# OPTIMAL FACTORIAL DESIGNS WITH ROBUST PROPERTIES 

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## Abstract

Fractional factorial designs are used in a wide variety of disciplines as a means of studying how changes in the settings of a set of factors influence a response variable. Two important considerations in choosing a fractional factorial design are identifying which effects can be jointly estimated and how the effects not estimated influence the estimation.

Orthogonal arrays with clear two-factor interactions provide a class of designs robust to nonnegligible effects. In the first part of this thesis, we introduce the concept of partially clear interactions which leads to a richer class of robust designs when specific interactions are known to be negligible a priori. We develop several methods to construct designs that allow for additional factors to be studied in comparison to designs with clear two-factor interactions. When used in conjunction with non-regular designs, the results become even more powerful as they provide additional flexibility and retain the robust properties.

In some situations, the experimenter would like to study factors at more than two levels, such as when curvature has the potential to occur within the experimental region. The second part of this thesis focuses on the estimation of main effects and specified interactions for designs with more than two levels. As designs with more than two levels have additional complications, results are provided that aid in the search for efficient designs that also have robust properties.

For two-level designs, the criteria of $\$ \mathrm{G} \$$ and $G_{2}$-aberration are based on $J$ -character-
istics and they provide measures of the projection properties of a design. For multilevel designs, extension to $G_{2}$ was previously done without the use of $J$-characteristics.

The $J$-characteristics for multi-level designs are introduced in the last part of this thesis as an intuitively appealing means to measure lower-dimensional properties, which leads to more natural definitions of $G$ and $G_{2}$-aberration. We show how the properties of a design can be gleaned by using an analysis of variance as taught in introductory statistics courses.

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## Chapter 1

## Introduction

### 1.1 Factorial Designs

Experimenters are often concerned with how changes of certain factors impact a process. If the experimenter has some control over the settings of these factors, then experimentation through changing these settings can aid in understanding the process. By using settings of the factors at a fixed number of values, referred to as levels, a factorial design is typically used as a plan to conduct such an experiment. Instead of focusing on one factor at a time, a factorial experiment varies the levels of these factors simultaneously. Depending on the different combinations of factors used in the experiment, not only can the experimenter study how the factors impact the response individually, but also their interaction. If enough attention is not placed in the design phase, the experimenter may not be able to meet the objectives of the experiment. A well-designed experiment can be analyzed in a valid and objective way, making it an important aspect of the experimental process.

Throughout this thesis, we assume that the experimental data can be adequately modeled by the general linear model. For a factorial design with $n$ runs and $m$ factors, the general linear model is written as

$$
\begin{equation*}
Y=X_{0} \beta_{0}+X_{1} \beta_{1}+X_{2} \beta_{2}+\cdots+X_{m} \beta_{m}+\epsilon \tag{1.1}
\end{equation*}
$$

where $Y$ is the vector of $n$ observations for the response, $X_{0}$ is the vector of 1 's, $\beta_{0}$ the
intercept, $X_{i}$ is the matrix of covariate values for the $i$-factor interactions with $\beta_{i}$ the corresponding effects, and $\epsilon$ the vector of independent random errors. For convenience, when discussing the use of (1.1), we assume the use of orthogonal contrasts and denote the effects as factorial effects throughout. This allows independent tests on the effects and estimates can still be related back to original values. We refer the reader to Chapter 1 of Wu and Hamada (2000) or an introductory text on design and analysis of experiments for more details.

Running all possible level combinations of the factors, called a full factorial design, allows for the estimation of all factorial effects. While this makes performing a full factorial design desirable, running all possible combinations of the factors may not be feasible for a variety of reasons including economic limitations, ethical concerns for certain combinations of factors, and situations where certain combinations do not make practical sense or are not possible to be run together. In such situations, a fractional factorial design (FFD) that uses a fraction of the runs of the full factorial design is frequently used. However, using a smaller subset of runs comes with a price; factorial effects become aliased. In regular designs, those determined by a defining relation, factorial effects are either orthogonal or fully aliased. If a design does not have the aforementioned property we refer to the design as non-regular. The question then becomes how to determine the desirability of a design based on the nature of this aliasing.

### 1.2 Choosing Fractional Factorial Designs

While the appropriateness of a design is ultimately based on the experimental objectives, there are some working assumptions that are generally accounted for in choosing designs. These are based on empirical evidence and play an important role in ranking designs and analyzing results. Three fundamental principles for factorial effects in fractional factorial designs (Wu and Hamada, 2000) are:

1. Hierarchical Ordering Principle: (i) Lower order effects are more likely to be important than higher order effects, and;
(ii) effects of the same order are equally likely to be important.
2. Effect Sparsity Principle: The number of relatively important effects in a factorial experiment is small (Box and Meyer, 1986).
3. Effect Heredity Principle: In order for an interaction to be active, at least one of its parent factors should be active (Hamada and Wu, 1992).

Using these principles, FFDs can be constructed and ranked. Following the hierarchical ordering principle, the main effects are the most important factorial effects, followed by two-factor interactions, three-factor interactions, etc. In the ANOVA model (1.1), these apply to $\beta_{1}, \beta_{2}, \beta_{3}, \ldots$. With this ranking of effects in mind, Box and Hunter (1961) introduced the criterion of maximum resolution for regular designs to measure the aliasing between lower order effects. As a means of further distinguishing between designs having the same resolution, Fries and Hunter (1980) proposed the minimum aberration criterion. These ideas were generalized to non-regular twolevel designs by Deng and Tang (1999) and Tang and Deng (1999) through $G_{2}$ and $G$-aberration. Generalized minimum aberration was derived for multi-level designs by Xu and $\mathrm{Wu}(2001)$. The notion of $G_{2}$ and $G$-aberration for multi-level designs will be returned to later in this thesis. We now discuss other considerations that may arise in choosing an appropriate design for an experiment.

### 1.2.1 Requirement Sets

By the hierarchical ordering principle, effects of the same order are equally likely to be important. However, there are situations where a certain subset of effects are of interest to the experimenter. For example, the experimenter's interest may be in the estimation of the main effects and a specified set of two-factor interactions. Considering a specific subset of interactions is important from a practical standpoint, but breaks the hierarchical ordering principle, meaning maximum resolution and minimum aberration designs will no longer be the best choice for these cases. The goal is then to find designs that allow estimation of the effects of interest, if such designs exist. This idea goes back to Addelman (1962). Franklin and Bailey (1977) presented
an algorithm for generating designs with a requirement set for two levels, and was extended by Franklin (1985) additional levels. In this thesis we call this set of effects a requirement set, as done by Greenfield (1976). If we are able to find more than one design that allows for estimation of the requirement set, we can differentiate between these designs using another criterion. Popular examples of these include D-optimality, which minimizes the volume of the confidence ellipsoid for the set of factorial effects, and $A$-optimality, which minimizes the sum of the variances of the regression coefficients.

### 1.2.2 Robustness to Nonnegligible Effects

If we can find a design that can estimate the requirement set, the estimates may be aliased with effects that are not estimated. If the effects not estimated are nonnegligible, this aliasing should be accounted for. For example, if the experimenter is interested in estimating the main effects of the factors, they may fit the main effects model given by

$$
\begin{equation*}
Y=X_{0} \beta_{0}+X_{1} \beta_{1}+\epsilon \tag{1.2}
\end{equation*}
$$

Using model (1.2), the least squares estimate of $\beta_{1}$ is calculated as $\hat{\beta}_{1}=\left(X_{1}^{T} X_{1}\right)^{-1} X_{1}^{T} Y$. This estimate is unbiased for model (1.2), but if model (1.1) is the true model, we have

$$
\begin{equation*}
E\left(\hat{\beta}_{1}\right)=\beta_{1}+C_{2} \beta_{2}+\cdots+C_{m} \beta_{m}, \tag{1.3}
\end{equation*}
$$

where the $C_{i}=\left(X_{1}^{T} X_{1}\right)^{-1} X_{1}^{T} X_{i}$ are called the aliasing matrices for $i=2, \ldots, m$. If elements of $\beta_{i}$ are nonnegligible, they can introduce bias on the estimation of $\beta_{1}$ as seen in (1.3). We will see later that, for a requirement set with some two-factor interactions, replacing $X_{1}$ in (1.2) by the model matrix for the requirement set can be easily adapted to (1.2) and (1.3).

This examination of the bias dates back to Box and Draper (1959) for response surface designs. If we are only interested in the main effects, the ranking of designs can be done with $G$ and $G_{2}$-aberration. When certain interactions are to be estimated, a different tactic needs to be taken in regards to bias. One approach is to treat all those effects not estimated as negligible, but this may not be a reasonable assumption.

Another typical approach is to treat higher-order interactions as negligible. This can still leave nonnegligible effects that are not estimated. If the effects that are estimated are aliased with these effects, we get biased estimates. To prevent this, we want a design that is robust to nonnegligible effects. Wu and Chen (1992) defined clear two-factor interactions as those which are orthogonal to all main effects and other two-factor interactions. Clear two-factor interactions are valuable because their estimates are robust to nonnegligible effects if interactions involving three or more factors are negligible.

One of the themes of this thesis is combining the requirement set problem with robustness to nonnegligible interactions. In one case, robustness will be achieved through prior information about some of the effects to ensure robustness to nonnegligible effects. In another, we look to efficient estimation of the requirement set firstly, and then try to establish robustness properties. Related work for two-level designs is seen in Ke and Tang (2003) and Wu (2009), and by Jones and Nachstheim (2011) for three-level designs.

### 1.2.3 Screening Experiments and Response Surface Exploration

Much of the discussion to this point has been concerned with a specific set of effects to be estimated. Without a requirement set, FFDs can be used for factor screening, with the intention of identifying a subset of the factors that have a large influence on the response. More detailed experimentation can then be performed on this smaller subset of factors. By the effect sparsity principle, this subset is expected to be small in size. If there is the possibility that one or more of the lower order interactions are significant, ideally we should be able to entertain as many of the smaller subsets that include some interactions as possible. For regular designs, the nature of the aliasing is well-defined, but leaves some effects indistinguishable from each other. Non-regular designs have a complex aliasing pattern, but have the potential to allow for the estimation of a greater number of models involving smaller subsets of factors. Some metrics for choosing designs for this purpose include estimation capacity (Sun,

1993 and Cheng, Steinberg and Sun, 1999), projection estimation capacity (Loeppky, Sitter and Tang, 2007), and average D-efficiency (Cheng, Deng and Tang, 2002).

In non-regular designs, complex aliasing can make it difficult to discriminate between sets of models. Some proposed solutions to identifying subsets of factors influencing the response include Hamada and Wu (1992), Chipman, Hamada, and Wu (1997) and Wolters and Bingham (2011). If the purpose of the experiment is to maximize or minimize a response, one strategy is to use response surface methodology on this smaller subset of factors. Response surface methodology is a two-phase process which involves using first order models to find an experimental region near the optimum and fit a second-order model that allows for curvature within that region. In Chapter 3 we will consider an alternative to standard response surface methodology, when estimating curvature is desirable but not follow-up experimentation.

### 1.3 Orthogonal Arrays

The fractional factorial designs considered in this thesis are orthogonal arrays. Orthogonal arrays date back to the 1940's to Rao (1947) and Plackett and Burman (1946) and form the basis for most factorial designs. An orthogonal array $O A\left(n, s_{1}, \ldots, s_{m}, t\right)$ is an $n \times m$ matrix in which column $i$ has $s_{i}$ levels for $i=1, \ldots, m$, such that for any subset of $t$ columns all possible level combinations occur equally often. In the experimental sense, the $n$ rows can be thought of as the experimental runs and each column represents the settings for a factor. We refer to $t$ as the strength of the orthogonal array. The strength of an orthogonal array is related to the estimability of the interaction terms in the linear regression model. The higher the strength of an orthogonal array, the more interaction terms can be estimated independently of each other. In terms of clear effects, for an orthogonal array of strength 2, the main effects and clear two-factor interactions can be estimated with the clear two-factor interactions robust to nonnegligible two-factor interactions, under the assumption that interactions involving three or more factors are negligible. Under the same assumption with interactions involving three or more factors being negligible, both the main
effects and clear two-factor interactions are estimable and robust to nonnegligible twofactor interactions in an orthogonal array of strength 3. Unfortunately, the higher the strength of an orthogonal array for a fixed set of factors, the larger the run size needs to be. Similarly, for a fixed run size, the higher strength implies fewer factors can be considered.

Regular designs presented in introductory design of experiments textbooks are a special case of orthogonal arrays. With the inclusion of non-regular designs, orthogonal arrays provide a much richer class of designs to consider in terms of run size economy and flexibility, while ensuring the main effect estimates are mutually orthogonal. Throughout this thesis, we will make use of catalogs such as those in Chen, Sun and Wu (1993) and Evangelaras, Koukouvinos and Lappas (2011) in searching for designs for different experimental objectives. An extensive look at orthogonal arrays is done in Hedayat, Sloane, and Stufken (1999). More recent work on designs using orthogonal arrays appears throughout this thesis.

### 1.4 Outline

We now give an outline for the remainder of the thesis with a brief description of each chapter and motivation behind it.

Orthogonal arrays with clear two-factor interactions provide a class of designs that are robust to nonnegligible effects. If certain prior knowledge is available, then robust designs may allow additional factors to be studied. This is done through partially clear two-factor interactions. In Chapter 2 we investigate the existence and construction of such robust designs and present an upper bound on the maximum number of clear two-factor interactions.

While two-level designs allow the estimation of linear effects, the experimenter may require the use of more than two levels per factor. The ideas of a requirement set and influence of those effects not estimated become more complicated when factors are studied at more than two levels. Chapter 3 studies the problem of requirement sets for factors with more than two levels and robustness to those effects not estimated.

Chapter 4 considers finding designs with desirable projection properties. In an
orthogonal array of strength $t$, in any subset of $t$ columns, all level permutations occur equally often. If strength $t$ is not attainable, ideally the lower-dimensional projections of a design should resemble this property for smaller values of $t$. With this idea, we introduce the $J$-characteristics for multi-level designs which are based on the frequency of design points and can be used to examine lower dimensions. We will use these $J$-characteristics to rank designs.

The thesis is concluded in Chapter 5 with a summary of Chapters 2, 3 and 4 and a general discussion on possible future work.

## Chapter 2

## Robust Designs Through Partially Clear Two-Factor Interactions

### 2.1 Introduction

In this chapter, we consider factorial designs with two levels, although the main ideas are generally applicable. Two-level factorial designs are widely used for screening experiments in industrial applications. Two-level orthogonal arrays enjoy run size economy and flexibility beyond those of regular factorial designs.

Orthogonal arrays with clear two-factor interactions are robust to nonnegligible two-factor interactions. In an orthogonal array, a two-factor interaction (2fi) is said to be clear if it is orthogonal to all main effects and all other two-factor interactions (2fis). Under the assumption that interactions containing three or more factors are negligible, an orthogonal array of strength 2 allows estimation of clear 2f's even if the other 2fi's are nonnegligible, although main effects may be aliased with some 2 fi 's. Under the same assumption regarding interactions of three or more factors, if an orthogonal array is of strength 3, main effects and clear 2fi's can all be estimated regardless of the nonnegligible 2 fi 's.

Clear 2fi's were introduced by Wu and Chen (1992) and their existence was examined by Chen and Hedayat (1998). Cheng, Steinberg and Sun (1999) and Wu and Wu (2002) explored the relationship between the criterion of minimum aberration and
that of the maximum number of clear 2fi's. Other work on clear 2 ff 's includes Tang, Ma, Ingram and Wang (2002), Ke, Tang and Wu (2005), Chen, Li, Liu, and Zhang (2006), and Lisonek (2006). The concept of clear 2fi's was generalized to orthogonal arrays by Tang (2006).

In this chapter, instead of treating all 2fi's that are not estimated as either negligible or nonnegligible, we consider the situation in which some 2 fi's are assumed to be negligible while others are not. Assuming all 2fi's that are not estimated are negligible such as in Addelman (1962), Greenfield (1976) and Sun (1993) may be too strong whereas assuming all are nonnegligible may be too restrictive. The assumption that some 2fi's are negligible leads to a class of robust designs that allow more factors to be studied as compared to designs with clear 2fi's. They are obtained using the concept of partially clear 2fi's. Partially clear 2fi's are orthogonal to nonnegligible 2fi's but allowed to be aliased with negligible 2 fi's.

Section 2.2 introduces basic concepts and presents some preliminary results. Section 2.3 establishes several theoretical results for the construction of robust designs and Section 2.4 provides tables of robust designs for 32 and 64 runs. We conclude this chapter with a result on the maximum number of clear 2fi's in an orthogonal array in Section 2.5.

### 2.2 Concepts and Preliminary Results

A two-level fractional factorial design $D$ of $n$ runs for $m$ factors is represented by an $n \times m$ matrix of $\pm 1$. Design $D$ is a two-level orthogonal array of strength $t$ if, in every $n \times t$ submatrix of $D$, each of $2^{t}$ level combinations appears with the same frequency. We denote such an orthogonal array by $O A\left(n, 2^{m}, t\right)$. The run size $n$ must be a multiple of 4 for $t=2$, and a multiple of 8 for $t=3$. Orthogonal arrays of strength 2 allow orthogonal estimation of all main effects when all interactions are negligible. For orthogonal arrays of strength 3, the main effects can be orthogonally estimated under the weaker assumption that interactions involving three or more factors are negligible. In this chapter, we assume that interactions involving three
or more factors are negligible. Under this assumption, orthogonal arrays of strength 3 allow unbiased estimation of all main effects regardless of two-factor interactions. Clear 2fi's carry this idea further. Orthogonal arrays of strength 3 with clear 2fi's allow unbiased estimation of all main effects and all clear 2fi's even if the other 2fi's are nonnegligible.

A 2fi in an orthogonal array of strength 3 is clear if it is orthogonal to all other 2 fi's. A clear 2fi can be estimated regardless of other 2 fi 's. In the situation where prior knowledge suggests that certain 2fi's are negligible, this property of robust estimation for a 2 fi remains intact so long as it is orthogonal to all nonnegligible 2 fi 's. Such a 2 fi is called partially clear. A partially clear 2fi is allowed to be aliased with negligible 2 fi's but this does not affect its robust estimation. The idea for partially clear 2 fi's is now formally developed.

Let $S_{1}$ denote the set of 2 fi 's that are to be estimated, $S_{2}$ the set of nonnegligible 2 fi's and $S_{3}$ the set of negligible 2fi's. Thus, the entire set of all 2fi's is partitioned into three mutually exclusive and exhaustive sets $S_{1}, S_{2}$ and $S_{3}$. The problem of interest here is to find designs that allow estimation of all main effects and the 2fi's in $S_{1}$ in the presence of the nonnegligible 2fi's in $S_{2}$. Designs considered for this problem are orthogonal arrays of strength 3, which guarantee that main effects are orthogonal to 2 fi 's. The 2fi's in $S_{1}$ may or may not be orthogonal under a given orthogonal array. For most of this chapter, we consider arrays such that the 2 fi 's in $S_{1}$ are orthogonal. In this case, we have orthogonal designs for estimating main effects and the 2 f 's in $S_{1}$. According to Greenfield (1976), the set of main effects together with $S_{1}$ is called a requirement set. We call $S_{1}$ a requirement set in this chapter. In Section 2.3.2, we will allow the requirement set $S_{1}$ to be non-orthogonal.

Definition 2.1. For a given orthogonal array of strength 3, a 2fi in the requirement set $S_{1}$ is said to be partially clear if it is orthogonal to all 2fi's in the nonnegligible set $S_{2}$.

When the negligible set $S_{3}$ is empty, a partially clear 2 fi becomes clear. Definition 2.1 allows prior knowledge as summarized by $S_{3}$ to be utilized when finding designs for a given requirement set. If an orthogonal array of strength 3 is such that all 2 fi's
in the requirement set $S_{1}$ are partially clear, then this array provides a robust design for the requirement set $S_{1}$. This chapter studies the existence and construction of such robust designs.

We now introduce some structures for $S_{1}, S_{2}$ and $S_{3}$ which will allow us to gain insights into and obtain theoretical results on robust designs. The factors to be studied are divided into two groups of factors, $G_{1}$ and $G_{2}$. Let $G_{1} \times G_{1}$ denote the set of 2fi's within the factors in $G_{1}, G_{2} \times G_{2}$ the set of 2 fi 's within the factors in $G_{2}$, and $G_{1} \times G_{2}$ the set of 2 f 's between the factors in $G_{1}$ and those in $G_{2}$. We consider the following three cases:

Case 1: $S_{1}=G_{1} \times G_{1}, S_{2}=G_{1} \times G_{2}$ and $S_{3}=G_{2} \times G_{2} ;$
Case 2: $S_{1}=G_{1} \times G_{1}, S_{2}=G_{2} \times G_{2}$ and $S_{3}=G_{1} \times G_{2}$;
Case 3: $S_{1}=G_{1} \times G_{2}, S_{2}=G_{1} \times G_{1}$ and $S_{3}=G_{2} \times G_{2}$.
For convenience, robust designs for Cases 1,2 and 3 are referred to as robust designs of types 1,2 and 3 , respectively.

### 2.2.1 Preliminary Results

Simple characterization of robust designs can be obtained using $J$-characteristics. For an arbitrary number of vectors $x_{j}=\left(x_{1 j}, \ldots, x_{n j}\right)^{\mathrm{T}}$, where $j=1, \ldots, k$, their $J$-characteristic is defined as

$$
J\left(x_{1}, \ldots, x_{k}\right)=\sum_{i=1}^{n} x_{i 1} \cdots x_{i k}
$$

Let $D=\left(d_{1}, \ldots, d_{m}\right)$ be a two-level factorial design of $n$ runs for $m$ factors, where $d_{j}=\left(d_{1 j}, \ldots, d_{n j}\right)^{\mathrm{T}}$ is the $j$ th column of $D$. Then design $D$ is an orthogonal array of strength $t$ if and only if $J\left(d_{j_{1}}, \ldots, d_{j_{k}}\right)=0$ for all $j_{1}, \ldots, j_{k}$ such that $1 \leq j_{1}<\cdots<$ $j_{k} \leq m$ and $k \leq t$.

Let $a_{1}, \ldots, a_{m_{1}}$ be the columns of design $D$ corresponding to the factors in $G_{1}$, and $b_{1}, \ldots, b_{m_{2}}$ be the columns of design $D$ corresponding to the factors in $G_{2}$. Thus $D=$ $\left(a_{1}, \ldots, a_{m_{1}}, b_{1}, \ldots, b_{m_{2}}\right)$. We also write $G_{1}=\left(a_{1}, \ldots, a_{m_{1}}\right)$ and $G_{2}=\left(b_{1}, \ldots, b_{m_{2}}\right)$. Define

$$
\begin{aligned}
& A_{40}=\sum_{1 \leq j_{1}<j_{2}<j_{3}<j_{4} \leq m_{1}}\left[J\left(a_{j_{1}}, a_{j_{2}}, a_{j_{3}}, a_{j_{4}}\right)\right]^{2}, \\
& A_{31}=\sum_{1 \leq j_{1}<j_{2}<j_{3} \leq m_{1}} \sum_{j=1}^{m_{2}}\left[J\left(a_{j_{1}}, a_{j_{2}}, a_{j_{3}}, b_{j}\right)\right]^{2} \text { and } \\
& A_{22}=\sum_{1 \leq j_{1}<j_{2} \leq m_{1}} \sum_{1 \leq i_{1}<i_{2} \leq m_{2}}\left[J\left(a_{j_{1}}, a_{j_{2}}, b_{i_{1}}, b_{i_{2}}\right)\right]^{2} .
\end{aligned}
$$

The following result is immediate.
Lemma 2.1. Let $D=\left(G_{1}, G_{2}\right)$ be an orthogonal array of strength 3. We have that (i) design $D$ is a robust design of type 1 if and only if $A_{40}=A_{31}=0$;
(ii) design $D$ is a robust design of type 2 if and only if $A_{40}=A_{22}=0$;
(iii) design $D$ is a robust design of type 3 if and only if $A_{31}=A_{22}=0$.

In Lemma 2.1(i), condition $A_{40}=0$ ensures an orthogonal requirement set $S_{1}=$ $G_{1} \times G_{1}$. Condition $A_{31}=0$ makes the design robust. More precisely, that the 2fi's in $S_{1}=G_{1} \times G_{1}$ are partially clear is captured by $A_{31}=0$. As seen from parts (ii) and (iii) of Lemma 2.1, robust designs of types 2 and 3 both require that $A_{22}=0$. This is quite a strong condition to impose on a design, as shown by the following lemma.

Lemma 2.2. Let $D=\left(G_{1}, G_{2}\right)$ be an orthogonal array of strength 3 such that $2 \leq$ $m_{1} \leq m_{2} \leq m-2$.

1. If $A_{22}=0$, then it must be true that $m \leq n / 4+1$ for every $m_{1}$ except for $m_{1}=3$.
2. When $m_{1}=3$, we have that $m \leq n / 4+2$. If $D$ is regular, we have $m \leq n / 4+1$.

Proof. For the case of $m_{1}=2$, condition $A_{22}=0$ implies that $a_{1} a_{2}$ is a clear 2fi. By Proposition 2 of Tang (2006), it must be true that $m \leq n / 4+1$.

Let $m_{1} \geq 3$. Since $D=\left(G_{1}, G_{2}\right)$ is an orthogonal array of strength 3, all 2 fi's are orthogonal to main effects. Because $A_{22}=0$, the 2 fi 's $a_{i} b_{j}$ for $i=1, \ldots, m_{1}$
and $j=1, \ldots, m_{2}$ are mutually orthogonal. Thus, the following vectors $a_{1}, \ldots, a_{m_{1}}$, $b_{1}, \ldots, b_{m_{2}}$, and all $a_{i} b_{j}$ for $i=1, \ldots, m_{1}$ and $j=1, \ldots, m_{2}$ are mutually orthogonal, and all are orthogonal to the column of all plus ones. This establishes that

$$
\begin{equation*}
1+m+m_{1}\left(m-m_{1}\right) \leq n \tag{2.1}
\end{equation*}
$$

Solving for $m$ gives

$$
\begin{equation*}
m \leq n /\left(m_{1}+1\right)+m_{1}-1 . \tag{2.2}
\end{equation*}
$$

Taking $m_{1}=3$, we obtain $m \leq n / 4+2$.
Now consider $m_{1} \geq 4$, in which case we must have $n>16$. (It would be impossible to have $A_{22}=0$ if $n \leq 16$ and $m_{2} \geq m_{1} \geq 4$.) We prove Lemma 2.2 for this case by contradication. Assume that $m \geq n / 4+2$. Then from (2.2), we obtain $n / 4+2 \leq n /\left(m_{1}+1\right)+m_{1}-1$. Solving for $n$ gives $n \leq 4\left(m_{1}+1\right)$, implying that $m_{1} \geq n / 4-1$. Since $m_{2}=m-m_{1} \geq m_{1} \geq n / 4-1$, by (2.1) we obtain $1+2(n / 4-1)+(n / 4-1)^{2} \leq n$, which leads to $n \leq 16$, a contradiction. The proof of Lemma 2.2 is completed.

We next prove that $m \leq n / 4+1$ for $m_{1}=3$ if $D$ is a regular design. Now assume that $A_{22}=0$ holds for a strength 3 array $D=\left(a_{1}, a_{2}, a_{3}, b_{1}, \ldots, b_{m_{2}}\right)$ where $m=$ $n / 4+2$ and $m_{2}=n / 4-1$. Consider design $D^{*}=\left(a_{1}, a_{2}, b_{1}, \ldots, b_{m_{2}}\right)$. Since $A_{22}=0$, the $2 \mathrm{fi} a_{1} a_{2}$ is clear in design $D^{*}$. This implies that $a_{1}, a_{2}, b_{1}, \ldots, b_{m_{2}}, a_{1} b_{1}, \ldots, a_{1} b_{m_{2}}$, $a_{2} b_{1}, \ldots, a_{2} b_{m_{2}}, a_{1} a_{2}, a_{1} a_{2} b_{1}, \ldots, a_{1} a_{2} b_{m_{2}}$ together form a saturated regular design, as $m_{2}=n / 4-1$. Noting that $a_{1}, a_{2}, b_{1}, \ldots, b_{m_{2}}, a_{1} b_{1}, \ldots, a_{1} b_{m_{2}}, a_{2} b_{1}, \ldots, a_{2} b_{m_{2}}$, $a_{3}, a_{3} b_{1}, \ldots, a_{3} b_{m_{2}}$ are all mutually orthogonal, we conclude that the set of columns $a_{1} a_{2}, a_{1} a_{2} b_{1}, \ldots, a_{1} a_{2} b_{m_{2}}$ is identical to the set of columns $a_{3}, a_{3} b_{1}, \ldots, a_{3} b_{m_{2}}$. Thus $a_{3}$ must equal to $a_{1} a_{2} b_{i}$ for some $i$. For simplity, we take $a_{3}=a_{1} a_{2} b_{1}$. Now consider $a_{3} b_{2}$. Then there must be $b_{j}$ such that $a_{3} b_{2}=a_{1} a_{2} b_{j}$. This shows that $b_{1}=b_{2} b_{j}$, contradicting that $D$ is of strength 3 .

Using the catalog of regular designs of 64 runs in Chen, Sun and Wu (1993), we conduct a complete search for robust designs. Table 2.1 gives the maximum value of $m$ for given $m_{1}$ for each of the three types of robust designs of 64 runs. Ke, Tang
and Wu (2005) studied clear compromise plans. In Table 2.1, the maximum number of factors given by clear compromise plans with the same requirement set is provided under $\max \left(m_{c}\right)$.

Table 2.1: Robust designs of 64 runs and corresponding clear compromise plans.

|  | type 1 |  | type 2 |  | type 3 |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $m_{1}$ | $\max (m)$ | $\max \left(m_{c}\right)$ | $\max (m)$ | $\max \left(m_{c}\right)$ | $\max (m)$ | $\max \left(m_{c}\right)$ |
| 7 | 17 | 8 | 10 | 8 | 12 | 8 |
| 6 | 22 | 9 | 10 | 9 | 12 | 12 |
| 5 | 22 | 10 | 11 | 10 | 11 | 11 |
| 4 | 28 | 11 | 11 | 11 | 11 | 11 |
| 3 | 31 | 17 | 17 | 17 | 17 | 11 |

For an orthogonal array of strength 3 to have clear 2fis, it is necessary that $m \leq n / 4+1$; see Chen and Hedayat (1998) and Tang (2006). Lemma 2.2 reveals that robust designs of types 2 and 3 are unable to break this barrier and the gain from considering such designs is unlikely to be substantial. This is all confirmed by Table 2.1. However, much gain can be achieved by robust designs of type 1 , as shown in Table 2.1. In the next section, we present several general methods for constructing this type of robust design.

### 2.3 Some Theoretical Results

This section focuses on robust designs of type 1. For simplicity, when we speak of robust designs in this section, we actually mean robust designs of type 1 . We present five methods for constructing robust designs in this section. The first and fifth methods are applicable to regular designs.

Regular fractional factorials are linear orthogonal arrays and treated in almost all design textbooks. See Dey and Mukerjee (1999), Hedayat, Sloane and Stufken (1999), and Wu and Hamada (2000). Regular fractional factorials are specified by their defining relations. The resolution of a regular factorial provides a simple characterization
of the aliasing properties of the design. A regular fractional factorial of resolution $R$ is an orthogonal array of strength $R-1$; in particular regular designs of resolution IV are special orthogonal arrays of strength 3. The saturated resolution IV design has $n=2^{k}$ runs and $m=2^{k-1}$ factors where $k$ is a positive integer. Our first method of constructing robust designs makes use of this satuarated resolution IV design.

While the first method and fifth methods only apply to regular designs, our other three methods can be used to construct robust orthogonal arrays, regular or nonregular. The scope of application is in fact even greater as these three methods also allow robust designs to be constructed for non-orthogonal requirement sets. This will be discussed in Secion 2.3.2.

Some of the construction methods are best presented using Kronecker products, which we now introduce. Let $x=\left(x_{1}, \ldots, x_{n_{1}}\right)^{\mathrm{T}}$ and $y=\left(y_{1}, \ldots, y_{n_{2}}\right)^{\mathrm{T}}$. The Kronecker product of two vectors $x$ and $y$ is defined as

$$
x \otimes y=\left(x_{1} y_{1}, \ldots, x_{1} y_{n_{2}}, \ldots, x_{n_{1}} y_{1}, \ldots, x_{n_{1}} y_{n_{2}}\right)^{\mathrm{T}}
$$

Tang (2006) provided a simple way for calculating the $J$-characteristic of Kronecker products.

Lemma 2.3. We have that

$$
J\left(a_{1} \otimes b_{1}, \ldots, a_{k} \otimes b_{k}\right)=J\left(a_{1}, \ldots, a_{k}\right) J\left(b_{1}, \ldots, b_{k}\right),
$$

where $a_{j}=\left(a_{1 j}, \ldots, a_{n_{1} j}\right)^{T}$ and $b_{j}=\left(b_{1 j}, \ldots, b_{n_{2} j}\right)^{T}$ for $j=1, \ldots, k$.

### 2.3.1 Construction of Robust Designs

Method 1. Let $X$ be a saturated resolution IV design with $n=2^{k}$ runs and $m=2^{k-1}$ factors. Then for any three distinct columns $x, y, z$ of $X$, their Hadamard product $x y z$ must also belong to $X$. The Hadamard product of two vectors $x=\left(x_{1}, \ldots, x_{n}\right)^{T}$ and $y=\left(y_{1}, \ldots, y_{n}\right)^{T}$ is defined as $x y=\left(x_{1} y_{1}, \ldots, x_{n} y_{n}\right)^{T}$. We obtain the following result.

Proposition 2.1. Design $D=\left(G_{1}, G_{2}\right)$ is a robust design, where
(i) $G_{1}$ is a subset of the columns of $X$ and $G_{1}$ has resolution VI or higher;
(ii) $G_{2}$ is obtained by removing columns xyz from $X \backslash G_{1}$ for all three distinct $x, y, z$ in $G_{1}$.

Proposition 2.1 is immediate from part (i) of Lemma 2.1, as it is obvious that $A_{40}=$ $A_{31}=0$ for design $D=\left(G_{1}, G_{2}\right)$.

As before, let $m_{1}$ and $m_{2}$ denote the numbers of columns in $G_{1}$ and $G_{2}$, respectively. When $m_{1} \leq 5$, the number of columns that need to be removed from $X \backslash G_{1}$ is $\binom{m_{1}}{3}$. Thus we have that $m_{2}=2^{k-1}-m_{1}-\binom{m_{1}}{3}$. For $m_{1} \geq 6$, it is advantageous to choose $G_{1}$ to have resolution VI as the number of columns that have to be removed to obtain $G_{2}$ will be less than $\binom{m_{1}}{3}$ due to the existence of the relation $x_{1} y_{1} z_{1}=x_{2} y_{2} z_{2}$ when $G_{1}$ is of resolution VI.

We next look at the application of the above construction method to designs of 32 and 64 runs. For designs of 32 runs, we obtain $m_{2}=12$ for $m_{1}=3, m_{2}=8$ for $m_{1}=4$, and $m_{2}=1$ for $m_{1}=5$. Our computer search of robust designs from among all resolution IV designs shows that the maximum number of columns in $G_{2}$ for given $m_{1}$ is $m_{2}=12$ for $m_{1}=3, m_{2}=8$ for $m_{1}=4$, and $m_{2}=5$ for $m_{1}=5$. This reveals that our method of construction provides the best results for $m_{1}=3,4$. However, the method is not very effective for $m_{1}=5$. We will return to this example when we present other construction methods.

For designs of 64 runs, our construction method gives $m_{2}=28$ for $m_{1}=3, m_{2}=24$ for $m_{1}=4$, and $m_{2}=17$ for $m_{1}=5$. For $m_{1}=6$, choosing $G_{1}$ to be of resolution VI gives $m_{2}=16$. In all theses cases, the method maximizes $m_{2}$ for given $m_{1}$, as is clear from Table 2.1. This method does not produce a useful design for $m_{1}=7$ as it gives $m_{2}=0$. This case will be further examined later.

Method 2. Let $D^{*}$ and $D_{0}$ be two orthogonal arrays of $n$ runs such that $D^{*}$ has strength 4 and $D_{0}$ has strength 2 and that $D^{*} \subset D_{0}$. We use $I$ to denote a column of all plus ones of length $n$. Further let $a_{1}=(-1,-1,1,1)^{T}, a_{2}=(-1,1,-1,1)^{T}$ and $a_{1} a_{2}=(1,-1,-1,1)^{T}$.

Proposition 2.2. Design $D=\left(G_{1}, G_{2}\right)$ is a robust design, where

$$
G_{1}=a_{1} \otimes\left(I, D^{*}\right) \text { and } G_{2}=a_{1} a_{2} \otimes\left(I, D_{0}\right)
$$

The proof is straightforward. To prove Proposition 2.2, we need to show that $D=$ $\left(G_{1}, G_{2}\right)$ is of strength 3 and that $A_{40}=A_{31}=0$. All these can easily be verified using Lemma 2.3.

When $D_{0}$ is regular, the construction in Proposition 2.2 can be thought as a special case of that in Proposition 2.1. This can be seen by noting that $X=\left(a_{1}, a_{1} a_{2}\right) \otimes\left(I, D_{0}\right)$ is a saturated resolution IV design of $4 n$ runs when $D_{0}$ is a saturated resolution III design of $n$ runs, and that $G_{1}$, a subset of $X$, has resolution VI or higher when $D^{*}$ has resolution V.

The usefulness of Proposition 2.2 lies in the fact that it is applicable to non-regular designs. More importantly, this method provides robust designs for non-orthogonal requirement sets, a topic we will discuss in Section 2.3.2.

Method 3. This method is a modification of Method 2 but allows one extra column in $G_{1}$. Let $D_{0}, D^{*}, a_{1}, a_{2}$ be defined as in Method 2. Then Method 3 chooses

$$
\begin{equation*}
G_{1}=\left(a_{2} \otimes I, a_{1} \otimes I, a_{1} \otimes D^{*}\right) \text { and } G_{2}=a_{1} a_{2} \otimes\left(D_{0} \backslash D^{*}\right) \tag{2.3}
\end{equation*}
$$

Proposition 2.3. Let $G_{1}$ and $G_{2}$ be defined as in (2.3). Then $D=\left(G_{1}, G_{2}\right)$ is a robust design.

Similar to Method 2, Method 3 is applicable to both regular and non-regular designs and also allows non-orthogonal requirement sets. When applied to regular designs, Method 3 is not a special case of Method 1 as design $D=\left(G_{1}, G_{2}\right)$ given by Method 3 is not an even design. Even designs are those that can be obtained by selecting columns from a saturated resolution IV design. For designs of 32 runs, Method 3 gives a robust design with $m_{1}=5$ and $m_{2}=4$. For designs of 64 runs, Method 3 gives a robust design with $m_{1}=7$ and $m_{2}=10$, the same result as that
obtained by computer search given in Table 2.1. Note that Method 1 produces $m_{2}=0$ when $m_{1}=7$ for designs of 64 runs.

Method 4. This method can be viewed as a generalization of doubling (Chen and Cheng 2006; Cheng, Mee and Yee 2008). Let $D^{*}$ be an orthogonal array of strength 4 and $D_{0}$ be an orthogonal array of strength 3 such that $D^{*} \subset D_{0}$. Let

$$
\begin{equation*}
G_{1}=\binom{D^{*}}{D^{*}} \text { and } G_{2}=\binom{D_{0}}{-D_{0}} \tag{2.4}
\end{equation*}
$$

Proposition 2.4. Let $G_{1}$ and $G_{2}$ be defined as in (2.4). Then $D=\left(G_{1}, G_{2}\right)$ is a robust design.

If $D_{0}$ is chosen to be a saturated resolution IV design, then Method 4 is quite similar to Method 1. The generality of Method 4 is that $D_{0}$ can be chosen to be any other maximal design (Chen and Cheng 2006) than the saturated resolution IV design. Let $D^{*}$ and $D_{0}$ be the same maximal design with 16 runs and 5 factors. Then Method 4 gives $m_{1}=m_{2}=5$ for designs of 32 runs. Finally, Method 4 applies to non-regular designs and non-orthogonal requirement sets as well.

We present Method 5 through an example. Method 5 is similar to Method 3, but is only valid for regular designs. The purpose of Method 5 is to increase $m_{1}$ over that obtained with Method 3.

Example 2.1. Let $n=128$ and denote the columns for the independent factors from the full factorial design as $1, \ldots, 7$. Let $D^{*}$ be a resolution V design for 8 factors with generators from factors $2, \ldots, 7$, such as $8=2345$ and $9=4567$. We form $G_{1}$ with columns 1,2 , and all the columns of $D^{*}$, where if the column of $D^{*}$ is not generated by column 2, we take the Hadamard product of column 1 with that column. In this example, this gives $G_{1}=\{1,2,13,14,15,16,17,2345,14567\}$. Let $D_{0}$ be a saturated resolution III design based on factors $3, \ldots, 7$ including the column of 1 's, and $D_{0(12)}$ the Hadamard product of columns 1 and 2 with the columns of $D_{0}$. To form $G_{2}$, we will use a subset of the columns of $D_{0(12)}$. That $A_{40}=0$ and $A_{31}=0$ is immediate from the construction. To make $D=\left(G_{1}, G_{2}\right)$ strength 3 , we remove the columns $x_{i} x_{j}$ from $D_{0(12)}$, where $x_{i}, x_{j}$ are columns of $G_{1}$ such that $x_{i}$ is generated by factor 1
and $x_{j}$ generated by factor 2 . This corresponds to the removal of $7 \times 2=14$ columns. Then Method 5 gives a design with $m_{1}=9$ and $m_{2}=18$. In comparison, Method 1 gives $m_{1}=9$ and $m_{2}=1$. For $n=128$, the maximum $m_{1}$ for Method 3 gives a design with $m_{1}=8$ and $m_{2}=25$.

Method 5 follows the approach in Example 2.1, and is applicable when the resolution V design $D^{*}$ can be formed by more than one generator, as this provides the additional columns for $G_{1}$. In general, column 2 should be chosen as a factor that appears in the fewest number of generators in $D^{*}$. This minimizes the number of factors to be removed from $D_{0(12)}$ through the number of $x_{i} x_{j}$ terms.

### 2.3.2 Non-orthogonal Requirement Sets

A robust design, as constructed in Methods 1-5, is an orthogonal array $D=\left(G_{1}, G_{2}\right)$ of strength 3 such that $A_{40}=A_{31}=0$. The condition $A_{31}=0$ ensures robustness in that the 2 fi 's in the requirement set $G_{1} \times G_{1}$ are orthogonal to the set $G_{1} \times G_{2}$ of nonnegligible 2fi's. The condition $A_{40}=0$ says that $G_{1}$ is an orthogonal array of strength 4 , which guarantees orthogonal estimation of the requirement set $G_{1} \times G_{1}$. Instead of requiring $G_{1}$ to be an orthogonal array of strength 4 , we now introduce a weaker condition on $G_{1}$. A design is said to have property $A$ if it allows estimation of all main effects and all 2fi's. Property $A$ is weaker than strength 4. An orthogonal array of strength 4 has property $A$ as it allows orthogonal estimation of all main effects and all 2fis.

If $D=\left(G_{1}, G_{2}\right)$ is an orthogonal array of strength 3 that satisfies $A_{31}=0$ and $G_{1}$ has property $A$, then $D=\left(G_{1}, G_{2}\right)$ provides a design for the requirement set $G_{1} \times G_{1}$ that is robust to the nonnegligible 2fi's in $G_{1} \times G_{2}$. In Methods 2,3 and 4 , if we choose $D^{*}$ to have property $A$, then it can be easily checked that $G_{1}$ also has property $A$. We summarize this result in the following theorem.

Theorem 2.1. In Methods 2, 3 and 4, if $D^{*}$ has property $A$, then design $D=$ $\left(G_{1}, G_{2}\right)$
(i) allows estimation of the requirement set $G_{1} \times G_{1}$, and
(ii) is robust to the set $G_{1} \times G_{2}$ of nonnegligible 2fi's.

Note that the full requirement set is $G_{1} \times G_{1}$ plus all main effects. Design $D=$ $\left(G_{1}, G_{2}\right)$ in Theorem 2.1 supports this full requirement set follows from the fact $D$ is an orthogonal array of strength 3 . Theorem 2.1 is a powerful result. We give two examples to illustrate.

Example 2.2. Let $D_{0}$ be an orthogonal array of 12 runs for 11 factors and let $D^{*}$ consist of any four columns of $D_{0}$. According to Cheng (1995), design $D^{*}$ has property $A$. Using $D^{*}$ and $D_{0}$, Method 2 gives a robust design $D=\left(G_{1}, G_{2}\right)$ of 48 runs with $m_{1}=5$ and $m_{2}=12$ and Method 3 constructs a robust design $D=\left(G_{1}, G_{2}\right)$ of 48 runs with $m_{1}=6$ and $m_{2}=7$.

Example 2.3. Let $D_{0}$ be an orthogonal array of 20 runs for 19 factors and $D^{*}$ be a subarray of $D_{0}$ with five columns that has property $A$. The existence of such $D^{*}$ follows from the results in Loeppky, Sitter and Tang (2007). Applying Methods 2 and 3 , we obtain a robust design $D=\left(G_{1}, G_{2}\right)$ of 80 runs with $m_{1}=6$ and $m_{2}=20$, and a robust design $D=\left(G_{1}, G_{2}\right)$ of 80 runs with $m_{1}=7$ and $m_{2}=14$, respectively.

By introducing non-orthogonality, we gain in terms of estimation and robustness. The trade-off here is the loss of estimation efficiency. One common measure of the $D$-efficiency is given by $\left(X^{T} X / n\right)^{1 / p}$, where $X$ denotes the full model matrix and $p$ is the number of parameters. For the designs in Examples 2.2 and 2.3, we have found that their D-efficiencies are all well above $90 \%$. So at least for these examples, the efficiency loss is quite minor.

### 2.4 Catalog of Designs for 32 and 64 Runs

Chen, Sun, and Wu (1993) gave a complete catalog for non-isomorphic designs of resolution IV having 32 and 64 runs. We use these designs to search for robust designs of types 1,2 , and 3 . While the results in the previous section provide a method for constructing robust designs of type 1, we present here which designs from the catalog of Chen, Sun, and Wu (1993) these robust designs can be obtained from. In Tables 2.2 to 2.5, "parent design" refers to the design from Chen, Sun, and Wu (1993) which

Table 2.2: 32-Run Partially Clear Designs of Type 1

| $\left\|G_{1}\right\|$ | $\max (\operatorname{ma})$ | $\max \left(m_{c}\right)$ | parent design | Columns in $G_{1}$ | Columns in $G_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :--- |
| 6 | 6 | 6 | $6-1.1,6-1.2$ |  |  |
| 5 | 10 | 6 | $10-5.1$ | $(1,4,8,16,29)$ | $(2,7,11,19,30)$ |
| 4 | 12 | 7 | $12-7.1$ | $(16,19,21,25)$ | $(1,2,4,7,8,11$, <br> $13,14)$ |
| 3 | 15 | 9 | $15-10.1$ | $(7,8,16)$ | $(1,2,4,11,13$, <br> $14,19,21,22$, <br> $25,26,28)$ |

allows for the construction of the robust design and $\max (m)$ the maximum number of columns in a robust design for given $m_{1}$. For robust designs of types 2 and 3 , the results come from multiple parent designs, which can then be chosen according to some secondary criterion. Tables $2.2-2.5$ can be used to construct robust designs, as they specify a set of columns to form both $G_{1}$ and $G_{2}$. For 32 runs, only robust designs of type 1 provide useful results, as given in Table 2.2, which also lists the maximum number $\left(\max \left(m_{c}\right)\right)$ of columns allowable for a clear design.

For a design of type 1, additional type 1 designs can be created by moving an element from $G_{1}$ to $G_{2}$. If we have a design in which two-factor interactions in $G_{1} \times G_{2}$ are clear, moving an element from $G_{2}$ to $G_{1}$ results in a partially clear design of type 3. For the designs listed, further partially clear designs of the same type can be created by removing factors from either $G_{1}$ or $G_{2}$.

For $n=64$ and $m_{1}=6$, the partially clear design of type 1 is unique in that for all 15 non-isomorphic designs and all possible choices for columns in $G_{1}$, there is only one design that is partially clear of type 1 . In terms of minimum aberration of these designs, it is the 13 th out of 15 . Also of note for $n=64$ and $m_{1}=5$, the only choice for $G_{1}$ that results in a partially clear design of type 1 is achieved by moving an element from $G_{1}$ to $G_{2}$ from the partially clear design for $m_{1}=6$.

Table 2.3: 64-Run Partially Clear Designs of Type 1

| $\left\|G_{1}\right\|$ | $\max (\mathrm{m})$ | parent design | Columns in $G_{1}$ | Columns in $G_{2}$ |
| :---: | :---: | :---: | :--- | :--- |
| 7 | 17 | $17-11.2$ | $(16,19,29,32$, <br> $37,41,47)$ | $1,2,4,7,8,11,49$, <br> $55,59,62)$ |
| 6 |  |  | 22 | $22-16.13$ |
|  |  |  | $(32,35,37,41$, <br> $49,62)$ | $(1,2,4,7,8,11,13$, <br> $14,16,19,21,22,25$, <br> $26,28,31)$ |
| 5 | 22 | $22-16.13$ | $(32,35,37,41$, | $(1,2,4,7,8,11,13$, <br> $14,16,19,21,22,25$, <br> $26,28,31,62)$ |
|  |  |  | $49)$ | $(1,2,4,7,8,11,13$, <br>  |
|  | 28 | $28-22.1$ | $(49,50,52,56)$ |  |
|  |  |  |  | $14,16,19,21,22,25$, |
| $26,28,31,32,35,37$, |  |  |  |  |
| $38,41,42,44,47)$ |  |  |  |  |
| 3 | 31 | $31-25.1$ | $(14,16,32)$ | $(1,2,4,7,8,11,13$, <br>  |
|  |  |  |  | $19,21,22,25,26,28$, |
| $31,35,37,38,41,42$, |  |  |  |  |
| $44,47,49,50,52,55$, |  |  |  |  |
|  |  |  |  | $56,59,61)$ |

Table 2.4: 64 -Run Partially Clear Designs of Type 2

| $\left\|G_{1}\right\|$ | $\max (\mathrm{m})$ | parent design | Columns in $G_{1}$ | Columns in $G_{2}$ |
| :---: | :---: | :---: | :--- | :--- |
| 7 | 10 | $10-4.1$ | $(1,2,4,8,16,27,32)$ | $(7,43,53)$ |
|  |  | $10-4.2$ | $(2,4,7,8,16,25,32)$ | $(1,42,53)$ |
| 6 | 10 | $10-4.1$ | $(1,2,4,16,27,32)$ | $(7,8,43,53)$ |
|  |  | $10-4.2$ | $(1,2,4,8,16,32)$ | $(7,25,42,53)$ |
|  |  | $10-4.7$ | $(1,2,4,8,16,32)$ | $(7,25,42,52)$ |
| 5 | 11 | $11-5.1$ | $(1,16,29,32,51)$ | $(2,4,7,8,11,45)$ |
|  |  | $11-5.3$ | $(2,16,29,32,49)$ | $(1,4,7,8,11,46)$ |
|  |  | $11-5.4$ | $(11,16,32,46,56)$ | $(1,2,4,7,8,21)$ |
|  |  | $11-5.11$ | $(2,16,30,32,49)$ | $(1,4,7,8,11,13)$ |
| 4 | 11 | $11-5.1$ | $(1,16,29,32)$ | $(2,4,7,8,11,45,51)$ |
|  |  | $11-5.3$ | $(1,16,29,32)$ | $(2,4,7,8,11,46,49)$ |
|  |  | $11-5.4$ | $(8,16,32,46)$ | $(1,2,4,7,11,21,56)$ |
|  |  | $11-5.6$ | $(1,29,32,62)$ | $(2,4,7,8,11,16,19)$ |
|  |  | $11-5.7$ | $(8,16,32,57)$ | $(1,2,4,7,11,21,38)$ |
|  |  | $11-5.9$ | $(1,29,32,45)$ | $(2,4,7,8,11,16,19)$ |
|  |  | $11-5.11$ | $(1,16,30,32)$ | $(2,4,7,8,11,13,49)$ |
|  |  | $11-5.13$ | $(1,16,30,32)$ | $(2,4,7,8,11,13,46)$ |
|  |  | $11-5.20$ | $(13,16,32,53)$ | $(1,2,4,7,8,11,19)$ |
|  |  | $11-5.22$ | $(13,16,32,46)$ | $(1,2,4,7,8,11,19)$ |
|  |  | $11-5.24$ | $(13,19,32,61)$ | $(1,2,4,7,8,11,16)$ |
|  |  | $11-5.30$ | $(1,16,32,51)$ | $(2,4,7,8,11,13,14)$ |
|  |  | $11-5.31$ | $(2,16,32,49)$ | $(1,4,7,8,11,13,14)$ |
| 3 | 17 | $17-11.6$ | $(1,32,63)$ | $(2,4,7,8,11,13,14$, |
|  |  |  |  | $16,19,21,22,25,26$, |
|  |  |  |  | $28)$ |

Table 2.5: 64-Run Partially Clear Designs of Type 3

| $\left\|G_{1}\right\|$ | $\max (\mathrm{m})$ | parent design | Columns in $G_{1}$ | Columns in $G_{2}$ |
| :---: | :---: | :---: | :--- | :--- |
| 7 | 12 | $12-6.1$ | $(1,2,4,7,8,11,16)$ | $(29,32,45,51,62)$ |
|  |  | $12-6.8$ | $(1,2,4,7,8,11,13)$ | $(16,30,32,46,49)$ |
| 6 | 12 | $12-6.1$ | $(1,2,4,7,8,11)$ | $(16,29,32,45,51,62)$ |
|  |  | $12-6.2$ | $(8,16,32,46,54,56)$ | $(1,2,4,7,11,21)$ |
|  |  | $12-6.8$ | $(1,16,30,32,46,49)$ | $(2,4,7,8,11,13)$ |
| 5 | 11 | $11-5.1$ | $(1,16,29,32,45)$ | $(2,4,7,8,11,51)$ |
|  |  | $11-5.2$ | $(1,2,4,7,63)$ | $(8,16,25,32,42,52)$ |
|  |  | $11-5.4$ | $(8,16,32,46,56)$ | $(1,2,4,7,11,21)$ |
|  |  | $11-5.11$ | $(1,16,30,32,49)$ | $(2,4,7,8,11,13)$ |
|  |  | $11-5.13$ | $(1,16,30,32,46)$ | $(2,4,7,8,11,13)$ |
| 4 | 11 | $11-5.1$ | $(16,29,32,45)$ | $(1,2,4,7,8,11,51)$ |
|  |  | $11-5.2$ | $(1,2,4,7)$ | $(8,16,25,32,42,52$, |
|  |  |  |  | $63)$ |
|  |  | $11-5.3$ | $(1,16,32,49)$ | $(2,4,7,8,11,29,46)$ |
|  |  | $11-5.4$ | $(11,21,32,46)$ | $(1,2,4,7,8,16,56)$ |
|  |  | $11-5.6$ | $(1,29,32,62)$ | $(2,4,7,8,11,16,19)$ |
|  |  | $11-5.7$ | $(8,16,32,57)$ | $(1,2,4,7,11,21,38)$ |
|  |  | $11-5.9$ | $(16,29,32,45)$ | $(1,2,4,7,8,11,19)$ |
|  |  | $11-5.11$ | $(1,16,32,49)$ | $(2,4,7,8,11,13,30)$ |
|  |  | $11-5.13$ | $(16,30,32,46)$ | $(1,2,4,7,8,11,13)$ |
|  |  | $11-5.20$ | $(13,16,32,53)$ | $(1,2,4,7,8,11,19)$ |
| 3 | 17 | $17-11.6$ | $(1,32,63)$ | $(2,4,7,8,11,13,14$, |
|  |  |  |  | $16,19,21,22,25,26$, |
|  |  |  |  | $28)$ |

### 2.5 Discussion and Further Results

The following two problems of theoretical importance arise from robust designs of type 1. One is to construct such robust designs that maximize $m_{2}$ for given $m_{1}$, and the other is to construct robust designs with $m_{1}$ maximized. For designs of 32 and 64 runs, we have shown in Section 2.3 that our methods allow such designs to be found. Although completely solving these problems is likely to be quite nontrivial, it would be possible to obtain some useful general results. We leave the problems to future research.

We conclude this chapter with a result on the maximum number of clear 2 f 's in an orthogonal array of strength 3 .

Theorem 2.2. We have that

$$
\beta(n, m) \leq m(n-2 m) /(m-2)
$$

where $\beta(n, m)$ denotes the number of clear 2fi's in an orthogonal array of strength 3 with $n$ runs for $m$ factors.

Proof. Let $D=\left(a_{1}, \ldots, a_{m}\right)$ be an orthogonal array of strength 3 with $n$ runs for $m$ factors. Suppose that it has $\beta$ clear 2 fi 's and we use $c_{1}, \ldots, c_{\beta}$ to denote the column vectors corresponding to these clear 2 fi 's. Further let $R_{0}$ denote the $n-1$ dimensional vector space that collects all vectors with $n$ real entries that are orthogonal to the vector of all plus ones. Then $a_{1}, \ldots, a_{m}, c_{1}, \ldots, c_{\beta}$ are mutually orthogonal and all belong to $R_{0}$. Let $R_{1}$ denote the linear subspace of $R_{0}$ that consists of all vectors that are orthogonal to $a_{1}, \ldots, a_{m}, c_{1}, \ldots, c_{\beta}$. Then $R_{1}$ is a $p$ dimentional subspace with $p=n-1-m-\beta$. Let $d_{1}, \ldots, d_{p}$ be an orthonormal basis of $R_{1}$. That is, $d_{1}, \ldots, d_{p}$ are orthogonal vectors with length unity in $R_{1}$. Since $a_{1} a_{2}, \ldots, a_{1} a_{m}$ are mutually orthogonal, for any given $k=1, \ldots, p$, we have

$$
\left|<a_{1} a_{2} / \sqrt{n}, d_{k}>\left.\right|^{2}+\cdots+\left|<a_{1} a_{m} / \sqrt{n}, d_{k}>\right|^{2} \leq 1\right.
$$

where $\langle x, y\rangle$ denotes the inner product of two vectors $x$ and $y$. Equivalently, we have

$$
\left|<a_{1} a_{2}, d_{k}>\left.\right|^{2}+\cdots+\left|<a_{1} a_{m}, d_{k}>\right|^{2} \leq n\right.
$$

Noting that $<a_{1} a_{1}, d_{k}>=0$, we obtain $\sum_{j=1}^{m}\left|<a_{1} a_{j}, d_{k}>\right|^{2} \leq n$. In general, we have $\sum_{j=1}^{m}\left|<a_{i} a_{j}, d_{k}>\right|^{2} \leq n$, for every $i=1, \ldots, m$. Upon combining, we obtain

$$
\sum_{1 \leq i<j \leq m}\left|<a_{i} a_{j}, d_{k}>\right|^{2} \leq n m / 2
$$

Summing over $k$ and re-arranging, we obtain

$$
\begin{equation*}
\sum_{1 \leq i<j \leq m} \sum_{k=1}^{p}\left|<a_{i} a_{j}, d_{k}>\right|^{2} \leq p n m / 2 . \tag{2.5}
\end{equation*}
$$

Now consider $a_{i} a_{j}$ for fixed $i, j$. This 2 fi is orthogonal to $R_{1}$ if it is clear and belongs to $R_{1}$ otherwise. Therefore, we have that $\sum_{k=1}^{p}\left|<a_{i} a_{j}, d_{k}>\right|^{2}$ is equal to 0 if $a_{i} a_{j}$ is clear and is equal to $n$ otherwise. We thus obtain $\binom{m}{2}-\beta \leq p m / 2$. Substituting $p=n-1-m-\beta$ in and solving for $\beta$, we obtain $\beta \leq m(n-2 m) /(m-2)$.

The same result for regular designs was obtained earlier in Tang, Ma, Ingram and Wang (2002). However, the proof of Theorem 2.2 does not follow from the idea of proving the result for regular designs.

In the case of regular designs, the bound in Theorem 2.2 can be slightly improved; see Wu and Wu (2002) and Yang and Butler (2008).

## Chapter 3

## Multi-Level Orthogonal Arrays for Estimating Main Effects and Specified Interactions

### 3.1 Introduction

In this chapter, we examine the requirement set problem for orthogonal arrays with more than two levels. There are two key differences when handling more than two levels. Firstly, main effects and interaction terms can be broken up into orthogonal components, of which only some may be of interest. Secondly, level permutations of the factors become an issue when using these orthogonal components. This chapter investigates how these differences impact searching for designs. In Chapter 2, we enforced orthogonality between the nonnegligible effects and the effects in the requirement set. This chapter relaxes the requirement of orthogonality, but we will still attempt to find designs with robust properties.

Given a factor with $q$ levels, the main effect for this factor has $q-1$ degrees of freedom and can be broken up into $q-1$ orthogonal contrasts. In this chapter, we adopt the approach from Xu and Wu (2001) which uses normalized main effect contrasts so that all factorial effects have the same variance if a full factorial design
were used. For example, for a three-level factor, we have linear and quadratic main effects coded as $(-1,0,1) \times \sqrt{3} / \sqrt{2}$ and $(1,-2,1) / \sqrt{2}$, respectively. For convenience, our discussion refers to the linear and quadratic contrasts for the three-level coding, but the results presented hold for any set of orthogonal contrasts.

A two-factor interaction has $(q-1)^{2}$ orthogonal components, each corresponding to one degree of freedom. In a two-level design, this implies that a two-factor interaction corresponds to one component and, as such, one degree of freedom. For a three-level design, a two-factor interaction has four orthogonal components corresponding to linear-by-linear, linear-by-quadratic, quadratic-by-linear, and quadratic-by-quadratic effects.

The experimenter may be interested in only a subset of the two-factor interaction components. In this chapter we consider experiments in which the experimenter is interested in a model with the grand mean, all main effects, and certain twofactor interaction components, the set of which we call the requirement set. For a requirement set $S$, define the core set, $C(S)$, as the subset of $S$ which includes all two-factor interaction components in $S$ and all main effects for which a main effect component occurs in one of the two-factor interaction components in $S$. For example, consider a special case of the second-order model as in Cheng and Wu (2001) and Xu , Cheng and Wu (2004) in which some or all of the linear-by-linear two-factor interaction components are of interest:

$$
\begin{equation*}
y=\beta_{0}+\sum_{i=1}^{r+k} \beta_{i} x_{i l}+\sum_{i=1}^{r+k} \beta_{i i} x_{i q}+\sum_{1 \leq i<j \leq r} \beta_{i j} x_{i l} x_{j l}+\epsilon, \tag{3.1}
\end{equation*}
$$

where $y$ is the response, $\epsilon$ the error term, $x_{i l}$ and $x_{i q}$ the linear and quadratic contrast coefficients for factor $i$ and $\beta_{i}, \beta_{i i}$ their corresponding effects, $\beta_{i j}$ the linear-by-linear two-factor interaction for factors $i$ and $j$, and $\beta_{0}$ is the grand mean. Cheng and Wu (2001) and Xu , Cheng and Wu (2004) consider the case where $k=0$. If $r=3$ and $k=2$, the requirement set is $\left\{\beta_{0}, \beta_{1}, \beta_{11}, \beta_{2}, \beta_{22}, \beta_{3}, \beta_{33}, \beta_{4}, \beta_{44}, \beta_{5}, \beta_{55}, \beta_{12}, \beta_{13}, \beta_{23}\right\}$, while the core set is $\left\{\beta_{0}, \beta_{1}, \beta_{11}, \beta_{2}, \beta_{22}, \beta_{3}, \beta_{33}, \beta_{12}, \beta_{13}, \beta_{23}\right\}$. We will consider Model (3.1) again in Section 3.6. A design is said to support a requirement set if all effects in the requirement set are estimable. If we have multiple non-isomorphic designs that
support a given requirement set, we often distinguish between these designs using one or more optimality criteria, some of which will be discussed in the next section.

If the experimenter believes that curvature may exist in the experimental region and only wants to run one experiment, this can be explored with the designs discussed in this chapter. One common approach for testing the presence of curvature is the addition of center points to a two-level design. The addition of center points can only detect if there is curvature. Here we want to estimate the effects with the design. Other designs for this purpose include Box-Behnken designs (Box and Behnken, 1960) and central composite designs (Box and Wilson, 1951).

In Section 3.2 we introduce various design criteria and Section 3.3 examines how these are affected by level permutations. Section 3.4 investigates the existence of designs for a requirement set when a saturated orthogonal array exists. Section 3.5 uses the results to propose a method of searching for designs from a saturated orthogonal array, which is then applied to 27 -run designs under a second-order model in Section 3.6. We conclude the chapter with a brief discussion in Section 3.7.

### 3.2 Optimality Criteria

In this section we look at some optimality criteria which can be used to differentiate between designs that support a given requirement set $S$. Consider a model that includes all effects in the requirement set:

$$
\begin{equation*}
Y=X_{M} \beta_{M}+\epsilon \tag{3.2}
\end{equation*}
$$

where $Y$ is the vector of $n$ observations, $\beta_{M}$ is the vector of effects in our requirement set, $X_{M}$ is the corresponding matrix of contrast coefficients, and $\epsilon$ is the vector of $n$ independent random errors. The matrix $X_{M}$ is also referred to as the model matrix. Define

$$
\begin{equation*}
M=X_{M}^{T} X_{M} / n=\left[m_{i j}\right] \tag{3.3}
\end{equation*}
$$

as the moment matrix. A D-optimal design maximizes $|M|$, the determinant of $M$, and minimizes the volume of the joint confidence region on the vector of regression coefficients. For a design that supports $S$, its D-efficiency is defined as

$$
D_{e f f}=\left|X_{M}^{T} X_{M} / n\right|^{1 / p}
$$

where $p$ is the number of parameters in the model (ie. the size of the requirement set). If $D_{\text {eff }}=1$, the columns of $X_{M}$ are orthogonal.

There are other optimality criteria that use the moment matrix. These include the $E\left(s^{2}\right)$ criterion, where

$$
E\left(s^{2}\right)(M)=\sum_{1 \leq i<j \leq p} m_{i j}^{2} /\binom{p}{2},
$$

and $E$-optimality, which maximizes the minimum eigenvalue of the moment matrix and minimizes the maximum possible variance of a normalized linear function of $\beta_{M}$. Another common criterion is the $A$-optimality which minimizes $\operatorname{tr}\left(M^{-1}\right)$, the trace of the inverse of the moment matrix, and minimizes the average variance of the estimates of the regression coefficients.

In this chapter, our first means of differentiating between designs is to look at the D-efficiency. After D-efficiency, we further distinguish designs by measuring their robustness to nonnegligible effects outside of the requirement set.

To get an estimate for $\beta_{M}$ in Model (3.2), we use $\hat{\beta}_{M}=\left(X_{M}^{T} X_{M}\right)^{-1} X_{M}^{T} Y$. This estimate is unbiased for $\beta_{M}$ if Model (3.2) is true. While the model we fit is (3.2), assume that the true model is actually

$$
\begin{equation*}
Y=X_{0} \beta_{0}+X_{1} \beta_{1}+X_{2} \beta_{2}+\boldsymbol{\epsilon} \tag{3.4}
\end{equation*}
$$

where $X_{1}$ and $X_{2}$ correspond to the matrix of contrast coefficients for the main effects, $\beta_{1}$, and two-factor interactions, $\beta_{2}, X_{0}$ the vector of ones with $\beta_{0}$ being the grand mean, and $\epsilon$ the error term. Note that $X_{M}$ contains the columns of $X_{0}, X_{1}$, and a subset of columns from $X_{2}$. An alternative way to write (3.4) is then

$$
\begin{equation*}
Y=X_{M} \beta_{M}+X_{2 o} \beta_{2 o}+\epsilon, \tag{3.5}
\end{equation*}
$$

where $\beta_{2 o}$ refers to the two-factor interaction components from $\beta_{2}$ outside of the requirement set, and $X_{2 o}$ the corresponding matrix of contrast coefficients.

In order to differentiate between designs having the same D-efficiency, we can minimize the contamination of nonnegligible two-factor interactions outside of the model. Due to restrictions on the run size, Model (3.5) may not be estimable. If we fit Model (3.2) when (3.5) is the truth,

$$
\begin{aligned}
E\left(\hat{\beta_{M}}\right) & =E\left(\left(X_{M}^{T} X_{M}\right)^{-1} X_{M}^{T} Y\right) \\
& =\left(X_{M}^{T} X_{M}\right)^{-1} X_{M}^{T} E(Y) \\
& =\left(X_{M}^{T} X_{M}\right)^{-1} X_{M}^{T}\left(X_{M} \beta_{M}+X_{2 o} \beta_{2 o}\right) \\
& =\beta_{M}+\left(X_{M}^{T} X_{M}\right)^{-1} X_{M}^{T} X_{2 o} \beta_{2 o} \\
& =\beta_{M}+C_{2 o} \beta_{2 o} .
\end{aligned}
$$

In this chapter we call $C_{2 o}=\left[c_{i j}\right]=\left(X_{M}^{T} X_{M}\right)^{-1} X_{M}^{T} X_{2 o}$ the alias matrix. Because of $\beta_{2 o}$, estimation of $\beta_{M}$ is contaminated by these nonnegligible two-factor interactions outside of the model, so ideally $C_{2 o}$ should be small. One measure of this contamination is through the minimum contamination criterion which minimizes $\left\|C_{2 o}\right\|^{2}=\sum c_{i j}^{2}$ (Tang and Deng, 1999, Xu and Wu, 2001, and Steinberg and Bursztyn, 2001).

In Equation (3.5), $X_{2 o}$ is the matrix of contrast coefficients corresponding to all two-factor interactions outside of the requirement set. In practice, linear-by-linear two-factor interactions are more often active than other interactions (Xu, Cheng and $\mathrm{Wu}, 2004$ ), so one could consider the contamination of linear-by-linear two-factor interactions not in the requirement set. In this case, we would minimize $\left\|C_{2 l}\right\|^{2}$, with

$$
\begin{equation*}
\left\|C_{2 l}\right\|^{2}=\left\|\left(X_{M}^{T} X_{M}\right)^{-1} X_{M}^{T} X_{2 l}\right\|^{2}, \tag{3.6}
\end{equation*}
$$

where $X_{2 l}$ is the subset of columns from $X_{2 o}$ that correspond to the linear-by-linear components outside of the model. If we let $X_{2 q}$ be the matrix consisting of the remaining columns from $X_{2 o}$ whose components contain a quadratic effect (ie. $X_{2 q}=$ $X_{2 o} \backslash X_{2 l}$ ), we have

$$
\begin{equation*}
\left\|C_{2 q}\right\|^{2}=\left\|\left(X_{M}^{T} X_{M}\right)^{-1} X_{M}^{T} X_{2 q}\right\|^{2} . \tag{3.7}
\end{equation*}
$$

To minimize the contamination from the two-factor interactions containing a quadratic main effect, we would minimize $\left\|C_{2 q}\right\|^{2}$. Combining (3.6) and (3.7), we have

$$
\begin{equation*}
\left\|C_{2 o}\right\|^{2}=\left\|C_{2 l}\right\|^{2}+\left\|C_{2 q}\right\|^{2} \tag{3.8}
\end{equation*}
$$

In our discussion of contamination of nonnegligible effects, our consideration has only been placed on two-factor interactions. In general, we can consider $i$-factor interactions through $C_{i}=\left(X_{M}^{T} X_{M}\right)^{-1} X_{M}^{T} X_{i}$, where $X_{i}$ is the matrix of contrast coefficients for the $i$-factor interactions and measure the contamination of these effects with $\left\|C_{i}\right\|^{2}$. If we use the hierarchical ordering principle, on the effects outside of the requirement set, we can sequentially minimize $\left\|C_{2 o}\right\|^{2},\left\|C_{3}\right\|^{2}, \ldots$ as a way to distinguish between designs.

### 3.3 Level Permutations

In this section we look at the effect of permuting the levels of a factor on the properties of a design. For multi-level designs, because of the multiple components for main effects and interactions, level permutations of factors can impact a design to the point that one set of level permutations supports a requirement set while another does not for the same design. Remark 2.3.1 from Dey and Mukerjee (1999) states that their main results do not depend on the choice of level permutations. While this holds in situations where we are interested in all components of interactions, this invariance to level permutations may no longer hold if our requirement set contains only certain interaction components. In this section, we examine the influence of level permutations on the optimality criteria of the previous section. The results focus on level permutations on factors outside of the core set, as it will be shown that such level permutations retain many design properties. Throughout this chapter we assume that our design is an orthogonal array, implying main effects are orthogonal and the lack of orthogonality with effects in the requirement set comes through the two-factor interaction components in the requirement set.

Throughout this chapter, we will make use of the fact that this level permutation can be performed via an orthonormal matrix $Q_{1}$, where $Q_{1}^{T} Q_{1}=I$. Let $A$ denote the $n \times(q-1)$ matrix containing the $q-1$ vectors of contrast coefficients for the main effects of a factor. Let $A^{*}$ consist of the vectors of contrast coefficients for the same factor after a level permutation. The relationship between $A^{*}$ and $A$ can be expressed
with the $(q-1) \times(q-1)$ orthonormal matrix $Q_{1}$ as

$$
\begin{equation*}
A^{*}=A Q_{1} \tag{3.9}
\end{equation*}
$$

For example, consider the case of three levels and six runs in which we have linear and quadratic main effects for a factor such that $A$ is

$$
A=\left(\begin{array}{cc}
-\sqrt{3} / \sqrt{2} & 1 / \sqrt{2} \\
-\sqrt{3} / \sqrt{2} & 1 / \sqrt{2} \\
0 & -2 / \sqrt{2} \\
0 & -2 / \sqrt{2} \\
\sqrt{3} / \sqrt{2} & 1 / \sqrt{2} \\
\sqrt{3} / \sqrt{2} & 1 / \sqrt{2}
\end{array}\right)
$$

and we consider a level permutation to $A^{*}$ given by

$$
A^{*}=\left(\begin{array}{cc}
0 & -2 / \sqrt{2} \\
0 & -2 / \sqrt{2} \\
\sqrt{3} / \sqrt{2} & 1 / \sqrt{2} \\
\sqrt{3} / \sqrt{2} & 1 / \sqrt{2} \\
-\sqrt{3} / \sqrt{2} & 1 / \sqrt{2} \\
-\sqrt{3} / \sqrt{2} & 1 / \sqrt{2}
\end{array}\right) .
$$

We can then obtain $Q_{1}$ as

$$
Q_{1}=A^{T} A^{*} / n=\left(\begin{array}{cc}
-1 / 2 & \sqrt{3} / 2 \\
-\sqrt{3} / 2 & -1 / 2
\end{array}\right)
$$

From equation (3.9), we have a matrix $Q_{1}$ to perform a level permutation on a factor in the model. Consider a level permutation on a factor outside of the core set. Denote $X_{M}$ as the model matrix before the level permutation, and $X_{M}^{*}$ as the model matrix after the level permutation. If we let

$$
Q=\left(\begin{array}{cc}
Q_{1} & 0 \\
0 & I_{p-(q-1)}
\end{array}\right)
$$

we can relate $X_{M}$ and $X_{M}^{*}$ through

$$
\begin{equation*}
X_{M}^{*}=X_{M} Q \tag{3.10}
\end{equation*}
$$

where, without loss of generality, the first $q-1$ columns of $X_{M}$ correspond to the main effect components of the level permuted factor. The moment matrix after level permutations, $M^{*}$, is then

$$
\begin{equation*}
M^{*}=\left(X_{M} Q\right)^{T}\left(X_{M} Q\right) / n=Q^{T} M Q \tag{3.11}
\end{equation*}
$$

where $M$ is the moment matrix before the level permutation.
At this point, it is important to make the distinction between level permutations of factors inside the core set versus factors outside of the core set. A level permutation of a factor inside the core set causes permutations of the two-factor interaction components in the requirement set, which may not allow us to write $X_{M}^{*}$ in the form given in (3.10). With this formulation of level permutations in mind, we can now look at the effect on the optimality criteria from the previous section.

Theorem 3.1. When the levels of a factor outside of the core set are permuted, the $D, A, E$, and $E\left(s^{2}\right)$ criteria remain the same.

Proof. In order to establish that the level permutations outside of the core set preserve the criteria, we will start by showing that the eigenvalues of $M^{*}$ and $M$ are the same. Let $Q$ be the orthonormal matrix that performs the level permutation as in (3.10). We have $M^{*}=Q^{T} M Q$ from (3.11). Recall that the characteristic equation of a square matrix $W$ is: $0=|W-\lambda I|$. The roots of the characteristic equation are the eigenvalues of $W$. The characteristic equation of $M^{*}$ is $0=\left|M^{*}-\lambda I\right|=$ $\left|\left(Q^{T} M Q\right)-\lambda I\right|=\left|Q^{T}(M-\lambda I) Q\right|=\left|Q^{T}\right||M-\lambda I||Q|=|M-\lambda I|$. So $M^{*}$ and $M$ have the same characteristic equation, and as such the same eigenvalues.

Recalling that the determinant of a matrix is the product of its eigenvalues, and the trace equal to the sum of its eigenvalues, we have that the $\mathrm{D}, A$, and $E$ criteria are the same after level permutation.

For $E\left(s^{2}\right)$, note that $E\left(s^{2}\right)(M)=\operatorname{tr}\left(M^{T} M\right) / 2-\sum m_{i i}^{2}$. Since we assume normalized length for contrast coefficients, $m_{i i}^{2}=n$ for all $i$, which remains true after a level
permutation. Examining $M^{T} M$, we have $\operatorname{tr}\left(M^{* T} M^{*}\right)=\operatorname{tr}\left(\left(Q^{T} M Q\right)^{T}\left(Q^{T} M Q\right)\right)=$ $\operatorname{tr}\left(M^{T} M\right)$, implying $E\left(s^{2}\right)(M)=E\left(s^{2}\right)\left(M^{*}\right)$.

Theorem 3.1 tells us that we can make level permutations to factors outside of the core set without impacting many criteria. What Theorem 3.1 does not address is if level permutations affect the contamination of the two-factor interaction components outside of the requirement set. This connection is not immediately obvious, since measurement of this contamination is based on $X_{M}$ and $X_{2 o}$, both of which are changed after level permutations. We address this relationship in the next theorem.

Theorem 3.2. When the levels of a factor outside of the core set are permuted, the contamination of the two-factor interaction components outside of the model, measured by $\left\|C_{2 o}\right\|^{2}$, remains the same.

Proof. After the level permutation, $\left\|C_{2 o}\right\|^{2}=\left\|\left(X_{M}^{T} X_{M}\right)^{-1} X_{M}^{T} X_{2 o}\right\|^{2}$ becomes $\left\|C_{2 o}^{*}\right\|^{2}=\left\|\left(X_{M}^{* T} X_{M}^{*}\right)^{-1} X_{M}^{* T} X_{2 o}^{*}\right\|^{2}$. In order to show $\left\|C_{2 o}\right\|^{2}=\left\|C_{2 o}^{*}\right\|^{2}$, we will look at the relationship between $X_{M}^{*}$ and $X_{M}$ and $X_{2 o}^{*}$ and $X_{2 o}$. We will make use of the fact that

$$
\|C\|^{2}=\operatorname{tr}\left(C^{T} C\right)
$$

Let $Q$ be the orthonormal matrix from (3.10) such that $X_{M}^{*}=X_{M} Q$. Since $Q^{T}=Q^{-1}$, we can simplify the terms of $\left\|C_{2 o}^{*}\right\|^{2}$ involving $X_{M}^{*}$ to

$$
\left(X_{M}^{* T} X_{M}^{*}\right)^{-1} X_{M}^{* T}=Q^{T}\left(X_{M}^{T} X_{M}\right)^{-1} X_{M}^{T}
$$

As the matrix $X_{2 o}$ corresponds to the contrast coefficients for all two-factor interaction components outside of the model, it can be partitioned into sets of two-factor interaction components corresponding to pairs of the $m$ factors in the model. We denote this partition as $X_{2 o p}$ for $p \in\left(1, \ldots,\binom{m}{2}\right)$, where $p$ is an index for a pair of factors from the design. For $p$ indexing a pair of factors in the core set, $X_{2 o p}$ may not contain the full set of $(q-1)^{2}$ two-factor interaction components, as some of these components may be in the requirement set. From this partition of $X_{2 o}$, we have that

$$
\begin{equation*}
\left\|C_{2 o}\right\|^{2}=\sum_{p}\left\|\left(X_{M}^{T} X_{M}\right)^{-1} X_{M}^{T} X_{2 o p}\right\|^{2} \tag{3.12}
\end{equation*}
$$

Consider $i$ for which $X_{2 o i}$ corresponds to a set of two-factor interaction components for a pair of factors that have not had levels permuted. After the level permutation, we have $X_{2 o i}^{*}=X_{2 o i}$. Looking at the effect of this subset of $X_{2 o}$ on $\left\|C_{2 o}^{*}\right\|^{2}$ after level permutation,

$$
\begin{equation*}
\left\|Q^{T}\left(X_{M}^{T} X_{M}\right)^{-1} X_{M}^{T} X_{2 o i}\right\|^{2}=\left\|\left(X_{M}^{T} X_{M}\right)^{-1} X_{M}^{T} X_{2 o i}\right\|^{2} \tag{3.13}
\end{equation*}
$$

showing that the contamination is unchanged for the pair of factors indexed by $i$.
The remaining columns of $X_{2 o}$ correspond to two-factor interaction components a pairs of columns in which one involves a level permutation. Let $X_{2 o j}$ correspond to the $n \times(q-1)^{2}$ subset for such a pair (all of the interaction components are outside of the requirement set, otherwise the factor with a level permutation would be in the core set). Let $X_{2 o j}^{*}$ be the same matrix after level permutations, and in a similar fashion to (3.9), there is a $(q-1)^{2} \times(q-1)^{2}$ orthonormal matrix $Q_{2}$ such that $X_{2 o j}^{*}=X_{2 o j} Q_{2 j}$. The influence on $\left\|C_{o 2}^{*}\right\|^{2}$ is then

$$
\begin{equation*}
\left\|Q^{T}\left(X_{M}^{T} X_{M}\right)^{-1} X_{M}^{T} X_{2 o j} Q_{2 j}\right\|^{2}=\left\|\left(X_{M}^{T} X_{M}\right)^{-1} X_{M}^{T} X_{2 o j}\right\|^{2} \tag{3.14}
\end{equation*}
$$

which is also unchanged from the level permutation. Combining (3.13) and (3.14) into (3.12), we have the required result.

From the proof of Theorem 3.2, we have not only that the overall contamination from two-factor interaction components remains the same, but also the componentwise contamination from each pair of factors. As a result, if we were to sequentially minimize these pairwise contaminations starting from the largest, in a manner analogous to G-aberration, the ranking would not change after level permutations outside of the core set. Likewise, if contamination is only considered for interactions involving one or more factors from the core set, nothing is changed. One situation Theorem 3.2 does not apply to is when contamination is to be considered for certain twofactor interaction components, such as all linear-by-linear components outside of the requirement set.

Our discussion in this section assumes that we have a design to support a requirement set. The next section discusses the existence of such designs from orthogonal arrays.

### 3.4 Existence of Designs for a Given Requirement Set through Orthogonal Arrays

Tang and Zhou (2009) showed that for two-level designs, the existence of an orthogonal array that supports a requirement set is equivalent to that of an orthogonal array that supports its core set. For designs with greater than two levels, a similar result holds, but the fact that main effects and interactions have multiple components presents some restrictions. While only some components of interactions may be in the requirement set, all main effect components are in the requirement set. In this section, we consider removing columns representing main effect components from a saturated orthogonal array that occupy the same space as the interaction components in the core set. If any one of the main effect components of a factor occupies the space of the interaction components, we remove all remaining main effect components for that factor. The results are presented based on three-level designs, as they are most powerful in this case, but the arguments are applicable for greater than three levels.

Theorem 3.3. For a requirement set $S$ with $m$ factors and e two-factor interaction components, if an orthogonal array that supports the core set $C(S)$ with $m_{1}$ factors exists, then an orthogonal array that supports $S$ exists, provided $m \leq(n-1) / 2-e$.

Proof. Consider a matrix $D$ constructed from a saturated three-level orthogonal array where each column is replaced by normalized main effect contrasts, such as the coding for linear and quadratic main effects, and a column of all 1's added to the array. We will show that so long as there is a subset $D_{1}$ of $D$ that supports $C(S)$, there exists a partition $D=\left(D_{1}, D_{2}, D_{3}\right)$, where $D_{2}$ is the set of columns in $S$ that are not in $C(S)$ and $D_{3}$ are the columns to be removed from $D$ that occupy the same space as the vectors corresponding to interaction components. We will denote the number of factors in $D_{1}, D_{2}$, and $D_{3}$ as $m_{1}, m_{2}$, and $m_{3}$, respectively, and $m=m_{1}+m_{2}$ is the total number of factors in the requirement set.

Let $X_{1 c}=\left(\mathbf{1}, D_{1}\right)$, where $\mathbf{1}$ is the column of all 1 s corresponding to the grand mean and $D_{1}$ a subset of $D$ that supports $C(S)$. The model matrix for $C(S)$ is then $\left(X_{1 c}, X_{2 c}\right)$, where $X_{2 c}$ corresponds to the interaction components in the core set. Since
$D$ is a saturated orthogonal array, the partition $D=\left(X_{1 c}, D_{2}, D_{3}\right)$ forms the column space of $D$. If we let $D^{*}=\left(D_{2}, D_{3}\right)$, by the properties of block matrices,

$$
\begin{align*}
& \operatorname{det}\left[\left(X_{1 c}, X_{2 c}\right)^{T}\left(X_{1 c}, X_{2 c}\right)\right] \\
= & \operatorname{det}\left(X_{1 c}^{T} X_{1 c}\right) \operatorname{det}\left(X_{2 c}^{T} X_{2 c}-X_{2 c}^{T} X_{1 c}\left(X_{1 c}^{T} X_{1 c}\right)^{-1} X_{1 c}^{T} X_{2 c}\right) \\
= & \operatorname{det}\left(X_{1 c}^{T} X_{1 c}\right) \operatorname{det}\left(X_{2 c}^{T} D^{*} D^{* T} X_{2 c}\right) \\
= & n^{2 m_{1}+1-e} \operatorname{det}\left(X_{2 c}^{T} D^{*} D^{* T} X_{2 c}\right) . \tag{3.15}
\end{align*}
$$

Since $D_{1}$ supports $C(S), \operatorname{det}\left(X_{2 c}^{T} D^{*} D^{* T} X_{2 c}\right)>0$, so $X_{2 c}^{T} D^{*} D^{* T} X_{2 c}$ must be of rank $e$. Recalling that $\operatorname{rank}\left(A A^{T}\right)=\operatorname{rank}(A)$, this also implies that $X_{2 c}^{T} D^{*}$ has rank $e$.

The dimensions of $X_{2 c}^{T} D^{*}$ is $e \times\left(2 m_{2}+2 m_{3}\right)$. In order for $X_{2 c}$ to be estimated, it must be that $\left(2 m_{2}+2 m_{3}\right) \geq 2 m_{3} \geq e$. Then $X_{2 c}^{T} D^{*}$ must contain a subset of $2 m_{3}$ columns that has rank $e$, corresponding to $m_{3}$ factors (each with two main effect components) from $D^{*}$. As the columns in $D_{3}$ are pairs of main effect components for factors, the preceding bound is $m_{3} \geq\lceil e / 2\rceil$. It is possible that there exists a subset of $m_{3}^{*}<m_{3}$ columns such that $X_{2 c}^{T} D^{*}$ has a subset of $2 m_{3}^{*}$ columns with rank $e$.

Let $D_{3}^{*}$ be a subset of $2 m_{3}$ columns from $D^{*}$, so that $\operatorname{rank}\left(X_{2 c}^{T} D_{3}^{*}\right)=e$. This also gives us $\operatorname{rank}\left(X_{2 c}^{T} D_{3}^{*} D_{3}^{* T} X_{2 c}\right)=e$ and

$$
\begin{equation*}
\operatorname{det}\left(X_{2 c}^{T} D_{3}^{*} D_{3}^{* T} X_{2 c}\right)>0 \tag{3.16}
\end{equation*}
$$

With $D^{*}=\left(D_{2}^{*}, D_{3}^{*}\right)$, let $X=\left(X_{1}, D_{2}^{*}\right)$ be be the model matrix for the requirement set $S$. Similar to (3.15), we have

$$
\begin{equation*}
\operatorname{det}\left[\left(X, X_{2 c}\right)^{T}\left(X, X_{2 c}\right)\right]=n^{2 m+1-e} \operatorname{det}\left(X_{2 c}^{T} D_{3}^{*} D_{3}^{* T} X_{2 c}\right) \tag{3.17}
\end{equation*}
$$

which is greater than 0 by (3.16), and implies that ( $D_{1}, D_{2}^{*}$ ) supports $S$. If there exists $m_{3}^{*}$ as described above, $S$ can accommodate additional factors.

As an alternative to the proof from Theorem 3.1, (3.17) shows that level permutations on factors outside of the core set, as represented by $D_{2}$, do not effect the D-efficiency of the design. The connection to D-efficiency and $D_{3}$ is also seen through the following theorem.

Theorem 3.4. The $D$-efficiency is maximized when $\operatorname{det}\left(X_{2 c}^{T} D_{3}^{*} D_{3}^{* T} X_{2 c}\right)$ is maximized.
Theorem 3.3 provides a lower bound on the maximum number of factors outside of the core set that can be supported. In the formation of $D_{3}^{*}$ in the proof of Theorem 3.3 , the worst case scenario is that the $e$ two-factor interaction components take up an $e$-dimensional space that corresponds to $e$ different factors occupying a $2 e$-dimensional space. However, it is possible that the $e$ two-factor interaction components occupy the $2 e^{*}$-dimensional space from $e^{*}<e$ different factors, a consideration we return to in Section 3.6 when searching for designs. Theorems 3.3 and 3.4 provide insight which will be useful in the next section to aid in searching for designs.

It is clear that level permutations for factors outside of the core set or model, such as in $D_{2}$ and $D_{3}$, have no effect on the results of Theorems 3.3 and 3.4. This is not true for factors in the core set, as level permutations change $X_{2 c}$. Even if the core set is supported by a $D_{1}$ after a level permutation of a factor within it, the corresponding $D_{2}$ and $D_{3}$ will not be the same. In addition, it is possible that level permutations on the core set allow additional factors to be considered outside of the core set or an improvement to the D-efficiency. This implies that when searching for designs, different level permutations of factors in the core set should be considered, as will be done in the next section.

### 3.5 Searching for D-efficient Designs with Robust Properties from Orthogonal Arrays

In this section, the previous results are applied to search for designs that support a requirement set, $S$, if a saturated orthogonal array of the required run size is available. As Theorems 3.3 and 3.4 are applicable when a design that supports the core set, $C(S)$, is specified, the algorithm will consider different combinations of factors from the orthogonal array to form the core set, and find the best choices among the remaining columns for the selection of the factors outside of the core set. Level permutations are considered, but only those on the factors within the core set need to be considered, based on the results in Section 3.3.

Let $r$ be the number of factors in the core set, and $k$ the number of factors outside of the core set. For an orthogonal array with $n$ runs and $m$ columns having $q$ levels each, and replace each column by a set of $q-1$ orthogonal contrasts to create the matrix $D$. When we refer to choosing a factor from $\{1, \ldots, m\}$, we mean the set of orthogonal contrasts of that factor. The search proceeds as follows:

1. Let $c=\left(c_{1}, \ldots, c_{r}\right)$, a subset of size $r$ from $\{1, \ldots, m\}$, denote the factors in the core set.
2. Let $p_{c}=\left(p_{c_{1}}, \ldots, p_{c_{r}}\right)$ denote a set of level permutations on the factors in the core set.
3. Let $D_{1}$ be the set of columns from $D$ formed by the factors given by $c$ after performing the level permutations upon the factors as given by $p_{c}$. With the main effect components in the core set specified, $X_{2 c}$ is the set of contrast coefficients for the interaction components in the requirement set.
4. If $C(S)$ is supported by $D_{1}$,
(a) Take a subset of size $m-r-k$ from $\{1, \ldots, m\} \backslash c$ to be the factors not to be used in the design, and use these to form $D_{3}^{*}$ from $D$. The design under consideration is ( $D_{1}, D_{2}^{*}$ ), where $D_{2}^{*}$ is comprised of the remaining columns from $D$ not in $D_{1}$ or $D_{3}^{*}$.
(b) Calculate $\operatorname{det}\left(X_{2 c}^{T} D_{3}^{*} D_{3}^{* T} X_{2 c}\right)$.
(c) Repeat Steps 4 a and 4 b for all $\binom{m-r}{m-r-k}$ possible subsets from the remaining columns.
5. Repeat Steps 2 to 4 for all possible level permutations $p_{c}$.
6. Repeat Steps 1 to 5 for all $c \subset\{1, \ldots, m\}$ such that $|c|=r$.

By Theorem 3.4, the D-optimal design among those from the saturated orthogonal array is the design for which $\operatorname{det}\left(X_{2 c}^{T} D_{3}^{*} D_{3}^{* T} X_{2 c}\right)$ in Step 4 b is maximized. Recall that Theorem 3.3 gives a lower bound on the number of factors that can be supported
outside of the core set. Provided there are sufficient degrees of freedom, the algorithm presented allows us to search for designs in which $k$ is greater than the bound in Theorem 3.3.

The algorithm as presented is applicable to search for D-optimal designs. If we want to minimize the contamination from the two-factor interaction components outside of the requirement set, $\left\|C_{2 o}\right\|^{2}$ can be calculated in Step 4b, or calculated for those designs with the largest D-efficiencies after the search is complete. Theorems 3.1 and 3.2 imply that we do not need to consider any level permutations for the factors outside of the core set, as this will have no influence on the D-efficiency or contamination from the two-factor interaction components outside of the model.

This method for searching for designs is applied to designs of 27 runs in the next section.

### 3.6 Efficient Designs of 27 Runs Robust to Twofactor Interactions

In this section, we consider a special case of the second order model with requirement sets that contain two-factor interaction components for a subset of the factors in the requirement set. The algorithm from the previous section will be used to search for D-efficient designs that minimize the contamination from the two-factor interaction components outside of the model.

The model of interest in this section is the second order model (3.1) as introduced in Section 3.1:

$$
\begin{equation*}
y=\beta_{0}+\sum_{i=1}^{r+k} \beta_{i} x_{i l}+\sum_{i=1}^{r+k} \beta_{i i} x_{i q}+\sum_{1 \leq i<j \leq r} \beta_{i j} x_{i l} x_{j l}+\epsilon, \tag{3.18}
\end{equation*}
$$

where we are interested only in the linear-by-linear two-factor interaction effects from a subset of $r$ factors out of $m=r+k$ factors.

The results presented concentrate on 27-run three-level designs. We use the catalog of Evangelaras, Koukouvinos and Lappas (2011) in which they identified 129 nonisomorphic 27-run saturated orthogonal arrays with three levels. We refer to a design
from this catalog as $D i$ where $i$ refers to the $i^{\prime}$ th design as ordered in the catalog from Evangelaras, Koukouvinos and Lappas (2011). We only consider cases of $r$ and $k$ in which there are sufficient degrees of freedom to estimate all effects in (3.18). A saturated design from which we take a subset to form a design is referred to as a parent design.

To gauge the impact of the results in searching for designs, consider the case $r=3$ and $k=8$. If we were to consider all possible column choices and level permutations, we would need $\binom{13}{3} \times\binom{ 10}{8} \times 3^{11}=2279881890$ different combinations. Theorems 3.1 and 3.2 allow us to restrict level permutations to the core set, giving $\binom{13}{3} \times\binom{ 10}{8} \times 3^{3}=$ 347490 different combinations. Only three level permutations per factor in the core set are considered, as reflection of a factor around the center level retains the projection properties of the design, including D-efficiency, as discussed in Xu, Cheng and Wu (2004). With $\mapsto$ indicating the levels before and after level permutation, we refer to permutation 0 as $(012) \mapsto(012)$, permutation 1 as $(012) \mapsto(120)$ and permutation 2 as $(012) \mapsto(201)$.

Even with the reduction of possible searches, in some situations an exhaustive search is still computationally intensive. We will consider a means of reducing the search in Section 3.6.2.

### 3.6.1 Example of a Search for a Requirement Set from a Saturated Orthogonal Array

We now examine a complete search of two saturated orthogonal arrays to look for efficient designs. For parent designs $D 24$ and $D 52$, we use the algorithm to search for designs with $r=3$ and $k=8$. That is, we use the 24 th and 52 nd designs from the catalog of Evangelaras, Koukouvinos and Lappas (2011) to search for designs in which the requirement set contains the linear and quadratic main effect components for 11 factors, and a subset of size 3 of these factors in which the linear-by-linear two-factor interaction components are in the requirement set.

Tables 3.1 and 3.2 show the top 20 D-efficient designs found for each parent design. We rank the designs firstly by D-efficiency, followed by contamination of the two-factor
interaction components outside of the model. We also include the contamination from only the linear-by-linear two-factor interaction components outside of the model. The design column refers to the parent design $D i$, core refers to the set of columns from $D i$ that form the $r$ factors in the core set, and outside the columns used for the $k$ remaining factors. The set of level permutations for the factors in the core set is denoted by perm, the D-efficiency by D-eff, and the contamination from the twofactor interaction components outside of the model $\left\|C_{2 o}\right\|^{2}$, with the linear-by-linear contribution being $\left\|C_{2 l}\right\|^{2}$.

Comparing Tables 3.1 and 3.2 , we see that $D 24$ and $D 52$ provide different Defficient designs for the second-order model (3.18). The highest D-efficiency comes from $D 52$, and the highest D-efficiency for $D 24$ has less contamination from the twofactor interaction components measured by $\left\|C_{2 o}\right\|^{2}$. Also, the number of possible designs that give the same D-efficiency differs between the parent designs. For D24, there tends to be a large number of level permutations and factor assignments that give the same D-efficiency in comparison to $D 52$. If we were to restrict ourselves to a random selection of core sets rather than all possibilities, we are more likely to miss D-efficient designs among $D 52$ versus $D 24$.

Xu , Cheng and Wu (2004) suggested using $G_{2}$ and $G$-aberration in choosing designs for second-order models for factor screening and interaction detection. These will be discussed in more detail in the next chapter, but we mention them here to make the distinction between the approaches. The criteria of $G$ and $G_{2}$-aberration do not take a specific requirement set into consideration. While minimum $G$ and $G_{2}$-aberration designs may perform well when we consider finding designs that are applicable under a variety of models with two-factor interaction components involving fewer factors, in this situation there are specific two-factor interaction components to be estimated. The use of $G$ and $G_{2}$-aberration is to give designs that remain efficient over many subsets of factors. However, when we have a particular model in mind, they may not necessarily lead to a good design.

### 3.6.2 Searching among all Saturated Orthogonal Arrays

Even with the preceding results, a complete search of all possible designs among all 129 saturated designs is still a time-consuming undertaking. For one, we have to consider all $\binom{13}{r}$ possibilities for the core set and level permutations for factors in the core set. For each of these core set choices and level permutations, there is a large amount of time spent in step 4 when searching for the best choice of factors to be removed from the design, particularly if the number of these subsets is large. In order to reduce the time spent on searching, one possibility is to consider only a subset of the core set and level permutation combinations in steps 5 and 6 of the algorithm. While this random sample of the different combinations may be used, ideally we would like a means of choosing factors and level permutations for the core sets that look promising and investigate these further.

Wu (2009) showed that for two-level designs, consideration of D-efficient designs for the core set lead to D-efficient designs for the entire requirement set. In essence, this is due to the two-factor interactions occupying the same space as a subset of columns from a Hadamard matrix that are outside of the core set. In the three-level situation, this approach becomes more complicated due to the multiple components attached to a factor - instead of removing individual columns, we remove sets of columns corresponding to the main effect components for a factor. Choosing a D-efficient design for the core set does not take into consideration the multiple componenents for a factor nor which factors should be removed from the remaining.

Letting $W=D_{3}^{* T} X_{2 c}$, where $D_{3}^{* T}$ and $X_{2 c}$ are defined as in Section 3.4, by Theorem 3.4 we want to maximize $\operatorname{det}\left(W^{T} W\right)$. This requires evaluation of all factor assignments to $D_{3}^{*}$. We now consider an alternative approach to the D-efficiency of the core set. The (M.S)-criterion can be used as a surrogate for D-efficiency and is less computationally intensive to calculate. The (M.S)-criterion firstly maximizes $\operatorname{tr}\left(W^{T} W\right)$, followed by minimizing $\operatorname{tr}\left(\left(W^{T} W\right)^{2}\right)$. For a given core set and level permutations, let $D_{o}$ be the columns from the saturated orthogonal array not in the core set that correspond to $m_{o}=m-r$ factors. We want a $W=D_{3}^{* T} X_{2 c}$ for some $D_{3}^{*} \subset D_{o}$ for $m_{3}=m-r-k$ factors ( $2 m_{3}$ columns) that maximizes $\operatorname{det}\left(W^{T} W\right)$. Following the
(M.S)-criterion, an alternative is to find $D_{3}^{+} \subset D_{o}$ by firstly maximizing $\operatorname{tr}\left(W^{T} W\right)$, followed by minimizing $\operatorname{tr}\left(\left(W^{T} W\right)^{2}\right)$, where $W=D_{3}^{+T} X_{2 c}$. Looking at the matrix $W_{o}=D_{o}^{T} X_{2 c}$, the rows correspond to main effect components for all factors not in the core set and the columns the two-factor interaction components in the core set. We want to pick the best $2 m_{3}$ rows from $W_{o}$ which would correspond to the $m_{3}$ factors needed for $D_{3}^{+}$. Since $\operatorname{tr}\left(\left(W^{T} W\right)\right)$ is the sum of the squared components of $W$, taking $W_{o}^{2}$ as the squared components of $W_{o}$, we can collect the row sums of $W_{o}^{2}$ corresponding to the $m_{o}$ factors. The top $m_{3}$ of these correspond to $D_{3}^{+}$and their sum gives $\operatorname{tr}\left(W^{T} W\right)$ with $W=D_{3}^{+T} X_{2 c}$. We can then calculate $\operatorname{tr}\left(\left(W^{T} W\right)^{2}\right)$ as the sum of the squared column sums of $W_{o}^{2}$, over the rows corresponding the the factors in $D_{3}^{+}$just chosen.

If a complete search of all combinations of level permutations and factor assignments is infeasible, using the (M.S)-approach is appealing for several reasons. Firstly, $W_{o}$ only needs to be calculated once for each core set and level permutation, and can calculate the quantities $\operatorname{tr}\left(W^{T} W\right)$ and $\operatorname{tr}\left(\left(W^{T} W\right)^{2}\right)$ to rank those core sets that warrant further investigation while taking into account the nature of the factors to be removed. Perhaps more importantly, the (M.S)-approach also identifies a potential set of factors to form $D_{3}$. Instead of searching through all possible factor assignments to $D_{3}$ for a given core set, we can restrict the search to a subset of those identified as best during the maximization of $\operatorname{tr}\left(W^{T} W\right)$. This is particularly appealing if the number of combinations among the remaining columns to be removed is large. We will come back to this point in Section 3.6.6.

Use of this (M.S)-approach for $D 24$ and $D 52$ for $r=3$ and $k=8$ provides the same D-efficient designs as the complete search in Tables 3.1 and 3.2. Tables 3.3 and 3.4 show the top designs from a complete search for $r=4$ and $k=6$ for D24 and D52. Using the (M.S)-approach, Tables 3.5 and 3.6 show the best D-efficient designs using the top 500 core sets for each parent design. While we do not get the top D-efficient designs, the D-efficiencies are close to those found through a complete search. It is worth noting, that if we use the top 1000 designs using the (M.S)-criterion, we do obtain the designs as given in Tables 3.3 and 3.4.

### 3.6.3 Searching for efficient designs with fewer factors outside of the core set

In Sections 3.6.1 and 3.6.2, we examined a complete search for the cases of $r=3$ and $k=8$ and $r=4$ and $k=6$. In both situations, the number of factors outside of the core set is the maximum allowable for the given $r$. With the search algorithm, designs with fewer factors outside of the core set than the maximum allowable tend to take longer to search for. The reason for this is that the number of combinations to choose the $k$ factors outside of the core set increases, in addition to extra dimensions in the matrix $D_{3}^{*}$ for evaluating the determinant. As $k$ decreases, the advantage of using $D_{3}$ over the model matrix decreases, as step 4 of the algorithm can be done by calculating $\operatorname{det}\left(X_{M}^{T} X_{M}\right)$. The algorithm in Section 3.5 can also be used with the (M.S)-criterion as discussed in Section 3.6.2.

In Section 3.6.4 we provide tables with D-efficient designs with the maximum number of allowable factors outside of the core set for different values of $r$ using the (M.S)-criterion. We can use these designs and remove factors outside of the core set in an attempt to find D-efficient designs.

Table 3.7 presents the results from a complete search for $r=3$ and $k=7$. We get the same results using the top 500 designs with the (M.S)-criterion, or by using a search of the top 400 D-efficient designs from $r=3$ and $k=8$ and checking all subsets that remove one of the factors outside of the core set. We see that at least for this case, we do not lose any efficiency taking a simpler approach. While this approach may not always find the D-optimal design, we do have

$$
\operatorname{det}\left(X_{\text {Mnew }}^{T} X_{\text {Mnew }}\right) \geq \operatorname{det}\left(X_{M}^{T} X_{M}\right)
$$

where $X_{M n e w}$ is the model matrix after removing factors from outside the core set. This suggests that so long as the D-efficiency is high for larger $k$, it should remain so when removing additional factors.

### 3.6.4 Tables of Efficient Designs

Using the (M.S)-approach, we now search for D-efficient designs that minimize contamination of the two-factor interaction components outside of the model. Table 3.8 presents the top 20 designs in terms of D-efficiency followed by contamination from the two-factor interaction components. Table 3.9 gives the same results where parent designs are presented only once for a unique D-efficiency. The top designs have Defficiencies better than was found with the complete search of $D 24$ and D52. For the top designs, the core set and level permutations from the parent design that lead to the best D-efficiencies are few. In addition, the designs coming from the same parent design are not typically found from the same subset of factors. Table 3.10 gives the top 20 designs for $r=4$ and $k=6$. Many of the observations from the previous case hold here as well. Also of note, is that the top D-efficiencies occur in less frequency than in comparison to $r=3$; there appear to be fewer parent design and core set choices that lead to D-efficient designs.

### 3.6.5 Minimizing Contamination from Two-factor Interaction Components Outside of the Requirement Set

As discussed previously, if all two-factor interaction components outside of the requirement set are treated equally, Theorem 3.2 tells us we no longer need to consider level permutations for the factors outside of the core set. However, if we differentiate among two-factor interaction components by considering linear-by-linear components as more likely to be nonnegligible, we can first find a D-optimal design and then consider level permutations of the factors outside of the core set to minimize the contamination from the linear-by-linear components. By Theorem 3.1, the D-efficiency will be unchanged.

Tables 3.11 and 3.12 show level permutations of factors outside of the core set for two D-efficient designs for $r=4$ and $k=2$. Table 3.11 is from $D 46$ and 3.12 from $D 110$. In this case, the core set has level permutations 0 for all factors in the core set, and the perm out refers to the level permutations for the factors outside of the core set. For each design, we report the D-efficiency, the contamination of the two-factor
interaction components given by $\left\|C_{2 l}\right\|^{2},\left\|C_{2 q}\right\|^{2}$, and $\left\|C_{2 o}\right\|^{2}$ as defined by equations (3.6), (3.7), and (3.8), respectively. Note that within each table, $\left\|C_{2 o}\right\|^{2}$ is the same, as expected via Theorem 3.2.

In general, level permutations of factors outside of the core set influence the impact of the linear-by-linear two-factor interaction components outside of the requirement set. In this example, the D-efficiency of the designs in Table 3.11 are better than those in Table 3.12, as is the minimum of $\left\|C_{2 l}\right\|^{2}$. We do have $\left\|C_{2 o}\right\|^{2}$ smaller in Table 3.12. Since $\left\|C_{2 o}\right\|^{2}$ is invariant to level permutations outside of the core set, it would be ideal to use this value before considering level permutations if choosing among designs with the same D-efficiency. As we have seen in this example, looking at $\left\|C_{2 o}\right\|^{2}$ before considering level permutations outside of the core set has no guarantee of finding the minimum $\left\|C_{2 l}\right\|^{2}$. The number of all possible level permutations of the factors outside of the core set can still take a significant amount of time to search. However, this minimization of $\left\|C_{2 l}\right\|^{2}$ is essentially the last step in the search, completed after finding D-efficient designs. Using this search for only a few top D-efficient designs, the experimenter might trade off a slight reduction in D-efficiency for less contamination from the linear-by-linear two-factor interaction components.

### 3.6.6 Searching for an 81-Run Design

We conclude this section with an example where a complete search is not feasible. Taking the 81-run, 40-factor design from Neil Sloane's website (Sloane, 2011), consider the case where $r=5$ and $k=25$. According to Theorem 3.3, if we have a subset of 5 factors that supports the core set, then a design that supports the requirement set exists. However, for such a core set, there are $\binom{35}{25}=183579396$ different combinations for the factors outside of the core set. This is for a single choice of columns that supports a core set - aiming for D-efficient designs becomes even more cumbersome. The (M.S)-approach not only identifies core sets to be investigated further, but can also be used to select possible choices for factors outside of the core set. While full enumeration of all core sets and level permutations is not possible, we are still able to consider a subset of these and restrict our search for the factors outside of the core
set.
For a small sample of core sets and level permutations, we used the (M.S)-approach to select the top 12 factors and search among those $\binom{12}{10}$ to be the columns removed from the design. The (M.S)-approach found designs with D-efficiencies of about $90 \%$, whereas a random sample of several thousands out of the $\binom{35}{25}$ possibilities did not find a design that supports the requirement set.

### 3.7 Discussion

While much of the attention in this chapter has been placed on three-level designs, all of the results hold generally for designs with more than three levels. Depending on the nature of the interaction components in the core set, it is possible that the search can be simplified even further if an orthonormal matrix can be used on the interactions, such as when we consider all interaction components for a pair of factors. The results also apply to mixed level designs. This is particularly useful if the effects of interest vary among the factors. For example, in the second order model studied in this chapter, if we are not interested in curvature for the factors outside of the core set, these factors can be set at two levels. Our searches were based on a saturated orthogonal array. If we have an orthogonal array available that is not saturated, the results of this chapter still apply. In this case, the columns in the null space of the design are automatically in $D_{3}$.


| Rank | design | core |  |  | outside |  |  |  |  |  |  |  | perm |  |  | D-eff | $\mid C_{2 o} \\|^{2}$ | $\mid C_{2 l} \\|^{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 24 | 2 | 3 | 7 | 4 | 5 | 6 | 8 | 9 | 10 | 11 | 13 | 1 | 1 | 0 | 0.90751 | 210.3599 | 126.0924 |
| 2 | 24 | 2 | 11 | 12 | 1 | 3 | 4 | 5 | 7 | 9 | 10 | 13 | 1 | 1 | 0 | 0.90751 | 211.7624 | 121.6943 |
| 3 | 24 | 3 | 5 | 12 | 1 | 2 | 4 | 6 | 7 | 9 | 11 | 13 | 0 | 2 | 1 | 0.90751 | 211.9441 | 124.1502 |
| 4 | 24 | 2 | 5 | 8 | 1 | 3 | 6 | 9 | 10 | 11 | 12 | 13 | 1 | 2 | 2 | 0.90751 | 211.9999 | 127.5223 |
| 5 | 24 | 7 | 8 | 13 | 1 | 2 | 3 | 4 | 6 | 9 | 11 | 12 | 2 | 0 | 1 | 0.90751 | 212.1622 | 121.0393 |
| 6 | 24 | 1 | 11 | 12 | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 10 | 1 | 2 | 0 | 0.90751 | 212.3426 | 128.7172 |
| 7 | 24 | 4 | 6 | 11 | 1 | 2 | 3 | 8 | 9 | 10 | 12 | 13 | 2 | 0 | 2 | 0.90751 | 212.6203 | 116.1502 |
| 8 | 24 | 3 | 12 | 13 | 1 | 4 | 5 | 6 | 7 | 8 | 9 | 11 | 0 | 2 | 0 | 0.90751 | 212.7159 | 128.2826 |
| 9 | 24 | 5 | 7 | 10 | 1 | 3 | 4 | 6 | 8 | 11 | 12 | 13 | 0 | 1 | 1 | 0.90751 | 212.7527 | 124.0225 |
| 10 | 24 | 3 | 8 | 11 | 1 | 2 | 4 | 5 | 6 | 9 | 10 | 12 | 1 | 1 | 0 | 0.90751 | 212.7751 | 123.4190 |
| 11 | 24 | 4 | 6 | 13 | 1 | 2 | 5 | 8 | 9 | 10 | 11 | 12 | 1 | 0 | 2 | 0.90751 | 212.7822 | 122.4536 |
| 12 | 24 | 1 | 6 | 10 | 2 | 3 | 4 | 7 | 9 | 11 | 12 | 13 | 0 | 1 | 0 | 0.90751 | 212.8084 | 121.4064 |
| 13 | 24 | 2 | 10 | 13 | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 12 | 2 | 1 | 2 | 0.90751 | 212.8944 | 117.4062 |
| 14 | 24 | 1 | 4 | 9 | 2 | 3 | 5 | 6 | 7 | 8 | 10 | 13 | 2 | 0 | 2 | 0.90751 | 212.9416 | 121.6931 |
| 15 | 24 | 1 | 3 | 6 | 2 | 4 | 5 | 7 | 9 | 10 | 12 | 13 | 0 | 2 | 2 | 0.90751 | 212.9611 | 122.0260 |
| 16 | 24 | 5 | 10 | 11 | 1 | 3 | 4 | 6 | 7 | 8 | 9 | 13 | 0 | 0 | 2 | 0.90751 | 212.9723 | 124.3190 |
| 17 | 24 | 2 | 9 | 13 | 1 | 3 | 5 | 7 | 8 | 10 | 11 | 12 | 2 | 2 | 0 | 0.90751 | 213.0855 | 130.5637 |
| 18 | 24 | 8 | 10 | 12 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 13 | 1 | 2 | 2 | 0.90751 | 213.1078 | 127.5364 |
| 19 | 24 | 1 | 2 | 12 | 3 | 4 | 5 | 6 | 7 | 9 | 10 | 11 | 2 | 0 | 0 | 0.90751 | 213.1779 | 113.1100 |
| 20 | 24 | 2 | 6 | 8 | 1 | 3 | 4 | 5 | 10 | 11 | 12 | 13 | 0 | 1 | 2 | 0.90751 | 213.1784 | 122.3386 |

Table 3.2: Full Search for D52 with $r=3, k=8$.

| Rank | design | core |  |  | outside |  |  |  |  |  |  |  | perm |  |  | D-eff | $\mid C_{2 o} \\|^{2}$ | $\mid C_{2 l} \\|^{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 52 | 1 | 3 | 11 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 13 | 2 | 0 | 2 | 0.91642 | 212.5225 | 119.0060 |
| 2 | 52 | 1 | 3 | 11 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 13 | 2 | 0 | 2 | 0.91642 | 212.6413 | 122.6324 |
| 3 | 52 | 3 | 7 | 9 | 1 | 2 | 4 | 6 | 8 | 10 | 12 | 13 | 0 | 2 | 2 | 0.91642 | 212.6723 | 127.5303 |
| 4 | 52 | 3 | 7 | 9 | 1 | 2 | 4 | 6 | 8 | 11 | 12 | 13 | 0 | 2 | 2 | 0.91642 | 212.6723 | 130.0322 |
| 5 | 52 | 2 | 3 | 10 | 4 | 5 | 7 | 8 | 9 | 11 | 12 | 13 | 2 | 0 | 2 | 0.91642 | 212.8388 | 121.9882 |
| 6 | 52 | 2 | 3 | 10 | 1 | 4 | 5 | 8 | 9 | 11 | 12 | 13 | 2 | 0 | 2 | 0.91642 | 212.9387 | 126.4317 |
| 7 | 52 | 1 | 4 | 7 | 3 | 5 | 8 | 9 | 10 | 11 | 12 | 13 | 0 | 1 | 0 | 0.91018 | 211.0688 | 114.7868 |
| 8 | 52 | 1 | 4 | 7 | 2 | 3 | 5 | 8 | 9 | 11 | 12 | 13 | 0 | 1 | 0 | 0.91018 | 211.0787 | 123.9678 |
| 9 | 52 | 4 | 10 | 11 | 1 | 2 | 3 | 6 | 8 | 9 | 12 | 13 | 1 | 1 | 1 | 0.91018 | 212.2993 | 130.6163 |
| 10 | 52 | 4 | 10 | 11 | 1 | 2 | 3 | 6 | 7 | 8 | 12 | 13 | 1 | 1 | 1 | 0.91018 | 212.3142 | 133.0460 |
| 11 | 52 | 2 | 4 | 9 | 3 | 5 | 6 | 7 | 8 | 10 | 11 | 13 | 1 | 1 | 0 | 0.91018 | 214.3700 | 125.4501 |
| 12 | 52 | 2 | 4 | 9 | 1 | 3 | 5 | 6 | 7 | 8 | 10 | 13 | 1 | 1 | 0 | 0.91018 | 214.5439 | 127.2921 |
| 13 | 52 | 1 | 2 | 13 | 3 | 4 | 6 | 8 | 9 | 10 | 11 | 12 | 2 | 2 | 1 | 0.90474 | 211.5843 | 105.1457 |
| 14 | 52 | 1 | 2 | 13 | 3 | 4 | 6 | 7 | 8 | 10 | 11 | 12 | 2 | 2 | 1 | 0.90474 | 211.5843 | 107.5666 |
| 15 | 52 | 7 | 10 | 13 | 2 | 3 | 4 | 5 | 6 | 8 | 9 | 11 | 0 | 1 | 1 | 0.90474 | 212.1420 | 124.6016 |
| 16 | 52 | 7 | 10 | 13 | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 9 | 0 | 1 | , | 0.90474 | 212.1420 | 125.6868 |
| 17 | 52 | 9 | 11 | 13 | 1 | 3 | 4 | 5 | 7 | 8 | 10 | 12 | 0 | 1 | 1 | 0.90474 | 212.9785 | 125.3592 |
| 18 | 52 | 9 | 11 | 13 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 12 | 0 | 1 | 1 | 0.90474 | 212.9785 | 126.7352 |
| 19 | 52 | 1 | 2 | 13 | 3 | 4 | 6 | 8 | 9 | 10 | 11 | 12 | 0 | 1 |  | 0.90474 | 213.8150 | 121.8106 |
| 20 | 52 | 1 | 2 | 13 | 3 | 4 | 6 | 7 | 8 | 10 | 11 | 12 | 0 | 1 | 1 | 0.90474 | 213.8150 | 122.5939 |

Table 3.3: Full Search for D24 with $r=4, k=6$.

| Rank | design | core |  |  |  | outside |  |  |  |  |  | perm |  |  |  | D-eff | $\left\\|C_{2 o}\right\\|^{2}$ | $\left\\|C_{2 l}\right\\|^{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 24 | 1 | 4 | 5 | 9 | 3 | 6 | 7 | 10 | 11 | 12 | 0 | 0 | 0 | 1 | 0.81496 | 162.3934 | 89.64550 |
| 2 | 24 | 3 | 5 | 12 | 13 | 1 | 2 | 7 | 8 | 10 | 11 | 0 | 0 | 0 | 1 | 0.81496 | 162.3934 | 93.77505 |
| 3 | 24 | 5 | 7 | 10 | 11 | 2 | 3 | 4 | 8 | 9 | 12 | 0 | 0 | 2 | 0 | 0.81496 | 162.6169 | 94.57458 |
| 4 | 24 | 1 | 3 | 6 | 10 | 4 | 5 | 8 | 11 | 12 | 13 | 0 | 0 | 0 | 2 | 0.81496 | 162.6169 | 95.74347 |
| 5 | 24 | 4 | 8 | 10 | 12 | 1 | 2 | 5 | 6 | 9 | 11 | 0 | 2 | 2 | 0 | 0.81496 | 162.8417 | 99.40190 |
| 6 | 24 | 1 | 2 | 11 | 12 | 3 | 6 | 8 | 9 | 10 | 13 | 0 | 2 | 0 | 2 | 0.81496 | 163.6982 | 81.27873 |
| 7 | 24 | 1 | 7 | 8 | 13 | 2 | 3 | 4 | 5 | 6 | 10 | 0 | 0 | 2 | 1 | 0.81496 | 163.9439 | 97.01499 |
| 8 | 24 | 6 | 7 | 9 | 12 | 1 | 2 | 3 | 4 | 11 | 13 | 0 | 0 | 1 | 0 | 0.81496 | 164.0664 | 94.07995 |
| 9 | 24 | 4 | 6 | 11 | 13 | 1 | 3 | 5 | 7 | 8 | 9 | 0 | 0 | 0 | 1 | 0.81496 | 164.0664 | 94.51895 |
| 10 | 24 | 5 | 7 | 10 | 11 | 2 | 3 | 4 | 8 | 9 | 12 | 2 | 0 | 2 | 0 | 0.81496 | 164.2559 | 88.50985 |
| 11 | 24 | 1 | 2 | 11 | 12 | 3 | 6 | 8 | 9 | 10 | 13 | 0 | 2 | 0 | 0 | 0.81496 | 164.2900 | 76.67385 |
| 12 | 24 | 2 | 3 | 4 | 7 | 1 | 5 | 9 | 10 | 12 | 13 | 2 | 0 | 0 | 0 | 0.81496 | 164.2900 | 86.31036 |
| 13 | 24 | 2 | 5 | 6 | 8 | 4 | 7 | 9 | 10 | 11 | 13 | 2 | 0 | 0 | 2 | 0.81496 | 164.5147 | 78.08652 |
| 14 | 24 | 3 | 5 | 12 | 13 | 1 | 2 | 7 | 8 | 10 | 11 | 1 | 0 | 0 | 1 | 0.81496 | 165.0664 | 97.94988 |
| 15 | 24 | 2 | 9 | 10 | 13 | 1 | 4 | 6 | 7 | 8 | 12 | 2 | 1 | 2 | 1 | 0.81496 | 165.1687 | 101.91022 |
| 16 | 24 | 3 | 8 | 9 | 11 | 2 | 5 | 6 | 7 | 12 | 13 | 0 | 2 | 1 | 0 | 0.81496 | 165.6169 | 91.77722 |
| 17 | 24 | 1 | 7 | 8 | 13 | 2 | 3 | 4 | 5 | 6 | 10 | 0 | 0 | 2 | 2 | 0.81496 | 165.8136 | 92.62801 |
| 18 | 24 | 2 | 5 | 6 | 8 | 4 | 7 | 9 | 10 | 11 | 13 | 2 | 0 | 0 | 0 | 0.81496 | 165.9630 | 81.70841 |
| 19 | 24 | 2 | 3 | 4 | 7 | 1 | 5 | 9 | 10 | 12 | 13 | 2 | 0 | 0 | 1 | 0.81496 | 165.9975 | 81.86348 |
| 20 | 24 | 3 | 8 | 9 | 11 | 2 | 5 | 6 | 7 | 12 | 13 | 0 | 2 | 1 | 1 | 0.81496 | 166.2218 | 96.82280 |

Table 3.4: Full Search for D52 with $r=4, k=6$.

| Rank | design | core |  |  |  | outside |  |  |  |  |  | perm |  |  |  | D-eff | $\mid C_{2 o} \\|^{2}$ | $\mid C_{2 l} \\|^{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 52 | 2 | 5 | 11 | 13 | 3 | 4 | 6 | 8 | 9 | 12 | 0 | 1 | 2 | 0 | 0.83792 | 162.4883 | 97.00187 |
| 2 | 52 | 6 | 7 | 11 | 13 | 3 | 4 | 5 | 8 | 10 | 12 | 0 | 2 | 0 | 0 | 0.83792 | 163.0312 | 88.83047 |
| 3 | 52 | 9 | 10 | 12 | 13 | 3 | 4 | 5 | 6 | 8 | 11 | 2 | 0 | 0 | 0 | 0.83792 | 164.0350 | 88.85841 |
| 4 | 52 | 1 | 5 | 10 | 13 | 3 | 4 | 6 | 7 | 8 | 12 | 1 | 0 | 2 | 0 | 0.83792 | 165.1516 | 95.71076 |
| 5 | 52 | 9 | 10 | 12 | 13 | 3 | 4 | 5 | 6 | 8 | 11 | 0 | 0 | 1 | 2 | 0.83792 | 165.1856 | 91.56477 |
| 6 | 52 | 1 | 6 | 9 | 13 | 2 | 3 | 4 | 5 | 8 | 12 | 2 | 1 | 1 | 0 | 0.83792 | 165.4647 | 93.95153 |
| 7 | 52 | 2 | 7 | 12 | 13 | 1 | 3 | 4 | 5 | 6 | 8 | 2 | 1 | 1 | 0 | 0.83792 | 166.0109 | 87.22564 |
| 8 | 52 | 6 | 7 | 11 | 13 | 3 | 4 | 5 | 8 | 10 | 12 | 1 | 0 | 0 | 2 | 0.83792 | 166.1894 | 98.86032 |
| 9 | 52 | 1 | 6 | 9 | 13 | 2 | 3 | 4 | 5 | 8 | 12 | 0 | 0 | 1 | 2 | 0.83792 | 167.0529 | 90.68273 |
| 10 | 52 | 2 | 5 | 11 | 13 | 3 | 4 | 6 | 8 | 9 | 12 | 0 | 0 | 1 | 2 | 0.83792 | 167.3160 | 93.69656 |
| 11 | 52 | 2 | 7 | 12 | 13 | 1 | 3 | 4 | 5 | 6 | 8 | 1 | 1 | 0 | 2 | 0.83792 | 167.3691 | 89.23705 |
| 12 | 52 | 1 | 5 | 10 | 13 | 3 | 4 | 6 | 7 | 8 | 12 | 1 | 1 | 1 | 2 | 0.83792 | 169.3557 | 96.54786 |
| 13 | 52 | 2 | 3 | 6 | 7 | 4 | 5 | 8 | 9 | 12 | 13 | 0 | 1 | 1 | 2 | 0.82637 | 164.8869 | 98.48818 |
| 14 | 52 | 1 | 3 | 9 | 12 | 4 | 5 | 6 | 7 | 8 | 13 | 1 | 1 | 2 | 1 | 0.82637 | 164.9460 | 95.64872 |
| 15 | 52 | 2 | 3 | 11 | 12 | 4 | 5 | 6 | 8 | 10 | 13 | 2 | 1 | 0 | 0 | 0.82637 | 166.1974 | 87.74581 |
| 16 | 52 | 1 | 3 | 6 | 10 | 4 | 5 | 8 | 11 | 12 | 13 | 2 | 1 | 0 | 0 | 0.82637 | 166.5262 | 86.34181 |
| 17 | 52 | 3 | 5 | 7 | 11 | 1 | 4 | 6 | 8 | 12 | 13 | 1 | 0 | 1 | 2 | 0.82637 | 167.4242 | 91.53866 |
| 18 | 52 | 3 | 5 | 9 | 10 | 2 | 4 | 6 | 8 | 12 | 13 | 1 | 1 | 1 | 2 | 0.82637 | 169.0701 | 95.40400 |
| 19 | 52 | 1 | 4 | 5 | 10 | 3 | 6 | 7 | 8 | 12 | 13 | 2 | 0 | 0 | 1 | 0.82474 | 164.6268 | 86.66257 |
| 20 | 52 | 2 | 4 | 5 | 11 | 3 | 6 | 8 | 9 | 12 | 13 | 2 | 0 | 1 | 1 | 0.82474 | 164.8185 | 89.98030 |



| Rank | design | core |  |  |  | outside |  |  |  |  |  | perm |  |  |  | D-eff | $\left\\|C_{2 o}\right\\|^{2}$ | $\mid C_{2 l} \\|^{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 24 | 3 | 8 | 9 | 11 | 1 | 2 | 4 | 7 | 10 | 13 | 2 | 1 | 0 | 2 | 0.80346 | 159.1687 | 87.52976 |
| 2 | 24 | 4 | 6 | 11 | 13 | 2 | 3 | 5 | 7 | 10 | 12 | 1 | 1 | 2 | 0 | 0.80346 | 160.7192 | 91.35896 |
| 3 | 24 | 6 | 7 | 9 | 12 | 1 | 3 | 5 | 8 | 10 | 13 | 1 | 2 | 0 | 1 | 0.80346 | 160.7192 | 96.42893 |
| 4 | 24 | 2 | 9 | 10 | 13 | 1 | 3 | 4 | 5 | 6 | 11 | 1 | 0 | 1 | 0 | 0.80346 | 161.6181 | 95.31477 |
| 5 | 24 | 2 | 3 | 4 | 7 | 1 | 6 | 8 | 10 | 11 | 12 | 1 | 2 | 1 | 2 | 0.80346 | 162.1687 | 92.97066 |
| 6 | 24 | 1 | 2 | 11 | 12 | 4 | 5 | 6 | 7 | 8 | 13 | 2 | 1 | 2 | 1 | 0.80346 | 162.1687 | 94.72860 |
| 7 | 24 | 1 | 7 | 8 | 13 | 4 | 5 | 9 | 10 | 11 | 12 | 2 | 2 | 1 | 0 | 0.80346 | 162.5147 | 90.03270 |
| 8 | 24 | 3 | 5 | 12 | 13 | 2 | 4 | 6 | 8 | 9 | 10 | 2 | 1 | 1 | 0 | 0.80346 | 164.0652 | 89.76182 |
| 9 | 24 | 1 | 4 | 5 | 9 | 2 | 3 | 8 | 11 | 12 | 13 | 2 | 1 | 1 | 0 | 0.80346 | 164.0652 | 93.12114 |
| 10 | 24 | 5 | 7 | 10 | 11 | 1 | 2 | 6 | 9 | 12 | 13 | 1 | 2 | 1 | 2 | 0.80346 | 165.5147 | 87.52230 |
| 11 | 24 | 1 | 3 | 6 | 10 | 2 | 5 | 7 | 8 | 9 | 11 | 2 | 2 | 1 | 1 | 0.80346 | 165.5147 | 94.30790 |
| 12 | 24 | 2 | 5 | 6 | 8 | 1 | 3 | 4 | 7 | 9 | 12 | 1 | 1 | 1 | 1 | 0.80346 | 165.6181 | 101.34181 |
| 13 | 24 | 4 | 8 | 10 | 12 | 3 | 6 | 7 | 9 | 11 | 13 | 1 | 1 | 1 | 1 | 0.80346 | 168.9642 | 102.52963 |
| 14 | 24 | 1 | 2 | 11 | 12 | 5 | 6 | 8 | 9 | 10 | 13 | 0 | 0 | 0 | 2 | 0.78800 | 156.9516 | 88.78948 |
| 15 | 24 | 5 | 7 | 10 | 11 | 1 | 2 | 3 | 8 | 9 | 12 | 2 | 0 | 2 | 1 | 0.78800 | 158.2443 | 84.49759 |
| 16 | 24 | 5 | 7 | 10 | 11 | 2 | 3 | 4 | 6 | 9 | 12 | 2 | 1 | 2 | 0 | 0.78800 | 160.9944 | 88.21398 |
| 17 | 24 | 1 | 2 | 11 | 12 | 3 | 6 | 7 | 8 | 10 | 13 | 1 | 2 | 0 | 2 | 0.78800 | 161.3952 | 99.95896 |
| 18 | 24 | 1 | 2 | 11 | 12 | 3 | 4 | 6 | 8 | 9 | 13 | 0 | 2 | 1 | 2 | 0.78800 | 162.1376 | 89.17005 |
| 19 | 24 | 3 | 8 | 9 | 11 | 2 | 4 | 5 | 7 | 12 | 13 | 1 | 2 | 1 | 1 | 0.78800 | 162.1542 | 96.29928 |
| 20 | 24 | 3 | 5 | 12 | 13 | 2 | 6 | 7 | 8 | 10 | 11 | 1 | 2 | 0 | 1 | 0.78800 | 163.2546 | 91.92878 |

Table 3.6: Using Search Method for D52 with $r=4, k=6$.

| Rank | design | core |  |  |  | outside |  |  |  |  |  | perm |  |  |  | D-eff | $\mid C_{2 o} \\|^{2}$ | $\mid C_{2 l} \\|^{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 52 | 1 | 5 | 8 | 11 | 2 | 3 | 4 | 6 | 10 | 13 | 1 | 2 | 1 | 0 | 0.81291 | 164.1246 | 93.66659 |
| 2 | 52 | 7 | 8 | 9 | 12 | 2 | 3 | 4 | 6 | 10 | 13 | 1 | 1 | 1 | 2 | 0.81291 | 165.7855 | 94.66795 |
| 3 | 52 | 6 | 7 | 8 | 9 | 1 | 3 | 4 | 11 | 12 | 13 | 2 | 1 | 1 | 1 | 0.81291 | 166.0298 | 88.08316 |
| 4 | 52 | 2 | 8 | 10 | 12 | 3 | 4 | 5 | 7 | 9 | 13 | 0 | 1 | 0 | 2 | 0.81291 | 166.5332 | 81.33443 |
| 5 | 52 | 1 | 6 | 8 | 11 | 3 | 4 | 5 | 7 | 9 | 13 | 1 | 2 | 1 | 0 | 0.81291 | 168.1113 | 81.56717 |
| 6 | 52 | 2 | 5 | 8 | 10 | 1 | 3 | 4 | 11 | 12 | 13 | 0 | 2 | 1 | 0 | 0.81291 | 168.2393 | 85.19275 |
| 7 | 52 | 1 | 2 | 10 | 11 | 3 | 5 | 6 | 7 | 9 | 12 | 1 | 2 | 1 | 1 | 0.80571 | 163.2840 | 96.50605 |
| 8 | 52 | 1 | 2 | 10 | 11 | 3 | 5 | 6 | 7 | 9 | 12 | 0 | 1 | 0 | 2 | 0.80571 | 163.2840 | 97.72632 |
| 9 | 52 | 1 | 7 | 9 | 11 | 2 | 3 | 5 | 6 | 10 | 12 | 2 | 1 | 0 | 1 | 0.80571 | 164.3994 | 91.60579 |
| 10 | 52 | 1 | 7 | 9 | 11 | 2 | 3 | 5 | 6 | 10 | 12 | 0 | 0 | 2 | 0 | 0.80571 | 164.3994 | 92.71080 |
| 11 | 52 | 2 | 7 | 9 | 10 | 1 | 3 | 5 | 6 | 11 | 12 | 2 | 0 | 1 | 1 | 0.80571 | 165.9570 | 89.16467 |
| 12 | 52 | 2 | 7 | 9 | 10 | 1 | 3 | 5 | 6 | 11 | 12 | 1 | 2 | 0 | 0 | 0.80571 | 165.9570 | 93.73994 |
| 13 | 52 | 1 | 2 | 10 | 11 | 3 | 5 | 6 | 7 | 9 | 12 | 0 | 1 | 2 | 0 | 0.80571 | 166.0724 | 90.65135 |
| 14 | 52 | 1 | 2 | 10 | 11 | 3 | 5 | 6 | 7 | 9 | 12 | 2 | 0 | 1 | 1 | 0.80571 | 166.0724 | 92.51222 |
| 15 | 52 | 2 | 7 | 9 | 10 | 1 | 3 | 5 | 6 | 11 | 12 | 1 | 1 | 0 | 2 | 0.80571 | 167.0724 | 87.89259 |
| 16 | 52 | 2 | 7 | 9 | 10 | 1 | 3 | 5 | 6 | 11 | 12 | 0 | 0 | 2 | 1 | 0.80571 | 167.0724 | 91.03420 |
| 17 | 52 | 1 | 7 | 9 | 11 | 2 | 3 | 5 | 6 | 10 | 12 | 1 | 2 | 0 | 1 | 0.80571 | 168.3031 | 86.96101 |
| 18 | 52 | 1 | 7 | 9 | 11 | 2 | 3 | 5 | 6 | 10 | 12 | 0 | 0 | 1 | 2 | 0.80571 | 168.3031 | 90.47473 |
| 19 | 52 | 1 | 5 | 7 | 11 | 2 | 3 | 6 | 9 | 10 | 12 | 2 | 0 | 1 | 1 | 0.79871 | 164.7085 | 89.94885 |
| 20 | 52 | 1 | 6 | 10 | 11 | 2 | 3 | 5 | 7 | 9 | 12 | 0 | 0 | 0 | 2 | 0.79871 | 164.8753 | 94.37449 |

Table 3.7: Results for D24 with $r=3, k=7$.

| Rank | design | core |  |  | outside |  |  |  |  |  |  | perm |  |  | D-eff | $\left\\|C_{2 o}\right\\|^{2}$ | $\mid C_{2 l} \\|^{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 24 | 4 | 5 | 11 | 1 | 2 | 3 | 7 | 9 | 10 | 13 | 2 | 1 | 0 | 0.93056 | 170.7317 | 96.33509 |
| 2 | 24 | 2 | 4 | 10 | 1 | 3 | 8 | 9 | 11 | 12 | 13 | 0 | 0 | 1 | 0.93056 | 171.4650 | 100.31228 |
| 3 | 24 | 6 | 7 | 13 | 1 | 3 | 4 | 8 | 10 | 11 | 12 | 2 | 0 | 0 | 0.93056 | 171.4923 | 103.73614 |
| 4 | 24 | 3 | 7 | 12 | 1 | 2 | 4 | 5 | 6 | 8 | 13 | 2 | 1 | 0 | 0.93056 | 171.9793 | 96.92269 |
| 5 | 24 | 5 | 9 | 11 | 1 | 3 | 7 | 8 | 10 | 12 | 13 | 2 | 1 | 2 | 0.93056 | 172.1675 | 102.59238 |
| 6 | 24 | 3 | 6 | 11 | 2 | 4 | 5 | 8 | 9 | 10 | 13 | 0 | 2 | 2 | 0.93056 | 172.3346 | 107.04504 |
| 7 | 24 | 2 | 6 | 9 | 3 | 4 | 5 | 7 | 10 | 12 | 13 | 0 | 0 | 0 | 0.93056 | 172.4373 | 96.41721 |
| 8 | 24 | 4 | 5 | 6 | 1 | 2 | 7 | 9 | 11 | 12 | 13 | 1 | 0 | 2 | 0.93056 | 172.4417 | 103.91636 |
| 9 | 24 | 4 | 7 | 13 | 1 | 2 | 3 | 5 | 8 | 11 | 12 | 0 | 2 | 2 | 0.93056 | 172.5734 | 100.79833 |
| 10 | 24 | 2 | 8 | 9 | 1 | 3 | 5 | 6 | 7 | 10 | 13 | 1 | 0 | 1 | 0.93056 | 172.6229 | 95.31995 |
| 11 | 24 | 1 | 6 | 12 | 2 | 3 | 5 | 7 | 10 | 11 | 13 | 2 | 0 | 2 | 0.93056 | 172.7086 | 97.76794 |
| 12 | 24 | 1 | 3 | 4 | 2 | 5 | 6 | 8 | 9 | 10 | 12 | 2 | 0 | 2 | 0.93056 | 172.7302 | 96.89505 |
| 13 | 24 | 1 | 3 | 5 | 2 | 6 | 9 | 10 | 11 | 12 | 13 | 1 | 2 | 0 | 0.93056 | 172.7638 | 101.14507 |
| 14 | 24 | 6 | 8 | 10 | 2 | 3 | 4 | 5 | 7 | 11 | 12 | 0 | 1 | 0 | 0.93056 | 172.7808 | 97.31855 |
| 15 | 24 | 1 | 9 | 12 | 2 | 3 | 5 | 6 | 7 | 8 | 11 | 0 | 2 | 1 | 0.93056 | 172.8053 | 100.21616 |
| 16 | 24 | 8 | 11 | 12 | 1 | 2 | 3 | 4 | 6 | 7 | 9 | 2 | 2 | 2 | 0.93056 | 172.8703 | 102.99662 |
| 17 | 24 | 3 | 11 | 13 | 4 | 5 | 7 | 8 | 9 | 10 | 12 | 2 | 1 | 1 | 0.93056 | 172.8861 | 104.21742 |
| 18 | 24 | 2 | 8 | 11 | 1 | 3 | 4 | 5 | 6 | 9 | 13 | 2 | 1 | 1 | 0.93056 | 173.0412 | 105.75809 |
| 19 | 24 | 4 | 6 | 8 | 2 | 3 | 5 | 9 | 11 | 12 | 13 | 0 | 1 | 0 | 0.93056 | 173.0958 | 99.07562 |
| 20 | 24 | 2 | 3 | 12 | 1 | 4 | 5 | 6 | 10 | 11 | 13 | 2 | 1 | 1 | 0.93056 | 173.4056 | 98.98429 |

Table 3.8: Using Search Method for $r=3, k=8$.

| Rank | design | core |  |  | outside |  |  |  |  |  |  |  | perm |  |  | D-eff | $\left\\|C_{2 o}\right\\|^{2}$ | $\mid C_{2 l} \\|^{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 129 | 2 | 10 | 11 | 1 | 4 | 5 | 6 | 7 | 9 | 12 | 13 | 2 | 1 | 1 | 0.93953 | 210.8255 | 117.1609 |
| 2 | 127 | 2 | 9 | 11 | 1 | 4 | 6 | 7 | 8 | 10 | 12 | 13 | 2 | 1 | 1 | 0.93953 | 210.8255 | 118.2195 |
| 3 | 125 | 4 | 9 | 11 | 1 | 2 | 6 | 7 | 8 | 10 | 12 | 13 | 2 | 1 | 1 | 0.93953 | 210.8255 | 119.9672 |
| 4 | 126 | 5 | 10 | 13 | 1 | 3 | 4 | 7 | 8 | 9 | 11 | 12 | 0 | 0 | 2 | 0.93953 | 210.8255 | 122.2694 |
| 5 | 95 | 5 | 9 | 11 | 1 | 3 | 4 | 7 | 8 | 10 | 12 | 13 | 0 | 0 | 2 | 0.93953 | 210.8255 | 122.2694 |
| 6 | 129 | 7 | 9 | 11 | 1 | 3 | 4 | 5 | 6 | 8 | 10 | 13 | 1 | 0 | 0 | 0.93953 | 212.0503 | 118.9779 |
| 7 | 97 | 4 | 5 | 9 | 1 | 3 | 6 | 7 | 8 | 10 | 11 | 12 | 2 | 2 | 0 | 0.93953 | 212.0503 | 119.7267 |
| 8 | 124 | 3 | 7 | 11 | 1 | 2 | 5 | 6 | 9 | 10 | 12 | 13 | 2 | 1 | 2 | 0.93953 | 212.0503 | 121.6325 |
| 9 | 114 | 2 | 10 | 11 | 1 | 4 | 5 | 7 | 8 | 9 | 12 | 13 | 0 | 0 | 0 | 0.93953 | 212.0503 | 122.7779 |
| 10 | 128 | 10 | 11 | 12 | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 13 | 1 | 0 | 0 | 0.93953 | 212.0503 | 124.2831 |
| 11 | 113 | 5 | 7 | 9 | 1 | 3 | 4 | 8 | 10 | 11 | 12 | 13 | 1 | 1 | 2 | 0.93953 | 212.2744 | 123.5556 |
| 12 | 128 | 7 | 12 | 13 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 11 | 0 | 2 | 2 | 0.93953 | 212.3291 | 123.8918 |
| 13 | 95 | 8 | 11 | 12 | 1 | 3 | 4 | 5 | 6 | 7 | 9 | 10 | 0 | 2 | 0 | 0.93953 | 212.4985 | 126.1110 |
| 14 | 126 | 9 | 11 | 13 | 1 | 3 | 4 | 5 | 6 | 7 | 8 | 10 | 0 | 0 | 2 | 0.93953 | 212.4985 | 126.1110 |
| 15 | 128 | 5 | 11 | 13 | 1 | 2 | 3 | 6 | 7 | 8 | 9 | 12 | 0 | 2 | 2 | 0.93953 | 212.8868 | 122.7850 |
| 16 | 97 | 5 | 11 | 12 | 2 | 3 | 4 | 6 | 7 | 8 | 9 | 10 | 2 | 1 | 1 | 0.93953 | 213.0562 | 124.5567 |
| 17 | 115 | 5 | 11 | 12 | 2 | 3 | 4 | 6 | 7 | 8 | 9 | 10 | 2 | 1 | 1 | 0.93953 | 213.0562 | 124.6173 |
| 18 | 127 | 2 | 7 | 10 | 1 | 4 | 5 | 6 | 8 | 9 | 11 | 13 | 2 | 2 | 2 | 0.93953 | 213.2750 | 128.2857 |
| 19 | 125 | 4 | 7 | 10 | 1 | 2 | 5 | 6 | 8 | 9 | 11 | 13 | 2 | 2 | 2 | 0.93953 | 213.2750 | 130.1711 |
| 20 | 96 | 4 | 7 | 11 | 1 | 2 | 6 | 8 | 9 | 10 | 12 | 13 | 0 | 1 | 1 | 0.93953 | 213.7233 | 118.9792 |

Table 3.9: Using Search Method for $r=3, k=8$ - unique D-efficiencies.

| Rank | design | core |  |  | outside |  |  |  |  |  |  |  | perm |  |  | D-eff | $\left\\|C_{20}\right\\|^{2}$ | $\mid C_{2 l} \\|^{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 129 | 2 | 10 | 11 | 1 | 4 | 5 | 6 | 7 | 9 | 12 | 13 | 2 | 1 | 1 | 0.93953 | 210.8255 | 117.1609 |
| 2 | 127 | 2 | 9 | 11 | 1 | 4 | 6 | 7 | 8 | 10 | 12 | 13 | 2 | 1 | 1 | 0.93953 | 210.8255 | 118.2195 |
| 3 | 125 | 4 | 9 | 11 | 1 | 2 | 6 | 7 | 8 | 10 | 12 | 13 | 2 | 1 | 1 | 0.93953 | 210.8255 | 119.9672 |
| 4 | 126 | 5 | 10 | 13 | 1 | 3 | 4 | 7 | 8 | 9 | 11 | 12 | 0 | 0 | 2 | 0.93953 | 210.8255 | 122.2694 |
| 5 | 95 | 5 | 9 | 11 | 1 | 3 | 4 | 7 | 8 | 10 | 12 | 13 | 0 | 0 | 2 | 0.93953 | 210.8255 | 122.2694 |
| 6 | 97 | 4 | 5 | 9 | 1 | 3 | 6 | 7 | 8 | 10 | 11 | 12 | 2 | 2 | 0 | 0.93953 | 212.0503 | 119.7267 |
| 7 | 124 | 3 | 7 | 11 | 1 | 2 | 5 | 6 | 9 | 10 | 12 | 13 | 2 | 1 | 2 | 0.93953 | 212.0503 | 121.6325 |
| 8 | 114 | 2 | 10 | 11 | 1 | 4 | 5 | 7 | 8 | 9 | 12 | 13 | 0 | 0 | 0 | 0.93953 | 212.0503 | 122.7779 |
| 9 | 128 | 10 | 11 | 12 | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 13 | 1 | 0 | 0 | 0.93953 | 212.0503 | 124.2831 |
| 10 | 113 | 5 | 7 | 9 | 1 | 3 | 4 | 8 | 10 | 11 | 12 | 13 | 1 | 1 | 2 | 0.93953 | 212.2744 | 123.5556 |
| 11 | 115 | 5 | 11 | 12 | 2 | 3 | 4 | 6 | 7 | 8 | 9 | 10 | 2 | 1 | 1 | 0.93953 | 213.0562 | 124.6173 |
| 12 | 96 | 4 | 7 | 11 | 1 | 2 | 6 | 8 | 9 | 10 | 12 | 13 | 0 | 1 | 1 | 0.93953 | 213.7233 | 118.9792 |
| 13 | 46 | 2 | 5 | 8 | 1 | 4 | 6 | 9 | 10 | 11 | 12 | 13 | 2 | 1 | 1 | 0.92919 | 209.3596 | 115.4164 |
| 14 | 49 | 2 | 7 | 10 | 1 | 4 | 6 | 8 | 9 | 11 | 12 | 13 | 2 | 1 | 1 | 0.92919 | 209.3596 | 115.4164 |
| 15 | 50 | 2 | 5 | 8 | 1 | 4 | 6 | 9 | 10 | 11 | 12 | 13 | 2 | 1 | 1 | 0.92919 | 209.3596 | 115.4164 |
| 16 | 13 | 2 | 7 | 9 | 1 | 4 | 6 | 8 | 10 | 11 | 12 | 13 | 2 | 1 | 1 | 0.92919 | 209.3596 | 115.4164 |
| 17 | 47 | 5 | 9 | 12 | 1 | 2 | 6 | 7 | 8 | 10 | 11 | 13 | 0 | 1 | , | 0.92919 | 210.5843 | 124.8164 |
| 18 | 14 | 7 | 9 | 11 | 1 | 2 | 5 | 6 | 8 | 10 | 12 | 13 | 0 | 1 | 1 | 0.92919 | 210.5843 | 124.8164 |
| 19 | 48 | 3 | 5 | 6 | 1 | 4 | 7 | 8 | 9 | 10 | 11 | 12 | 1 | 0 | 0 | 0.92919 | 211.1420 | 123.5673 |
| 20 | 127 | 5 | 10 | 11 | 1 | 2 | 6 | 7 | 8 | 9 | 12 | 13 | 0 | 2 | 0 | 0.92767 | 210.1307 | 119.4287 |

Table 3.10: Using Search Method for $r=4, k=6$.

| Rank | design | core |  |  |  | outside |  |  |  |  |  | perm |  |  |  | D-eff | $\left\\|C_{2 o}\right\\|^{2}$ | $\mid C_{2 l} \\|^{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 2 | 5 | 7 | 11 | 12 | 2 | 4 | 6 | 8 | 9 | 13 | 1 | 1 | 1 | 1 | 0.88509 | 161.1687 | 98.15536 |
| 2 | 76 | 7 | 10 | 11 | 13 | 1 | 2 | 5 | 6 | 8 | 9 | 0 | 2 | 0 | 0 | 0.86761 | 163.2900 | 93.78200 |
| 3 | 76 | 9 | 10 | 11 | 13 | 2 | 4 | 6 | 7 | 8 | 12 | 1 | 2 | 1 | 0 | 0.86761 | 163.5130 | 93.93308 |
| 4 | 76 | 7 | 9 | 10 | 13 | 1 | 4 | 5 | 6 | 11 | 12 | 2 | 2 | 2 | 0 | 0.86761 | 163.5984 | 92.42728 |
| 5 | 75 | 1 | 2 | 9 | 13 | 4 | 5 | 6 | 8 | 10 | 11 | 2 | 2 | 2 | 1 | 0.86761 | 164.1561 | 89.39627 |
| 6 | 75 | 1 | 4 | 9 | 13 | 2 | 5 | 6 | 7 | 11 | 12 | 1 | 0 | 2 | 1 | 0.86761 | 164.6894 | 96.30732 |
| 7 | 75 | 2 | 4 | 9 | 13 | 1 | 6 | 7 | 8 | 10 | 12 | 0 | 1 | 2 | 1 | 0.86761 | 169.2443 | 92.07690 |
| 8 | 66 | 9 | 10 | 11 | 13 | 1 | 2 | 5 | 6 | 7 | 12 | 1 | 0 | 1 | 1 | 0.85940 | 162.2075 | 92.83496 |
| 9 | 66 | 7 | 9 | 10 | 13 | 2 | 4 | 5 | 6 | 8 | 11 | 2 | 2 | 0 | 1 | 0.85940 | 165.3944 | 92.85750 |
| 10 | 66 | 7 | 10 | 11 | 13 | 1 | 4 | 6 | 8 | 9 | 12 | 0 | 0 | 0 | 1 | 0.85940 | 166.8548 | 89.62653 |
| 11 | 68 | 6 | 8 | 12 | 13 | 2 | 4 | 5 | 7 | 10 | 11 | 1 | 0 | 2 | 2 | 0.85302 | 164.7261 | 90.92066 |
| 12 | 68 | 6 | 7 | 12 | 13 | 1 | 2 | 8 | 9 | 10 | 11 | 1 | 0 | 2 | 1 | 0.85302 | 166.6398 | 94.51921 |
| 13 | 67 | 2 | 4 | 6 | 13 | 1 | 5 | 7 | 10 | 11 | 12 | 1 | 1 | 1 | 2 | 0.85302 | 167.1338 | 96.88900 |
| 14 | 67 | 1 | 2 | 6 | 13 | 4 | 7 | 8 | 9 | 10 | 12 | 0 | 2 | 1 | 2 | 0.85302 | 167.6260 | 86.36760 |
| 15 | 68 | 6 | 7 | 8 | 12 | 1 | 4 | 5 | 9 | 10 | 13 | 1 | 2 | 1 | 2 | 0.85302 | 167.8491 | 91.30406 |
| 16 | 67 | 1 | 4 | 6 | 13 | 2 | 5 | 8 | 9 | 10 | 11 | 1 | 2 | 1 | 2 | 0.85302 | 168.4627 | 88.96247 |
| 17 | 69 | 1 | 2 | 9 | 13 | 3 | 4 | 5 | 7 | 11 | 12 | 1 | 2 | 2 | 1 | 0.85045 | 163.6013 | 95.86586 |
| 18 | 70 | 7 | 8 | 10 | 12 | 2 | 3 | 4 | 9 | 11 | 13 | 2 | 1 | 2 | 0 | 0.85045 | 165.6422 | 93.10338 |
| 19 | 69 | 1 | 4 | 9 | 13 | 2 | 3 | 7 | 8 | 10 | 12 | 2 | 0 | 2 | 1 | 0.85045 | 166.2363 | 92.74255 |
| 20 | 70 | 7 | 10 | 12 | 13 | 1 | 2 | 3 | 5 | 8 | 9 | 1 | 2 | 0 | 2 | 0.85045 | 166.8771 | 90.22760 |

Table 3.11: Level Permutations of Factors Outside of the Core Set from Design D46 for $r=4$ and $k=2$.

| main |  |  |  | outside | perm $_{\text {out }}$ | Dopt | $\left\\|C_{2 l}\right\\|^{2}$ | $\left\\|C_{2 q}\right\\|^{2}$ | $\left\\|C_{2 o}\right\\|^{2}$ |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :---: | :---: | :---: | :---: |
| 2 | 4 | 5 | 12 | 9 | 10 | 0 | 0 | 0.960041 | 5.3802 | 31.6758 | 37.0559 |
| 2 | 4 | 5 | 12 | 9 | 10 | 0 | 1 | 0.960041 | 6.3898 | 30.6661 | 37.0559 |
| 2 | 4 | 5 | 12 | 9 | 10 | 1 | 0 | 0.960041 | 6.3898 | 30.6661 | 37.0559 |
| 2 | 4 | 5 | 12 | 9 | 10 | 2 | 0 | 0.960041 | 6.3898 | 30.6661 | 37.0559 |
| 2 | 4 | 5 | 12 | 9 | 10 | 0 | 2 | 0.960041 | 6.3898 | 30.6661 | 37.0559 |
| 2 | 4 | 5 | 12 | 9 | 10 | 2 | 1 | 0.960041 | 7.0795 | 29.9764 | 37.0559 |
| 2 | 4 | 5 | 12 | 9 | 10 | 1 | 1 | 0.960041 | 7.0795 | 29.9764 | 37.0559 |
| 2 | 4 | 5 | 12 | 9 | 10 | 1 | 2 | 0.960041 | 7.0795 | 29.9764 | 37.0559 |
| 2 | 4 | 5 | 12 | 9 | 10 | 2 | 2 | 0.960041 | 7.0795 | 29.9764 | 37.0559 |

Table 3.12: Level Permutations of Factors Outside of the Core Set from Design D110 for $r=4$ and $k=2$.

| main |  |  |  | outside |  | $\mathrm{perm}_{\text {out }}$ |  | Dopt | $\left\\|C_{2 l}\right\\|^{2}$ | $\left\\|C_{2 q}\right\\|^{2}$ | $\mid C_{2 o} \\|^{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 3 | 9 | 11 | 13 | 6 | 10 | 2 | 2 | 0.959229 | 6.6922 | 30.1542 | 36.8464 |
| 3 | 9 | 11 | 13 | 6 | 10 | 0 | 2 | 0.959229 | 6.8698 | 29.9766 | 36.8464 |
| 3 | 9 | 11 | 13 | 6 | 10 | 2 | 0 | 0.959229 | 6.8698 | 29.9766 | 36.8464 |
| 3 | 9 | 11 | 13 | 6 | 10 | 2 | 1 | 0.959229 | 6.9092 | 29.9372 | 36.8464 |
| 3 | 9 | 11 | 13 | 6 | 10 | 1 | 2 | 0.959229 | 6.9092 | 29.9372 | 36.8464 |
| 3 | 9 | 11 | 13 | 6 | 10 | 1 | 0 | 0.959229 | 7.1888 | 29.6575 | 36.8464 |
| 3 | 9 | 11 | 13 | 6 | 10 | 0 | 1 | 0.959229 | 7.1888 | 29.6575 | 36.8464 |
| 3 | 9 | 11 | 13 | 6 | 10 | 0 | 0 | 0.959229 | 7.2514 | 29.5950 | 36.8464 |
| 3 | 9 | 11 | 13 | 6 | 10 | 1 | 1 | 0.959229 | 7.3303 | 29.5161 | 36.8464 |

## Chapter 4

## $J$-Characteristics for Multi-Level Factorial Designs

### 4.1 Introduction

In Chapters 2 and 3, designs were studied to estimate a subset of the two-factor interactions based on our knowledge about certain effects. This chapter has no requirement sets, nor prior knowledge about effects. In the absence of a requirement set, the hierarchical ordering principle would suggest that we want an orthogonal array of higher strength. If we think of strength in terms of design points, the higher strength of an orthogonal array implies an even spread of the design points in lower dimensions. In this chapter, our goal is to measure the properties of a design when projected onto lower dimensions. We will do so by measuring the spread of the design points in lower dimensions, which will lead to robust properties for the design. This approach provides designs that are useful for screening and also appealing as 'nearly-orthogonal' arrays either on their own or as building blocks for other designs.

For two-level designs, the criterion of minimum $G$-aberration (Deng and Tang, 1999) provides a measure of the projection properties of a design, and a relaxed variant, minimum $G_{2}$-aberration, also has a projection justification (Tang, 2001). Xu and Wu (2001) studied minimum $G_{2}$-aberration for multi-level designs, but did so
without the use of $J$-characteristics. In this chapter we extend the concept of $J$ characteristics to multi-level designs and derive $G$ and $G_{2}$-aberration based on these $J$-characteristics. The $J$-characteristics are appealing because they characterize the distributions of design points in various dimensions and do not require the specification of orthonormal contrasts for factors as is done by other approaches.

Section 4.2 introduces the $J$-characteristics for multi-level designs and Section 4.3 uses them to define $G$ and $G_{2}$-aberration, and a new criterion that places consideration on individual factors. Section 4.4 provides an illustration of using the $J$-characteristics to evaluate designs using the standard analysis of variance (ANOVA) provided in statistical packages. We give some designs ranked according to $G$ and $G_{2}$-aberration in Section 4.5.

### 4.2 J-Characteristics and Their Properties

### 4.2.1 J-Characteristics for Multi-Level Designs

In this section, we define the $J$-characteristics for multi-level designs and give an explicit example, followed by some of the properties of $J$-characteristics in Section 4.2.2. The definition of the $J$-characteristics is based on decomposing the frequency of the design points analogous to the ANOVA decomposition taught in introductory statistics courses. The key idea is to treat the frequency of design points as the response variable.

Let $D$ be a design with $n$ runs and $m$ factors, where factor $j$ has $s_{j}$ levels, given by $0,1, \ldots, s_{j}-1$, for $j=1, \ldots, m$. Denote $N\left(x_{1}, \ldots, x_{m}\right)$ as the number of design points at level combination $\left(x_{1}, \ldots, x_{m}\right)$ where $x_{j}=0, \ldots, s_{j}-1$. Consider the ANOVA decomposition of $N\left(x_{1}, \ldots, x_{m}\right)$ given by

$$
\begin{equation*}
N\left(x_{1}, \ldots, x_{m}\right)=\sum_{u \subseteq Z_{m}} N_{u}\left(x_{1}, \ldots, x_{m}\right), \tag{4.1}
\end{equation*}
$$

where the summation is over all subsets of $Z_{m}=\{1, \ldots, m\}$,

$$
\begin{equation*}
N_{\emptyset}=\left(s_{1} \cdots s_{m}\right)^{-1} \sum_{x_{1}, \ldots, x_{m}} N\left(x_{1}, \ldots, x_{m}\right) \tag{4.2}
\end{equation*}
$$

is the grand mean, and

$$
\begin{equation*}
N_{u}\left(x_{1}, \ldots, x_{m}\right)=\left(\prod_{j \notin u} s_{j}\right)^{-1} \sum_{x_{j}, j \notin u} N\left(x_{1}, \ldots, x_{m}\right)-\sum_{v \subset u} N_{v}\left(x_{1}, \ldots, x_{m}\right) \tag{4.3}
\end{equation*}
$$

is the interaction involving the factors in $u$.
The $J$-characteristics are defined as

$$
\begin{equation*}
J_{u}\left(x_{1}, \ldots, x_{m}\right)=s_{1} \cdots s_{m} N_{u}\left(x_{1}, \ldots, x_{m}\right) \tag{4.4}
\end{equation*}
$$

The $J$-characteristics can then be related back to the design points through

$$
\begin{equation*}
s_{1} \cdots s_{m} N\left(x_{1}, \ldots, x_{m}\right)=\sum_{u \subseteq Z_{m}} J_{u}\left(x_{1}, \ldots, x_{m}\right) \tag{4.5}
\end{equation*}
$$

Ai and Zhang (2004) defined $J$-characteristics using orthonormal contrasts following the approach of Xu and Wu (2001). In their formulation, the $J$-characteristics require a set of orthonormal contrasts for each factor and are calculated based on these contrasts. In our formulation, the $J$-characteristics are based only on the design points and do not require the specification of contrasts.

The value of $J_{u}\left(x_{1}, \ldots, x_{m}\right)$ depends only on $x_{j}$ 's for $j \in u$. For the remainder of this chapter, unless otherwise specified, by referring to the $J$-characteristics we mean those as defined in equation (4.4). The ANOVA decomposition allows for standard statistical software to evaluate the properties of a design, a point which we return to in Section 4.4.

The multi-level $J$-characteristics are consistent with those used for two-level designs. For a two-level design, $D=\left(x_{i j}\right)$ with $x_{i j}= \pm 1$, Tang and Deng (1999) and Deng and Tang (1999) defined the $J$-characteristics as

$$
\begin{equation*}
J_{u}=\sum_{i=1}^{n} \prod_{j \in u} x_{i j} \tag{4.6}
\end{equation*}
$$

for any $u \subseteq Z_{m}=\{1, \ldots, m\}$. Tang (2001) related the distribution of design points as a vector, $N$, to the vector of $J$-characteristics of a design through

$$
\begin{equation*}
J_{u}=H^{T} N \tag{4.7}
\end{equation*}
$$

or equivalently $N=2^{-m} H J$, where $H$ is a Hadamard matrix that contains all possible Hadamard products of the column vectors of the full factorial design. For two-level designs, $J_{u}$ is characterized by a single number and the calculation is based on $\pm 1$. In the two-level case, for any of the $2^{m}$ combinations of $x_{1}, \ldots, x_{m},\left|J_{u}\left(x_{1}, \ldots, x_{m}\right)\right|$ $=\left|J_{u}\right|$. More precisely we have $J_{u}\left(x_{1}, \ldots, x_{m}\right)=\left(\prod_{i \in u} x_{i}\right) J_{u}$. For multi-level designs, the situation becomes more complicated, as $J_{u}\left(x_{1}, \ldots, x_{m}\right)$ can take on numerous values. We demonstrate this in the following example.

Example 4.1. Consider the 9-run, three-level design

$$
D=\left(\begin{array}{lll}
0 & 0 & 0 \\
0 & 1 & 2 \\
0 & 2 & 1 \\
1 & 0 & 0 \\
1 & 1 & 1 \\
1 & 2 & 2 \\
2 & 0 & 2 \\
2 & 1 & 0 \\
2 & 2 & 1
\end{array}\right) .
$$

For the above design $D$, we have $N\left(x_{1}, x_{2}, x_{3}\right)=1$ if $\left(x_{1}, x_{2}, x_{3}\right)$ is one of the nine rows and $N\left(x_{1}, x_{2}, x_{3}\right)=0$ otherwise. We use the decomposition in (4.1) to obtain the $N_{u}$ terms which will allow us to calculate the $J$-characteristics by Equation (4.4). Starting with the grand mean, we have $N_{\emptyset}=(1 / 27) * 9=1 / 3$. For the subsets $u$ with $|u|=1, N_{1}\left(x_{1}, x_{2}, x_{3}\right)=N_{2}\left(x_{1}, x_{2}, x_{3}\right)=N_{3}\left(x_{1}, x_{2}, x_{3}\right)=(1 / 9) * 3-(1 / 3)=0$,
for any $x_{1}, x_{2}, x_{3} \in\{0,1,2\}$. To clarify, examining the calculation of $N_{1}(0,0,0)$ in greater detail,
$N_{1}(0,0,0)=\left(1 /\left(s_{2} * s_{3}\right)\right) \sum_{x_{2}, x_{3}} N\left(0, x_{2}, x_{3}\right)-N_{\emptyset}=(1 /(3 * 3)) * 3-(1 / 3)=0$,
where $\sum_{x_{2}, x_{3}} N\left(0, x_{2}, x_{3}\right)=3$ is the number of times 0 occurs in column 1. For $u=\{1,2\}$ and $u=\{1,3\}$,
$N_{1,2}\left(x_{1}, x_{2}, x_{3}\right)=N_{1,3}\left(x_{1}, x_{2}, x_{3}\right)=(1 / 3) * 1-(1 / 3)=0$,
for all $x_{1}, x_{2}, x_{3} \in\{0,1,2\}$, while for $u=\{2,3\}$,
$N_{2,3}\left(x_{1}, 1,0\right)=N_{2,3}\left(x_{1}, 1,1\right)=N_{2,3}\left(x_{1}, 1,2\right)=$
$N_{2,3}\left(x_{1}, 0,2\right)=N_{2,3}\left(x_{1}, 2,2\right)=1 / 3-1 / 3=0$, for all $x_{1} \in\{0,1,2\}$,
and $N_{2,3}\left(x_{1}, 0,0\right)=N_{2,3}\left(x_{1}, 2,1\right)=(2 / 3-1 / 3)=1 / 3$,
$N_{2,3}\left(x_{1}, 0,1\right)=N_{2,3}\left(x_{1}, 2,0\right)=(0-1 / 3)=-1 / 3$ for all $x_{1} \in\{0,1,2\}$.
Table 4.1 gives the set of values for $N_{1,2,3}$ for all $\left(x_{1}, x_{2}, x_{3}\right)$. The entries correspond to the design points giving those values. For instance,

$$
\begin{aligned}
N_{1,2,3}(0,0,0)= & N(0,0,0)-N_{1,2}(0,0,0)-N_{1,3}(0,0,0)-N_{2,3}(0,0,0) \\
& -N_{1}(0,0,0)-N_{2}(0,0,0)-N_{3}(0,0,0)-N_{\emptyset} \\
= & 1-1 / 3-1 / 3=1 / 3 .
\end{aligned}
$$

The remaining values in Table 4.1 are calculated similarly.

Table 4.1: $N_{1,2,3}$ values over all combinations of $\left(x_{1}, x_{2}, x_{3}\right)$.

| $N_{1,2,3}$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| $-2 / 3$ | $-1 / 3$ | 0 | $1 / 3$ | $2 / 3$ |
| $(1,2,1)$ | $(0,0,2)$ | $(0,0,1)$ | $(0,0,0)$ | $(0,1,2)$ |
| $(2,0,0)$ | $(0,1,0)$ | $(0,2,0)$ | $(0,2,1)$ | $(1,1,1)$ |
|  | $(0,1,1)$ | $(1,0,1)$ | $(1,0,0)$ | $(1,2,2)$ |
|  | $(0,2,2)$ | $(1,2,0)$ | $(2,2,1)$ | $(2,0,2)$ |
|  | $(1,0,2)$ | $(2,0,1)$ |  | $(2,1,0)$ |
|  | $(1,1,0)$ | $(2,2,0)$ |  |  |
|  | $(1,1,2)$ |  |  |  |
|  | $(2,1,1)$ |  |  |  |
|  | $(2,1,2)$ |  |  |  |
|  | $(2,2,2)$ |  |  |  |

If we return to Equation (4.1), we have the decomposition of $N(0,0,0)$ as

$$
\begin{aligned}
N(0,0,0)= & N_{\emptyset}+N_{1}(0,0,0)+N_{2}(0,0,0)+N_{3}(0,0,0)+N_{1,2}(0,0,0)+ \\
& N_{1,3}(0,0,0)+N_{2,3}(0,0,0)+N_{1,2,3}(0,0,0) \\
= & 1 / 3+0+0+0+0+0+1 / 3+1 / 3 \\
= & 1 .
\end{aligned}
$$

The $J$-characteristics are calculated by Equation (4.4) as $J_{u}\left(x_{1}, x_{2}, x_{3}\right)=$ $27 N_{u}\left(x_{1}, x_{2}, x_{3}\right)$ for all $u \subseteq Z_{m}$ and $x_{1}, x_{2}, x_{3} \in\{0,1,2\}$. The $J$-characteristics for the
individual factors, $J_{1}\left(x_{1}, x_{2}, x_{3}\right), J_{2}\left(x_{1}, x_{2}, x_{3}\right)$ and $J_{3}\left(x_{1}, x_{2}, x_{3}\right)$, are 0 , which reflects the fact we have a balanced design. For the two-factor interactions, $J_{1,2}\left(x_{1}, x_{2}, x_{3}\right)$ and $J_{1,3}\left(x_{1}, x_{2}, x_{3}\right)$ are 0 for all $x_{1}, x_{2}$ and $x_{3}$, whereas $J_{2,3}\left(x_{1}, x_{2}, x_{3}\right)$ is non-zero for some $x_{1}, x_{2}$ and $x_{3}$. This indicates column 1 is orthogonal to columns 2 and 3 , but columns 2 and 3 are not orthogonal. In this example, $J_{1,2,3}\left(x_{1}, x_{2}, x_{3}\right)$ takes on five different values.

In the next subsection, we take a closer look at some of the properties of $J$ characteristics.

### 4.2.2 Properties of $J$-Characteristics

The results in the following Lemma provide some insight into the usefulness of the $J$-characteristics.

Lemma 4.1. (i) A design is uniquely determined by its J-characteristics, and vice versa.
(ii) A design is a full factorial or several replicates of it if and only if $J_{u}=0$ for all nonempty subsets $u$ of $Z_{m}$.
(iii) A design is an orthogonal array of strength $t$ if and only if $J_{u}=0$ for all nonempty subsets $u$ such that $|u| \leq t$.
(iv) when $s_{1}=\cdots=s_{m}=s$ and the design is an orthogonal array of strength $t$, then $J_{u}$ must be a multiple of $s^{t}$.

If our interest lies in the lower dimensions of the factors, Lemma 4.1 tells us that for smaller values of $|u|$, a good design should have $J_{u}$ values as close to zero as possible. As was seen in Example 4.1, we need to be mindful that unlike the two-level case, $J_{u}$ can take on a number of different values for a given $u$. The construction of the $J$-characteristics gives a convenient way to summarize the size of $J_{u}$. By using the ANOVA decomposition for the frequency of the design points, we have

$$
\begin{equation*}
\sum_{x_{1}=0}^{s_{1}-1} \cdots \sum_{x_{m}=0}^{s_{m}-1}\left[s_{1} \cdots s_{m} N\left(x_{1}, \ldots, x_{m}\right)\right]^{2}=\sum_{u \subseteq Z_{m}} \sum_{x_{1}=0}^{s_{1}-1} \cdots \sum_{x_{m}=0}^{s_{m}-1}\left[J_{u}\left(x_{1}, \ldots, x_{m}\right)\right]^{2} \tag{4.8}
\end{equation*}
$$

as $\sum_{x_{1}=0}^{s_{1}-1} \cdots \sum_{x_{m}=0}^{s_{m}-1} J_{u}\left(x_{1}, \ldots, x_{m}\right) J_{v}\left(x_{1}, \ldots, x_{m}\right)=0$ for $u \neq v$ because the ANOVA decomposition is an orthogonal decomposition.

If we let

$$
\begin{equation*}
S_{u}^{2}=\sum_{x_{1}=0}^{s_{1}-1} \cdots \sum_{x_{m}=0}^{s_{m}-1}\left[J_{u}\left(x_{1}, \ldots, x_{m}\right)\right]^{2} \tag{4.9}
\end{equation*}
$$

represent the size of $J_{u}$, then the decomposition becomes

$$
\begin{equation*}
\sum_{x_{1}=0}^{s_{1}-1} \cdots \sum_{x_{m}=0}^{s_{m}-1}\left[s_{1} \cdots s_{m} N\left(x_{1}, \ldots, x_{m}\right)\right]^{2}=\sum_{u \subseteq Z_{m}} S_{u}^{2} \tag{4.10}
\end{equation*}
$$

The values of $S_{u}^{2}$ in (4.9) will be used throughout the remainder of the chapter as a means of evaluating the properties of a design, and will be justified in the next section.

Example 4.1 (continued). Calculating the left-hand side of Equation (4.10),

$$
\sum_{x_{1}=0}^{2} \sum_{x_{2}=0}^{2} \sum_{x_{3}=0}^{2}\left[3^{3} N\left(x_{1}, x_{2}, x_{3}\right)\right]^{2}=6561 .
$$

Using Equation (4.9), $S_{\emptyset}^{2}=2187, S_{1}^{2}=S_{2}^{2}=S_{3}^{2}=0, S_{1,2}^{2}=S_{1,2}^{2}=0, S_{2,3}^{2}=972$ and $S_{1,2,3}^{2}=3402$. Taking the right-hand side of (4.10), we have $\sum_{u \subseteq Z_{3}} S_{u}^{2}=6561$.

The quantities of $S_{u}^{2}$ are the partitioned sums of squares for the factorial effects in an ANOVA. The $m$ factors are a full factorial design corresponding to all level combinations of $x_{1}, \ldots, x_{m}$ and the response variable is $s_{1} \cdots s_{m} N\left(x_{1}, \ldots, x_{m}\right)$. The sum of squares component $S_{u}^{2}$ corresponds to the factors in $u$. In the standard usage of ANOVA for a general factorial design, we typically want lower order effects with a large sum of squares. For our situation, the lower order effects should be as small as possible, ideally zero, as these represent the projections onto factors in $u$. An example of analyzing a design using this ANOVA approach on the frequency of design points will be given in Section 4.4. We now explore the justification for the use of $S_{u}^{2}$.

### 4.3 Criteria of Aberration and their Justification

In this section, we introduce a means of ranking designs that makes use of the $J$ characteristics through $S_{u}^{2}$. This will be done through the variability of design points and contamination from nonnegligible interactions.

### 4.3.1 $\quad G_{2}$-Aberration

If we consider Equation (4.10), moving the grand mean to the left-hand side, we have the total corrected sum of squares in an ANOVA. This quantity represents the total variability of the design points. The $S_{u}^{2}$ terms give the partitioned sums of squares corresponding to the different effects in the design. For any subset $v$ of $Z_{m}$, if we want to look at the variability of the projection onto the factors in $v$ we can look at the $S_{u}^{2}$ for $u \subseteq v$. To ensure good projections onto lower dimensions, we want the $S_{u}^{2}$ to be small when $|u|$ is small. Based on $J$-characteristics, we now define the criterion of minimum $G_{2}$-aberration. Let

$$
\begin{equation*}
B_{j}=\sum_{|u|=j} S_{u}^{2} \tag{4.11}
\end{equation*}
$$

Then minimum $G_{2}$-aberration is to sequentially minimize $B_{1}, \ldots, B_{m}$. Similar to Tang (2001), Ai and Zhang (2004) gave a projection justification for generalized minimum aberration, which we discuss shortly, based on their version of the $J$-characteristics. Their justification used the same idea of the variability of design points. Our measure of $S_{u}^{2}$ is equivalent to the sum of the squared $J$-characteristics used by Ai and Zhang (2004), but in our derivation, $S_{u}^{2}$ is directly linked to the distribution of design points.

The J-characteristics of Ai and Zhang (2004) are based on the orthonormal contrasts as used in Xu and Wu (2001) in defining generalized minimum aberration (GMA). Generalized minimum aberration sequentially minimizes $A_{3}, A_{4}, \ldots$ for

$$
\begin{equation*}
A_{j}=n^{-2} \sum_{k=1}^{m_{j}}\left|\sum_{i=1}^{n} x_{i k}^{(j)}\right|^{2} \tag{4.12}
\end{equation*}
$$

where $x_{i k}^{(j)}$ is the $i$ th component of the $k$ th factor effect contrast, of which there are $m_{j}$. We have

$$
\begin{equation*}
A_{j}=B_{j} /\left(s_{1} \cdots s_{m} n^{2}\right) \tag{4.13}
\end{equation*}
$$

for $j=1, \ldots, m$. The value of $A_{j}$ measures the overall aliasing between the $j$-factor effects and the grand mean.

In regards to robustness to nonnegligible interactions, consider the ANOVA model written as

$$
\begin{equation*}
Y=X_{0} \beta_{0}+X_{1} \beta_{1}+X_{2} \beta_{2}+\cdots+X_{m} \beta_{m}+\epsilon \tag{4.14}
\end{equation*}
$$

where $Y$ is the vector of $n$ observations, $X_{i}$ is the matrix of contrast coefficients for the $i$-factor interactions with $\beta_{i}$ the corresponding effects, $\beta_{0}$ the grand mean and $X_{0}$ the vector of 1's, and $\epsilon$ the vector of independent random errors. The main effects model for (4.14) is given by

$$
\begin{equation*}
Y=X_{0} \beta_{0}+X_{1} \beta_{1}+\epsilon \tag{4.15}
\end{equation*}
$$

Using model (4.15), the least squares estimate of $\beta_{1}$ is given by $\hat{\beta}_{1}=\left(X_{1}^{T} X_{1}\right)^{-1} X_{1}^{T} Y$. This estimate is unbiased for model (4.15), but if model (4.14) is the true model, we have

$$
E\left(\hat{\beta_{1}}\right)=\beta_{1}+C_{2} \beta_{2}+\cdots+C_{m} \beta_{m}
$$

where $C_{i}=\left(X_{1}^{T} X_{1}\right)^{-1} X_{1}^{T} X_{i}$ are the aliasing matrices for $i=2, \ldots, m$. In the situation our design $D$ is an orthogonal array, $C_{i}$ simplifies to $C_{i}=(1 / n) X_{1}^{T} X_{i}$. By the hierarchical ordering principle, lower order effects are more likely to be important than higher order effects, and effects of the same order are equally likely to be important. If we want to minimize the contamination from nonnegligible interactions, we would like to sequentially minimize $C_{2}, C_{3}, \ldots$. To sequentially minimize $\left\|C_{2}\right\|^{2},\left\|C_{3}\right\|^{2}, \ldots$ gives the minimum contamination criterion such as in Tang and Deng (1999), Xu and Wu (2001), and Steinberg and Bursztyn (2001). Xu, Cheng and Wu (2004) suggested using this approach for screening with three-level designs. In Xu and Wu (2001), they showed that sequentially minimizing $A_{3}, A_{4}, \ldots$ is equivalent to sequentially minimizing $\left\|C_{2}\right\|^{2},\left\|C_{3}\right\|^{2}, \ldots$. This means that for orthogonal arrays $\left(B_{1}=B_{2}=0\right), G_{2}$-aberration sequentially minimizes $\left\|C_{2}\right\|^{2},\left\|C_{3}\right\|^{2}, \ldots$.

The entries of $C_{i}$ are dependent on the choice of orthonormal contrasts, as are the components $x_{i k}^{(j)}$ in (4.12). Our definition of $B_{j}$ in (4.11) does not depend on orthonormal contrasts. As noted in Xu and $\mathrm{Wu}(2001),\left\|C_{i}\right\|^{2}$ is independent of the choice of orthonormal contrasts. If we look at the projection design onto the factors referenced by $v$, note that the projected $A_{j}$ values and $S_{u}^{2}$ remain the same regardless of the choice of orthonormal contrasts for all $u \subseteq v$.

In related work, Liu, Fang, and Hickernell (2006) studied the connection between different criteria for asymmetrical fractional factorial designs showing that generalized minimum aberration is equivalent to the minimum $\chi^{2}$-criterion (an extension of

Yamada and Matsui, 2002) and minimum projection uniformity (Hickernell and Liu, 2002).

### 4.3.2 $G$-Aberration

By the effect sparsity principle, few of the effects are expected to be significant. A more conservative approach than $G_{2}$-aberration is to minimize the worst case scenario. Assuming interactions involving three or more factors are negligible, if $C_{2}=\left[c_{i j}\right]$, in order to best protect estimation of the main effects from nonnegligible two-factor interactions, we can try to sequentially minimize the largest of the $c_{i j}^{2}$ 's. However, the $c_{i j}^{\prime} s$ are based on a set of orthonormal contrasts. Without prior knowledge or choice of contrasts, a good surrogate is to sequentially minimize the frequency of $S_{u}^{2}$ starting from the largest value among with smaller values of $|u|$, effectively collecting the $\sum c_{i j}^{2}$ for $i, j$ corresponding to factors in $u$. That is, we first sequentially minimize the largest $S_{u}^{2}$ values for the smallest nonzero $|u|$.

Sequentially minimizing the frequencies of $S_{u}^{2}$ can also be thought of from the projection standpoint. From this view, minimum $G$-aberration sequentially minimizes the worst of the lower-dimensional projections as measured by the variability of the design points. Whereas $B_{j}$ measures the overall projection properties, the projected $S_{u}^{2}$ values provide a more conservative approach. If we expect that the response is active in a smaller subset of the factors, $G$-aberration aims to minimize poor projections in smaller subsets. Our concentration for $G$-aberration is for $\left|u_{0}\right|$ with the first nonzero $S_{u_{0}}^{2}$ from $\left|u_{0}\right|=1,2, \ldots$. The reason is that if we consider a $\left|u^{*}\right|>\left|u_{0}\right|$, the projection properties are based on $S_{u}^{2}$ for all subsets $u \subseteq u^{*}$. This causes no problems for $G_{2}$-aberration.

Another possibility to define $G$-aberration is to sequentially minimize the frequency of $\left|J_{u}\left(x_{1}, \ldots x_{m}\right)\right|$ 's. In the two-level case, defining $G$-aberration with either $\left|J_{u}\left(x_{1}, \ldots x_{m}\right)\right|$ or $S_{u}^{2}$ are equivalent, but not so when there are more than two levels due to $\left|J_{u}\left(x_{1}, \ldots x_{m}\right)\right|$ taking on different values. This approach can also be used to further differentiate designs with the same $G$-aberration as defined using $S_{u}^{2}$. Based on the direct relationship to the variability of the design points, for the remainder of
this chapter we use $S_{u}^{2}$ in relation to $G$-aberration.
Our definition of minimum $G$-aberration is equivalent to the definition for twolevel designs. For multi-level designs, using the frequency of $S_{u}^{2}$ is more general than the projection aberration criterion in Xu, Cheng and Wu (2004), in which they sequentially minimized the projected $A_{3}$ values starting from the largest as a means of combining factor screening and interaction detection for three-level designs.

### 4.3.3 $G_{2(i)}$-Aberration

While minimum $G_{2}$-aberration considers an overall measure of lower dimensions and minimum $G$-aberration takes a conservative approach, the impact on individual factors is not taken into account from both the contamination and projection perspectives. Ideally, given any factor, we should have good projection properties for all those projections involving that factor. Equivalently, we want the contamination to the estimation of each factor minimized so that no individual main effect has a substantial amount of contamination from nonnegligible interactions. Minimum $G_{2(i)}$-aberration aims to address this by sequentially minimizing the largest values of

$$
\begin{equation*}
S_{i: k}^{2}=\sum_{\{u: i \in u,|u|=k\}} S_{u}^{2} \tag{4.16}
\end{equation*}
$$

for $i=1, \ldots, m$, beginning from $k=1,2, \ldots$. In general, we expect that a design with smaller $G_{2}$-aberration should also have smaller $G_{2(i)}$-aberration as $G_{2}$-aberration is based on the sum of the $S_{i, k}^{2}$ 's.

Using $G_{2(i)}$ can be thought of as somewhere between $G$ and $G_{2}$-aberration. Not as conservative as $G$-aberration, it still gives us the flexibility to distinguish between designs having the same $G_{2}$-aberration. When we reach a point that higher strength is not possible, the aim is to have the contamination or projection properties spread out evenly among the factors. The concept of $G_{2(i)}$-aberration can be extended to $G_{(i)}$-aberration whereby we use the frequency vectors of $S_{u}^{2}$ for each factor and rank these according to $G$-aberration.

### 4.3.4 Considerations from Previous Chapters

In Chapter 2, we considered a case where certain two-factor interactions were safely assumed to be negligible before conducting the experiment while other two-factor interactions were nonnegligible. If it were known ahead of time that certain factors do not interact, then $G, G_{2}$ and $G_{2(i)}$-aberration can all be adapted to account for this information. Instead of using $B_{j}$, we can give different weights to $S_{u}^{2}$ to account for this information.

As an example, similar to Chapter 2, consider an orthogonal array with a subset of factors $T_{2}$ such that the interactions within $T_{2} \times T_{2}$ are assumed negligible. For $|u|=1,2$, all $S_{u}^{2}=0$. For $|u|=3$, any $S_{u}^{2}$ where $u \subseteq T_{2}$ (ie. all three factors are in $T_{2}$ ) can be assigned zero weight. In addition, the $S_{u}^{2}$ where $u$ has two factors from $T_{2}$ would be considered less serious than the $S_{u}^{2}$ with only one or zero factors from $T_{2}$. There are then a number of ways to rank designs using this information, whether by creating a metric with a weighted sum of the $S_{u}^{2}$, or a partition based on the factors in the subset $u$. An example of this partitioning was used in Chapter 2 with the $A_{40}$, $A_{31}$ and $A_{22}$ terms.

One of the appeals of the multi-level $J$-characteristics and the use of $S_{u}^{2}$ is that they are independent of the choice of orthonormal contrasts. Our concern is with the general structure of the design in lower dimensions. In Chapter 3, we had a particular model in mind with a set of orthonormal contrasts and specified two-factor interaction components. In such a situation, the $J$-characteristics from Ai and Zhang (2004) may be more appropriate as they are defined based on a set of orthonormal contrasts. The $S_{u}^{2}$ terms can be thought of as summarizing all of the interaction components of the factors in $u$, when our interest may only be in a subset of these components.

### 4.4 An ANOVA Example

We now analyze two designs based on their $S_{u}^{2}$ values calculated using a standard ANOVA from the statistical software package R. Consider the $O A\left(27,3^{6}\right)^{\prime}$ 's denoted by $D_{1}$ and $D_{2}$ in Table 4.2. The design $D_{1}$ is a non-regular design, while $D_{2}$ is
regular. To analyze these designs using an ANOVA, our design matrix is the full factorial design for $m=6$ factors with 3 levels each, and the response $y$ is the number of occurrences of a particular row multiplied by $3^{6}$ (see the left-hand side of equation (4.5)). If we let $x_{1}, \ldots, x_{6}$ represent the six factors in the full factorial, and $y$ be as just described, we can use the R command
$\operatorname{aov}\left(\mathrm{y} \sim \mathrm{x} 1^{*} \mathrm{x} 2{ }^{*} \mathrm{x} 3{ }^{*} \mathrm{x} 4{ }^{*} \mathrm{x} 5{ }^{*} \mathrm{x} 6\right)$
to calculate $S_{u}^{2}$ for all $u \subseteq Z_{m}$ and investigate the properties of the design.
Table 4.3 gives the results of the ANOVA for the three-factor interactions for both designs and Table 4.4 for interactions involving four or more factors. The interaction listed refers to the set $u$. For example, x1:x2:x3 refers to $u=\{1,2,3\}$. The degrees of freedom for the interaction, represented by 'df' in the table, give the number of components these interactions have if main effects were broken into orthogonal components, as was done in the previous chapter. We present interactions involving three or more factors as for the main effects and two-factor interactions $(|u|=1,2)$ $S_{u}^{2}=0$. Then for both designs, $B_{j}$ from (4.11) is zero for $j=1,2$. In a typical ANOVA, we look for significant effects through larger sums of squares for lower-order effects. In this situation, we want the lower-order terms as close to zero as possible. Design $D_{2}$ has less $G_{2}$-aberration than $D_{1}$ as we have $B_{3}=3188646$ for $D_{2}$ versus $B_{3}=4015332$ for $D_{1}$. However, $D_{1}$ has less $G$-aberration than $D_{2}$ as seen in Table 4.5 with the frequency of $S_{u}^{2}$ values for $|u|=3$. Table 4.6 gives the $S_{i: 3}^{2}$ as calculated by equation (4.16). By these values, design $D_{1}$ has less $G_{2(i)}$-aberration than $D_{2}$. From Table 4.6, we see that while the sum of the $S_{u}^{2}$ terms is smaller for $D_{2}, D_{1}$ has more of an even spread of the contribution of each factor to $B_{3}$.

Table 4.2: Two $O A\left(27,3^{6}\right), D_{1}$ a non-regular design and $D_{2}$ regular.

$$
D_{1}=\left(\begin{array}{l}
001001 \\
011221 \\
022201 \\
000022 \\
010100 \\
021010 \\
002212 \\
012120 \\
020112 \\
100200 \\
112011 \\
122022 \\
102110 \\
111102 \\
120220 \\
101111 \\
111222 \\
120001 \\
200121 \\
212000 \\
221102 \\
201020 \\
210012 \\
222121 \\
202202 \\
210211 \\
221210
\end{array}\right) \quad D_{2}=\left(\begin{array}{l}
000000 \\
001101 \\
002202 \\
010120 \\
011221 \\
012022 \\
020210 \\
021011 \\
022112 \\
100111 \\
101212 \\
102010 \\
110201 \\
111002 \\
112100 \\
120021 \\
121122 \\
122220 \\
200222 \\
201020 \\
202121 \\
210012 \\
21110 \\
212211 \\
220102 \\
221200 \\
222001
\end{array}\right)
$$

For $D_{1}$, the vector of $B_{j}$ values are $(0,0,4015332,5078214,3306744,1417176)$ and for $D_{2}$ we have $(0,0,3188646,6377292,3188646,1062882)$. It should be noted that $\sum_{j=0}^{6} B_{j}=14348907$ for both designs. If we return to Equation (4.9), we see that $\sum_{j} B_{j}$ will be the same for two designs with the same factor and level combinations, provided they have no repeated runs. From this perspective, minimizing $B_{j}$ for smaller values of $j$ is equivalent to maximizing $B_{j}$ for larger values of $j$.

Table 4.3: $S_{u}^{2}(\mathrm{SS})$ for 3-factor interactions for $D_{1}$ and $D_{2}$.

| Interaction | df | $D_{1}$ | $D_{u}^{2}$ |
| :---: | ---: | :---: | :---: |
| x1:x2:x3 | 8 | 157464 | 0 |
| x1:x2:x4 | 8 | 236196 | 0 |
| x1:x3:x4 | 8 | 236196 | 0 |
| x2:x3:x4 | 8 | 157464 | 0 |
| x1:x2:x5 | 8 | 236196 | 1062882 |
| x1:x3:x5 | 8 | 157464 | 0 |
| x2:x3:x5 | 8 | 236196 | 0 |
| x1:x4:x5 | 8 | 157464 | 0 |
| x2:x4:x5 | 8 | 236196 | 0 |
| x3:x4:x5 | 8 | 157464 | 0 |
| x1:x2:x6 | 8 | 157464 | 0 |
| x1:x3:x6 | 8 | 236196 | 1062882 |
| x2:x3:x6 | 8 | 236196 | 0 |
| x1:x4:x6 | 8 | 236196 | 0 |
| x2:x4:x6 | 8 | 236196 | 1062882 |
| x3:x4:x6 | 8 | 236196 | 0 |
| x1:x5:x6 | 8 | 236196 | 0 |
| x2:x5:x6 | 8 | 157464 | 0 |
| x3:x5:x6 | 8 | 157464 | 0 |
| x4:x5:x6 | 8 | 157464 | 0 |

### 4.5 Searching for Designs

In this section we search for designs using $S_{u}^{2}$ as discussed in Section 4.3 to find minimum $G$ and $G_{2}$-aberration designs among orthogonal arrays. As was the approach in Chapter 3, we start with an existing orthogonal array and search among smaller subsets for the best designs. We firstly examine orthogonal arrays of 18 runs and then move to 27 runs. Because we are dealing with orthogonal arrays, our attention will be placed on 3-factor projections through $S_{u}^{2}$ with $|u|=3$. For convenience, throughout this section we report results in terms of projected $A_{3}$ values instead of $S_{u}^{2}$ to remain consistent with previous literature. We use projected $A_{3}$ and $S_{u}^{2}$ interchangeably in

Table 4.4: $S_{u}^{2}(\mathrm{SS})$ for interactions with greater than 3 factors for $D_{1}$ and $D_{2}$.

| Interaction | df | $D_{1}$ | $D_{2}$ |
| :--- | :---: | :---: | :---: |
| x1:x2:x3:x4 | 16 | 275562 | 1062882 |
| x1:x2:x3:x5 | 16 | 393660 | 0 |
| x1:x2:x4:x5 | 16 | 314928 | 0 |
| x1:x3:x4:x5 | 16 | 354294 | 1062882 |
| x2:x3:x4:x5 | 16 | 393660 | 1062882 |
| x1:x2:x3:x6 | 16 | 393660 | 0 |
| x1:x2:x4:x6 | 16 | 314928 | 0 |
| x1:x3:x4:x6 | 16 | 236196 | 0 |
| x2:x3:x4:x6 | 16 | 314928 | 0 |
| x1:x2:x5:x6 | 16 | 275562 | 0 |
| x1:x3:x5:x6 | 16 | 275562 | 0 |
| x2:x3:x5:x6 | 16 | 393660 | 1062882 |
| x1:x4:x5:x6 | 16 | 393660 | 1062882 |
| x2:x4:x5:x6 | 16 | 275562 | 0 |
| x3:x4:x5:x6 | 16 | 472392 | 1062882 |
| x1:x2:x3:x4:x5 | 32 | 551124 | 0 |
| x1:x2:x3:x4:x6 | 32 | 590490 | 1062882 |
| x1:x2:x3:x5:x6 | 32 | 551124 | 1062882 |
| x1:x2:x4:x5:x6 | 32 | 629856 | 1062882 |
| x1:x3:x4:x5:x6 | 32 | 551124 | 0 |
| x2:x3:x4:x5:x6 | 32 | 433026 | 0 |
| x1:x2:x3:x4:x5:x6 | 64 | 1417176 | 1062882 |

this section.
Before moving on, we clarify the calculations used in the search for designs in the remainder of this chapter. In Section 4.4, the calculation of $S_{u}^{2}$ was based upon an ANOVA analysis of a full factorial design with the frequency of design points as the response. Since we have only a fraction of the runs from the full factorial, many components of the response vector were zero. In the case of an $O A\left(27,3^{13}\right)$, the full factorial would have $3^{13}=1594323$ runs, but the response vector would have at most 27 nonzero entries. Because of the ANOVA decomposition, the calculation of $S_{u}^{2}$ can be done with simpler formulas through the definition in Section 4.2. The simplified

Table 4.5: Tabulation of $S_{u}^{2}$ for $|u|=3$ for $D_{1}$ and $D_{2}$.

|  | $S_{u}^{2}$ frequency |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| design | 0 | 157464 | 236196 | 1062882 |
| $D_{1}$ | 0 | 9 | 11 | 0 |
| $D_{2}$ | 17 | 0 | 0 | 3 |

Table 4.6: Tabulation of $S_{i, 3}^{2}$ for $D_{1}$ and $D_{2}$.

|  | $S_{1,3}^{2}$ | $S_{2,3}^{2}$ | $S_{3,3}^{2}$ | $S_{4,3}^{2}$ | $S_{5,3}^{2}$ | $S_{6,3}^{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $D_{1}$ | 2047032 | 2047032 | 1968300 | 2047032 | 1889568 | 2047032 |
| $D_{2}$ | 2125764 | 2125764 | 1062882 | 1062882 | 1062882 | 2125764 |

formulation allows easier calculation of $S_{u}^{2}$ without creating the full factorial and can be thought of as the manual computing formulas for the sum of squares components. Since many of the terms are 0 , the computational savings is substantial as the manual computation formulas allow us to ignore any of the terms that are 0 in the calculation.

We examine designs by presenting our results beginning with the maximum number of factors from the larger orthogonal array. With the $S_{u}^{2}$ calculated for all $u \subseteq Z_{m}$, we can analyze a subset $v$ of $m^{*} \leq m$ factors by simply looking at the $S_{u}^{2}$ for $u \subseteq v$. If we relate this approach back to linear regression, we are dealing with a best subset selection process. One could then use forward or backwards selection using the error sum of squares (SSE) such that the number of factors in the model is of size $m^{*}$. In our situation, the SSE would be formed by taking the sum of all $S_{u}^{2}$ such that $u$ contains a factor outside of the current design. Instead of trying to minimize the SSE, we want to maximize it. By starting with the larger design and calculating the full set of $S_{u}^{2}$ values, it is convenient to start with the larger designs and sequentially remove factors.

### 4.5.1 Designs with 18 Runs

Table 4.7 provides the three non-isomorphic 18-run designs with 7 three-level factors as identified by Evangelaras, Koukouvinos and Lappas (2007). We refer to these designs as $N 18.1, N 18.2$ and $N 18.3$. Schoen (2009) also identified the three non-isomorphic 18-run designs and presented the projected $A_{3}$ frequency, which is equivalent to our definition of $G$-aberration.

For $m=7$, all three designs are the same by minimum $G_{2}$-aberration with an overall $A_{3}=22$. Table 4.8 shows the frequencies for the projected $A_{3}$ values. According to minimum $G$-aberration, we would rank the designs in order from best to worst as $N 18.2, N 18.1$, and $N 18.3$. Table 4.9 gives the values for $A_{3(i)}$, where $A_{3(i)}$ corresponds to the sum of the projected $A_{3}$ values that involve factor $i$ (the equivalent of $S_{i: k}^{2}$ ). The ranking according to $G_{2(i)}$-aberration is the same as $G$-aberration.

Table 4.10 shows the results for the top designs found by removing one column from the parent designs. For each parent design, the design listed is best in terms of $G, G_{2}$, and $G_{2(i)}$-aberration. For $m=6$, the rank is reversed from the case of $m=7$. Table 4.11 gives the $A_{3(i)}$ values, where $i$ refers to the column in the resulting design with $m=6$. By a theoretical result in Xu (2003), any subdesign of $m \leq 6$ from $N 18.3$ with column 4 removed has minimum $G$ and $G_{2}$-aberration.

In terms of overall $A_{3}$, the change in the ranking was expected based on the results for $m=7$ from Table 4.9. For $N 18.3$, we find that the largest value for $A_{3(i)}$ occurs for $i=4$. If we remove factor 4 from this design, the overall $A_{3}$ value for the subdesign will be the overall $A_{3}$ from the full design minus all projected $A_{3}$ values that include factor 4 , which is exactly what is calculated by $A_{3(4)}$. That is, given a design $D$ with $k$ factors, if we remove factor $i$, the overall $A_{3}$ for the subdesign is $A_{3}(D)-A_{3(i)}$, where $A_{3}(D)$ is the overall $A_{3}$ for $D$.

The above observations have some practical implications. Intuitively, we might expect that selecting the best larger design as a parent design may lead to better subdesigns. As we have seen in this example, this is not necessarily the case, as all of the best subdesigns come from the parent design with the worst ranking for $m=7$. This also has implications in constructing designs in a columnwise manner. If our goal
is for $m=7$ factors, design $N 18.3$ would be selected as the best design for $m \leq 6$, and the worst for $m=7$. Computer algorithms that create designs by sequentially adding columns can fall prey to creating less appealing designs in such a situation.

We conclude our studies on 18-run designs with an interesting case. For N18.2, with $m=6$ with factor 1 removed, all projections onto $m=5$ factors have the same $A_{3}=6$, and projected $A_{3}$ frequency of 4 with 0.5 and 6 with $2 / 3$ (recall that $A_{3(i)}$ is the same for all $i$ in Table 4.11), as shown in Table 4.12. At $m=6$, the $A_{3(i)}$ values make no distinction between any of the remaining factors in terms of $G, G_{2}$, or $G_{2(i)}$. However, looking at the $A_{3(i)}$ for $m=5$ in Table 4.13, $G_{2(i)}$-aberration does make a separation between designs. Table 4.14 shows the projected $A_{3}$ frequencies for $m=4$. For design $N 18.2$, if we want a design for $m=5$, we should remove columns 1 and 2 , whereas if we want $m=4$, we would be removing columns 1 and 3 at $m=5$. If we were only using $G$ and $G_{2}$-aberration and the ultimate goal was $m=4$, the removal of columns at $m=6$ to $m=5$ would find no difference between removing columns 1 and 2 versus columns 1 and 3 . In this case, $G_{2(i)}$ provides insight into the lower dimensions, and depending on the desired $m$, we want to choose designs with worst $G_{2(i)}$. Before we move on to 27 run designs, we state this more precisely.

Proposition 4.1. Let $D_{1}$ and $D_{2}$ be two designs with $k$ factors having the same $G_{2}$ aberration such that $A_{3}\left(D_{1}\right)=A_{3}\left(D_{2}\right)$. If $D_{1}$ has less $G_{2(i)}$-aberration than $D_{2}$, then the best subdesigns of $D_{1}$ and $D_{2}$ with $k-1$ factors in terms of $A_{3}, D_{1}^{-}$and $D_{2}^{-}$, are such that $A_{3}\left(D_{1}^{-}\right) \geq A_{3}\left(D_{2}^{-}\right)$. That is, $D_{2}^{-}$is at least as good as $D_{1}^{-}$in terms of $G_{2}$-aberration.

Proposition 4.2. Let $D_{1}$ and $D_{2}$ be two designs with $k$ factors having the same $G$ aberration such that they have the same frequency of projected $A_{3}$ values. If $D_{1}$ has smaller $G_{(i)}$-aberration than $D_{2}$, then the best subdesigns of $D_{1}$ and $D_{2}$ with $k-1$ factors in terms of $G$-aberration, $D_{1}^{-}$and $D_{2}^{-}$, are such that $D_{2}^{-}$is at least as good as $D_{1}^{-}$in terms of $G$-aberration.

Table 4.7: Three nonisomorphic $O A\left(18,3^{7}\right), N 18.1, N 18.2$, and $N 18.3$.


Table 4.8: Frequency of projected $A_{3}$ values for $N 18.1, N 18.2$ and $N 18.3$ and $m=7$.

|  | $\frac{1}{2}$ | $\frac{2}{3}$ | 1 | 2 | $A_{3}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $N 18.1$ | 20 | 12 | 2 | 1 | 22 |
| N18.2 | 16 | 18 | 0 | 1 | 22 |
| $N 18.3$ | 28 | 0 | 6 | 1 | 22 |

Table 4.9: Tabulation of $A_{3(i)}$ for $N 18.1, N 18.2$ and $N 18.3$ and $m=7$.

|  | $A_{3(1)}$ | $A_{3(2)}$ | $A_{3(3)}$ | $A_{3(4)}$ | $A_{3(5)}$ | $A_{3(6)}$ | $A_{3(7)}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| N18.1 | 9.67 | 9 | 10.67 | 9.67 | 9 | 9 | 9 |
| N18.2 | 10 | 9 | 10 | 10 | 9 | 9 | 9 |
| N18.3 | 9 | 9 | 9 | 12 | 9 | 9 | 9 |

Table 4.10: Frequency of projected $A_{3}$ values for $N 18.1, N 18.2$ and $N 18.3$ and $m=6$.

|  | removed | $\frac{1}{2}$ | $\frac{2}{3}$ | 1 | 2 | $A_{3}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $N 18.1$ | 3 | 12 | 8 | 0 | 0 | 11.33 |
| $N 18.2$ | 1 | 8 | 12 | 0 | 0 | 12 |
| $N 18.3$ | 4 | 20 | 0 | 0 | 0 | 10 |

### 4.5.2 Designs with 27 Runs

In this Section we search through the design catalog of Evangelaras, Koukouvinos and Lappas (2011) of 129 saturated orthogonal arrays with 27 runs to find the minimum $G$ and $G_{2}$-aberration designs among these. In this chapter, we refer to the $i$ 'th design from the catalog as D27.i. Among the saturated designs, we look for the best subdesigns in terms of $G$ and $G_{2}$-aberration. As discussed previously, since we are starting with saturated designs and looking for subdesigns from these, our results will be presented starting from $m=13$ and moving down sequentially. Xu, Cheng and Wu (2004) did a complete search of 27-run orthogonal arrays and found there are eight distinct values that the projected $A_{3}$ 's can take: $(0,8 / 27,4 / 9,14 / 27,2 / 3,20 / 27,10 / 9,2)$. Our results will be presented with the projected $A_{3}$ frequency vector for each design.

Table 4.15 summarizes the saturated orthogonal arrays ordered in terms of $G$ aberration. All saturated orthogonal arrays have an overall $A_{3}=104$. The worst design in terms of $G$-aberration is the regular design. Among all these designs, only $D 27.1$ and $D 27.24$ do not contain a projected $A_{3}$ value of 2 , corresponding to full aliasing. All of the designs have the same $A_{3}$ values of 80 and 60 for $m=12$ and

Table 4.11: Tabulation of $A_{3(i)}$ for $N 18.1, N 18.2$ and $N 18.3$ and $m=6$.

|  | removed | $A_{3(1)}$ | $A_{3(2)}$ | $A_{3(3)}$ | $A_{3(4)}$ | $A_{3(5)}$ | $A_{3(6)}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| N18.1 | 3 | 5.67 | 5.67 | 5.67 | 5.67 | 5.67 | 5.67 |
| N18.2 | 1 | 6 | 6 | 6 | 6 | 6 | 6 |
| N18.3 | 4 | 5 | 5 | 5 | 5 | 5 | 5 |

Table 4.12: Frequency of projected $A_{3}$ values for $N 18.1, N 18.2$ and $N 18.3$ and $m=5$.

|  | removed | $\frac{1}{2}$ | $\frac{2}{3}$ | 1 | 2 | $A_{3}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| $N 18.1$ | $(1,3)$ | 6 | 4 | 0 | 0 | 5.67 |
| $N 18.2$ | $(1,2)$ | 4 | 6 | 0 | 0 | 6 |
| $N 18.2$ | $(1,3)$ | 4 | 6 | 0 | 0 | 6 |
| $N 18.3$ | $(1,4)$ | 10 | 0 | 0 | 0 | 5 |

$m=11$, respectively. Tables 4.16 and 4.17 give the top 30 designs in terms of $G$ aberration. The numbers under removed refer to the columns to be removed from the parent design to get a resulting design with the given projected $A_{3}$ frequency vector and overall $A_{3}$ value as provided. As was the case for $m=13$, only designs $D 27.1$ and $D 27.24$ do not have a projected $A_{3}$ with a value of 2 . The frequency vector of projected $A_{3}$ values for each factor is not necessarily the same, as we see some differences in the frequency vectors for $m=12$ from the same parent design.

At $m=10$, we begin to see differentiation between designs in terms of $G_{2^{-}}$ aberration. Table 4.18 provides the top designs in terms of $G$-aberration, while Table 4.19 in terms of $G_{2}$-aberration. For designs having the same overall $A_{3}$, our secondary ranking is that of $G$-aberration. Deng and Tang (2002) showed that for two-level designs with small run sizes, the ranking between $G$ and $G_{2}$-aberration is generally consistent. As we see at $m=10$, this is not the case for 27 run designs. The best $G_{2^{-}}$ aberration designs have an overall $A_{3}$ value of 42 , and the best of the $G_{2}$-aberration designs in terms of $G$-aberration does not appear on the list of top $G$-aberration designs. In particular, all contain projected $A_{3}$ values of 2 .

We continue to see this gap between $G$ and $G_{2}$-aberration designs for $9 \geq m \geq 5$

Table 4.13: Tabulation of $A_{3(i)}$ for $N 18.1, N 18.2$ and $N 18.3$ and $m=5$.

|  | removed | $A_{3(1)}$ | $A_{3(2)}$ | $A_{3(3)}$ | $A_{3(4)}$ | $A_{3(5)}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $N 18.1$ | $(1,3)$ | 3.33 | 3.67 | 3.33 | 3.33 | 3.33 |
| $N 18.2$ | $(1,2)$ | 3.5 | 3.5 | 3.67 | 3.67 | 3.67 |
| $N 18.2$ | $(1,3)$ | 3.5 | 4 | 3.5 | 3.5 | 3.5 |
| $N 18.3$ | $(1,4)$ | 3 | 3 | 3 | 3 | 3 |

Table 4.14: Frequency of projected $A_{3}$ values for $N 18.1, N 18.2$ and $N 18.3$ and $m=4$.

|  | removed | $\frac{1}{2}$ | $\frac{2}{3}$ | 1 | 2 | $A_{3}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $N 18.1$ | $(1,3,4)$ | 4 | 0 | 0 | 0 | 2 |
| $N 18.2$ | $(1,2,5)$ | 2 | 2 | 0 | 0 | 2.33 |
| $N 18.2$ | $(1,3,4)$ | 4 | 0 | 0 | 0 | 2 |
| $N 18.3$ | $(1,2,4)$ | 4 | 0 | 0 | 0 | 2 |

as seen in tables 4.20-4.29. At $m=4$, the best designs for both $G$ and $G_{2}$-aberration are the strength three $O A\left(27,3^{4}\right)$, and not presented here. For $m \geq 6$, we find that the top $G_{2}$ designs always have at least one projected $A_{3}$ value of 2 . The best $G$-aberration designs tend to come from the same parent designs as we reduce the number of factors. This is also true of the $G_{2}$-aberration designs. However, we do not generally see overlap between the top parent designs.

Using an algorithm from Xu (2002), Xu, Cheng and Wu (2004) found designs for $5 \leq m \leq 10$ with less $G$-aberration than those found as subdesigns from the saturated orthogonal arrays. It is interesting to note however, that they did not find designs having less $G_{2}$-aberration than those found through the saturated orthogonal array.

Before concluding this section, we make some comments on $G_{2 i}$-aberration. For $m=13$ and $m=12$, all designs are identical by $G_{2(i)}$-aberration. In both cases, the $A_{3(i)}$ values are the same for all $i$ for each design. For $m=11$ and $m=10$, the top $G$-aberration designs are ranked above the $G_{2}$-aberration designs. This can also be seen through Proposition 4.1 and looking at the overall $A_{3}$ values for $m=10$ and $m=9$ factors. For $m \leq 9$, we tend to see a reversal of this ranking. That is, after
this point the better $G_{2}$-aberration designs rank above the $G$-aberration ones. This is due in large part to the fact that the overall $A_{3}$ values are much smaller.

Based on these results, in designing a 27-run experiment, the choice between $G$ and $G_{2}$-aberration becomes much more of an issue as their ranking is not generally consistent. One particular point of interest is that the minimum $G_{2}$-aberration designs tend to have at least one projected $A_{3}$ value of 2 , corresponding to full aliasing. If $G$-aberration is more of a priority, one compromise is to take the smallest overall $A_{3}$ value from a list of the top ranked $G$-aberration designs. If $G_{2}$-aberration is considered more important to the experimenter, one might take the smallest overall $A_{3}$ value such that there are no projected $A_{3}$ values of 2 .

### 4.6 Discussion

In this chapter, we introduced $J$-characteristics for multi-level designs that lead to the definition of $G$ and $G_{2}$-aberration. Using the ANOVA decomposition has a broad appeal in that it is easily accessible by a wide range of audiences. The algorithmic construction of nearly-orthogonal arrays (Xu, 2002 and Nguyen and Liu, 2008) typically use an equivalent of $G_{2}$-aberration and focus on one dimension. With the use of $J$-characteristics, considerations can be placed on $G$-aberration and additional lower dimensional properties. By taking the viewpoint of trying to spread out the design points in lower dimensions, we get intuitively good projection properties that also have robustness properties if we return to the linear model. By viewing $G$ and $G_{2^{-}}$ aberration from a projection standpoint, we get designs that are appealing as building blocks for other designs such as those described in Bingham, Sitter and Tang (2009) and Lin, Bingham, Sitter and Tang (2010).

Table 4.15: All 129 designs ranked by $G$-aberration.

| design | $\left(0, \frac{8}{27}, \frac{4}{9}, \frac{14}{27}, \frac{2}{3}, \frac{20}{27}, \frac{10}{9}, 2\right)$ | $A_{3}$ |
| :--- | :---: | :---: |
| 1 | $(78,0,156,0,52,0,0,0)$ | 104 |
| 24 | $(52,130,0,52,0,52,0,0)$ | 104 |
| 53 | $(85,27,108,0,63,0,0,3)$ | 104 |
| 73 | $(92,30,80,24,50,0,7,3)$ | 104 |
| 74 | $(92,30,80,24,50,0,7,3)$ | 104 |
| 95 | $(98,26,80,28,37,0,14,3)$ | 104 |
| 96 | $(98,26,80,28,37,0,14,3)$ | 104 |
| 97 | $(98,26,80,28,37,0,14,3)$ | 104 |
| 113 | $(94,46,56,36,29,8,14,3)$ | 104 |
| 114 | $(94,46,56,36,29,8,14,3)$ | 104 |
| 115 | $(94,46,56,36,29,8,14,3)$ | 104 |
| 123 | $(108,0,126,0,28,0,21,3)$ | 104 |
| 124 | $(98,34,94,12,16,8,21,3)$ | 104 |
| 125 | $(98,34,94,12,16,8,21,3)$ | 104 |
| 126 | $(98,34,94,12,16,8,21,3)$ | 104 |
| 127 | $(94,56,68,18,10,16,21,3)$ | 104 |
| 128 | $(94,56,68,18,10,16,21,3)$ | 104 |
| 129 | $(94,56,68,18,10,16,21,3)$ | 104 |
| 2 | $(91,73,45,28,0,25,21,3)$ | 104 |
| 9 | $(117,27,0,54,81,0,0,7)$ | 104 |
| 10 | $(117,27,0,54,81,0,0,7)$ | 104 |
| 3 | $(108,27,45,36,45,18,0,7)$ | 104 |
| 4 | $(108,27,45,36,45,18,0,7)$ | 104 |
| 5 | $(108,27,45,36,45,18,0,7)$ | 104 |
| 5 | $(108,27,45,36,45,18,0,7)$ | 104 |
| 6 | $(108,27,45,36,45,18,0,7)$ | 104 |
| 7 | $(108,27,45,36,45,18,0,7)$ | 104 |
| 8 | $(117,27,12,54,63,0,6,7)$ | 104 |
| 11 | $(117,27,12,54,63,0,6,7)$ | 104 |
| 12 | $(120,0,90,0,60,0,9,7)$ | 104 |
| 28 | $(120,0,90,0,60,0,9,7)$ | 104 |
| 29 | $(120,0,90,0,60,0,9,7)$ | 104 |
| 29 | $(120,0,90,0,60,0,9,7)$ | 104 |
| 30 | $(120,0,90,0,60,0,9,7)$ | 104 |
| 31 | $(126,0,72,0,72,0,9,7)$ | 104 |
| 32 | $(126,0,72,0,72,0,9,7)$ | 104 |
| 33 | $(108,18,84,24,24,12,9,7)$ | 104 |
| 34 | $(108,18,84,24,24,12,9,7)$ | 104 |
| 15 | $108,18,84,24,24,12,9,7)$ | 104 |
| 16 | $(108,18,84,24,24,12,9,7)$ | 104 |
| 17 |  |  |
| 18 |  |  |

Table 4.15: All 129 designs ranked by $G$-aberration.

| design | $\left(0, \frac{8}{27}, \frac{4}{9}, \frac{14}{27}, \frac{2}{3}, \frac{20}{27}, \frac{10}{9}, 2\right)$ | $A_{3}$ |
| :--- | :---: | :---: |
| 19 | $(108,18,84,24,24,12,9,7)$ | 104 |
| 20 | $(108,18,84,24,24,12,9,7)$ | 104 |
| 21 | $(108,18,84,24,24,12,9,7)$ | 104 |
| 22 | $(108,18,84,24,24,12,9,7)$ | 104 |
| 23 | $(108,18,84,24,24,12,9,7)$ | 104 |
| 25 | $(111,18,75,24,30,12,9,7)$ | 104 |
| 26 | $(111,18,75,24,30,12,9,7)$ | 104 |
| 27 | $(111,18,75,24,30,12,9,7)$ | 104 |
| 13 | $(120,30,36,12,60,12,9,7)$ | 104 |
| 14 | $(120,30,36,12,60,12,9,7)$ | 104 |
| 41 | $(120,18,42,36,51,0,12,7)$ | 104 |
| 42 | $(120,18,42,36,51,0,12,7)$ | 104 |
| 43 | $(120,18,42,36,51,0,12,7)$ | 104 |
| 44 | $(120,18,42,36,51,0,12,7)$ | 104 |
| 45 | $(120,18,42,36,51,0,12,7)$ | 104 |
| 37 | $(114,18,72,24,27,12,12,7)$ | 104 |
| 38 | $(114,18,72,24,27,12,12,7)$ | 104 |
| 39 | $(114,18,72,24,27,12,12,7)$ | 104 |
| 40 | $(114,18,72,24,27,12,12,7)$ | 104 |
| 35 | $(114,18,72,24,27,12,12,7)$ | 104 |
| 36 | $(114,18,72,24,27,12,12,7)$ | 104 |
| 58 | $(117,18,57,36,36,0,15,7)$ | 104 |
| 59 | $(117,18,57,36,36,0,15,7)$ | 104 |
| 54 | $(114,18,78,24,18,12,15,7)$ | 104 |
| 55 | $(114,18,78,24,18,12,15,7)$ | 104 |
| 56 | $(114,18,78,24,18,12,15,7)$ | 104 |
| 57 | $(114,18,78,24,18,12,15,7)$ | 104 |
| 51 | $(114,18,78,24,18,12,15,7)$ | 104 |
| 52 | $(114,18,78,24,18,12,15,7)$ | 104 |
| 52 | $(120,30,48,12,42,12,15,7)$ | 104 |
| 46 | $120,30,48,12,42,12,15,7)$ | 104 |
| 47 | $120,30,48,12,42,12,15,7)$ | 104 |
| 48 | $120,30,48,12,42,12,15,7)$ | 104 |
| 49 | $(120,30,48,12,42,12,15,7)$ | 104 |
| 50 | $(120,18,54,36,33,0,18,7)$ | 104 |
| 62 | $(120,18,54,36,33,0,18,7)$ | 104 |
| 63 | $(117,18,75,24,15,12,18,7)$ | 104 |
| 60 | $(117,18,75,24,15,12,18,7)$ | 104 |
| 61 | $(126,0,96,0,36,0,21,7)$ | 104 |
| 64 | $(126,0,96,0,36,0,21,7)$ | 104 |
| 65 |  |  |
|  |  |  |

Table 4.15: All 129 designs ranked by $G$-aberration.

| design | $\left(0, \frac{8}{27}, \frac{4}{9}, \frac{14}{27}, \frac{2}{3}, \frac{20}{27}, \frac{10}{9}, 2\right)$ | $A_{3}$ |
| :--- | :---: | :---: |
| 66 | $(120,30,60,12,24,12,21,7)$ | 104 |
| 67 | $(120,30,60,12,24,12,21,7)$ | 104 |
| 68 | $(120,30,60,12,24,12,21,7)$ | 104 |
| 71 | $(126,0,102,0,27,0,24,7)$ | 104 |
| 72 | $(126,0,102,0,27,0,24,7)$ | 104 |
| 69 | $(129,0,93,0,33,0,24,7)$ | 104 |
| 70 | $(129,0,93,0,33,0,24,7)$ | 104 |
| 79 | $(132,0,84,0,39,0,24,7)$ | 104 |
| 80 | $(132,0,84,0,39,0,24,7)$ | 104 |
| 81 | $(132,0,84,0,39,0,24,7)$ | 104 |
| 77 | $(120,30,66,12,15,12,24,7)$ | 104 |
| 78 | $(120,30,66,12,15,12,24,7)$ | 104 |
| 75 | $(123,30,57,12,21,12,24,7)$ | 104 |
| 76 | $(123,30,57,12,21,12,24,7)$ | 104 |
| 82 | $(126,0,108,0,18,0,27,7)$ | 104 |
| 83 | $(126,0,108,0,18,0,27,7)$ | 104 |
| 86 | $(120,30,72,12,6,12,27,7)$ | 104 |
| 87 | $(120,30,72,12,6,12,27,7)$ | 104 |
| 84 | $(126,30,54,12,18,12,27,7)$ | 104 |
| 85 | $(126,30,54,12,18,12,27,7)$ | 104 |
| 88 | $(123,30,69,12,3,12,30,7)$ | 104 |
| 89 | $(123,30,69,12,3,12,30,7)$ | 104 |
| 90 | $(132,0,102,0,12,0,33,7)$ | 104 |
| 91 | $(132,0,102,0,12,0,33,7)$ | 104 |
| 92 | $(132,0,54,0,90,0,0,10)$ | 104 |
| 93 | $(132,0,54,0,90,0,0,10)$ | 104 |
| 94 | $(132,18,30,36,45,0,15,10)$ | 104 |
| 98 | $(138,0,66,0,57,0,15,10)$ | 104 |
| 99 | $(138,0,66,0,57,0,15,10)$ | 104 |
| 100 | $(138,0,96,0,12,0,30,10)$ | 104 |
| 101 | $(132,30,60,12,0,12,30,10)$ | 104 |
| 102 | $(132,30,60,12,0,12,30,10)$ | 104 |
| 104 | $(126,0,108,0,36,0,0,16)$ | 104 |
| 105 | $(126,0,108,0,36,0,0,16)$ | 104 |
| 106 | $(135,0,81,0,54,0,0,16)$ | 104 |
| 107 | $(135,0,81,0,54,0,0,16)$ | 104 |
| 103 | $(162,0,0,0,108,0,0,16)$ | 104 |
| 108 | $(144,0,72,0,45,0,9,16)$ | 104 |
| 111 | $(144,0,90,0,18,0,18,16)$ | 104 |
| 109 | $(162,0,36,0,54,0,18,16)$ | 104 |
|  |  |  |
|  |  |  |
| 10 |  |  |

Table 4.15: All 129 designs ranked by $G$-aberration.

| design | $\left(0, \frac{8}{27}, \frac{4}{9}, \frac{14}{27}, \frac{2}{3}, \frac{20}{27}, \frac{10}{9}, 2\right)$ | $A_{3}$ |
| :--- | :---: | :---: |
| 110 | $(162,0,36,0,54,0,18,16)$ | 104 |
| 112 | $(153,0,81,0,9,0,27,16)$ | 104 |
| 116 | $(162,0,54,0,27,0,27,16)$ | 104 |
| 117 | $(162,0,54,0,27,0,27,16)$ | 104 |
| 118 | $(162,0,72,0,0,0,36,16)$ | 104 |
| 119 | $(162,0,72,0,0,0,36,16)$ | 104 |
| 120 | $(180,0,0,0,81,0,0,25)$ | 104 |
| 121 | $(180,0,54,0,0,0,27,25)$ | 104 |
| 122 | $(234,0,0,0,0,0,0,52)$ | 104 |

Table 4.16: Top $G$-aberration designs for $m=12$.

| design | removed | $\left(0, \frac{8}{27}, \frac{4}{9}, \frac{14}{27}, \frac{2}{3}, \frac{20}{27}, \frac{10}{9}, 2\right)$ | $A_{3}$ |
| :---: | :---: | :---: | :---: |
| 1 | $(1)$ | $(60,0,120,0,40,0,0)$ | 80 |
| 24 | $(1)$ | $(40,100,0,40,0,40,0,0)$ | 80 |
| 53 | $(1)$ | $(66,18,84,0,50,0,0,2)$ | 80 |
| 73 | $(3)$ | $(66,24,70,12,44,0,2,2)$ | 80 |
| 74 | $(3)$ | $(66,24,70,12,44,0,2,2)$ | 80 |
| 73 | $(2)$ | $(71,21,58,24,39,0,5,2)$ | 80 |
| 74 | $(8)$ | $(71,21,58,24,39,0,5,2)$ | 80 |
| 73 | $(1)$ | $(70,30,46,24,43,0,5,2)$ | 80 |
| 74 | $(13)$ | $70,30,46,24,43,0,5,2)$ | 80 |
| 73 | $(8)$ | $(73,18,63,18,40,0,6,2)$ | 80 |
| 74 | $(1)$ | $73,18,63,18,40,0,6,2)$ | 80 |
| 95 | $(2)$ | $70,26,54,28,33,0,7,2)$ | 80 |
| 96 | $(3)$ | $70,26,54,28,33,0,7,2)$ | 80 |
| 97 | $(13)$ | $70,26,54,28,33,0,7,2)$ | 80 |
| 73 | $(7)$ | $72,20,66,16,37,0,7,2)$ | 80 |
| 74 | $(5)$ | $(72,20,66,16,37,0,7,2)$ | 80 |
| 113 | $(2)$ | $(68,36,42,32,29,4,7,2)$ | 80 |
| 114 | $(3)$ | $(68,36,42,32,29,4,7,2)$ | 80 |
| 115 | $(13)$ | $(68,36,42,32,29,4,7,2)$ | 80 |
| 95 | $(3)$ | $(72,20,70,16,31,0,9,2)$ | 80 |
| 96 | $(13)$ | $72,20,70,16,31,0,9,2)$ | 80 |
| 97 | $(3)$ | $72,20,70,16,31,0,9,2)$ | 80 |
| 113 | $(3)$ | $72,28,54,24,23,8,9,2)$ | 80 |
| 114 | $(7)$ | $72,28,54,24,23,8,9,2)$ | 80 |
| 115 | $(2)$ | $(72,28,54,24,23,8,9,2)$ | 80 |
| 95 | $(1)$ | $(77,19,54,26,31,0,11,2)$ | 80 |
| 96 | $(6)$ | $77,19,54,26,31,0,11,2)$ | 80 |
| 97 | $(8)$ | $77,19,54,26,31,0,11,2)$ | 80 |
| 113 | $(1)$ | $74,34,36,32,25,6,11,2)$ | 80 |
| 114 | $(8)$ | $74,34,36,32,25,6,11,2)$ | 80 |

Table 4.17: Top $G$-aberration designs for $m=11$.

| design | removed | $\left(0, \frac{8}{27}, \frac{4}{9}, \frac{14}{27}, \frac{2}{3}, \frac{20}{27}, \frac{10}{9}, 2\right)$ | $A_{3}$ |
| :---: | :---: | :---: | :---: |
| 1 | $(1,2)$ | $(45,0,90,0,30,0,0,0)$ | 60 |
| 24 | $(1,2)$ | $(30,75,0,30,0,30,0,0)$ | 60 |
| 53 | $(1,4)$ | $(49,12,64,0,39,0,0,1)$ | 60 |
| 73 | $(3,7)$ | $(49,16,56,8,33,0,2,1)$ | 60 |
| 74 | $(3,5)$ | $(49,16,56,8,33,0,2,1)$ | 60 |
| 73 | $(2,3)$ | $(49,18,50,12,33,0,2,1)$ | 60 |
| 74 | $(3,8)$ | $(49,18,50,12,33,0,2,1)$ | 60 |
| 73 | $(1,3)$ | $(49,24,40,12,37,0,2,1)$ | 60 |
| 74 | $(3,13)$ | $(49,24,40,12,37,0,2,1)$ | 60 |
| 95 | $(2,3)$ | $(49,20,48,16,27,, 4,4,1)$ | 60 |
| 96 | $(3,13)$ | $(49,20,48,16,27,, 4,1)$ | 60 |
| 97 | $(3,13)$ | $(49,20,48,16,27,0,4,1)$ | 60 |
| 73 | $(2,8)$ | $(54,12,45,18,31,0,4,1)$ | 60 |
| 74 | $(1,8)$ | $(54,12,45,18,31,0,4,1)$ | 60 |
| 73 | $(1,8)$ | $(53,18,38,18,33,0,4,1)$ | 60 |
| 74 | $(1,13)$ | $(53,18,38,18,33,0,4,1)$ | 60 |
| 113 | $(2,3)$ | $(49,24,40,20,23,4,4,1)$ | 60 |
| 114 | $(3,7)$ | $(49,24,40,20,23,4,4,1)$ | 60 |
| 115 | $(2,13)$ | $(49,24,40,20,23,4,4,1)$ | 60 |
| 73 | $(2,7)$ | $(53,14,48,16,28,0,5,1)$ | 60 |
| 74 | $(5,8)$ | $(53,14,48,16,28,, 5,1)$ | 60 |
| 73 | $(1,7)$ | $(53,20,38,16,32,, 5,5,1)$ | 60 |
| 74 | $(5,13)$ | $(53,20,38,16,32,0,5,1)$ | 60 |
| 95 | $(2,6)$ | $(53,16,44,20,25,0,6,1)$ | 60 |
| 96 | $(3,5)$ | $(53,16,44,20,25,0,6,1)$ | 60 |
| 97 | $(1,13)$ | $(53,16,44,20,25,0,6,1)$ | 60 |
| 73 | $(7,8)$ | $(55,12,50,12,29,0,6,1)$ | 60 |
| 74 | $(1,5)$ | $(55,12,50,12,29,0,6,1)$ | 60 |
| 113 | $(2,6)$ | $(51,24,36,22,23,2,6,1)$ | 60 |
| 114 | $(3,5)$ | $(51,24,36,22,23,2,6,1)$ | 60 |

Table 4.18: Top $G$-aberration designs for $m=10$.

| design | removed | $\left(0, \frac{8}{27}, \frac{4}{9}, \frac{14}{27}, \frac{2}{3}, \frac{20}{27}, \frac{10}{9}, 2\right)$ | $A_{3}$ |
| :---: | :---: | :---: | :---: |
| 1 | $(1,2,4)$ | $(33,0,66,0,21,0,0,0)$ | 43.3333 |
| 1 | $(1,2,3)$ | $(33,0,65,0,22,0,0,0)$ | 43.5556 |
| 1 | $(1,2,11)$ | $(32,0,66,0,22,0,0,0)$ | 44.0000 |
| 53 | $(1,4,5)$ | $(34,8,48,0,30,0,0,0)$ | 43.7037 |
| 24 | $(1,2,5)$ | $(22,55,0,22,0,21,0,0)$ | 43.2593 |
| 24 | $(1,2,3)$ | $(22,55,0,21,0,22,0,0)$ | 43.4815 |
| 24 | $(1,2,4)$ | $(22,54,0,22,0,22,0,0)$ | 43.7037 |
| 24 | $(1,2,11)$ | $(21,55,0,22,0,22,0,0)$ | 44.0000 |
| 73 | $(2,3,7)$ | $(34,12,40,8,24,0,2,0)$ | 43.7037 |
| 74 | $(3,5,8)$ | $(34,12,40,8,24,0,2,0)$ | 43.7037 |
| 73 | $(1,3,7)$ | $(34,16,32,8,28,0,2,0)$ | 44.0000 |
| 74 | $(3,5,13)$ | $(34,16,32,8,28,0,2,0)$ | 44.0000 |
| 95 | $(2,3,7)$ | $(34,12,40,12,18,0,4,0)$ | 44.0000 |
| 96 | $(1,3,13)$ | $(34,12,40,12,18,0,4,0)$ | 44.0000 |
| 97 | $(3,6,13)$ | $(34,12,40,12,18,0,4,0)$ | 44.0000 |
| 95 | $(2,3,11)$ | $(34,16,36,8,22,0,4,0)$ | 44.0000 |
| 96 | $(3,4,13)$ | $(34,16,36,8,22,0,4,0)$ | 44.0000 |
| 97 | $(3,5,13)$ | $(34,16,36,8,22,0,4,0)$ | 44.0000 |
| 73 | $(2,7,8)$ | $(38,8,36,12,22,0,4,0)$ | 43.7037 |
| 74 | $(1,5,8)$ | $(38,8,36,12,22,0,4,0)$ | 43.7037 |
| 73 | $(1,7,8)$ | $(38,12,30,12,24,0,4,0)$ | 43.5556 |
| 74 | $(1,5,13)$ | $(38,12,30,12,24,0,4,0)$ | 43.5556 |
| 113 | $(2,3,10)$ | $(34,18,30,16,14,4,4,0)$ | 43.7037 |
| 114 | $(1,3,7)$ | $(34,18,30,16,14,4,4,0)$ | 43.7037 |
| 115 | $(2,6,13)$ | $(34,18,30,16,14,4,4,0)$ | 43.7037 |
| 113 | $(2,3,7)$ | $(34,20,28,12,18,4,4,0)$ | 44.0000 |
| 114 | $(2,3,7)$ | $(34,20,28,12,18,4,4,0)$ | 44.0000 |
| 115 | $(2,5,13)$ | $(34,20,28,12,18,4,4,0)$ | 44.0000 |
| 95 | $(2,6,7)$ | $(38,10,36,14,16,0,6,0)$ | 43.5556 |
| 96 | $(1,3,5)$ | $(38,10,36,14,16,0,6,0)$ | 43.5556 |

Table 4.19: Top $G_{2}$-aberration designs for $m=10$.

| design | removed | $\left(0, \frac{8}{27}, \frac{4}{9}, \frac{14}{27}, \frac{2}{3}, \frac{20}{27}, \frac{10}{9}, 2\right)$ | $A_{3}$ |
| :---: | :---: | :---: | :---: |
| 2 | $(1,2,4)$ | $(49,20,22,8,0,8,11,2)$ | 42 |
| 101 | $(1,9,10)$ | $(49,20,22,8,0,8,11,2)$ | 42 |
| 102 | $(1,2,4)$ | $(49,20,22,8,0,8,11,2)$ | 42 |
| 113 | $(7,10,11)$ | $(49,20,22,8,0,8,11,2)$ | 42 |
| 114 | $(1,2,4)$ | $(49,20,22,8,0,8,11,2)$ | 42 |
| 115 | $(5,6,7)$ | $(49,20,22,8,0,8,11,2)$ | 42 |
| 3 | $(1,2,4)$ | $(54,9,0,18,36,0,0,3)$ | 42 |
| 4 | $(5,7,13)$ | $(54,9,0,18,36,0,0,3)$ | 42 |
| 5 | $(5,6,13)$ | $(54,9,0,18,36,0,0,3)$ | 42 |
| 6 | $(5,7,13)$ | $(54,9,0,18,36,0,0,3)$ | 42 |
| 7 | $(1,2,3)$ | $(54,9,0,18,36,0,0,3)$ | 42 |
| 8 | $(5,7,13)$ | $(54,9,0,18,36,0,0,3)$ | 42 |
| 9 | $(1,2,3)$ | $(54,9,0,18,36,0,0,3)$ | 42 |
| 10 | $(5,6,13)$ | $(54,9,0,18,36,0,0,3)$ | 42 |
| 11 | $(1,2,4)$ | $(54,9,0,18,36,0,0,3)$ | 42 |
| 12 | $(2,7,9)$ | $(54,9,0,18,36,0,0,3)$ | 42 |
| 9 | $(5,6,12)$ | $(60,9,0,18,27,0,0,6)$ | 42 |
| 10 | $(1,2,3)$ | $(60,9,0,18,27,0,0,6)$ | 42 |
| 11 | $(5,9,10)$ | $(60,9,0,18,27,0,0,6)$ | 42 |
| 12 | $(1,11,12)$ | $(60,9,0,18,27,0,0,6)$ | 42 |
| 41 | $(5,8,11)$ | $(60,9,0,18,27,0,0,6)$ | 42 |
| 42 | $(5,8,11)$ | $(60,9,0,18,2,0,0,6)$ | 42 |
| 43 | $(1,11,12)$ | $(60,9,0,18,27,0,0,6)$ | 42 |
| 44 | $(1,6,7)$ | $(60,9,0,18,27,0,0,6)$ | 42 |
| 45 | $(4,9,11)$ | $(60,9,0,18,27,0,0,6)$ | 42 |
| 3 | $(5,9,10)$ | $(54,9,24,12,9,6,0,6)$ | 42 |
| 4 | $(1,2,3)$ | $(54,9,24,12,9,6,0,6)$ | 42 |
| 5 | $(1,2,3)$ | $(54,9,24,12,9,6,0,6)$ | 42 |
| 6 | $(1,2,3)$ | $(54,9,24,12,9,6,0,6)$ | 42 |
| 7 | $(5,9,10)$ | $(54,9,24,12,9,6,0,6)$ | 42 |

Table 4.20: Top $G$-aberration designs for $m=9$.

| design | removed | $\left(0, \frac{8}{27}, \frac{4}{9}, \frac{14}{27}, \frac{2}{3}, \frac{20}{27}, \frac{10}{9}, 2\right)$ | $A_{3}$ |
| :---: | :---: | :---: | :---: |
| 1 | $(1,2,4,10)$ | $(24,0,48,0,12,0,0,0)$ | 29.3333 |
| 1 | $(1,2,3,7)$ | $(24,0,45,0,15,0,0,0)$ | 30.0000 |
| 1 | $(1,2,3,11)$ | $(23,0,46,0,15,0,0,0)$ | 30.4444 |
| 1 | $(1,2,3,4)$ | $(22,0,47,0,15,0,0,0)$ | 30.8889 |
| 1 | $(1,2,3,5)$ | $(23,0,45,0,16,0,0,0)$ | 30.6667 |
| 1 | $(1,2,5,12)$ | $(22,0,46,0,16,0,0,0)$ | 31.1111 |
| 1 | $(1,2,11,13)$ | $(20,0,48,0,16,0,0,0)$ | 32.0000 |
| 53 | $(1,4,5,9)$ | $(22,8,36,0,18,0,0,0)$ | 30.3704 |
| 53 | $(1,2,4,5)$ | $(25,4,32,0,23,0,0,0)$ | 30.7407 |
| 24 | $(1,2,5,10)$ | $(16,40,0,16,0,12,0,0)$ | 29.0370 |
| 24 | $(1,2,3,13)$ | $(16,39,0,14,0,15,0,0)$ | 29.9259 |
| 24 | $(1,2,3,5)$ | $(16,38,0,15,0,15,0,0)$ | 30.1481 |
| 24 | $(1,2,3,6)$ | $(15,39,0,15,0,15,0,0)$ | 30.4444 |
| 24 | $(1,2,4,10)$ | $(16,37,0,16,0,15,0,0)$ | 30.3704 |
| 24 | $(1,2,4,9)$ | $(15,38,0,16,0,15,0,0)$ | 30.6667 |
| 24 | $(1,2,3,8)$ | $(16,38,0,14,0,16,0,0)$ | 30.3704 |
| 24 | $(1,2,4,6)$ | $(16,37,0,15,0,16,0,0)$ | 30.5926 |
| 24 | $(1,2,3,4)$ | $(15,38,0,15,0,16,0,0)$ | 30.8889 |
| 24 | $(1,2,7,9)$ | $(16,36,0,16,0,16,0,0)$ | 30.8148 |
| 24 | $(1,2,4,7)$ | $(15,37,0,16,0,16,0,0)$ | 31.1111 |
| 24 | $(1,2,11,12)$ | $(12,40,0,16,0,16,0,0)$ | 32.0000 |
| 73 | $(2,3,5,7)$ | $(22,10,31,6,14,0,1,0)$ | 30.2963 |
| 74 | $(3,4,5,10)$ | $(22,10,31,6,14,0,0,0)$ | 30.2963 |
| 73 | $(2,3,7,10)$ | $(22,10,29,6,16,0,1,0)$ | 30.7407 |
| 74 | $(3,4,5,8)$ | $(22,10,29,6,16,0,1,0)$ | 30.7407 |
| 73 | $(1,3,5,7)$ | $(22,14,23,6,18,0,1,0)$ | 30.5926 |
| 74 | $(3,4,5,13)$ | $(22,14,23,6,18,0,1,0)$ | 30.5926 |
| 73 | $(2,3,7,8)$ | $(25,6,28,4,20,0,1,0)$ | 30.7407 |
| 74 | $1,3,5,8)$ | $(25,6,28,4,20,0,1,0)$ | 30.7407 |
| 73 | $(1,3,7,8)$ | $(25,8,24,4,22,0,1,0)$ | 30.8889 |

Table 4.21: Top $G_{2}$-aberration designs for $m=9$.

| design | removed | $\left(0, \frac{8}{27}, \frac{4}{9}, \frac{14}{27}, \frac{2}{3}, \frac{20}{27}, \frac{10}{9}, 2\right)$ | $A_{3}$ |
| :---: | :---: | :---: | :---: |
| 3 | $(1,2,3,4)$ | $(54,0,0,0,27,0,0,3)$ | 24 |
| 4 | $(5,6,7,13)$ | $(54,0,0,0,27,0,0,3)$ | 24 |
| 5 | $(5,6,7,13)$ | $(54,0,0,0,27,0,0,3)$ | 24 |
| 6 | $(5,6,7,13)$ | $(54,0,0,0,27,0,0,3)$ | 24 |
| 7 | $(1,2,3,4)$ | $(54,0,0,0,27,0,0,3)$ | 24 |
| 8 | $(5,6,7,13)$ | $(54,0,0,0,27,0,0,3)$ | 24 |
| 9 | $(1,2,3,4)$ | $(54,0,0,0,27,0,0,3)$ | 24 |
| 10 | $(5,6,7,13)$ | $(54,0,0,0,27,0,0,3)$ | 24 |
| 11 | $(1,2,3,4)$ | $(54,0,0,0,27,0,0,3)$ | 24 |
| 12 | $(2,7,9,13)$ | $(54,0,0,0,27,0,0,3)$ | 24 |
| 13 | $(3,6,10,13)$ | $54,0,0,0,27,0,0,3)$ | 24 |
| 14 | $(3,6,10,13)$ | $(54,0,0,0,27,0,0,3)$ | 24 |
| 28 | $(1,2,3,4)$ | $(54,0,0,0,27,0,0,3)$ | 24 |
| 29 | $(1,2,3,4)$ | $(54,0,0,0,27,0,0,3)$ | 24 |
| 30 | $(5,6,7,13)$ | $(54,0,0,0,27,0,0,3)$ | 24 |
| 31 | $(5,6,7,13)$ | $(54,0,0,0,27,0,0,3)$ | 24 |
| 32 | $(5,6,7,13)$ | $(54,0,0,0,27,0,0,3)$ | 24 |
| 33 | $(1,2,3,4)$ | $(54,0,0,0,27,0,0,3)$ | 24 |
| 34 | $(5,6,7,13)$ | $544,0,0,0,27,0,0,3)$ | 24 |
| 46 | $(3,6,9,13)$ | $54,0,0,0,27,0,0,3)$ | 24 |
| 47 | $(3,7,8,13)$ | $54,0,0,0,27,0,0,3)$ | 24 |
| 48 | $(1,11,12,13)$ | $(54,0,0,0,27,0,0,3)$ | 24 |
| 49 | $(3,6,9,13)$ | $(54,0,0,0,27,0,0,3)$ | 24 |
| 50 | $(3,6,10,13)$ | $(54,0,0,0,27,0,0,3)$ | 24 |
| 79 | $(3,6,9,13)$ | $(54,0,0,0,27,0,0,3)$ | 24 |
| 80 | $(3,6,10,13)$ | $(54,0,0,0,27,0,0,3)$ | 24 |
| 81 | $(3,6,10,13)$ | $(54,0,0,0,27,0,0,3)$ | 24 |
| 92 | $(1,2,3,4)$ | $(54,0,0,0,27,0,0,3)$ | 24 |
| 93 | $(3,8,10,13)$ | $544,0,0,0,27,0,0,3)$ | 24 |
| 98 | $(3,6,9,13)$ | $(54,0,0,0,27,0,0,3)$ | 24 |

Table 4.22: Top $G$-aberration designs for $m=8$.

| design | removed | $\left(0, \frac{8}{27}, \frac{4}{9}, \frac{14}{27}, \frac{2}{3}, \frac{20}{27}, \frac{10}{9}, 2\right)$ | $A_{3}$ |
| :---: | :---: | :---: | :---: |
| 95 | $(2,3,5,8,10)$ | $(14,8,20,8,6,0,0,0)$ | 19.4074 |
| 96 | $(1,3,11,12,13)$ | $(14,8,20,8,6,0,0,0)$ | 19.4074 |
| 97 | $(3,4,6,11,13)$ | $(14,8,20,8,6,0,0,0)$ | 19.4074 |
| 1 | $(1,2,3,7,13)$ | $(17,0,31,0,8,0,0,0$ | 19.1111 |
| 1 | $(1,2,3,4,10)$ | $(16,0,32,0,8,0,0,0)$ | 19.5556 |
| 1 | $(1,2,4,6,10)$ | $(15,0,33,0,8,0,0,0)$ | 20.0000 |
| 73 | $(2,3,5,6,7)$ | $(14,8,22,4,8,0,0,0)$ | 19.5556 |
| 74 | $(3,4,5,8,11)$ | $(14,8,22,4,8,0,0,0)$ | 19.5556 |
| 53 | $(1,4,5,9,10)$ | $(14,8,24,0,10,0,0,0)$ | 19.7037 |
| 1 | $(1,2,3,6,7)$ | $(17,0,29,0,10,0,0,0)$ | 19.5556 |
| 1 | $(1,2,3,5,11)$ | $(16,0,30,0,10,0,0,0)$ | 20.0000 |
| 1 | $(1,2,3,4,5)$ | $(15,0,31,0,10,0,0,0)$ | 20.4444 |
| 1 | $(1,2,3,4,12)$ | $(14,0,32,0,10,0,0,0)$ | 20.8889 |
| 1 | $(1,2,3,11,13)$ | $(13,0,33,0,10,0,0,0)$ | 21.3333 |
| 73 | $(1,3,5,7,12)$ | $(14,12,16,4,10,0,0,0)$ | 19.4074 |
| 74 | $(3,4,5,11,13)$ | $(14,12,16,4,10,0,0,0)$ | 19.4074 |
| 95 | $(2,3,5,8,11)$ | $(14,12,16,4,10,0,0,0)$ | 19.4074 |
| 96 | $(3,4,7,8,13)$ | $14,12,16,4,10,0,0,0)$ | 19.4074 |
| 97 | $(3,4,5,11,13)$ | $(14,12,16,4,10,0,0,0)$ | 19.4074 |
| 95 | $(2,3,5,7,8)$ | $(14,8,16,8,10,0,0,0)$ | 20.2963 |
| 96 | $(1,3,7,8,13)$ | $(14,8,16,8,10,0,0,0)$ | 20.2963 |
| 97 | $(3,4,7,11,13)$ | $(14,8,16,8,10,0,0,0)$ | 20.2963 |
| 1 | $(1,2,3,5,6)$ | $(16,0,29,0,11,0,0,0)$ | 20.2222 |
| 1 | $(1,2,3,4,8)$ | $(15,0,30,0,11,0,0,0)$ | 20.6667 |
| 1 | $(1,2,3,5,8)$ | $(14,0,31,0,11,0,0,0)$ | 21.1111 |
| 1 | $(1,2,3,4,6)$ | $(13,0,32,0,11,0,0,0)$ | 21.5556 |
| 73 | $(2,3,5,7,13)$ | $(15,4,22,4,11,0,0,0)$ | 20.3704 |
| 74 | $(1,3,4,5,10)$ | $(15,4,22,4,11,0,0,0)$ | 20.3704 |
| 73 | $(1,3,5,6,7)$ | $(14,12,14,4,12,0,0,0)$ | 19.8519 |
| 74 | $(3,4,5,12,13)$ | $(14,12,14,4,12,0,0,0)$ | 19.8519 |

Table 4.23: Top $G_{2}$-aberration designs for $m=8$.

| design | removed | $\left(0, \frac{8}{27}, \frac{4}{9}, \frac{14}{27}, \frac{2}{3}, \frac{20}{27}, \frac{10}{9}, 2\right)$ | $A_{3}$ |
| :---: | :---: | :---: | :---: |
| 3 | $(1,2,3,4,5)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 4 | $(1,5,6,7,13)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 5 | $(1,5,6,7,13)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 6 | $(1,5,6,7,13)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 7 | $(1,2,3,4,5)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 8 | $(1,5,6,7,13)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 9 | $(1,2,3,4,5)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 10 | $(1,5,6,7,13)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 11 | $(1,2,3,4,5)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 12 | $(1,2,7,9,13)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 13 | $(1,3,6,10,13)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 14 | $(1,3,6,10,13)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 28 | $(1,2,3,4,5)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 29 | $(1,2,3,4,5)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 30 | $(1,5,6,7,13)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 31 | $(1,5,6,7,13)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 32 | $(1,5,6,7,13)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 33 | $(1,2,3,4,5)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 34 | $(1,5,6,7,13)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 46 | $(1,3,6,9,13)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 47 | $(1,3,7,8,13)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 48 | $(1,2,11,1,13)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 49 | $(1,3,6,9,13)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 50 | $(1,3,6,10,13)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 79 | $(1,3,6,9,13)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 80 | $(1,3,6,10,13)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 81 | $(1,3,6,10,13)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 92 | $(1,2,3,4,5)$ | $36,0,0,0,18,0,0,2)$ | 16 |
| 93 | $(1,3,8,10,13)$ | $(36,0,0,0,18,0,0,2)$ | 16 |
| 98 | $(1,3,6,9,13)$ | $(36,0,0,0,18,0,0,2)$ | 16 |

Table 4.24: Top $G$-aberration designs for $m=7$.

| design | removed | $\left(0, \frac{8}{27}, \frac{4}{9}, \frac{14}{27}, \frac{2}{3}, \frac{20}{27}, \frac{10}{9}, 2\right)$ | $A_{3}$ |
| :---: | :---: | :---: | :---: |
| 95 | $(2,3,5,6,8,10)$ | $(9,4,15,4,3,0,0,0)$ | 11.9259 |
| 96 | $(1,3,5,11,12,13)$ | $(9,4,15,4,3,0,0,0)$ | 11.9259 |
| 97 | $(1,3,4,6,11,13)$ | $(9,4,15,4,3,0,0,0)$ | 11.9259 |
| 95 | $(2,3,5,7,9,12)$ | $(9,6,11,6,3,0,0,0)$ | 11.7778 |
| 96 | $(1,3,7,11,12,13)$ | $(9,6,11,6,3,0,0,0)$ | 11.7778 |
| 97 | $(3,4,6,9,11,13)$ | $(9,6,11,6,3,0,0,0)$ | 11.7778 |
| 73 | $(2,3,5,6,7,10)$ | $(9,6,14,2,4,0,0,0)$ | 11.7037 |
| 74 | $(3,4,5,8,9,11)$ | $(9,6,14,2,4,0,0,0)$ | 11.7037 |
| 95 | $(2,3,5,8,10,11)$ | $(9,6,14,2,4,0,0,0)$ | 11.7037 |
| 96 | $(1,3,4,11,12,13)$ | $(9,6,14,2,4,0,0,0)$ | 11.7037 |
| 97 | $(3,4,5,6,11,13)$ | $(9,6,14,2,4,0,0,0)$ | 11.7037 |
| 73 | $(2,3,5,6,7,9)$ | $(9,4,16,2,4,0,0,0)$ | 12.0000 |
| 74 | $(3,4,5,6,8,11)$ | $(9,4,16,2,4,0,0,0)$ | 12.0000 |
| 95 | $(2,3,5,7,8,10)$ | $(9,2,14,6,4,0,0,0)$ | 12.5926 |
| 96 | $(1,2,3,7,8,13)$ | $(9,2,14,6,4,0,0,0)$ | 12.5926 |
| 97 | $(1,3,4,6,9,13)$ | $(9,2,14,6,4,0,0,0)$ | 12.5926 |
| 73 | $(2,4,5,7,10,13)$ | $(11,0,12,8,4,0,0,0)$ | 12.1481 |
| 74 | $(1,4,5,8,9,10)$ | $(11,0,12,8,4,0,0,0)$ | 12.1481 |
| 95 | $(1,2,5,7,9,13)$ | $(10,3,9,9,4,0,0,0)$ | 12.22222 |
| 96 | $(1,3,5,6,8,12)$ | $(10,3,9,9,4,0,0,0)$ | 12.2222 |
| 97 | $(1,4,6,8,9,13)$ | $(10,3,9,9,4,0,0,0)$ | 12.2222 |
| 1 | $(1,2,3,5,6,11)$ | $(12,0,18,0,5,0,0,0)$ | 11.3333 |
| 1 | $(1,2,3,4,8,10)$ | $(11,0,19,0,5,0,0,0)$ | 11.7778 |
| 1 | $(1,2,3,4,5,9)$ | $(10,0,20,0,5,0,0,0)$ | 12.2222 |
| 1 | $(1,2,3,5,8,13)$ | $(9,0,21,0,5,0,0,0)$ | 12.6667 |
| 1 | $(1,2,3,4,6,10)$ | $(8,0,22,0,5,0,0,0)$ | 13.1111 |
| 95 | $(2,3,5,8,9,11)$ | $(9,10,9,2,5,0,0,0)$ | 11.3333 |
| 96 | $(3,4,7,8,11,13)$ | $(9,10,9,2,5,0,0,0)$ | 11.3333 |
| 97 | $(3,4,5,9,11,13)$ | $(9,10,9,2,5,0,0,0)$ | 11.3333 |
| 73 | $(2,3,5,7,8,12)$ | $(9,4,15,2,5,0,0,0)$ | 12.2222 |

Table 4.25: Top $G_{2}$-aberration designs for $m=7$.

| design | removed | ( $\left.0, \frac{8}{27}, \frac{4}{9}, \frac{14}{27}, \frac{2}{3}, \frac{20}{27}, \frac{10}{9}, 2\right)$ | $A_{3}$ |
| :---: | :---: | :---: | :---: |
| , | (1,2,3,4, 5, 6) | ( $22,0,0,0,12,0,0,1)$ | 10 |
| 4 | (1,4, 5, 6, 7, 13) | (22, 0, 0, 0, 12, 0, 0, 1) | 10 |
| 5 | (1, 4, 5, 6, 7, 13) | (22, 0, 0, 0, 12, 0, 0, 1) | 10 |
| 6 | (1, 4, 5, 6, 7, 13) | (22, 0, 0, 0, 12, 0, 0, 1) | 10 |
| 7 | (1, 2, 3, 4, 5, 6) | (22, 0, 0, 0, 12, 0, 0, 1) | 10 |
| 8 | (1,4, 5, 6, 7, 13) | (22, 0, 0, 0, 12, 0, 0, 1) | 10 |
| 9 | (1, 2, 3, 4, 5, 7) | (22, 0, 0, 0, 12, 0, 0, 1) | 10 |
| 10 | (1, 4, 5, 6, 7, 13) | (22, 0, 0, 0, 12, 0, 0, 1) | 10 |
| 11 | (1, 2, 3, 4, 5, 6) | (22, 0, 0, 0, 12, 0, 0, 1) | 10 |
| 12 | (1,2,3,7,9,13) | (22, 0, 0, 0, 12, 0, 0, 1) | 10 |
| 13 | (1,3, 5, 6, 10, 13) | (22, 0, 0, 0, 12, 0, 0, 1) | 10 |
| 14 | $(1,3,5,6,10,13)$ | (22, 0, 0, 0, 12, 0, 0, 1) | 10 |
| 28 | (1, 2, 3, 4, 5, 6) | (22, 0, 0, 0, 12, 0, 0, 1) | 10 |
| 29 | (1,2,3, 4, 5, 6) | (22, 0, 0, 0, 12, 0, 0, 1) | 10 |
| 30 | (1,4, 5, 6, 7, 13) | (22, 0, 0, 0, 12, 0, 0, 1) | 10 |
| 31 | (1,4, 5, 6, 7, 13) | (22, 0, 0, 0, 12, 0, 0, 1) | 10 |
| 32 | (1, 4, 5, 6, 7, 13) | (22, 0, 0, 0, 12, 0, 0, 1) | 10 |
| 33 | (1,2,3,4, 5, 6) | (22, 0, 0, 0, 12, 0, 0, 1) | 10 |
| 34 | (1,4, 5, 6, 7, 13) | (22, 0, 0, 0, 12, 0, 0, 1) | 10 |
| 46 | (1, 3, 5, 6, 9, 13) | (22, 0, 0, 0, 12, 0, 0, 1) | 10 |
| 47 | (1, 3, 5, 7, 8, 13) | (22, 0, 0, 0, 12, 0, 0, 1) | 10 |
| 48 | (1,2,3,11, 12, 13) | (22, 0, 0, 0, 12, 0, 0, 1) | 10 |
| 49 | (1,3, 5, 6, 9, 13) | (22, 0, 0, 0, 12, 0, 0, 1) | 10 |
| 50 | $(1,3,5,6,10,13)$ | (22, 0, 0, 0, 12, 0, 0, 1) | 10 |
| 53 | (1,2, 3, 4, 8, 12) | $(22,0,0,0,12,0,0,1)$ | 10 |
| 73 | (1,2, 3, 4, 8, 13) | (22, 0, 0, 0, 12, 0, 0, 1) | 10 |
| 74 | (1, 2, 3, 5, 6, 7) | (22, 0, 0, 0, 12, 0, 0, 1) | 10 |
| 79 | (1,3, 5, 6, 9, 13) | (22, 0, 0, 0, 12, 0, 0, 1) | 10 |
| 80 | $(1,3,5,6,10,13)$ | $(22,0,0,0,12,0,0,1)$ | 10 |
| 81 | (1,3, 5, 6, 10, 13) | $(22,0,0,0,12,0,0,1)$ | 10 |

Table 4.26: Top $G$-aberration designs for $m=6$.

| design | removed | $\left(0, \frac{8}{27}, \frac{4}{9}, \frac{14}{27}, \frac{2}{3}, \frac{20}{27}, \frac{10}{9}, 2\right)$ | $A_{3}$ |
| :---: | :---: | :---: | :---: |
| 64 | $(1,3,5,7,10,11,12)$ | $(6,0,14,0,0,0,0,0)$ | 6.2222 |
| 65 | $(1,2,3,7,9,10,11)$ | $(6,0,14,0,0,0,0,0)$ | 6.2222 |
| 95 | $(2,3,5,6,7,9,12)$ | $(5,4,9,2,0,0,0,0)$ | 6.2222 |
| 96 | $(1,3,5,7,11,12,13)$ | $(5,4,9,2,0,0,0,0)$ | 6.2222 |
| 97 | $(1,3,4,6,11,12,13)$ | $(5,4,9,2,0,0,0,0)$ | 6.2222 |
| 113 | $(2,3,5,6,8,9,11)$ | $(5,4,8,3,0,0,0,0)$ | 6.2963 |
| 114 | $(1,3,5,7,9,10,12)$ | $(5,4,8,3,0,0,0,0)$ | 6.2963 |
| 115 | $(1,2,6,10,11,12,13)$ | $(5,4,8,3,0,0,0,0)$ | 6.2963 |
| 73 | $(2,3,5,6,7,9,10)$ | $(5,3,10,1,1,0,0,0)$ | 6.5185 |
| 74 | $(3,4,5,6,8,9,11)$ | $(5,3,10,1,1,0,0,0)$ | 6.5185 |
| 15 | $(1,4,5,6,7,8,13)$ | $(6,2,9,2,1,0,0,0)$ | 6.2963 |
| 16 | $(1,4,7,10,11,12,13)$ | $(6,2,9,2,1,0,0,0)$ | 6.2963 |
| 17 | $(1,2,5,6,8,10,13)$ | $6,2,9,2,1,0,0,0)$ | 6.2963 |
| 18 | $(1,4,6,8,11,12,13)$ | $(6,2,9,2,1,0,0,0)$ | 6.2963 |
| 19 | $(2,4,5,6,8,9,12)$ | $(6,2,9,2,1,0,0,0)$ | 6.2963 |
| 20 | $(1,2,4,6,11,12,13)$ | $(6,2,9,2,1,0,0,0)$ | 6.2963 |
| 21 | $(2,4,5,6,8,9,12)$ | $(6,2,9,2,1,0,0,0)$ | 6.2963 |
| 22 | $(1,2,4,6,11,12,13)$ | $(6,2,9,2,1,0,0,0)$ | 6.2963 |
| 23 | $(1,4,6,7,10,11,12)$ | $(6,2,9,2,1,0,0,0)$ | 6.2963 |
| 15 | $(2,4,5,6,8,9,12)$ | $(6,4,5,4,1,0,0,0)$ | 6.1481 |
| 16 | $(1,5,7,8,9,11,12)$ | $(6,4,5,4,1,0,0,0)$ | 6.1481 |
| 19 | $(1,3,5,6,11,12,13)$ | $(6,4,5,4,1,0,0,0)$ | 6.1481 |
| 20 | $(1,2,5,7,8,9,11)$ | $(6,4,5,4,1,0,0,0)$ | 6.1481 |
| 113 | $(1,2,5,6,9,11,13)$ | $(6,3,6,4,1,0,0,0)$ | 6.2963 |
| 114 | $(1,3,5,6,8,9,12)$ | $(6,3,6,4,1,0,0,0)$ | 6.2963 |
| 115 | $(1,3,4,7,9,10,13)$ | $(6,3,6,4,1,0,0,0)$ | 6.2963 |
| 113 | $1,2,5,6,9,10,13)$ | $(5,4,6,4,1,0,0,0)$ | 6.5926 |
| 114 | $(1,3,5,6,9,12,13)$ | $(5,4,6,4,1,0,0,0)$ | 6.5926 |
| 115 | $(1,3,4,6,9,10,13)$ | $5,4,6,4,1,0,0,0)$ | 6.5926 |
| 95 | $(1,2,5,6,7,9,13)$ | $(6,2,7,4,1,0,0,0)$ | 6.4444 |

Table 4.27: Top $G_{2}$-aberration designs for $m=6$.

| design | removed | (0, $\left.\frac{8}{27}, \frac{4}{9}, \frac{14}{27}, \frac{2}{3}, \frac{20}{27}, \frac{10}{9}, 2\right)$ | $A_{3}$ |
| :---: | :---: | :---: | :---: |
| 3 | (1, 2, 3, 4, 5, 9, 10) | (18, 0, 0, 0, 0, 0, 0, 2) | 4 |
| 4 | (1, 2, 3, 5, 6, 7, 13) | (18, 0, 0, 0, 0, 0, 0, 2) | 4 |
| 5 | (1,2, 3, 5, 6, 7, 13) | (18, 0, 0, 0, 0, 0, 0, 2) | 4 |
| 6 | (1,2,3, 5, 6, 7, 13) | (18, 0, 0, 0, 0, 0, 0, 2) | 4 |
| 7 | (1,2,3, 4, 5, 9, 10) | (18, 0, 0, 0, 0, 0, 0, 2) | 4 |
| 8 | (1,2,3, 5, 6, 7, 13) | (18, 0, 0, 0, 0, 0, 0, 2) | 4 |
| 9 | (1, 2, 3, 4, 5, 6, 12) | $(18,0,0,0,0,0,0,2)$ | 4 |
| 10 | (1, 2, 3, 5, 6, 7, 13) | (18, 0, 0, 0, 0, 0, 0, 2) | 4 |
| 11 | (1,2, 3, 4, 5, 9, 10) | (18, 0, 0, 0, 0, 0, 0, 2) | 4 |
| 12 | (1, 2, 7, 9, 11, 12, 13) | (18, 0, 0, 0, 0, 0, 0, 2) | 4 |
| 13 | (1, 2, 3, 4, 6, 10, 13) | (18, 0, 0, 0, 0, 0, 0, 2) | 4 |
| 14 | (1, 2, 3, 4, 6, 10, 13) | (18, 0, 0, 0, 0, 0, 0, 2) | 4 |
| 15 | (1, 2, 3, 4, 5, 8, 11) | (18, 0, 0, 0, 0, 0, 0, 2) | 4 |
| 16 | (1, 2, 3, 5, 6, 7, 13) | (18, 0, 0, 0, 0, 0, 0, 2) | 4 |
| 17 | (1, 2, 3, 4, 5, 9, 10) | (18, 0, 0, 0, 0, 0, 0, 2) | 4 |
| 18 | (1, 2, 3, 5, 6, 7, 13) | (18, 0, 0, 0, 0, 0, 0, 2) | 4 |
| 19 | $(1,2,3,4,5,8,11)$ | $(18,0,0,0,0,0,0,2)$ | 4 |
| 20 | (1, 2, 3, 5, 6, 7, 13) | (18, 0, 0, 0, 0, 0, 0, 2) | 4 |
| 21 | (1, 2, 3, 4, 5, 8, 11) | (18, 0, 0, 0, 0, 0, 0, 2) | 4 |
| 22 | (1, 2, 3, 5, 6, 7, 13) | (18, 0, 0, 0, 0, 0, 0, 2) | 4 |
| 23 | (1, 2, 3, 5, 6, 7, 13) | (18, 0, 0, 0, 0, 0, 0, 2) | 4 |
| 25 | $(1,2,3,4,5,8,11)$ | (18, 0, 0, 0, 0, 0, 0, 2) | 4 |
| 26 | (1, 2, 3, 5, 6, 7, 13) | (18, $0,0,0,0,0,0,2)$ | 4 |
| 27 | (1, 2, 3, 5, 6, 7, 13) | (18, 0, 0, 0, 0, 0, 0, 2) | 4 |
| 28 | (1, 2, 3, 4, 5, 8, 11) | (18, 0, 0, 0, 0, 0, 0, 2) | 4 |
| 29 | (1,2, 3, 4, 5, 8, 11) | (18, 0, 0, 0, 0, 0, 0, 2) | 4 |
| 30 | (1, 2, 3, 5, 6, 7, 13) | (18, 0, 0, 0, 0, 0, 0, 2) | 4 |
| 31 | (1, 2, 3, 5, 6, 7, 13) | (18, 0, 0, 0, 0, 0, 0, 2) | 4 |
| 32 | (1, 2, 3, 5, 6, 7, 13) | (18, 0, 0, 0, 0, 0, 0, 2) | 4 |
| 33 | $(1,2,3,4,5,8,11)$ | $(18,0,0,0,0,0,0,2)$ | 4 |

Table 4.28: Top $G$-aberration designs for $m=5$.

| design | removed | $\left(0, \frac{8}{27}, \frac{4}{9}, \frac{14}{27}, \frac{2}{3}, \frac{20}{27}, \frac{10}{9}, 2\right)$ | $A_{3}$ |
| :---: | :---: | :---: | :---: |
| 95 | $(2,3,5,6,8,9,11,12)$ | $(3,4,3,0,0,0,0)$ | 2.5185 |
| 96 | $(3,4,5,7,8,11,12,13)$ | $(3,4,3,0,0,0,0,0)$ | 2.5185 |
| 97 | $(1,3,4,5,9,11,12,13)$ | $(3,4,3,0,0,0,0,0)$ | 2.5185 |
| 113 | $(2,3,5,6,7,8,9,12)$ | $(3,2,5,0,0,0,0,0)$ | 2.8148 |
| 114 | $(2,3,5,7,9,10,11,12)$ | $(3,2,5,0,0,0,0,0)$ | 2.8148 |
| 115 | $(1,2,5,9,10,11,12,13)$ | $(3,2,5,0,0,0,0,0)$ | 2.8148 |
| 95 | $(2,3,5,6,7,9,11,12)$ | $(2,3,5,0,0,0,0,0)$ | 3.1111 |
| 96 | $(1,3,4,5,7,11,12,13)$ | $(2,3,5,0,0,0,0,0)$ | 3.1111 |
| 97 | $(1,3,4,5,6,11,12,13)$ | $(2,3,5,0,0,0,0,0)$ | 3.1111 |
| 13 | $(1,2,3,5,7,8,12,13)$ | $4,0,6,0,0,0,0,0)$ | 2.6667 |
| 14 | $1,2,3,4,5,8,10,11)$ | $(4,0,6,0,0,0,0,0)$ | 2.6667 |
| 25 | $(1,4,5,6,7,8,10,12)$ | $(4,0,6,0,0,0,0,0)$ | 2.6667 |
| 26 | $(1,2,5,7,8,9,10,12)$ | $4,0,6,0,0,0,0,0)$ | 2.6667 |
| 27 | $(1,2,3,4,5,7,8,9)$ | $4,0,6,0,0,0,0,0)$ | 2.6667 |
| 28 | $(3,4,5,6,7,10,11,13)$ | $4,0,6,0,0,0,0,0)$ | 2.6667 |
| 29 | $(1,4,5,6,7,8,10,13)$ | $4,0,6,0,0,0,0,0)$ | 2.6667 |
| 30 | $(1,2,3,4,6,7,8,10)$ | $4,0,6,0,0,0,0,0)$ | 2.6667 |
| 31 | $(1,2,5,7,8,9,10,12)$ | $4,0,6,0,0,0,0,0)$ | 2.6667 |
| 32 | $(1,2,6,8,9,10,12,13)$ | $4,0,6,0,0,0,0,0)$ | 2.6667 |
| 35 | $(2,3,5,7,8,9,10,12)$ | $4,0,6,0,0,0,0,0)$ | 2.6667 |
| 36 | $(1,2,3,5,6,7,10,13)$ | $4,0,6,0,0,0,0,0)$ | 2.6667 |
| 37 | $(3,4,5,6,7,10,11,12)$ | $(4,0,6,0,0,0,0,0)$ | 2.6667 |
| 38 | $1,2,3,5,6,10,11,13)$ | $4,0,6,0,0,0,0,0)$ | 2.6667 |
| 39 | $(2,4,5,6,7,10,11,12)$ | $(4,0,6,0,0,0,0,0)$ | 2.6667 |
| 40 | $(1,2,3,5,6,9,10,13)$ | $4,0,6,0,0,0,0,0)$ | 2.6667 |
| 46 | $(1,2,3,5,7,9,10,11)$ | $4,0,6,0,0,0,0,0)$ | 2.6667 |
| 47 | $(1,2,3,4,5,9,11,13)$ | $4,0,6,0,0,0,0,0)$ | 2.6667 |
| 48 | $(1,2,3,4,5,7,9,13)$ | $4,0,6,0,0,0,0,0)$ | 2.6667 |
| 64 | $(1,2,3,5,7,9,10,11)$ | $4,0,6,0,0,0,0,0)$ | 2.6667 |
| 65 | $(1,2,3,4,5,8,10,11)$ | $(4,0,6,0,0,0,0,0)$ | 2.6667 |

Table 4.29: Top $G_{2}$-aberration designs for $m=5$.

| design | removed | $\left(0, \frac{8}{27}, \frac{4}{9}, \frac{14}{27}, \frac{2}{3}, \frac{20}{27}, \frac{10}{9}, 2\right)$ | $A_{3}$ |
| :---: | :---: | :---: | :---: |
| 80 | $(1,2,6,8,9,11,12,13)$ | $(6,0,3,0,1,0,0,0)$ | 2 |
| 116 | $(1,3,5,6,7,8,12,13)$ | $(6,0,3,0,1,0,0,0)$ | 2 |
| 117 | $(1,3,5,6,7,8,10,12)$ | $(6,0,3,0,1,0,0,0)$ | 2 |
| 73 | $(1,2,7,8,9,10,11,12)$ | $(7,0,0,0,3,0,0,0)$ | 2 |
| 74 | $(1,4,5,6,7,8,12,13)$ | $(7,0,0,0,3,0,0,0)$ | 2 |
| 92 | $(1,2,5,6,7,9,10,11)$ | $(7,0,0,0,3,0,0,0)$ | 2 |
| 93 | $(1,4,5,6,7,8,9,10)$ | $(7,0,0,0,3,0,0,0)$ | 2 |
| 94 | $(1,3,5,6,7,8,10,13)$ | $(7,0,0,0,3,0,0,0)$ | 2 |
| 95 | $(1,3,5,6,7,8,11,13)$ | $(7,0,0,0,3,0,0,0)$ | 2 |
| 96 | $(1,4,5,6,7,8,9,13)$ | $(7,0,0,0,3,0,0,0)$ | 2 |
| 97 | $(1,2,3,4,5,7,8,11)$ | $(7,0,0,0,3,0,0,0)$ | 2 |
| 98 | $(1,2,6,8,9,10,11,12)$ | $7,0,0,0,3,0,0,0)$ | 2 |
| 99 | $(1,2,6,7,9,11,12,13)$ | $7,0,0,0,3,0,0,0)$ | 2 |
| 113 | $(1,3,5,6,7,8,10,13)$ | $7,0,0,0,3,0,0,0)$ | 2 |
| 114 | $(1,2,5,6,7,8,9,11)$ | $(7,0,0,0,3,0,0,0)$ | 2 |
| 115 | $(1,2,3,4,5,6,9,11)$ | $(7,0,0,0,3,0,0,0)$ | 2 |
| 120 | $(1,3,5,6,7,8,9,10)$ | $7,0,0,0,3,0,0,0)$ | 2 |
| 95 | $(1,2,6,7,8,9,10,11)$ | $(7,0,2,0,0,0,1,0)$ | 2 |
| 96 | $(1,2,3,4,5,6,7,12)$ | $(7,0,2,0,0,0,1,0)$ | 2 |
| 97 | $(1,4,5,, 6,7,8,12,13)$ | $(7,0,2,0,0,0,1,0)$ | 2 |
| 98 | $(1,2,3,5,6,7,10,11)$ | $7,0,0,0,0,0,1,0)$ | 2 |
| 99 | $(1,2,3,4,5,6,7,11)$ | $7,0,2,0,0,0,1,0)$ | 2 |
| 100 | $(1,2,3,4,5,6,7,9)$ | $7,0,2,0,0,0,1,0)$ | 2 |
| 101 | $(1,2,3,5,6,7,8,12)$ | $(7,0,2,0,0,0,1,0)$ | 2 |
| 102 | $(2,3,5,6,8,9,11,13)$ | $(7,0,2,0,0,0,1,0)$ | 2 |
| 121 | $(1,2,3,4,5,6,7,8)$ | $(7,0,2,0,0,0,1,0)$ | 2 |
| 2 | $(1,2,3,4,5,6,7,13)$ | $(9,0,0,0,0,0,0,1)$ | 2 |
| 3 | $(1,2,3,4,5,6,7,8)$ | $(9,0,0,0,0,0,0,1)$ | 2 |
| 4 | $(1,2,3,4,5,6,7,13)$ | $(9,0,0,0,0,0,0,1)$ | 2 |
| 5 | $(1,2,3,4,5,6,7,13)$ | $(9,0,0,0,0,0,0,1)$ | 2 |

## Chapter 5

## Summary and Future Work

We conclude this thesis with a brief summary of the work in each chapter, and some general discussion about potential directions for future work.

### 5.1 Partially Clear Two-factor Interactions

In Chapter 2 we introduced the concept of robust designs through partially clear two-factor interactions. We allowed effects of interest to be aliased with two-factor interactions that prior knowledge supported as being negligible, while still remaining robust to nonnegligible two-factor interactions. By allowing this aliasing, the goal was to entertain more factors in comparison to designs with clear two-factor interactions. The set of two-factor interactions were separated into three distinct categories: the requirement set, the set of nonnegligible interactions and the set of negligible interactions. By breaking the set of factors into two groups, we looked at different configurations of the sets of two-factor interactions within and between groups. In particular, one of these configurations, which we denoted type 1, showed substantial gain in terms of the number of factors versus designs with clear two-factor interactions. We gave constructions for robust designs of type 1, and showed that these constructions match with the optimal results for designs of 32 and 64 runs. The constructions are even more powerful, as they also apply if we allow nonorthogonality within the requirement set. The Chapter concluded with a bound on the number of
clear two-factor interactions in an orthogonal array.
From a practical standpoint, the concept of partially clear two-factor interactions is appealing as it allows prior knowledge to be used to gain more from a design, while still keeping the robustness properties of clear designs. In general, if we have knowledge about some interactions being negligible, we can separate the interactions into the three categories and do a computer search for robust designs. One future direction this research can take is to look at more than two groups of factors which allows more flexibility for the two-factor interactions between groups. With these additional groups, it would be useful to establish bounds on the number of factors that can be entertained depending on the configuration of factors, similar to Lemma 2.2. Another consideration is the D-efficiency for these robust designs when non-orthogonal requirement sets are used. For the robust designs in Chapter 2, we saw good performance in terms of D-efficiency, but it would be worthwhile to see what sacrifice towards the efficiency is made in general by ensuring robustness to the nonnegligible two-factor interactions. In the case of a non-orthogonal requirement set, the approaches in Chapters 2 and 3 take different directions. Here, our primary concern is the robustness to the nonnegligible two-factor interactions with D-efficiency as a secondary criterion, whereas in Chapter 3, D-efficiency was the first priority and then the robustness to nonnegligible two-factor interactions. This consideration depends on what the experimenter deems as more serious. Lemma 2.2 showed that requiring $A_{22}=0$ places quite a restriction on the factors in a design, and we focused our considerations to robust designs of type 1. For robust designs of type 3 , that $A_{22}=0$ is required for orthogonal estimation of the requirement set and $A_{31}=0$ gave the robustness to nonnegligible 2 fi 's. It is possible that introducing nonorthogonality into the requirement set could lead to robust designs of type 3 .

We conclude our discussion on partially clear designs by discussing robust parameter design. In robust parameter design, factors are separated into control and noise factors, where noise factors are those that are hard to control. The goal is to find optimal settings for the control factors that minimize the variability of a system by making it less sensitive to noise variation. Wu and Hamada (2000) give the following ordering for the importance of effects:

1. control-by-noise interactions, control main effects and noise main effects;
2. control-by-control interactions and control-by-control-by-noise interactions;
3. noise-by-noise interactions.

The setup is similar by having the two groups of factors and separating the interactions into these subsets. The idea of partially clear designs should be useful for robust parameter design problems but this warrants further investigation.

### 5.2 Multi-Level Orthogonal Arrays for Estimating Main Effects and Specified Interactions

Chapter 3 investigated the use of multi-level orthogonal arrays for estimating a requirement set. Designs were ranked firstly based on D-efficiency followed by contamination from the two-factor interactions not estimated. Focus was on designs with three-level factors to allow curvature of factors to be studied. These designs are an attractive alternative to other methods that require follow-up experimentation. Factors were grouped into two sets: the core set, which were those factors involved in an interaction in the requirement set, and the set of remaining factors in which only the main effects are of interest. We showed that level permutations of factors outside of the core set maintain many of the optimality measures, including D-efficiency and contamination from the interactions outside of the requirement set. A result giving a lower bound on the number of factors outside of the core set that can be entertained was provided, and a method was proposed to aid in searching for designs. The chapter was concluded with a search for three-level designs with 27 runs.

That the results from Chapter 3 hold for mixed level designs is particularly appealing, as it provides flexibility in how to treat some of the factors, such as only concerning ourselves with two levels for the factors outside of the core set (ie. if there is no interest in the curvature). It would be interesting to see how designs that are not a subset of columns from a saturated orthogonal array perform for the objectives of Chapter 3. The (M.S)-approach shows promise as an efficient means of improving the
search for good designs when the complete search is infeasible. This idea of studying the space occupied by the two-factor interactions rather than the model matrix for multi-level designs requires further study to see if anything else can be gleaned from it. Returning to the theme of Chapter 2, we can approach the problem by trying to make the design robust to certain interaction components, such as the linear-bylinear. Dealing with certain interaction components is likely to need consideration of level permutations - but it is worth noting that depending on the level assignments, the linear-by-linear components should contain many zero entries.

### 5.3 J-Characteristics for Multi-Level Factorial Designs

Whereas Chapters 2 and 3 focused on a specific set of effects, Chapter 4 made no such separation of effects. Instead, the focus was on the distribution of design points, and an ANOVA decomposition to study the uniformity in lower dimensions. This allowed a means of measuring the projection properties of a design through $G_{2}$-aberration that is equivalent to generalized minimum aberration as defined by Xu and Wu (2002). We also proposed more conservative approaches using $G$-aberration and $G_{2(i)}$-aberration. The $J$-characteristics were used to examine three-level designs of 18 and 27 runs. The ranking between $G$ and $G_{2}$-aberration was consistent for 18 runs, but not so for 27 runs.

In terms of measuring near-orthogonality, algorithmic approaches such as Xu (2002), Yamada and Matsui (2002), and Lekivetz, Sitter, Bingham, Hamada, Moore and Wendelberger (2011) are typically driven by $G_{2}$-aberration through some equivalent measure to $B_{2}$. The approach here has a natural extension to higher dimensions by using $B_{3}, B_{4}, \ldots$ if the experimenter wants to consider higher dimensional properties. The designs given in Xu , Cheng and Wu (2004) were found by constructing many orthogonal arrays and choosing the best among them. Using these algorithms to find designs for $G$-aberration rather than $G_{2}$ is currently being investigated. From Propositions 4.1 and 4.2, $G_{2(i)}$ and $G_{(i)}$-aberration may warrant some further research
as a tool for searching for designs through their connection to $G$ and $G_{2}$-aberration.
One of the most exciting potential uses for the $J$-characteristics is in measuring the projection properties of designs for computer experiments such as those described in Bingham, Sitter and Tang (2009) and Lin, Bingham, Sitter and Tang (2010). In terms of using designs as building blocks, particularly for nearly-orthogonal arrays, the $J$-characteristics can help select designs that have good lower-dimensional projection properties in terms of the spread of design points. As another use, we can create a grid over the lower dimensional projections and determine how many points fall within a partition. Ideally, we want to see the points spread out evenly within each of the partitions. We can use the $J$-characteristics to measure this spread, and still maintain properties such as generalized minimum aberration.

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