

Optimal Execution Size in Algorithmic Trading

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Abstract

Execution of a large trade by traders always comes at a price of market impact which can both help and hurt the effective execution trade. A trading signal may help attract counterparties to reduce the time it takes to complete the trade and the trading cost. On the other hand, it may also attract parasitic/opportunistic traders who make the completion less likely or more costly. One possible solution which market design offers may include dynamic order submission strategies or trading off the exchange because such strategies limit the amount of information that is revealed about their trading intentions. But in the age of sophisticated automated trading the efficient strategy to avoid adversely moving the share price due to negative market impact, one can use algorithmic logic to slice-up a Parent orders into tiny pieces (Child orders) across brief time bucket over execution horizon to make it look like they are retail. Also, to prevent pattern recognition and manipulation by parasitic traders, only part of the order is displayed at the trading space. We in this paper model optimal size of expected execution Child order which takes account of minimum market impact with reference to display size of the order in the dark pools environment.

Keywords: Algorithmic Trading, Optimal Execution, Market Impact, Child Order

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1. Introduction

Most individual investors around the world and especially in US, access the financial markets through their pension plan or mutual fund. Large financial intermediaries serve as their trustees and manage huge pools of money. Once these fund managers decide to buy or sell a stock to reposition their portfolio, they need to be able to trade in size. But the changed market structure didn't allow these investors to execute large trade orders. The proliferation of retail and algorithmic trading and recent government regulations have driven the average execution size on US exchanges from more than 2,000 shares down to about 340 shares. Also execution of large orders will lead to much fluctuation in the prices of equities. To avoid adversely moving the share price, institutions use algorithmic logic which slice-up a Parent orders into tiny pieces (Child orders) across brief time bucket over execution horizon to make it look like they are retail.

Trading in general markets can be defined as result of a successful bilateral search, in which sellers look for suitable buyers and buyers look for suitable sellers. The facilitation of trade becomes easy and efficient when traders display their trading intentions to the market. Thus the trader has to get 'proactive', exposing his interest actively to the market to attract possible trading counterparts. However, for large traders, disclosing trading interests can be quite costly as well. Sometimes, "parasite" traders (Harris, 1997) might somehow anticipate a big trader's desperateness to execute a position. They take away liquidity on the opposite side in the hope the big trader – in his urgent need to execute - will trade with them on better prices anyway. These cause great losses to big traders and market impact. Thus Traders, in particular big need to control their order exposure and its related market impact. The one possible solution to attract liquidity and encourage trading which markets provide to traders is the strategic freedom to hide their trading intentions.

2. Objective

The objective and contribution to academic space through this paper is to find optimal execution size and display. Here, without loss of generality we consider a trader who wants to sell a parent order with a definite display size at some price. With this initial background we lay down limit order book dynamics which is discussed in detail in subsequent section. We then specify the order precedence rules (Harris, 1997) to explain precise order flow in terms of sell order and a

market buy order. This helps us to model the placement order, which in turn give the display size.

3. Optimal Order Model

3.1 Limit Order Book Dynamics

Let us consider a trader who wants to sell an order of size, say N and display size Δ at the price level p_j . The corresponding sell-side of the limit order book with the price levels $(p_i)_{i \in N}$ is given by a sequence of positive real numbers. At time t_0 the limit order book state is described by the total sell-side hidden H_i and displayed depth D_i at the respective price level p_i . We denote the sum of both quantities by $C_i (= H_i + D_i)$ and call it the total depth at the price level p_i . Occasionally we will refer to the depths at the price levels as visible respectively “hidden liquidity”. We call the quantity $\hat{C}_i = \sum_{k < i} C_k$ the cumulative depth up to p_i . Now with the background of limit order book dynamics, we will first specify the precedence rules for orders in the limit order book, then we continue to explain the precise order flow dynamics in terms of sell orders arriving at time t_a and a market buy orders arriving at time $t_b = T$.

3.1.1 The Order Precedence Rules

Order-driven market uses trading rules to arrange their trades. In the limit order book markets trades among different orders are arranged according to a *Precedence Rule-based order-matching system* (Harris, 1997). Orders with highest precedence get executed first and stay in the order book until they get fully executed or canceled. In order to determine the precedence among orders, first orders get ranked according to their primary precedence rule. If two orders have the same precedence, then the secondary precedence rule is adopted and the priority ordering procedure continues in the same manner. In most markets the primary precedence rule is price priority that is buy orders that bid the highest prices and sell orders that offer the lowest prices rank highest on their respective sides. Notice that market orders always rank highest because they can trade at all prices. Display Precedence-rule is used as the secondary precedence rule for taking account for hidden or partially hidden order, which basically means that displayed orders have priority over hidden orders. If one order is partly undisclosed and partly displayed, the

market treats the two parts separately. In most cases, the third precedence rule obeys Time Priority that is orders with same primary and secondary precedence amounting to the same price level and same display status get precedence according to their submission time: Earlier submitted (orders) have precedence over the rest orders (First-in-First-Out). As per present market structure we will use precedence rule in the following order: price, display and time priority.

3.1.2 Order Submission

Let us consider that at initial time t_0 , the trader (seller) submits an order (Child) of size N , where its displayed share size by Δ at the price level p_j . As the trader cannot display more than the total order size, so $0 \leq \Delta \leq N$.

Order Submission

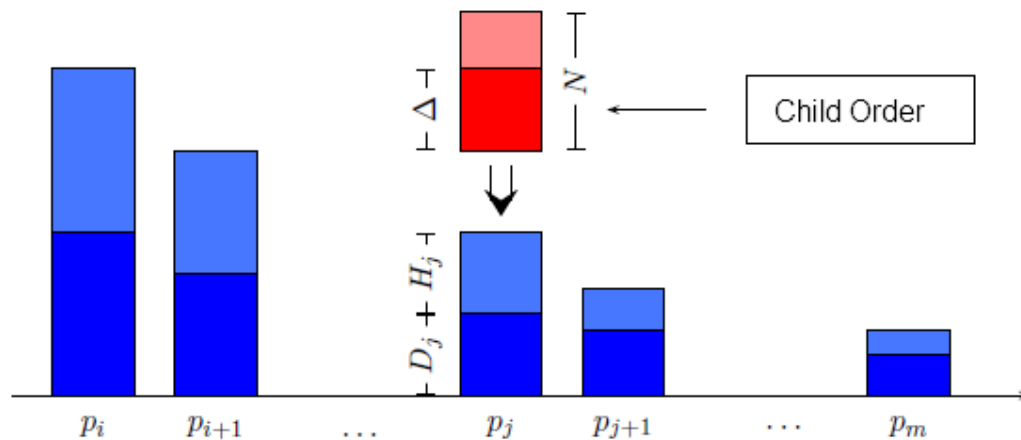


Figure 3.1: At time t_0 , the trader sells order at the price level p_j . The Child order size of the order is N share and its displayed (red colored) size is Δ shares. The remaining (light colored) $N - \Delta$ belongs to Child order hidden part. Liquidity before submission in limit order book: Hidden H_i (light blue colored) and Visible D_i (blue colored) depth at price level p_i . Displayed share (D_i, Δ) have priority over hidden one ($H_i, N - \Delta$).

After Order Submission

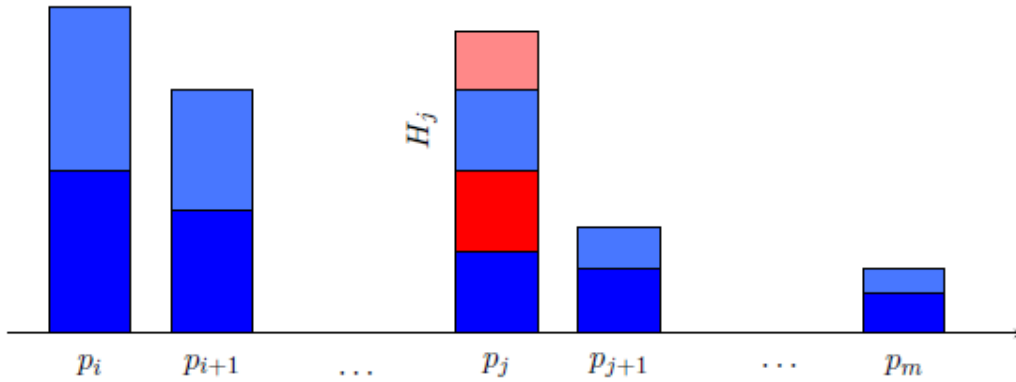


Figure 3.2: The Child order splits into two parts in presence hidden liquidity. At time t_0 , the trader sells order at the price level p_j . The hidden depth H_i (light blue colored) slips in between the visible and hidden part of Child order.

The hidden and visible parts of Child order arrange at the price level according to the precedence rule. More precisely, its visible part arranges behind the visible depth D_j (since it arrived later than the shares that already stand in the book, including D_i), then the hidden depth H_j and finally the hidden part of the Child order. This one arranges behind the hidden depth H_j because it has lower time priority, just as the displayed part of the Child order had against the visible depth D_i . Thus we can write the priority arrangement at the price level p_j at time t_0 as the sequence $(D_j, \Delta, H_j, N - \Delta)$.

3.1.3 Competing Orders

As our trader is certainly not the only one who wants to sell shares, he/she has to compete with other traders. We account for arrival of other sells order at time $t_a (> t_0)$ at each price p_i with total size of y_i share. Considering true market condition we allow arrival of hidden liquidity and denote q_i the ratio of displayed liquidity among the arriving sell orders at the price level p_i . Thus the respective visible share volume y_i^d arriving at price level p_i at time t_a can be written $y_i^d = q_i \cdot y_i$ (the superscript d denoting the “display” status) and accordingly the hidden share volume $y_i^h = y_i \cdot (1 - q_i)$.

Competing Sell Orders Arrival

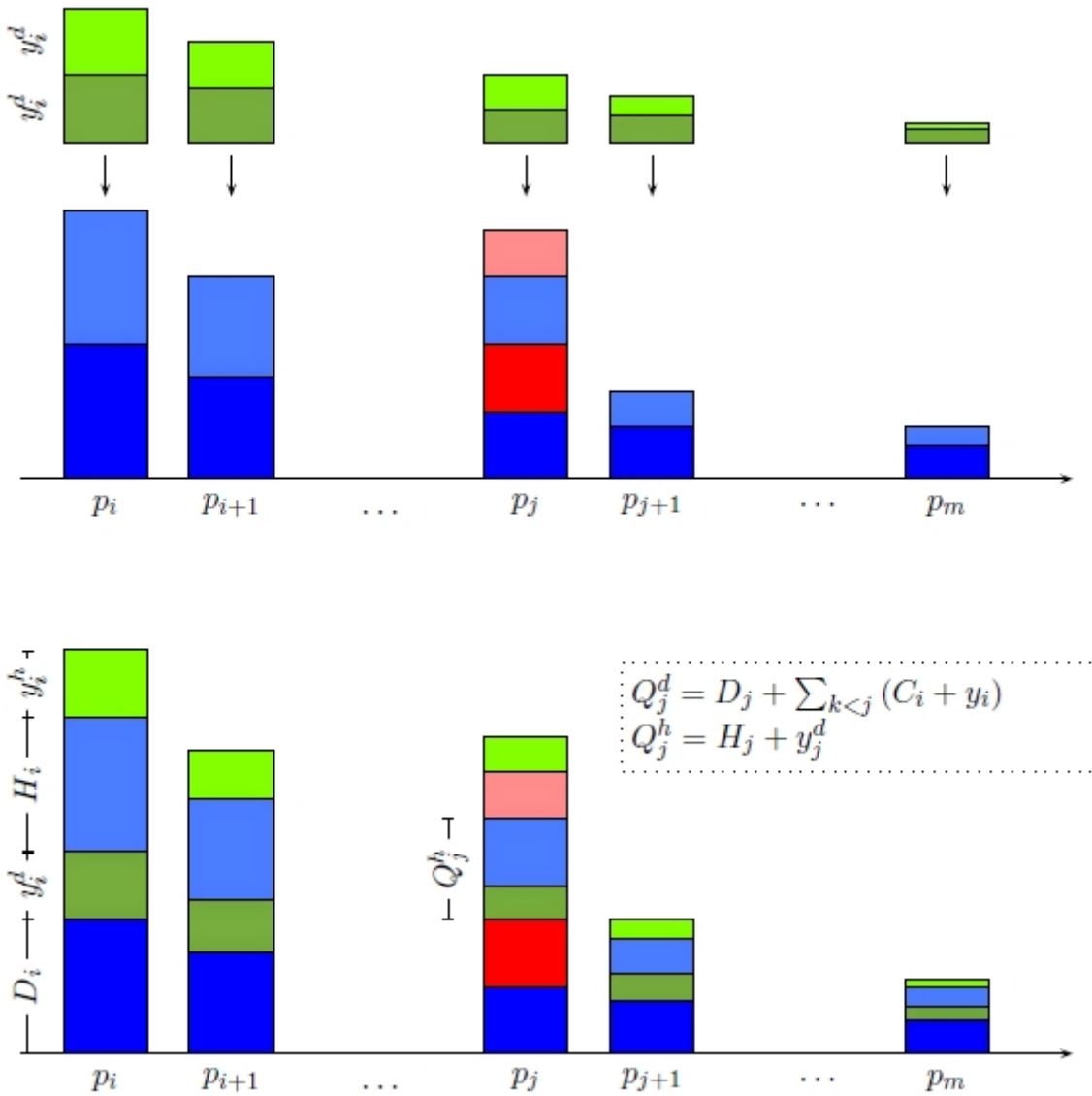


Figure 3.3: Competing sell orders arriving at time t_a , arrange according to the priority rules: Incoming (green-colored) visible orders with size y_i^d slip in front of the hidden depth at the respective price level. Incoming hidden liquidity (light green colored) has the worst priority and keeps to be the last in the priority queue at the respective price level. The “alien” liquidity between the Child order visible and hidden parts amounts to Q_j^d shares in total. Whereas the total volume of shares that have higher priority than the (visible) Child order equals Q_j^d shares

The precedence arrangement of the order arriving at t_a is done in total analogy with the arrival case. We already spotted that at time t_a the Child order splits in two parts in the sense that there is "alien" liquidity, that has higher priority than the visible Child order part, but less than the hidden Child order part. The total volume of this liquidity is denoted by $Q_j^h = H_i + y_i^d$. Thus the size of liquidity volume Q_j^d at time t_a consists of three parts, the initial liquidity at initial time $\hat{C}_j = \sum_{k < j} C_k$, the total size of orders that additional arrives at time t_a , $\hat{y}_j = \sum_{k < j} y_k$ and finally the visible depth D_j , hence $Q_j^d = \hat{C}_j + \hat{y}_j + D_j$.

Quantities	Descriptions
q_i	Ratio of displayed sell order volume arriving at the price level p_i at time t_a .
$y_i^h = y_i \cdot (1 - q_i)$	Total hidden order volume arriving at price level p_i at time t_a .
$y_i^d = q_i \cdot y_i$	Total displayed sell order volume arriving at price level p_i at time t_a .
$y_i = y_i^h + y_i^d$	Total sell order volume arriving at price level p_i at time t_a .
C_i	Total initial depth at time t_0 at price level p_j .
$\hat{C}_j = \sum_{k < j} C_k$	Cumulative depth "in front of" at time t_0 at price level p_j .
$\hat{y}_j = \sum_{k < j} y_k$	Arriving cumulative sell order volume "in front of" at time t_0 at price level p_j .
$Q_j^d = \hat{C}_j + \hat{y}_j + D_j$	Total size of the queue (share volume) at time t_a that has higher priority than the Child order.
$Q_j^h = H_i + y_i^d$	Total size of the queue at time t_a that has lower priority than the visible part of Child order, but higher priority than its hidden part.

Table 3.1: Main quantities in the limit order book.

3.1.4 Market Order Arrival and Execution

In the earlier section we took into account of arrival of competing (sell) orders by the liquidity provider-side at time t_a . As, no liquidity stays for longer time, in the next time step $t_b > t_a$ we now incorporate the arrival of the liquidity consumer.

Arriving Market Order

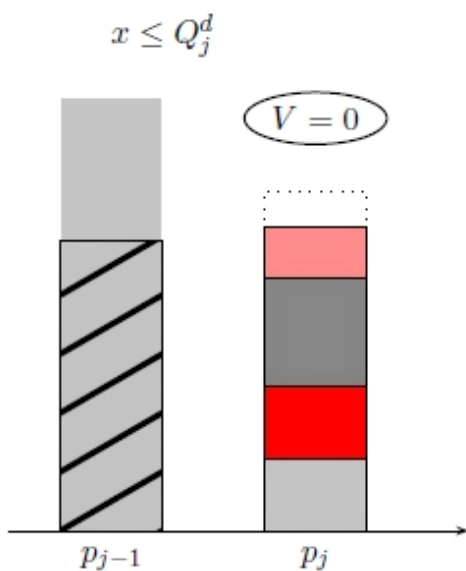


Figure 3.4: The Market Buy order size x is too small to execute shares of the Child order. Hence $V = 0$.

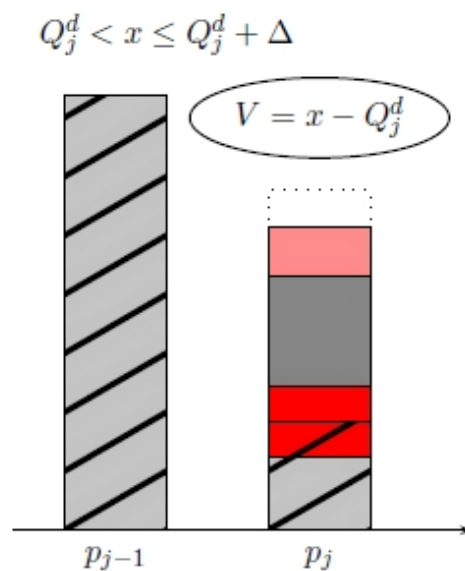


Figure 3.5: The Market Buy order size x is large enough to execute shares of the visible Child order. Hence $V = x - Q_j^d$.

Let us assume that at time t_b buy order consumption with the total size of x arrives. In this case limit order book situation is as follows: In the front of the visible Child order, Q_j^d shares have higher priority, hence they will be executed first. Thus if the market buy order size is sufficiently small, i.e. $x < Q_j^d$ no Child shares will get executed, i.e. $V = 0$ (see figure 3.4), V denoting the Child orders execution size at time t_b . If the market buys order obeys $Q_j^d < x < Q_j^d + \Delta$, it will execute $V = x - Q_j^d$ Child shares (see Figure 3.5), since out of x executed shares Q_j^d don't belong to the Child order as illustrated in Figure 3.5.

$$Q_j^d + \Delta < x \leq Q_j^d + \Delta + Q_j^h$$

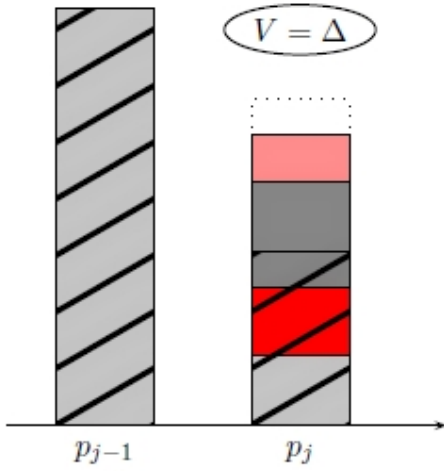


Figure 3.6: The Market Buy order size x is too small to execute shares of the Child order. Hence $V = \Delta$.

$$\Delta + Q_j^h + Q_j^d < x \leq Q_j^d + Q_j^h + N$$

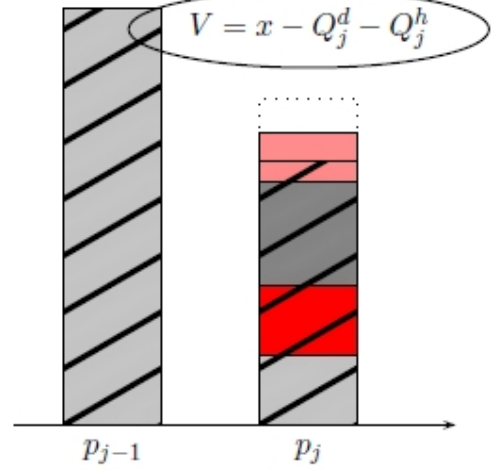


Figure 3.7: The Market Buy order size x is large enough to execute shares of the visible Child order. Hence $V = x - Q_j^d - Q_j^h$.

In case of $Q_j^d + \Delta < x < Q_j^d + Q_j^h + \Delta$ (see Figure 3.6), the buy order is able to execute all order up to visible Child order part, but not the following hidden order of size Q_j^h , since it has even lower priority than Q_j^h . For larger market buy order size, $Q_j^d + Q_j^h + \Delta < x < Q_j^d + Q_j^h + N$ (see Figure 3.7) the market buy order is able to execute also shares of the hidden part of the Child order; however it is not able to execute all hidden Iceberg shares, according to its size.

Child order gets executed, i.e. $N = V$, when buy order size obeys $x > Q_j^d + Q_j^h + N$. Thus, for every choice of the market buy order size $x \in [0, \infty)$ we can assign a value, the Child execution volume V , which we can formulate as follows

$$\begin{aligned}
& V(x) \\
& = \begin{cases} 0, & x \leq Q_j^d \\ x - Q_j^d, & Q_j^d < x < Q_j^d + Q_j^h + \Delta \\ \Delta, & Q_j^d + \Delta < x < Q_j^d + Q_j^h + \Delta \\ x - Q_j^d - Q_j^h, & Q_j^d + Q_j^h + \Delta < x < Q_j^d + Q_j^h + N \\ N, & x \geq Q_j^d + Q_j^h + N \end{cases} \quad (3.1)
\end{aligned}$$

V is the number of executed shares of the considered Child order at time $t_b = T$. We also notice that V depends on the market order size, which is a random variable ($x = x(\theta)$ for $\theta \in \Omega$) and the two order volumes Q_j^d and Q_j^h , which themselves depend on the random order arrival sizes $\hat{y}_j = \hat{y}_j(\theta)$ and $y_j = y_j(\theta)$, hence V is itself a random variable, i.e. $V = V(\theta)$. So, equation (2.1) is definition of the Child execution size V at time t_b , given that it was submitted at time t_0 . For sake of convenience and calculating the optimal execution order, we define subsets of the probability space Ω that are related to the cases that are distinguished in the definition of V in (3.1).

$$\begin{aligned}
S_0 &:= \{\theta \in \Omega | x(\theta) \leq Q_j^d(\theta)\} \\
S_1 &:= \{\theta \in \Omega | Q_j^d(\theta) < x < Q_j^d(\theta) + \Delta\} \\
S_2 &:= \{\theta \in \Omega | Q_j^d(\theta) + \Delta < x < Q_j^d(\theta) + Q_j^h(\theta) + \Delta\} \\
S_3 &:= \{\theta \in \Omega | Q_j^d(\theta) + Q_j^h(\theta) + \Delta < x < Q_j^d(\theta) + Q_j^h(\theta) + N\} \\
S_4 &:= \{\theta \in \Omega | x(\theta) \geq Q_j^d(\theta) + Q_j^h(\theta) + N\}
\end{aligned} \quad (3.2)$$

Also, we can see that

$$\begin{aligned}
\bigcup_{i=0}^4 S_i &= \left\{ \theta \in \Omega \mid \begin{aligned} & (x \leq Q_j^d) \vee (Q_j^d < x < Q_j^d + \Delta) \vee (Q_j^d + \Delta < x < Q_j^d + Q_j^h + \Delta) \\ & \vee (Q_j^d + Q_j^h + \Delta < x < Q_j^d + Q_j^h + N) \vee (x \geq Q_j^d + Q_j^h + N) \end{aligned} \right\} \\
&= \{\theta \in \Omega | x(\theta) < \infty\} \\
&= \Omega
\end{aligned} \quad (3.3)$$

and same time interval in the interval $(-\infty, Q_j^d], (Q_j^d, Q_j^d + \Delta), (Q_j^d + \Delta, Q_j^d + Q_j^h + \Delta), (Q_j^d + Q_j^h + \Delta, Q_j^d + Q_j^h + N), [Q_j^d + Q_j^h + N, \infty)$ are disjunct. Hence the market buy order size x can never be in two of these intervals at the same time. Consequently the sets S_i ($i \in \{0,1,2,3,4\}$) are disjunct as well. Hence the sets S_i establish a partition of the sample space Ω ,

$$\begin{aligned} S_i \cap S_j &= \emptyset & \text{for } i \neq j \\ S_1 \cup S_2 \cup S_3 \cup S_4 &= \Omega \end{aligned} \tag{3.4}$$

We can also infer from subset S_i ($i \in \{0,1,2,3,4\}$) and definition of V that,

$$\begin{aligned} S_0 &\subset \{\theta \in \Omega | V(\theta) = 0\} \\ S_1 &\subset \{\theta \in \Omega | 0 < V(\theta) < \Delta\} \\ S_2 &\subset \{\theta \in \Omega | V(\theta) = \Delta\} \\ S_3 &\subset \{\theta \in \Omega | \Delta < V(\theta) < N\} \\ S_4 &\subset \{\theta \in \Omega | V(\theta) = N\} \end{aligned} \tag{3.5}$$

It is now easy to draw analogous conclusion for the other sets in (3.5) as well. Therefore, all in all we have

$$\begin{aligned} S_0 &= \{\theta \in \Omega | V(\theta) = 0\} \\ S_1 &= \{\theta \in \Omega | 0 < V(\theta) < \Delta\} \\ S_2 &= \{\theta \in \Omega | V(\theta) = \Delta\} \\ S_3 &= \{\theta \in \Omega | \Delta < V(\theta) < N\} \\ S_4 &= \{\theta \in \Omega | V(\theta) = N\} \end{aligned} \tag{3.6}$$

3.1.5 Order Size

We assume that the order sizes x , \widehat{y}_j and y_j are random variables on a suitable probability space (Ω, \mathcal{H}, P) , where x denotes the market buy order size arriving at time t_b , y_i denotes the total sell share arriving at the price level p_i at time t_a and $\widehat{y}_j = \sum_{k < j} y_k$. The assumptions which

we will use to get optimal execution size, regarding the order sizes x , \hat{y}_j and y_j with respect to the Child order submission price level y_j are as follow: First, we assume that the order sizes x , \hat{y}_j and y_j are statistically independent. Second assumes order sizes to be exponentially distributed with corresponding densities $f_x(t)$, $f_{\hat{y}_j}(t)$, $f_{y_j}(t)$:

$$\begin{aligned}
 f_x(t) &= \frac{e^{-\frac{t}{\alpha}}}{\alpha} \\
 f_{\hat{y}_j}(t) &= \frac{e^{-\frac{t}{\hat{\beta}_j}}}{\hat{\beta}_j} \\
 f_{y_j}(t) &= \frac{e^{-\frac{t}{\beta_j}}}{\beta_j}
 \end{aligned} \tag{3.7}$$

Where, α , $\hat{\beta}_j$, β_j are order flow parameters.

The exponential assumption seems to be a reasonable and defensible first approach to order size distributions. Due to the presence of large traders for example, or herding behavior the relative frequency of large order sizes is much greater, than the exponential distribution can account for. Indeed it has been found by several studies, that the market order size for instance obeys a power law ((Gopikrishnan, 2003), (Maslow, 2001)).

3.1.6 Market Impact Parameters

Naturally, displaying trading intentions in the order book will in general have an impact on the market, since the limit order book is public and accessible to market participants. For example, consider the arrival of Child order with a big display size Δ . It might force other liquidity suppliers to be more aggressive in the face of a big displayed Child order, since posting orders behind (less aggressive) the big Child order amounts to correspondingly higher execution risk, which in turn causes liquidity supply to decrease.

In our model, a straightforward way of incorporating a mechanism of Market Impact is to make the parameters that control the liquidity supply and the liquidity consumption (the order flow respectively) dependent on the Child orders display size Δ . We consider the case where display of trading intention is penalized by the liquidity supplier side; more precisely we consider the (sell-) order flow in front of the Child order submission price level to depend on the display size Δ , i.e. $\hat{\beta}_j = \hat{\beta}_j(\Delta)$, while keeping other flow order parameters (α, β_j) constant. Therefore, functional dependency for the Market Impact model can be written as:

$$\hat{\beta}_j(\Delta) = \hat{\beta}_{j0} + \gamma\Delta^k \quad k \geq 1, \hat{\beta}_{j0}, \gamma \geq 0 \quad (3.8)$$

Where, γ is *Market Sensitivity or the Market Impact Parameter*

We can see that functional dependency indeed penalize display in the above mentioned case. So, if the issuer of the Child order intends to display more, he/she incurs more liquidity/shares arriving at better price levels in front of him, lowering his own Child order execution performance in turn.

The word Market Impact is used in the sense that conveying information to other market participants influences the markets behavior and in turn affects one's own trading outcome. This form of market impact is specifically related to traders on the liquidity supply side, since they incur market impact by mere displaying trading interests. Contemporary literature in Mathematical Finance that covers aspects of Market Impact considers mainly the liquidity consumer side ((Almgren, 2003), (Obizhaeva, 2006)).

3.2 The Child Order Expected Execution Size

A trader who is willing to execute (sell, buy) shares in a market will want to know how much he is able to execute within a prespecified trading horizon. Therefore the expected execution size (until the end of the trading horizon) is an important and first measure of the traders execution risk.

In the context of our model set up we will thus consider the expected execution volume $E(V)$ with respect to a Child order of total size N , display size Δ and within the given trading horizon $[t_0, T]$. To find expected execution volume $E(V)$, we consider the sets S_i ($i \in \{0,1,2,3,4\}$) where disjunct partition of the sample space Ω (see 2.4), hence we have

$$\begin{aligned}
E(V) &= \int_{\Omega} V(\theta) dP(\theta) \\
&= \sum_{i=0}^4 \int_{S_i} V(\theta) dP(\theta) \\
&= \sum_{i=0}^4 E_i(V)
\end{aligned} \tag{3.9}$$

where we used the fact that $V(\theta) = 0$ holds for all $\theta \in S_0$ (see 2.1). To simplify matters, we thus first go over to compute the $E(V)$'s in order to obtain the full execution volume $E(V)$.

Lemma 1.

$$E_1(V) = \frac{\alpha e^{-\frac{\hat{c}_j + D_j + \Delta}{\alpha}}}{\alpha + \hat{\beta}_j} \left(e^{\frac{\Delta}{\alpha}} \alpha - (\alpha + \Delta) \right) \tag{3.10}$$

Proof. We notice that, $\theta \in S_1$ the execution volume V obeys $V(\theta) = x - Q_j^d(\theta)$ (see 2.1) and $S_1 = \{\theta \in \Omega \mid Q_j^d(\theta) < x < Q_j^d(\theta) + \Delta\}$. Now,

$$\begin{aligned}
E_1(V) &= \int_{S_1} V(\theta) dP(\theta) \\
&= \int_{S_1} (x(\theta) - Q_j^d(\theta)) dP(\theta) \quad [\text{definition of execution volume } V(\text{see 3.1})]
\end{aligned}$$

$$\begin{aligned}
&= \int_{S_1} (x(\theta) - \widehat{C}_j - D_j - \widehat{y}_j(\theta)) dP(\theta) \quad [\text{definition of order volume (table 3.1)}] \\
&= \int_{\widehat{C}_j + D_j + \widehat{y}_j(\theta) < x(\theta) < \widehat{C}_j + D_j + \widehat{y}_j(\theta) + \Delta} (x - \widehat{C}_j - D_j - \widehat{y}_j(\theta)) dP(\theta) \quad [\text{subset (see 3.3)}] \\
&= \iint_{\widehat{C}_j + D_j + t < s < \widehat{C}_j + D_j + t + \Delta} (s - \widehat{C}_j - D_j - t) f_x(s) f_{\widehat{y}_j}(t) ds dt \quad [\text{statistical independence}] \\
&= \int_0^{\infty} \int_{\widehat{C}_j + D_j + t}^{\infty} (s - \widehat{C}_j - D_j - t) f_x(s) f_{\widehat{y}_j}(t) ds dt \\
&= \frac{1}{\alpha \widehat{\beta}_j} \int_0^{\infty} \int_{\widehat{C}_j + D_j + t}^{\infty} (s - \widehat{C}_j - D_j - t) e^{-\frac{s}{\alpha}} e^{-\frac{t}{\widehat{\beta}_j}} ds dt \quad [\text{distribution density (see 3.7)}] \\
&= \frac{1}{\alpha \widehat{\beta}_j} \int_0^{\infty} e^{-\frac{t}{\widehat{\beta}_j}} \int_{\widehat{C}_j + D_j + t}^{\infty} (s - \widehat{C}_j - D_j - t) e^{-\frac{s}{\alpha}} ds dt \\
&= \frac{1}{\alpha \widehat{\beta}_j} \int_0^{\infty} e^{-\frac{t}{\widehat{\beta}_j}} \left(\frac{e^{-\frac{1}{\alpha}(\widehat{C}_j + D_j + t + \Delta)} (-1 + e^{\frac{\Delta}{\alpha}} - \frac{\Delta}{\alpha})}{\frac{1}{\alpha^2}} \right) dt \tag{3.11} \\
&= \frac{\alpha}{\widehat{\beta}_j} e^{-\frac{1}{\alpha}(\widehat{C}_j + D_j + \Delta)} (-1 + e^{\frac{\Delta}{\alpha}} - \frac{\Delta}{\alpha}) \int_0^{\infty} e^{-\left(\frac{1}{\widehat{\beta}_j} + \frac{1}{\alpha}\right)t} dt \\
&= \frac{\alpha^2 e^{-\frac{1}{\alpha}(\widehat{C}_j + D_j + \Delta)}}{\widehat{\beta}_j (\alpha + \widehat{\beta}_j)} (e^{\frac{\Delta}{\alpha}} - (1 + \frac{\Delta}{\alpha})) \\
&= \frac{\alpha e^{-\frac{1}{\alpha}(\widehat{C}_j + D_j + \Delta)}}{\widehat{\beta}_j (\alpha + \widehat{\beta}_j)} (\alpha e^{\frac{\Delta}{\alpha}} - (\alpha + \Delta))
\end{aligned}$$

Lemma 2.

$$E_2(V) = \frac{\Delta \alpha e^{-\frac{\widehat{C}_j + D_j + \Delta}{\alpha}}}{\alpha + \widehat{\beta}_j} \left(1 - \frac{e^{-\frac{H_j}{\alpha} \alpha}}{q_j \beta_j} \right) \tag{3.12}$$

Proof. We notice that, $\theta \in S_2$ the execution volume V obeys $V(\theta) = \Delta$ (see 3.1) and $S_2 = \{\theta \in \Omega | Q_j^d(\theta) + \Delta < x < Q_j^d(\theta) + Q_j^h(\theta) + \Delta\}$. Now,

$$\begin{aligned}
E_2(V) &= \int_{S_2} V(\theta) dP(\theta) \\
&= \int_{S_2} \Delta dP(\theta) \\
&= \int_{Q_j^d(\theta)+\Delta < x < Q_j^d(\theta)+Q_j^h(\theta)+\Delta} \Delta dP(\theta) \\
&= \Delta \int_{\widehat{C}_j+D_j+\widehat{y}_j(\theta)+\Delta < x < \widehat{C}_j+D_j+H_j+\widehat{y}_j(\theta)+q_j \cdot y_j(\theta)+\Delta} dP(\theta) \\
&= \Delta \int_0^\infty f_{\widehat{y}_j}(t) \int_0^\infty f_{y_j}(u) \int_{\widehat{C}_j+D_j+\Delta+t}^{\widehat{C}_j+D_j+t+\Delta+q_j \cdot u+H_j+y_j} f_x(s) ds dt \\
&= \Delta \int_0^\infty f_{\widehat{y}_j}(t) \int_0^\infty f_{y_j}(u) \left(e^{-\frac{(\widehat{C}_j+D_j+\Delta+t)}{\alpha}} - e^{-\frac{(\widehat{C}_j+D_j+t+\Delta+q_j \cdot u+H_j+y_j)}{\alpha}} \right) dt du \quad (3.13) \\
&= \frac{\Delta}{\widehat{\beta}_j} \int_0^\infty f_{y_j}(u) \int_0^\infty e^{-\frac{t}{\widehat{\beta}_j}} \left(e^{-\frac{(\widehat{C}_j+D_j+\Delta+t)}{\alpha}} - e^{-\frac{(\widehat{C}_j+D_j+t+\Delta+q_j \cdot u+H_j+y_j)}{\alpha}} \right) dt du \\
&= \frac{\Delta e^{-\frac{\widehat{C}_j+D_j+\Delta}{\alpha}}}{\widehat{\beta}_j \beta_j} \int_0^\infty e^{-\frac{u}{\widehat{\beta}_j}} \int_0^\infty \left(e^{-\left(\frac{1}{\widehat{\beta}_j} + \frac{1}{\alpha}\right)t} - e^{-\left(\frac{1}{\widehat{\beta}_j} + \frac{1}{\alpha}\right)t - \frac{H_j+q_j \cdot u}{\alpha}} \right) dt du \\
&= \frac{\alpha \Delta e^{-\frac{\widehat{C}_j+D_j+\Delta}{\alpha}}}{\beta_j (\alpha + \widehat{\beta}_j)} \int_0^\infty e^{\frac{u}{\widehat{\beta}_j}} - e^{\frac{u}{\widehat{\beta}_j} - \frac{H_j+q_j \cdot u}{\alpha}} du \\
&= \frac{\Delta \alpha e^{-\frac{\widehat{C}_j+D_j+\Delta}{\alpha}}}{\alpha + \widehat{\beta}_j} \left(1 - \frac{e^{-\frac{H_j}{\alpha \alpha}}}{q_j \beta_j} \right)
\end{aligned}$$

where we used analogous argumentation for first five steps as in the proof to the previous lemmas (see lemma 1 for example).

Lemma 3.

$$E_3(V) = \frac{\alpha^3 e^{-\frac{\hat{C}_j + D_j + H_j + N + \Delta}{\alpha}}}{(\alpha + \hat{\beta}_j)(\alpha + q_j \cdot \beta_j)} \left[e^{\frac{N}{\alpha}} \left(1 + \frac{\Delta}{\alpha} \right) - e^{\frac{\Delta}{\alpha}} \left(1 + \frac{N}{\alpha} \right) \right] \quad (3.14)$$

Proof. To ease notational burden we introduce the abbreviation $G := \hat{C}_j + D_j + H_j$. Also, we can see that $\theta \in S_3$ the execution volume V obeys $V(\theta) = x(\theta) - Q_j^d(\theta) - Q_j^h(\theta)$ (see 3.1) and that $S_3 = \{\theta \in \Omega \mid Q_j^d(\theta) + Q_j^h(\theta) + \Delta < x < Q_j^d(\theta) + Q_j^h(\theta) + N\}$ holds. We have thus

$$\begin{aligned} E_3(V) &= \int_{S_3} V(\theta) dP(\theta) \\ &= \int_{S_3} (x(\theta) - Q_j^d(\theta) - Q_j^h(\theta)) dP(\theta) \\ &= \int_{Q_j^d(\theta) + Q_j^h(\theta) + \Delta < x < Q_j^d(\theta) + Q_j^h(\theta) + N} (x(\theta) - Q_j^d(\theta) - Q_j^h(\theta)) dP(\theta) \\ &= \int_{G + \hat{y}_j(\theta) + q_j \cdot y_j(\theta) + \Delta < x(\theta) < G + \hat{y}_j(\theta) + q_j \cdot y_j(\theta) + N} (x(\theta) - Q_j^d(\theta) - Q_j^h(\theta)) dP(\theta) \\ &= \iiint_{G + q_j \cdot u + t + \Delta < s < G + t + q_j \cdot u + N} (s - G - t - q_j \cdot u) f_x(s) f_{\hat{y}_j}(t) f_{y_j}(u) ds dt du \\ &= \int_0^\infty \int_0^\infty f_{y_j}(u) f_{\hat{y}_j}(t) \int_{G + q_j \cdot u + t + \Delta}^{G + t + q_j \cdot u + N} (s - G - t - q_j \cdot u) f_x(s) ds dt du \\ &= \frac{1}{\alpha} \int_0^\infty f_{\hat{y}_j}(t) \int_0^\infty f_{y_j}(u) e^{-\frac{G + q_j \cdot u + t + \Delta + N}{\alpha}} \left(e^{\frac{N}{\alpha}} \left(1 + \frac{\Delta}{\alpha} \right) - e^{\frac{\Delta}{\alpha}} \left(1 + \frac{N}{\alpha} \right) \right) dt du \quad (3.15) \\ &= \frac{\alpha e^{-\frac{G + \Delta + N}{\alpha}}}{\hat{\beta}_j \beta_j} \left(e^{\frac{N}{\alpha}} \left(1 + \frac{\Delta}{\alpha} \right) - e^{\frac{\Delta}{\alpha}} \left(1 + \frac{N}{\alpha} \right) \right) \int_0^\infty e^{-\left(\frac{1}{\beta_j} + \frac{1}{\alpha}\right)t} \int_0^\infty e^{-\left(\frac{1}{\beta_j} + \frac{q_j}{\alpha}\right)u} du dt \\ &= \frac{\alpha^3 e^{-\frac{G + \Delta + N}{\alpha}}}{(\alpha + \hat{\beta}_j)(\alpha + q_j \cdot \beta_j)} \left(e^{\frac{N}{\alpha}} \left(1 + \frac{\Delta}{\alpha} \right) - e^{\frac{\Delta}{\alpha}} \left(1 + \frac{N}{\alpha} \right) \right) \end{aligned}$$

$$= \frac{\alpha^3 e^{-\frac{\hat{c}_j + D_j + H_j + N + \Delta}{\alpha}}}{(\alpha + \hat{\beta}_j)(\alpha + q_j \cdot \beta_j)} \left(e^{\frac{N}{\alpha}} \left(1 + \frac{\Delta}{\alpha} \right) - e^{\frac{\Delta}{\alpha}} \left(1 + \frac{N}{\alpha} \right) \right)$$

Lemma 4.

$$E_4(V) = \frac{N \alpha^2 e^{-\frac{\hat{c}_j + D_j + H_j + N}{\alpha}}}{(\hat{\beta}_j + \alpha)(\alpha + q_j \cdot \beta_j)} \quad (3.16)$$

Proof. Like other lemmas we notice that $V(\theta) = N$ holds for $\theta \in S_4$, while the subset S_4 follows $S_4 = \{\theta \in \Omega | x(\theta) \geq Q_j^d(\theta) + Q_j^h(\theta) + N\}$. In the same fashion as in the previous lemmas we continue to write

$$\begin{aligned} E_4(V) &= \int_{S_4} V(\theta) dP(\theta) \\ &= N \int_{S_4} dP(\theta) \\ &= N \int_{x(\theta) \geq Q_j^d(\theta) + Q_j^h(\theta) + N} dP(\theta) \\ &= N \int_{x(\theta) \geq \hat{c}_j + D_j + H_j + \widehat{y}_j(\theta) + q_j \cdot y_j(\theta) + N} dP(\theta) \\ &= N \int_{x(\theta) \geq G + \widehat{y}_j(\theta) + q_j \cdot y_j(\theta) + N} dP(\theta) \\ &= N \iiint_{G + q_j \cdot u + t + N \leq s} f_x(s) f_{\widehat{y}_j}(t) f_{y_j}(u) ds dt du \\ &= N \int_0^\infty f_{\widehat{y}_j}(t) \int_0^\infty f_{y_j}(u) \int_{G + q_j \cdot u + t + N}^\infty f_x(s) ds dt du \\ &= \frac{N}{\alpha} \int_0^\infty f_{\widehat{y}_j}(t) \int_0^\infty f_{y_j}(u) \int_{G + q_j \cdot u + t + N}^\infty e^{-\frac{s}{\alpha}} ds dt du \end{aligned} \quad (3.17)$$

$$\begin{aligned}
&= \frac{N}{\alpha} \int_0^{\infty} f_{y_j}(t) \int_0^{\infty} f_{y_j}(u) e^{-\frac{G+q_j \cdot u+t+N}{\alpha}} dt du \\
&= \frac{N e^{-\frac{G+N}{\alpha}}}{\hat{\beta}_j \beta_j} \int_0^{\infty} \left(e^{-\left(\frac{1}{\hat{\beta}_j} + \frac{1}{\alpha}\right)t} \int_0^{\infty} \left(e^{-\left(\frac{1}{\hat{\beta}_j} + \frac{q_j}{\alpha}\right)u} \right) dt du \right. \\
&= \frac{N \alpha^2 e^{-\frac{\hat{c}_j + D_j + H_j + N}{\alpha}}}{(\hat{\beta}_j + \alpha)(\alpha + q_j \cdot \beta_j)}
\end{aligned}$$

where we used analogous argumentation for the first five steps as in the proof to the previous lemmas (see lemma 1 for example).

According to (3.9) and the lemmas 1, 2, 3 and 4 the following gives expected execution size for Child order.

$$\begin{aligned}
E(V) &= \sum_{i=0}^4 E_i(V) \\
&= E_1(V) + E_2(V) + E_3(V) + E_4(V) \\
&= \frac{\alpha e^{-\frac{\hat{c}_j + D_j + \Delta}{\alpha}}}{\alpha + \hat{\beta}_j} \left(e^{\frac{\Delta}{\alpha}} \alpha - (\alpha + \Delta) \right) + \frac{\Delta \alpha e^{-\frac{\hat{c}_j + D_j + \Delta}{\alpha}}}{\alpha + \hat{\beta}_j} \left(1 - \frac{e^{-\frac{H_j}{\alpha}}}{q_j \beta_j} \right) \\
&\quad + \frac{\alpha^3 e^{-\frac{\hat{c}_j + D_j + H_j + N + \Delta}{\alpha}}}{(\alpha + \hat{\beta}_j)(\alpha + q_j \cdot \beta_j)} \left(e^{\frac{N}{\alpha}} \left(1 + \frac{\Delta}{\alpha} \right) - e^{\frac{\Delta}{\alpha}} \left(1 + \frac{N}{\alpha} \right) \right) + \frac{N \alpha^2 e^{-\frac{\hat{c}_j + D_j + H_j + N}{\alpha}}}{(\hat{\beta}_j + \alpha)(\alpha + q_j \cdot \beta_j)} \\
&= \frac{\alpha^2 e^{-\frac{\hat{c}_j + D_j + H_j + N + \Delta}{\alpha}}}{(\alpha + \hat{\beta}_j)(\alpha + q_j \cdot \beta_j)} \left[(\alpha + q_j \cdot \beta_j) \left(\left(e^{\frac{\Delta}{\alpha}} - \left(1 - \frac{\Delta}{\alpha} \right) \right) e^{\frac{N+H_j}{\alpha}} + \frac{\Delta}{\alpha} e^{\frac{N+H_j}{\alpha}} \right) - \Delta e^{\frac{N}{\alpha}} \right. \\
&\quad \left. + \alpha \left(e^{\frac{N}{\alpha}} \left(1 + \frac{\Delta}{\alpha} \right) - e^{\frac{\Delta}{\alpha}} \left(1 + \frac{N}{\alpha} \right) \right) + N e^{\frac{\Delta}{\alpha}} \right] \\
&= \frac{\alpha^2 e^{-\frac{\hat{c}_j + D_j + H_j + N + \Delta}{\alpha}}}{(\alpha + \hat{\beta}_j)(\alpha + q_j \cdot \beta_j)} \left[(\alpha + q_j \cdot \beta_j) \left(e^{\frac{\Delta}{\alpha}} - 1 \right) e^{\frac{N+H_j}{\alpha}} + \alpha \left(e^{\frac{N}{\alpha}} - e^{\frac{\Delta}{\alpha}} \right) \right]
\end{aligned}$$

$$E(V) = \frac{\alpha^2 e^{-\frac{\hat{c}_j + D_j + H_j + N + \Delta}{\alpha}}}{(\alpha + \hat{\beta}_j)(\alpha + q_j \cdot \beta_j)} \left[(\alpha + q_j \cdot \beta_j) \left(e^{\frac{\Delta}{\alpha}} - 1 \right) e^{\frac{N + H_j}{\alpha}} + \alpha \left(e^{\frac{N}{\alpha}} - e^{\frac{\Delta}{\alpha}} \right) \right] \quad (3.18)$$

$E(V)$ is a measure-of-goodness for the execution performance of the child order under given market conditions and order-specific parameters (total order size, display size, submission level and trading time horizon).

3.3 Optimal Display

For the sake of simplicity, let us consider $Z := E(V)$ be traders objective function for determining optimal display size Δ^* , which requires solving $Z'(\Delta) = 0$ for Δ . In the face of the fact, that this equation incorporates exponentials terms in combination with rational expressions, solving this equation algebraically is not possible. To find optimal display, we, instead of considering objective function Z , consider the approximation of Z . For that, Taylor expansion up to order two is considered.

$$\begin{aligned} Z(\Delta) &= Z(0) + Z'(0)\Delta + \frac{1}{2}Z''(0)\Delta^2 + O(\Delta^3) \\ Z(\Delta) &= \hat{Z}(\Delta) + O(\Delta^3) \\ \hat{Z}(\Delta) &= Z(0) + Z'(0)\Delta + \frac{1}{2}Z''(0)\Delta^2 + O(\Delta^3) \end{aligned} \quad (3.19)$$

Where, \hat{Z} denotes approximate execution child order.

We denote Δ^{opt} as approximate optimal display, and $\Delta^{opt} \in (0, N)$. For any maximum located at $\Delta^{opt} \in (0, N)$ the following must hold.

$$\begin{aligned} \hat{Z}'(\Delta^{opt}) &= Z'(0) + Z''(0)\Delta^{opt} = 0 \\ \Leftrightarrow \Delta^{opt} &= -\frac{Z'(0)}{Z''(0)} \end{aligned} \quad (3.20)$$

4. Optimal Order and Display Property

In the sense of our optimal model, an order of size N submitted at time t_0 to the price level p_j , give rise to quantity called expected execution size Z . Under the given setting we derive optimal display size, with respect to traders trading objective, namely to execute as much possible within trading horizon. Since Z will naturally depends in multifarious ways on the market and its

property, so do the optimal display size Δ^{opt} . For the trader who uses slicing of the order, for instance to reduce his market impact, it is thus of utmost interest to understand properly the optimal display strategy's dependence on the market. In this section, we answer the above raised question through simulations.

4.1 Discrete Child Order

For understating the property of the optimal order which mathematically is discrete, we have run $n = 5,000$ simulation for the execution process of the Child order keeping market and the other parameters fixed. We then counted the frequency of the Child order execution. In this case (Figure 4.1), we assumed the display size $\Delta = 40$ and total order size $N = 100$. It is clearly observed that Child order is indeed executed at this size.

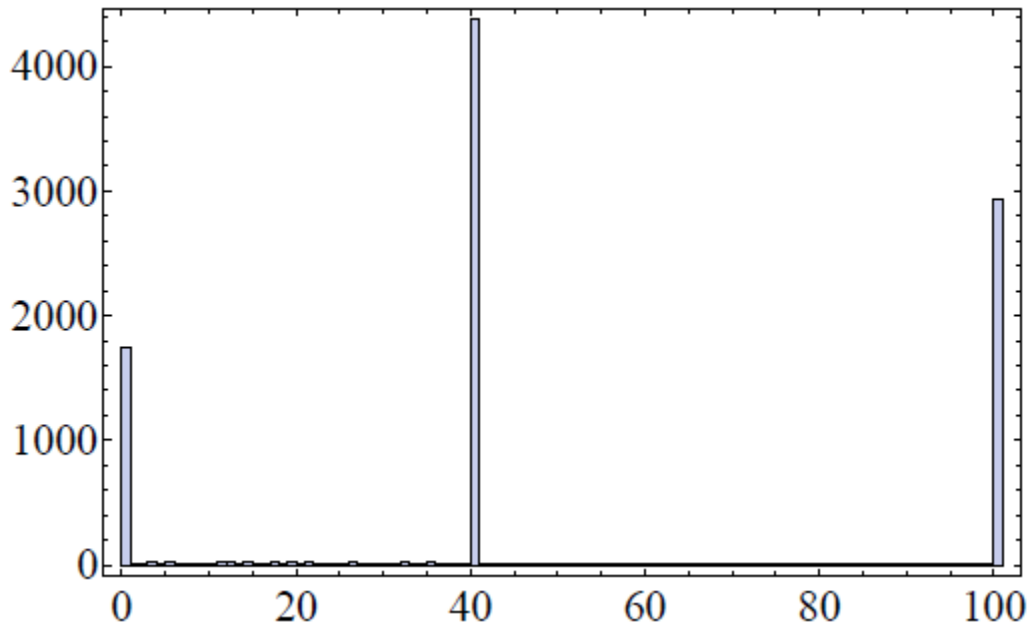


Figure 4.1: Discrete Child Order
Parameters: $\hat{C}_j=100, D_j=0, H_t=0, N=100, \Delta=40, \alpha=500, \hat{\beta}_{j_0}=600, \beta_j=100, q_j=0, k=5, \gamma=0.05$

4.2 Display Dependency on Liquidity

The very glance of the Z, tell us that in order to determine the optimal display Δ^{opt} at the time t_0 of the submission, the traders doesn't need to care about how much liquidity sits in front of the Child order price level. Traders can either ignore the displayed depth also. It also tells us that whole limit order book depth (at time t_0) doesn't affect the Δ^{opt} once the price is fixed. But all

this doesn't mean that liquidity in front of Child order will not have effect on the expected execution order. In fact the Z values clearly shows that the expected execution size decreases with increase in liquidity in \hat{C}_j and D_j (equally observed in Figure 4.2)

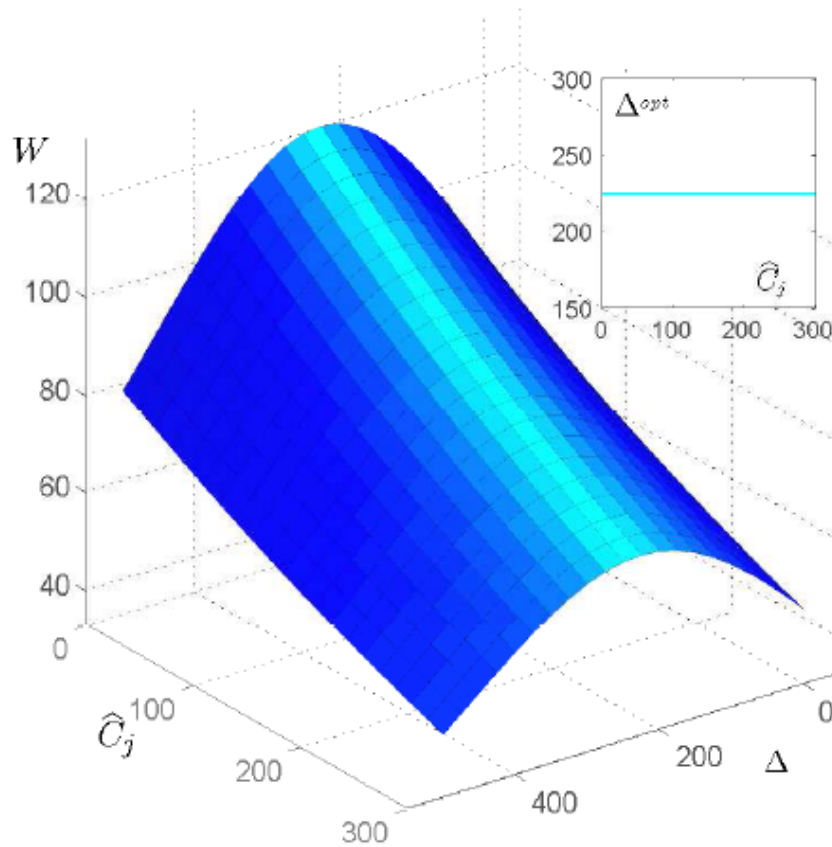


Figure 4.2: Liquidity Independence
Parameters: $D_j=0, N=500, \alpha=600, \hat{\beta}_{j0}=600, \beta_j=0, H_j=400, q_j=0, k=3, \gamma=0.00001$

But the above results doesn't capture the case of hidden liquidity H_j at the price level p_j . The mathematical intuition suggests that hidden liquidity forces the traders to increase his/her disclosure to the market (numerically Figure 4.3)

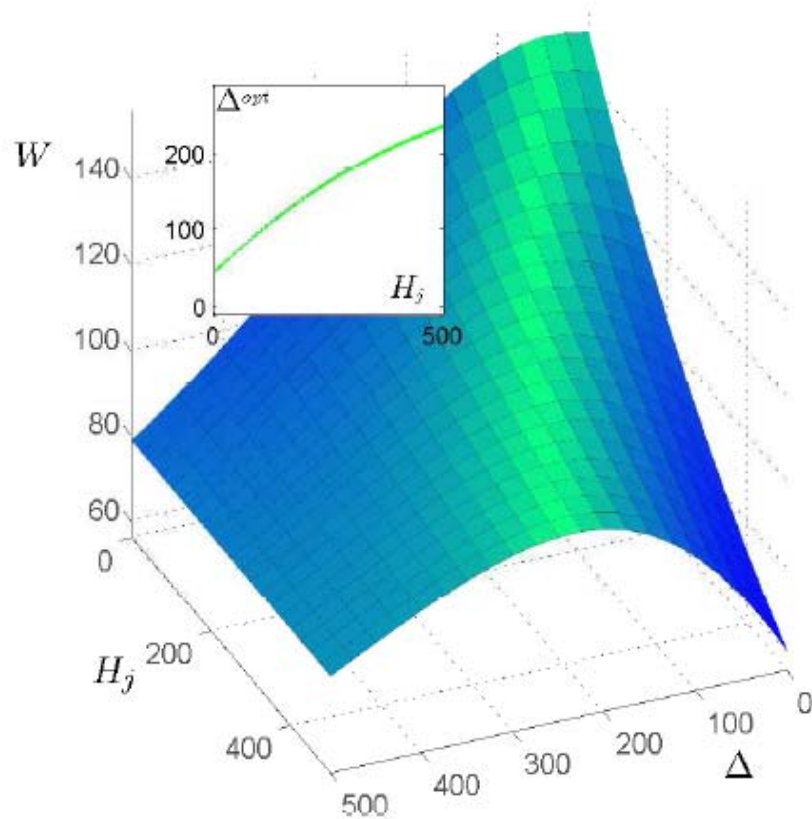


Figure 4.3: Hidden Liquidity forces Visibility

Parameters: $\hat{C}_j=0, D_j=0, H_j = 400, N=500, \alpha=600, \hat{\beta}_{j0}=600, \beta_j=0, q_j=0, k=2$

4.4 Market Sensitivity

The whole point of order slicing and optimal display was to mitigate ones trading intensions and as a results to avoid unfavorable market impact costs. Putting it in another way, optimal display will heavily depends on market sensitivity γ . We generally expect, the higher the market sensitivity the smaller the optimal display size. And indeed this notion is substantiated by the following Figure 4.4. In the given figure, the colored diagram shows expected execution size Z in dependence of the display size Δ and the market sensitivity γ (optimal display is colored in the dark red).

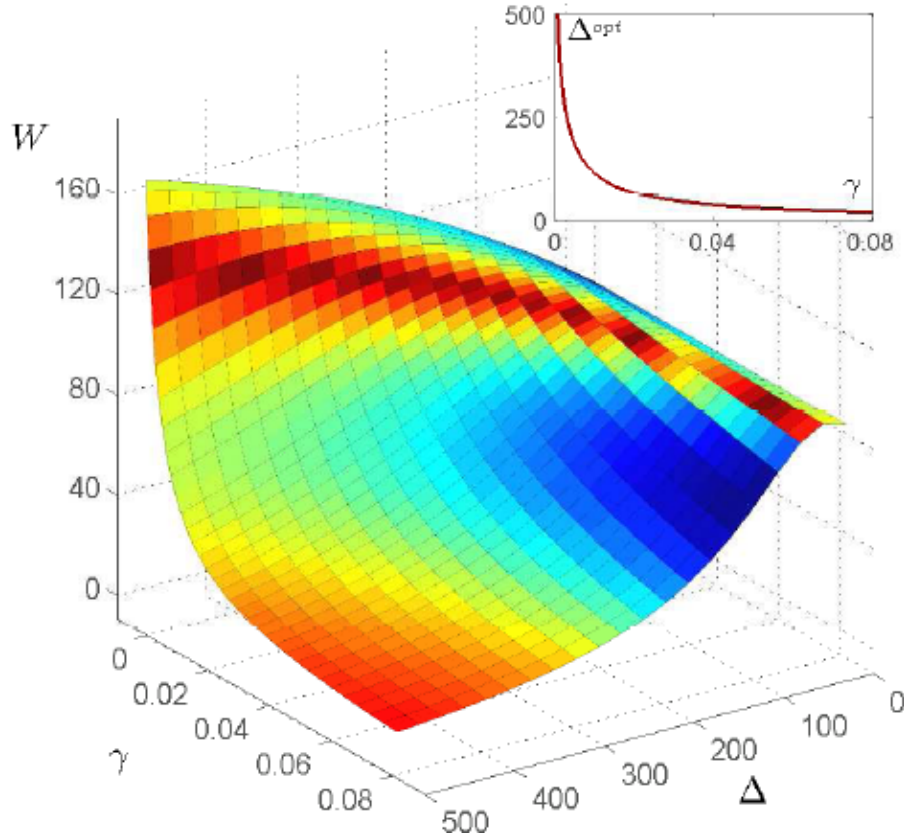


Figure 4.3: Market Sensitivity and Optimal Display
Parameters: $\hat{C}_j=0, D_j=0, N=500, \alpha=600, \hat{\beta}_{j0}=600, \beta_j=0, q_j=200.5, k=2, \gamma=0.005$

5. Conclusions

So far literature on theoretical model of child orders, that may appropriately answer the all the question for effective execution order is very scarce. The literature which talk about child order execution size, hidden liquidity and depth are is of empirical nature (Mak, 2000). This makes the work of designers of algorithmic traders really hard to find out execution size without any mathematical models. Though this is present in the professional world of algorithmic trading but quiet unknown in academic space. We in this thesis have extended the empirical studies to theoretical model which can be used to check execution size and display effect on hidden liquidity. While mathematically modeling the expected execution size of Child order, we skip the assumption of time-continuity (Poisson arrival), but we reduce the arrival of other traders orders to two time points. Hence we accounted for liquidity supply and liquidity demand, but by reducing this to two discrete time points, we believe that we simplified the model significantly.

We also took account of hidden liquidity in our model, but practically it is very difficult to access the same. Therefore we need to incorporate some parameters which can take account of accessible hidden liquidity. For the sake of simplicity we have relaxed many assumptions to emphasize the model's possible relevance for application purposes. The work can be further extended by simulating effect on order and display size relation to limit order parameters, market sensitivity, hidden and display liquidity, total order size etc. In our thesis though we have not taken into consideration of optimal time of execution in light of hidden liquidity, it is one important part of trading strategy.

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