^* \leq F(0+)$.

In this case,

$$\widehat{g}(y^*) = g(y^*) = y^* \psi(x^*) = 0$$

because $\psi \equiv 0$ on $[0, F(0+)]$. But $\widehat{g}(0) = 0$ and \widehat{g} is nondecreasing, so $\widehat{g} \equiv 0$ on $[0, y^*]$. Furthermore, x^* is positive, so $e^* < \overline{X}$. For $e \in (e^*, \overline{X})$, the number $\frac{\overline{X} - e}{T}$ is in $(0, y^*)$, and by (3.4), $D^+ \widehat{G}(e) = D^+ \Phi(e) = \psi(e+)$. On the other hand, e^* is the largest minimizer of \widehat{G} in $[0, \overline{X}]$, which implies $D^+ \widehat{G}(e) > 0$. This shows that $\psi(e+) > 0$ for every $e \in (e^*, \overline{X})$, which implies that $\psi(e) > 0$ for every $e \in (e^*, \overline{X})$ and

further implies that $e \geq F(0+)$ for every $e \in (e^*, \bar{X})$. We conclude that $e^* \geq F(0+)$. Applying h to this inequality and using the case assumption, we obtain

$$h(e^*) \geq h(F(0+)) \geq h(x^*) = \frac{\bar{X} - e^*}{T}. \quad (4.29)$$

In other words, $e^* + h(e^*)T \geq \bar{X}$, and by the defining equation (4.10) of \bar{e} , we conclude that $e^* \geq \bar{e}$.

Let X^A be the Type A strategy with $X_0^A = x^*$. According to (4.10),

$$0 \leq X_0^A = x^* = h^{-1} \left(\frac{\bar{X} - e^*}{T} \right) \leq h^{-1} \left(\frac{\bar{X} - \bar{e}}{T} \right) = \bar{e}.$$

Hence X_0^A satisfies the constraint (4.11) on the initial condition of Type A strategies. According to (4.8), $E_T^A = \bar{X} - h(x^*)T = e^*$. Equation (4.9) yields

$$C(X^A) = \Phi(e^*) + Tg(y^*) = \Phi(e^*) + T\hat{g}(y^*) = \hat{G}(e^*).$$

Case III. $\hat{g}(y^*) = g(y^*)$ and $x^* > F(0+)$.

Because $y^* = h(x^*)$ is positive, the left-hand derivative of \hat{g} at y^* is defined, and it satisfies

$$D^- \hat{g}(y^*) \geq \frac{\hat{g}(y^*) - \hat{g}(0)}{y^*} = \frac{g(y^*)}{y^*} = \psi(x^*). \quad (4.30)$$

In fact, the inequality in (4.30) is strict. If it were not, the affine function

$$\ell(y) = \psi(x^*)(y - y^*) + \hat{g}(y^*) = y\psi(x^*)$$

would describe a tangent line to the graph of \hat{g} at $(y^*, \hat{g}(y^*))$ lying below $\hat{g}(y)$, and hence below $g(y)$, for all $y \in [0, \bar{Y}]$. But the resulting inequality $y\psi(x^*) \leq g(y) = y\psi(h^{-1}(y))$ yields $\psi(x^*) \leq \psi(h^{-1}(y))$ for all $y \in (0, \bar{Y}]$, and letting $y \downarrow 0$, we would conclude $\psi(x^*) = 0$. This violates the case assumption $x^* > F(0+)$. We conclude that $D^- \hat{g}(y^*) > \psi(x^*)$. The strict inequality, the fact that e^* minimizes \hat{G} , and (3.4) further imply

$$0 \leq D^+ \hat{G}(e^*) = D^+ \Phi(e^*) - D^- \hat{g}(y^*) < \psi(e^*+) - \psi(x^*).$$

But $\psi(x^*) < \psi(e^*+)$ implies $x^* \leq e^*$. Consequently,

$$h(e^*) \geq h(x^*) = \frac{\bar{X} - e^*}{T}.$$

This is the essential part of inequality (4.29), and we conclude as in Case III, constructing an optimal Type A strategy with $X_0^A = x^*$.

Case IV. $\hat{g}(y^*) < g(y^*)$ and $x^* > 0$.

Recall from the beginning of the proof that this case can occur only if $0 < y^* < \bar{Y}$. We choose ℓ to be an affine function lying below g and satisfying (4.18), define α and β by (4.19) and (4.20), and recall the consequent equations (4.21)–(4.23). We

use these equations to construct a Type B strategy that satisfies (4.12). To do this, we define $t_0 \in (0, T)$ by

$$t_0 = \frac{(\beta - y^*)T}{\beta - \alpha},$$

so that $\alpha t_0 + \beta(T - t_0) = y^*T$. Consider the Type B strategy that makes an initial purchase $X_0^B = h^{-1}(\alpha)$, then purchases at rate $dX_t^B = \alpha dt$ for $0 \leq t < t_0$ (so $E_t^B = h^{-1}(\alpha)$ for $0 \leq t < t_0$), follows this with a purchase $\Delta X_{t_0}^B = h^{-1}(\beta) - h^{-1}(\alpha)$ at time t_0 , thereafter purchases at rate $dX_t^B = \beta dt$ for $t_0 \leq t < T$ (so $E_t^B = h^{-1}(\beta)$ for $t_0 \leq t < T$), and makes a final purchase $\bar{X} - X_{T-}^B$ at time T . According to (2.1),

$$X_t^B = \begin{cases} h^{-1}(\alpha) + \alpha t, & 0 \leq t < t_0, \\ h^{-1}(\beta) + \alpha t_0 + \beta(t - t_0), & t_0 \leq t < T, \\ \bar{X}, & t = T. \end{cases}$$

In particular,

$$\Delta X_T^B = \bar{X} - h^{-1}(\beta) - \alpha t_0 - \beta(T - t_0) = \bar{X} - h^{-1}(\beta) - y^*T = e^* - h^{-1}(\beta). \quad (4.31)$$

We show at the end of this proof that

$$h^{-1}(\beta) \leq e^*. \quad (4.32)$$

This will ensure that ΔX_T^B is nonnegative, and since X^B is obviously nondecreasing on $[0, T)$, this will establish that X^B is a feasible purchasing strategy.

Accepting (4.32) for the moment, we note that (4.31) implies

$$E_T^B = E_{T-}^B + \Delta E_T^B = h^{-1}(\beta) + \Delta X_T^B = e^*. \quad (4.33)$$

Using (4.4), (4.33), (4.22), the linearity of ℓ , and (4.18) in that order, we compute

$$\begin{aligned} C(X^B) &= \Phi(E_T^B) + \int_0^T g(h(E_t^B)) dt \\ &= \Phi(e^*) + g(\alpha)t_0 + g(\beta)(T - t_0) \\ &= \Phi(e^*) + \ell(\alpha)t_0 + \ell(\beta)(T - t_0) \\ &= \Phi(e^*) + T\ell\left(\frac{\alpha t_0 + \beta(T - t_0)}{T}\right) \\ &= \Phi(e^*) + T\ell(y^*) \\ &= \Phi(e^*) + T\hat{g}(y^*) \\ &= \hat{G}(e^*), \end{aligned}$$

This is (4.27).

Finally, still in Case IV, we turn to the proof of (4.32). Because e^* is the largest minimizer of the convex function \hat{G} in $[0, \bar{X}]$ and $e^* < \bar{X}$ (because $x^* > 0$), the right-hand derivative of \hat{G} at e^* must be nonnegative. Indeed, for all $e \in (e^*, \bar{X})$, this right-hand derivative must in fact be strictly positive. For e greater than but sufficiently close to e^* , $\frac{\bar{X}-e}{T}$ is in (α, y^*) , where \hat{g} is linear with slope $\frac{g(\beta)-g(\alpha)}{\beta-\alpha}$. For

such e ,

$$\begin{aligned}
0 &< D^+ \widehat{G}(e^*) \\
&= D^+ \Phi(e+) - D^- \widehat{g}(y) \Big|_{y=\frac{x-e}{T}} \\
&= \psi(e+) - \frac{g(\beta) - g(\alpha)}{\beta - \alpha} \\
&= \psi(e+) - \frac{\beta\psi(h^{-1}(\beta)) - \alpha\psi(h^{-1}(\alpha))}{\beta - \alpha} \\
&\leq \psi(e+) - \frac{\beta\psi(h^{-1}(\beta)) - \alpha\psi(h^{-1}(\beta))}{\beta - \alpha} \\
&= \psi(e+) - \psi(h^{-1}(\beta)).
\end{aligned}$$

This inequality $\psi(h^{-1}(\beta)) < \psi(e+)$ for all e greater than but sufficiently close to e^* implies (4.32). \square

4.3. Examples of Type B optimal strategies.

Example 2.2 (Modified block order book, continued) We continue Example 2.2 under the simplifying assumptions $T = 1$ and $h(x) = x$ for all $x \geq 0$, so $h^{-1}(y) = y$ for all $y \geq 0$ and $\overline{Y} = \overline{X}$. Recalling (2.4) and (4.1), we see that

$$g(y) = \begin{cases} y^2, & 0 \leq y \leq a, \\ y^2 + (b-a)y, & a < y < \infty. \end{cases}$$

The convex hull of g , given by (4.17), is

$$\widehat{g}(y) = \begin{cases} y^2, & 0 \leq y \leq a, \\ (2\beta + b - a)(y - a) + a^2, & a \leq y \leq \beta, \\ y^2 + (b-a)y, & \beta \leq y < \infty, \end{cases}$$

where

$$\beta = a + \sqrt{a(b-a)} \tag{4.34}$$

(see Fig. 4.1).

For $a < y^* < \beta$, we have $\widehat{g}(y^*) < g(y)$, α of (4.19) is a , and β of (4.20) is given by (4.34). In order to illustrate a case in which a Type B purchasing strategy is optimal, we assume

$$a + 2\beta < \overline{X} < 3\beta. \tag{4.35}$$

The function \widehat{G} of (4.26) is minimized over $[0, \overline{X}]$ at e^* if and only if

$$0 \in \partial \widehat{G}(e^*) = \partial \Phi(e^*) - \partial \widehat{g}(\overline{X} - e^*),$$

which is equivalent to $\partial \Phi(e^*) \cap \partial \widehat{g}(\overline{X} - e^*) \neq \emptyset$. We show below that the largest value of e^* satisfying this condition is $e^* = 2\beta$. According to (4.35), $e^* = 2\beta$ is in

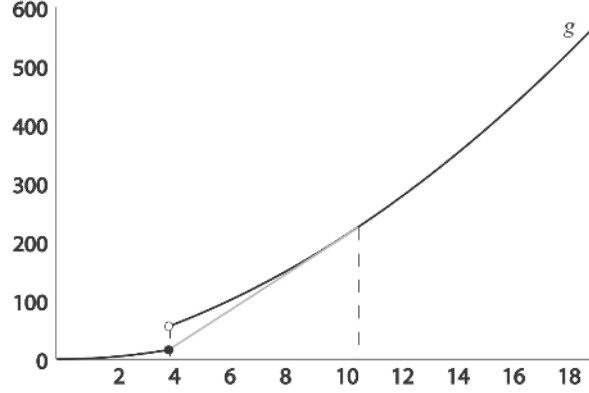


FIG. 4.1. Function g for the modified block order book with parameters $a = 4$ and $b = 14$. The convex hull \hat{g} is constructed by replacing a part $\{g(y), y \in (a, \beta)\}$ by a straight line connecting $g(a)$ and $g(\beta)$. Here $\beta = 10.324$

$(\bar{X} - \beta, \bar{X} - a)$. Because $\beta > a$, e^* is also in (a, ∞) . We compute (recall (2.12))

$$\partial\Phi(e) = \begin{cases} \{e\}, & 0 \leq e < a, \\ [a, b], & e = a, \\ \{e + b - a\}, & a < e < \infty, \end{cases}$$

$$\partial\hat{g}(\bar{X} - e) = \begin{cases} \{2(\bar{X} - e) + b - a\}, & 0 \leq e \leq \bar{X} - \beta, \\ \{2\beta + b - a\}, & \bar{X} - \beta \leq e < \bar{X} - a, \\ [2a, 2\beta + b - a], & e = \bar{X} - a, \\ \{2(\bar{X} - e)\}, & \bar{X} - a < e \leq \bar{X}, \end{cases}$$

and then evaluate

$$\partial\Phi(e^*) = \{e^* + b - a\} = \{2\beta + b - a\} = \partial\hat{g}(\bar{X} - e^*).$$

Therefore, \hat{G} attains its minimum at e^* .

To see that there is no $e \in (2\beta, \bar{X}]$ where \hat{G} attains its minimum, we observe that for $e \in (2\beta, \bar{X} - a)$, $\partial\Phi(e) \cap \partial\hat{g}(\bar{X} - e) = \{e + b - a\} \cap \{2\beta + b - a\} = \emptyset$. For $e \in [\bar{X} - a, \bar{X}]$, all points in $\partial\hat{g}(\bar{X} - e)$ lie in the interval $[0, 2a]$, whereas the only point in $\partial\Phi(e)$, which is $e + b - a$, lies in the interval $[\bar{X} + b - 2a, \bar{X} + b - a]$. Because of (4.35), we have $2a < \bar{X} + b - 2a$, and hence $\partial\Phi(e) \cap \partial\hat{g}(\bar{X} - e) = \emptyset$ for $e \in [\bar{X} - a, \bar{X}]$.

As in the proof of Theorem 4.4, we set $y^* = \bar{X} - e^* = \bar{X} - 2\beta$, $x^* = h^{-1}(y^*) = \bar{X} - 2\beta$. Condition (4.35) is equivalent to $a < y^* < \beta$, which in turn is equivalent to $\hat{g}(y^*) < g(y^*)$. The first inequality in (4.35) shows that $x^* > 0$, and we are thus in Case IV of the proof of Theorem 4.4. In this case, we define

$$t_0 = \frac{\beta - y^*}{\beta - a} = \frac{3\beta - \bar{X}}{\beta - a}.$$

The optimal purchasing strategy is

$$X_t^B = \begin{cases} a(t+1), & 0 \leq t < t_0, \\ at_0 + \beta(t+1-t_0), & t_0 \leq t < 1, \\ \bar{X}, & t = 1. \end{cases}$$

In particular,

$$\Delta X_0 = a, \quad \Delta X_{t_0} = \beta - a, \quad \Delta X_1 = \beta$$

(see (4.31) for the last equality). The corresponding E^B process is

$$E_t^B = \begin{cases} a, & 0 \leq t < t_0, \\ \beta, & t_0 < t < 1, \\ 2\beta, & t = 1. \end{cases}$$

The initial lump purchase moves the ask price to the left endpoint a of the gap in the order book. Purchasing is done to keep the ask price at a until time t_0 , when another lump purchase moves the ask price to β , beyond the right endpoint b of the gap in the order book. Purchasing is done to keep the ask price at β until the final time, when another lump purchase is executed. \square

Example 2.3 (Discrete order book, continued) We continue Example 2.3 under the simplifying assumptions that $T = 1$ and $h(x) = x$ for all $x \geq 0$, so that $h^{-1}(y) = y$ for all $y \geq 0$ and $\bar{Y} = \bar{X}$. From (2.6) and (4.1) we see that $g(0) = 0$, and $g(y) = ky$ for integers $k \geq 1$ and $k < y \leq k + 1$. In particular, $g(k) = (k - 1)k$ for nonnegative integers k . The convex hull of g interpolates linearly between the points $(k, (k - 1)k)$ and $(k + 1, k(k + 1))$, i.e.,

$$\hat{g}(y) = k(2y - (k + 1)) \text{ for } k \leq y \leq k + 1,$$

where k ranges over the nonnegative integers (see Fig. 4.2).

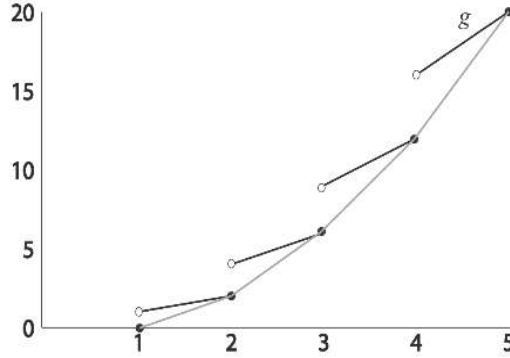


FIG. 4.2. Function g for the discrete order book. The convex hull \hat{g} interpolates linearly between the points $(k, (k - 1)k)$ and $(k + 1, k(k + 1))$

Therefore,

$$\partial \hat{g}(y) = \begin{cases} \{0\}, & y = 0, \\ [2(k - 1), 2k], & y = k \text{ and } k \text{ is a positive integer,} \\ \{2k\}, & k < y < k + 1 \text{ and } k \text{ is a nonnegative integer.} \end{cases}$$

Recall from the discussion following (2.14) that

$$\partial \Phi(y) = \begin{cases} \{0\}, & y = 0, \\ [k - 1, k], & y = k \text{ and } k \text{ is a positive integer,} \\ \{k\}, & k < y < k + 1 \text{ and } k \text{ is a nonnegative integer.} \end{cases}$$

We seek the largest number $e^* \in [0, \bar{X}]$ for which $\partial\Phi(e^*) \cap \partial\hat{g}(\bar{X} - e^*) \neq \emptyset$. This is the largest minimizer of $\hat{G}(e) = \Phi(e) + \hat{g}(\bar{X} - e)$ in $[0, \bar{X}]$. We define k^* to be the largest integer less than or equal to $\frac{\bar{X}}{3}$, so that

$$3k^* \leq \bar{X} < 3k^* + 3.$$

We divide the analysis into three cases.

Case A. $3k^* \leq \bar{X} < 3k^* + 1$.

We define $e^* = \bar{X} - k^*$, so that $2k^* \leq e^* < 2k^* + 1$ and $k^* = \bar{X} - e^*$. Then $2k^* \in \partial\Phi(e^*)$ and $\partial\hat{g}(\bar{X} - e^*) = [2(k^* - 1), 2k^*]$, so the intersection of $\partial\Phi(e^*)$ and $\partial\hat{g}(\bar{X} - e^*)$ is nonempty, as desired. On the other hand, if $e > e^*$, then $\partial\Phi(e) \subset [2k^*, \bar{X}]$ and $\partial\hat{g}(\bar{X} - e) \subset [0, 2(k^* - 1)]$, so the intersection of these two sets is empty.

In this case, y^* and x^* defined by (4.28) are both equal to k^* and hence $\hat{g}(y^*) = g(y^*)$. If $k^* = 0$, we are in Case I of the proof of Theorem 4.4. The optimal purchasing strategy is to do nothing until time 1, and then make a lump purchase of size \bar{X} . If $k^* = 1$, which is equal to $F(0+)$, we are in Case II of the proof of Theorem 4.4. We should make an initial purchase of size $x^* = 1$, purchase continuously over the time interval $(0, 1)$ at rate 1 so that that $E_t \equiv 1$ and $D_t \equiv 0$, and make a final purchase of size $\bar{X} - 2$. If $k^* \geq 2$, we are in Case III of the proof of Theorem 4.4. We should make an initial purchase of size k^* , purchase continuously over the time interval $(0, 1)$ at rate k^* so that $E_t \equiv k^*$ and $D_t \equiv k^* - 1$, and make a final purchase of size $\bar{X} - 2k^*$.

Case B. $3k^* + 1 \leq \bar{X} < 3k^* + 2$.

We define $e^* = 2k^* + 1$, so that $k^* \leq \bar{X} - e^* < k^* + 1$. Then $\partial\Phi(e^*) = [2k^*, 2k^* + 1]$ and $2k^* \in \partial\hat{g}(\bar{X} - e^*)$, so the intersection of $\partial\Phi(e^*)$ and $\partial\hat{g}(\bar{X} - e^*)$ is nonempty, as desired. On the other hand, if $e > e^*$, then $\partial\Phi(e) \subset [2k^* + 1, \bar{X}]$ and $\partial\hat{g}(\bar{X} - e) \subset [0, 2k^*]$, so the intersection of these two sets is empty.

In this case, y^* and x^* defined by (4.28) are both equal to $\bar{X} - e^*$. If $y^* = k^*$, then $\hat{g}(y^*) = g(y^*)$ and the optimal purchasing strategy falls into one of the three cases $k^* = 0$, $k^* = 1$ or $k^* \geq 2$ described in Case A. However, if $k^* < y^* < k^* + 1$, then $\hat{g}(y^*) < g(y^*)$ and we are in Case IV of the proof of Theorem 4.4 with $\alpha = k^*$ and $\beta = k^* + 1$ (see (4.19)–(4.23)). The optimal purchasing strategy is Type B. In particular, with $t_0 = \beta - y^* = k^* + 1 - x^*$, the optimal purchasing strategy makes an initial lump purchase $\alpha = k^*$, which executes the orders at prices $0, 1, \dots, k^* - 1$, then purchases continuously over the interval $(0, t_0)$ at rate k^* so that $E_t \equiv k^*$ and $D_t \equiv k^* - 1$, at time t_0 makes a lump purchase of size $\beta - \alpha = 1$, which consumes the order at price k^* , then purchases continuously over the interval $(t_0, 1)$ at rate $k^* + 1$ so that $E_t \equiv k^* + 1$ and $D_t \equiv k^*$, and finally executes a lump purchase of size $e^* - \beta = k^*$ (see (4.31)) at time 1. The total quantity purchased is

$$k^* + k^*t_0 + 1 + (k^* + 1)(1 - t_0) + k^* = \bar{X},$$

as required.

Case C. $3k^* + 2 \leq \bar{X} < 3k^* + 3$.

We define $e^* = \bar{X} - k^* - 1$, so that $2k^* + 1 \leq e^* < 2k^* + 2$ and $\bar{X} - e^* = k^* + 1$. Then $2k^* + 1 \in \partial\Phi(e^*)$ and $\hat{g}(\bar{X} - e^*) = [2k^*, 2k^* + 2]$, and the intersection of $\partial\Phi(e^*)$, so $\partial\hat{g}(\bar{X} - e^*)$ is nonempty, as desired. On the other hand, if $e > e^*$, then $\partial\Phi(e) \subset [2k^* + 1, \bar{X}]$ and $\partial\hat{g}(\bar{X} - e) \subset [0, 2k^*]$, so the intersection of these two sets

is empty. In this case, y^* and x^* are both equal to $k^* + 1$. The optimal purchasing strategy falls into either Case II (if $k^* = 0$) or Case III (if $k^* \geq 1$) of Theorem 4.4. \square

Appendix A. The process E . In this appendix we prove that there exists a unique adapted process E satisfying (2.1) pathwise, and we provide a list of its properties.

LEMMA A.1. *Let h be a nondecreasing, real-valued, locally Lipschitz function defined on $[0, \infty)$ such that $h(0) = 0$. Let X be an purchasing strategy. Then there exists a unique bounded adapted process E depending pathwise on X such that (2.1) is satisfied. Furthermore,*

- (i) E is right continuous with left limits;
- (ii) $\Delta E_t = \Delta X_t$ for all t ;
- (iii) E has finite variation on $[0, T]$;
- (iv) E takes values in $[0, \bar{X}]$.

PROOF: Because we do not know a priori that E is nonnegative, we extend h to all of \mathbb{R} by defining $h(x) = 0$ for $x < 0$. This extended h is nondecreasing and locally Lipschitz.

In Section 2 we introduced the filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$. The purchasing strategy X is right-continuous and adapted to this filtration, and hence is an optional process, i.e., $(t, \omega) \mapsto X_t(\omega)$ is measurable with respect to the optional σ -algebra, the σ -algebra generated by the right-continuous adapted processes. For any bounded optional process Y , $h(Y)$ and $\int_0^\cdot h(Y_s) ds$ are also bounded optional processes. Optional processes are adapted, and hence $\int_0^t h(Y_s) ds$ is \mathcal{F}_t -measurable for each $t \in [0, T]$.

We first prove uniqueness. If E and \widehat{E} are bounded processes satisfying (2.1), then there is a local Lipschitz constant K , chosen taking the bounds on E and \widehat{E} into account, such that

$$|E_t - \widehat{E}_t| = \left| \int_0^t (h(E_s) - h(\widehat{E}_s)) ds \right| \leq K \int_0^t |E_s - \widehat{E}_s| ds.$$

Gronwall's inequality implies $E = \widehat{E}$.

For the existence part of the proof, we assume for the moment that h is globally Lipschitz with Lipschitz constant K , and we construct E as a limit of a recursion. Let $E_t^0 \equiv X_t$. For $n = 1, 2, \dots$, define recursively

$$E_t^n = X_t - \int_0^t h(E_s^{n-1}) ds, \quad 0 \leq t \leq T.$$

Since X is bounded and optional, each E^n is bounded and optional. For $n = 1, 2, \dots$, let $Z_t^n = \sup_{0 \leq s \leq t} |E_s^n - E_s^{n-1}|$. A proof by induction shows that

$$Z_t^n \leq \frac{K^{n-1} t^{n-1}}{(n-1)!} \max \{ \bar{X}, Th(X_0) + X_0 \}.$$

Because this sequence of nonrandom bounds is summable, E^n converges uniformly in $t \in [0, T]$ and ω to a bounded optional process E that satisfies (2.1). In particular, E_t is \mathcal{F}_t -measurable for each t , and since X is nondecreasing and right-continuous with left limits and the integral in (2.1) is continuous, (i), (ii) and (iii) hold.

It remains to prove (iv). For $\epsilon > 0$, let $X_t^\epsilon = X_t + \epsilon t$ and define $t_0^\epsilon = \inf\{t \in [0, T] : E_t^\epsilon < 0\}$. Assume this set is not empty. Then the right-continuity of E^ϵ combined with the fact that E^ϵ has no negative jumps implies that $E_{t_0^\epsilon}^\epsilon = 0$. Let $t_n^\epsilon \downarrow t_0^\epsilon$ be such that $E_{t_n^\epsilon}^\epsilon < 0$ for all n . Then

$$\int_{t_0^\epsilon}^{t_n^\epsilon} h(E_s^\epsilon) ds = X_{t_n^\epsilon}^\epsilon - X_{t_0^\epsilon}^\epsilon - (E_{t_n^\epsilon}^\epsilon - E_{t_0^\epsilon}^\epsilon) > X_{t_n^\epsilon}^\epsilon - X_{t_0^\epsilon}^\epsilon \geq \epsilon(t_n^\epsilon - t_0^\epsilon).$$

But since

$$\int_{t_0^\epsilon}^{t_n^\epsilon} h(E_s^\epsilon) ds \leq K(\max_{t_0^\epsilon \leq s \leq t_n^\epsilon} E_s^\epsilon)(t_n^\epsilon - t_0^\epsilon),$$

there must exist $s_n^\epsilon \in (t_0^\epsilon, t_n^\epsilon)$ such that $E_{s_n^\epsilon}^\epsilon \geq \frac{\epsilon}{K}$. This contradicts the right continuity of E^ϵ at t_0^ϵ . Consequently, the set $\{t \in [0, T] : E_t^\epsilon < 0\}$ must be empty. We conclude that $E_t^\epsilon \geq 0$ for all $t \in [0, T]$.

Now notice that for $0 \leq t \leq T$,

$$E_t^\epsilon - E_t = \epsilon t - \int_0^t (h(E_s^\epsilon) - h(E_s)) ds,$$

and hence

$$|E_t^\epsilon - E_t| \leq \epsilon t + K \int_0^t |E_s^\epsilon - E_s| ds.$$

Gronwall's inequality implies that $E^\epsilon \rightarrow E$ as $\epsilon \downarrow 0$. Since $E_t^\epsilon \geq 0$, we must have $E_t \geq 0$ for all t . Equation (2.1) now implies that $E_t \leq X_t$, and therefore $E_t \leq \bar{X}$. The proof of (iv) is complete.

When h is locally but not globally Lipschitz, we let \tilde{h} be equal to h on $[0, \bar{X}]$, $\tilde{h}(x) = 0$ for $x < 0$, and $\tilde{h}(x) = h(\bar{X})$ for $x > \bar{X}$. We apply the previous arguments to \tilde{h} , and we observe that the resulting \tilde{E} satisfies the equation corresponding to h . \square

REMARK A.2. The pathwise construction of E in the proof of Lemma A.1 shows that if X is deterministic, then so is E .

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