Optimal Filtering in Fractional Fourier Domains

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***Abstract*— For time-invariant degradation models and stationary signals and noise, the classical Fourier domain Wiener filter, which can be implemented in $O(N \log N)$ time, gives the minimum mean-square-error estimate of the original undistorted signal. For time-varying degradations and nonstationary processes, however, the optimal linear estimate requires $O(N^2)$ time for implementation. We consider filtering in fractional Fourier domains, which enables significant reduction of the error compared with ordinary Fourier domain filtering for certain types of degradation and noise (especially of chirped nature), while requiring only $O(N \log N)$ implementation time. Thus, improved performance is achieved at no additional cost. Expressions for the optimal filter functions in fractional domains are derived, and several illustrative examples are given in which significant reduction of the error (by a factor of 50) is obtained.**

I. Introduction

IN MANY practical applications, desired signals are degraded by a known system and/or by a noise term. It is desirable to apply an estimation operator to the resulting (observed) signals to minimize the effect of degradation and noise. The problem is then to find the optimal estimation operator with respect to some design criteria that removes or minimizes these degradations. Appropriate solutions to this problem depend on the observation model and the design criteria used, the prior knowledge available about the desired signal, and the degradation process and/or noise. The most commonly used observation model, which is also easy to handle mathematically, is

$$\boldsymbol{y} = \mathcal{H}(\boldsymbol{x}) + \boldsymbol{n}$$

where $\mathcal{H}(\cdot)$ is a linear system that degrades the desired signal $\boldsymbol{x}$, and $\boldsymbol{n}$ is an additive noise term [1]. A frequently used design criteria is the mean square error (MSE). We consider a linear estimation operator $\mathcal{G}$ of the form

$$\hat{\boldsymbol{x}} = \mathcal{G}(\boldsymbol{y}).$$

For a time-invariant degradation model $\mathcal{H}$ with stationary processes $\boldsymbol{x}$ and $\boldsymbol{n}$, the linear operator $\mathcal{G}_{\text{opt}}$ that minimizes the error corresponds to the classical optimal Wiener filter [1]. This operator is time-invariant and can be expressed as a convolution and implemented with a multiplicative filter in the conventional Fourier domain. For an arbitrary degradation model or nonstationary processes, the resulting optimal recovery operator will not, in general, be time-invariant and, thus, cannot be expressed as a convolution and cannot be realized by filtering in the conventional Fourier domain. We can still seek the optimal Fourier domain filter, but this operation will not give the most satisfactory result. (When we speak of "filtering in the conventional Fourier domain," we simply mean multiplying the Fourier transform of a function with a filter function in that domain.)

Recently, we have discussed how various time-variant operations can be performed by multiplying with a filter function in a fractional Fourier domain [2], [3]. A related concept ("swept-frequency filters") has been discussed by Almeida [4]. Filtering in a fractional Fourier domain can be implemented as efficiently as filtering in the conventional Fourier domain since the fractional Fourier transformation has a fast digital algorithm [5], [6], and can also be optically realized much like the usual Fourier transform [7]–[12]. The problem considered in this paper is to minimize the MSE for arbitrary degradation models and nonstationary processes by filtering in fractional Fourier domains. It will be shown in Section V that for noise and degradation models of a chirped nature, much smaller MSE's are obtained as compared with filtering in the conventional Fourier domain at about the same computational cost. We also note, however, that since we present a general method and since the class of fractional Fourier filters is by definition a broader class than conventional Fourier domain filters, we expect that certain problems involving time-varying degradations and noise other than those given in our examples should also benefit from this method [46].

As a motivation for the concept of filtering in fractional Fourier domains, we first note that signals with significant overlap in both the space and frequency domains may have little or no overlap in a fractional Fourier domain. To understand the basic idea, consider the simple example shown in Fig. 1, where Wigner distributions of a desired signal and undesired distortion term are superimposed on the single plot. We observe that they overlap in both the zeroth and first domains (consider the projections onto the t and f axes), but they do not overlap in the 0.5th domain. (The concept of fractional Fourier domains was developed in [2] and [3] and will be reviewed in Section III). Thus, we can eliminate undesired signal components by using a simple unit amplitude mask in the 0.5th domain.

The remaining sections of the paper are organized as follows. In Section II, we define the problem and discuss related work. We introduce the concept of fractional Fourier transform in Section III. The analysis and solution of the problem posed will be given in Section IV. Section V includes simulations that show the applications and performance of the proposed

Manuscript received February 24, 1995; revised October 24, 1996. The associate editor coordinating the review of this paper and approving it for publication was Dr. Patrick Flandrin.

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Publisher Item Identifier S 1053-587X(97)02580-4.

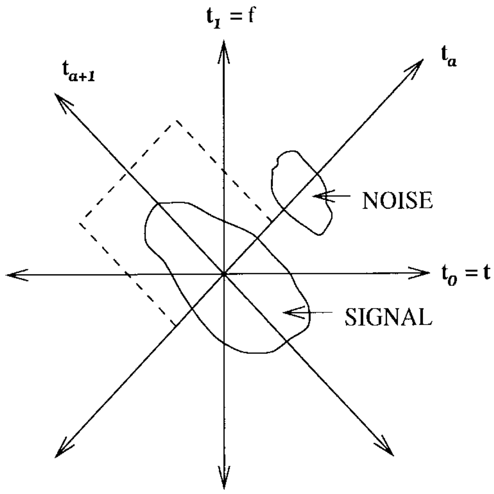


Fig. 1. Noise separation in the ath domain.

filtering scheme. In Section VI, we address the discrete-time problem and derive the solution. The remainder of the paper constitutes concluding sections.

II. Problem Definition and Related Work

In this section, the mathematical definition of the problem will be given and our approach to its solution will be formulated. The solution for the case of a linear time-invariant degradation model with stationary processes is the well-known optimal Wiener filter,[1] which can be implemented efficiently with the fast Fourier transform. For time-varying degradations and/or nonstationary signals and noise, the optimal recovery operator will, in general, be time varying. We will be reviewing below some time-varying filtering algorithms that are generally based on mixed time-frequency (TF) signal representations.

A. Problem Statement

Our signal observation model can be written as

$$y(t) = \int_{-\infty}^{\infty} h(t,t')x(t')\,dt' + n(t) \tag{1}$$

where $h(t,t')$ is the kernel of the degradation model, and $n(t)$ is the additive noise term. We will assume that as a prior knowledge, we know the correlation functions

$$\begin{aligned} R_{xx}(t,t') &= E[x(t)x^*(t')], \\ R_{nn}(t,t') &= E[n(t)n^*(t')] \end{aligned}$$

of the input signal (desired signal) $\boldsymbol{x}$ and the noise. We will further assume that the noise is independent of the input $\boldsymbol{x}$ and it is zero mean for all time, i.e., $E[n(t)] = 0\ \forall t$ and that we know the degradation model. Under these assumptions, we can also find the cross correlation function

$$R_{xy}(t,t') = E[x(t)y^*(t')]$$

of the processes $\boldsymbol{x}$ and $\boldsymbol{y}$ and the correlation function

$$R_{yy}(t,t') = E[y(t)y^*(t')]$$

using (1).

[1] This term is sometimes extended to refer to the more general case, but we use it to refer to the time-invariant case only.

First, consider the most general linear estimate of the form

$$\hat{x}(t) = \int_{-\infty}^{\infty} g(t,t')y(t')\,dt'. \tag{2}$$

Our design criteria is the mean square error (MSE), which is defined as

$$\sigma_e^2 = E[\|\boldsymbol{x} - \hat{\boldsymbol{x}}\|^2] \tag{3}$$

where $E[\cdot]$ denotes the expectation operator, and $\|\cdot\|$ denotes the L_2 norm:

$$\|\boldsymbol{x}\|^2 = \int_{-\infty}^{\infty} x(t)x^*(t)\,dt. \tag{4}$$

This definition (3) of the MSE is appropriate for nonstationary signals whose functional representations are square integrable (that are of finite energy). (For stationary processes, the MSE may be defined as the expected value of the magnitude squared of the difference term [1].)

The recovery operator kernel is the solution to the minimization problem

$$g_{\text{opt}}(t,t') = \arg\min_g \sigma_e^2. \tag{5}$$

That is, it is the function that minimizes the MSE. The solution to this problem, with the linear estimate defined in (2), is known. It is the solution of the following equation [13]:

$$R_{xy}(t,t') = \int_{-\infty}^{+\infty} g_{\text{opt}}(t,t'')R_{yy}(t'',t')\,dt'' \quad \forall t,t'.$$

The above equation can be solved numerically to obtain the kernel of the optimal linear recovery operator. However, application of this estimation operator [cf. (2)] on a given distorted and noisy signal would require $O(N^2)$ time, where N is the time-bandwidth product of the signals. In this paper, we restrict our estimate so that it corresponds to a multiplication with a filter function in the ath fractional Fourier domain. This estimate can be written in operator notation as

$$\hat{\boldsymbol{x}} = \mathcal{F}^{-a}(\boldsymbol{g} \cdot \mathcal{F}^a(\boldsymbol{y})) \tag{6}$$

where $\mathcal{F}^a$ is the ath-order fractional Fourier transformation operator, and $\boldsymbol{g}$ is the multiplicative filter. We note that for $a = 1$, this estimate corresponds to filtering in the conventional Fourier domain. With this form of estimation operator, the minimization problem considered in this paper can be defined as

$$\boldsymbol{g}_{\text{opt}} = \arg\min_g \sigma_e^2 \tag{7}$$

where σ_e^2 is as defined in (3), and $\hat{\boldsymbol{x}}$ is given by (6).

The class of fractional Fourier domain filters is a subclass of the class of all linear operators; therefore, the linear filter we find will not correspond to the global optimum among all linear operators. However, it is a much broader class than (time-invariant) Fourier domain filters, and in many problems involving time-varying degradation models and nonstationary processes, it is possible to obtain smaller MSE's when compared with filtering in the conventional Fourier domain. This reduction in MSE comes at no additional cost because the resulting filter can be implemented in $O(N \log N)$ time, just like the ordinary Fourier transform [5], or can be implemented optically with the same kind of hardware as the ordinary Fourier transform [7]–[12].

B. Related Work

There are a variety of time-varying digital signal processing algorithms based on mixed TF signal representations [14]–[19]. A conceptually simple time-varying estimation method consists of masking some regions of the signals' TF representations. However, this process is, in general, highly nonlinear [15], [17], resulting in difficulties especially when dealing with multicomponent signals. There are linear estimation operators based on linear time-frequency representations, but their performance generally depends on the window and wavelet used. Recently, another algorithm based on the Wigner distribution has been proposed [19]. TF projection and TF subspaces are the basic ideas behind it. It overcomes the nonlinearity problem as well as the problems of choosing wavelet and window. However, no efficient algorithm for its implementation has yet been proposed. The examples given in the above works have been limited to noise elimination, and the performance of the algorithms for time-varying degradation models have not been exploited.

III. Fractional Fourier Transform

In this section, we review the concept of fractional Fourier transforms and its relation to TF representations, especially the Wigner distribution. Its relation to the Wigner distribution function and its properties form the basis of our approach to time-varying filtering.

The fractional Fourier transform was defined in [20] and [21]. Several applications of the fractional Fourier transform have been suggested or explored to varying degrees. These include optical diffraction and beam propagation, and optical signal processing [2], [7]–[12], [22]–[25], quantum optics [26]–[30], phase retrieval, signal detection, pattern recognition, noise representation, time-variant filtering and multiplexing, data compression, study of space/TF distributions, and swept-frequency filters [2], [4], [31], [32]. The fractional Fourier transform is also related to a number of recently developed signal processing tools [33]–[35].

The ath-order fractional Fourier transform of a function, denoted by $\{\mathcal{F}^a x\}(t_a)$, is defined for $a \in [-2, 2]$ by [2], [21]

$$\begin{aligned}\{\mathcal{F}^a x\}(t_a) &= \int_{-\infty}^{\infty} B_a(t_a, t')x(t')\,dt',\\ B_a(t_a, t') &= A_\phi \exp[j\pi(t_a^2 \cot\phi - 2t_a t' \csc\phi + t'^2 \cot\phi)],\\ A_\phi &= (|\sin\phi|)^{-1/2} \exp[-j\pi\ \mathrm{sgn}(\sin\phi)/4 + j\phi/2]\end{aligned} \tag{8}$$

where $\phi \equiv a\pi/2$. The kernel $B_a(t_a, t')$ approaches $\delta(t_a - t')$ or $\delta(t_a + t')$ when a approaches 0 or ± 2, respectively. We will use t_a as the variable of the the ath-order transform, and the ath fractional Fourier transform $\{\mathcal{F}^a x\}(t_a)$ of the function $x(t)$ will be abbreviatedly denoted by $x_a(t_a)$.

The definition is easily extended outside the interval $[-2, 2]$ since $\mathcal{F}^4$ is the identity operation and the transform is additive in index $\mathcal{F}^{a_1}\mathcal{F}^{a_2} f = \mathcal{F}^{a_1+a_2} f$ [21]. Other essential properties are the following:

i) It is linear.
ii) The first order transform ($a = 1$) corresponds to the common Fourier transform.
iii) It is an unitary transform so that it preserves norms.

Other properties may be found in [2], [4], [9], [11], [20], [21].

An important property of fractional Fourier transform is its relation to the Wigner distribution. The Wigner distribution $W_x(t, f)$ of the signal $\boldsymbol{x}$ is defined as [36]

$$W_x(t, f) = \int_{-\infty}^{\infty} x\Big(t + \frac{\tau}{2}\Big)x^*\Big(t - \frac{\tau}{2}\Big)e^{-j2\pi f\tau}\,d\tau. \tag{9}$$

It can be interpreted as the signals TF energy distribution. It is well known that the projection of $W_x(t, f)$ onto the t axis gives the magnitude squared of the time domain representation, and the projection onto the f axis gives the magnitude squared of the frequency domain representation of the signal

$$\int_{-\infty}^{\infty} W_x(t, f)df = |x(t)|^2, \int_{-\infty}^{\infty} W_x(t, f)dt = |x_1(f)|^2. \tag{10}$$

A generalization of the above involving fractional transforms is [2], [37]

$$\{\mathcal{R}_\phi[W_x(t, f)]\}(t_a) = |x_a(t_a)|^2 \tag{11}$$

where $\mathcal{R}_\phi$ is the Radon transform operator. $\mathcal{R}_\phi$ takes the integral projection of the two dimensional function $W_x(t, f)$ onto an axis making angle $\phi = a\pi/2$ with the t axis. We will refer to this axis (t_a axis) as the ath fractional Fourier domain (Fig. 1). The t_0 axis is the usual time domain t, and the t_1 axis is the usual frequency domain f. Notice that (10) is a special case of (11).

An understanding of the Wigner distribution and its relation to the fractional Fourier transform is essential for a complete understanding of some applications of the fractional transform such as filtering and multiplexing. The reader is encouraged to consult [2], [3], and the references therein for further discussion.

We conclude this section with some remarks about the implementation of fractional Fourier transform. Its optical implementation is discussed in [9]–[12], and it is much like the implementation of the usual Fourier transform. Recently, a fast digital algorithm has been proposed [5]. With this algorithm, the cost of evaluating a fractional Fourier transform becomes comparable to evaluating the ordinary Fourier transform.

IV. Solution of the Estimation Problem

Based on the concepts developed in the previous sections, we will solve the minimization problem defined by (7). The calculus of variations method will be employed to the minimization problem.

We define the cost function J to be equal to the mean square error (MSE) [see (3)] with the estimate given by (6). Since fractional Fourier transform is unitary and it preserves norms, J is also equal to the MSE in the ath domain

$$J = \sigma_e^2 = E[\|\boldsymbol{x} - \hat{\boldsymbol{x}}\|^2] = E[\|\boldsymbol{x_a} - \hat{\boldsymbol{x}}_{\boldsymbol{a}}\|^2] \tag{12}$$

where from (6)

$$\hat{\boldsymbol{x}}_{\boldsymbol{a}} = \boldsymbol{g}\boldsymbol{y_a}.$$

J varies with the choice of $\boldsymbol{g}$ since $\hat{\boldsymbol{x}}_{\boldsymbol{a}}$ varies. This functional J is to be minimized with respect to $\boldsymbol{g}$. Let us substitute $\boldsymbol{g} = \boldsymbol{g_o} + \alpha\boldsymbol{\delta g_o}$, where

- α complex scalar parameter,
- $\boldsymbol{g_o}$ optimum filter,
- $\boldsymbol{\delta g_o}$ *arbitrary* perturbation term.

Since α is a complex parameter, we can express it as $\alpha = \alpha_{re} + j\alpha_{im}$. Now, $\hat{\boldsymbol{x}}_{\boldsymbol{a}}$, and J varies with α for each fixed $\boldsymbol{\delta g_o}$

$$\begin{aligned}\hat{x}_a(t_a, \alpha) &= (g_o(t_a) + (\alpha_{re} + j\alpha_{im})\delta g_o(t_a))y_a(t_a)\\ J(\alpha) &= E\bigg[\int_{-\infty}^{\infty} (x_a(t_a) - \hat{x}_a(t_a, \alpha))\\ &\qquad \cdot (x_a(t_a) - \hat{x}_a(t_a, \alpha))^* \, dt_a\bigg].\end{aligned} \tag{13}$$

The optimum value of J will be obtained from the conditions [38]

$$\left.\frac{\partial J(\alpha)}{\partial \alpha_{re}}\right|_{\alpha=0} = 0, \quad \left.\frac{\partial J(\alpha)}{\partial \alpha_{im}}\right|_{\alpha=0} = 0. \tag{14}$$

The above differentials are given by

$$\begin{aligned}\frac{\partial J(\alpha)}{\partial \alpha_{re}} &= -\bigg(E\bigg[\int_{-\infty}^{\infty} \frac{\partial \hat{x}_a^*(t_a, \alpha)}{\partial \alpha_{re}}\\ &\qquad \cdot (x_a(t_a) - \hat{x}_a(t_a, \alpha))\, dt_a\\ &\qquad + \int_{-\infty}^{\infty} \frac{\partial \hat{x}(t_a, \alpha)}{\partial \alpha_{re}} (x_a(t_a) - \hat{x}_a(t_a, \alpha))^* \, dt_a\bigg]\bigg)\\ \frac{\partial J(\alpha)}{\partial \alpha_{im}} &= -\bigg(E\bigg[\int_{-\infty}^{\infty} \frac{\partial \hat{x}_a^*(t_a, \alpha)}{\partial \alpha_{im}}\\ &\qquad \cdot (x_a(t_a) - \hat{x}_a(t_a, \alpha))\, dt_a\\ &\qquad + \int_{-\infty}^{\infty} \frac{\partial \hat{x}_a(t_a, \alpha)}{\partial \alpha_{im}} (x_a(t_a) - \hat{x}_a(t_a, \alpha))^* \, dt_a\bigg]\bigg)\end{aligned} \tag{15}$$

where by virtue of (13), we have

$$\begin{aligned}\frac{\partial \hat{x}_a(t_a, \alpha)}{\partial \alpha_{re}} &= \delta g_o(t_a) y_a(t_a)\\ \frac{\partial \hat{x}_a^*(t_a, \alpha)}{\partial \alpha_{re}} &= \left(\frac{\partial \hat{x}_a(t_a, \alpha)}{\partial \alpha_{re}}\right)^*,\\ \frac{\partial \hat{x}_a(t_a, \alpha)}{\partial \alpha_{im}} &= j\frac{\partial \hat{x}_a(t_a, \alpha)}{\partial \alpha_{re}},\\ \frac{\partial \hat{x}_a^*(t_a, \alpha)}{\partial \alpha_{im}} &= -j\left(\frac{\partial \hat{x}_a(t_a, \alpha)}{\partial \alpha_{re}}\right)^*.\end{aligned} \tag{16}$$

We define two variables $u(t_a, \alpha)$ and $v(t_a, \alpha)$

$$u(t_a, \alpha) = x(t_a) - \hat{x}_a(t_a, \alpha), \quad v(t_a, \alpha) = \frac{\partial \hat{x}_a(t_a, \alpha)}{\partial \alpha_{re}} \tag{17}$$

so that with the relations in (16), (15) becomes

$$\begin{aligned}\frac{\partial J(\alpha)}{\partial \alpha_{re}} &= -2E\bigg[\int_{-\infty}^{\infty} \mathrm{Re}(u^*(t_a, \alpha)v(t_a, \alpha))\, dt_a\bigg],\\ \frac{\partial J(\alpha)}{\partial \alpha_{im}} &= 2E\bigg[\int_{-\infty}^{\infty} \mathrm{Im}(u^*(t_a, \alpha)v(t_a, \alpha))\, dt_a\bigg].\end{aligned} \tag{18}$$

Based on the last equations, the conditions of optimality defined in (14) imply

$$E\left[\int_{-\infty}^{\infty} u^*(t_a, \alpha)v(t_a, \alpha)\, dt_a\right]\bigg|_{\alpha=0} = 0. \tag{19}$$

Evaluating at $\alpha = 0$ gives

$$E\bigg[\int_{-\infty}^{\infty} (x_a(t_a) - \hat{x}_a(t_a, 0))^* \delta g_o(t_a) y_a(t_a)\, dt_a\bigg] = 0 \tag{20}$$

where we use the definitions of $u(t, \alpha)$ and $v(t, \alpha)$. Since $\delta g_o(t_a)$ is an *arbitrary* term and (20) is true for all $\delta g_o(t_a)$, we infer that

$$E[(x_a(t_a) - \hat{x}_a(t_a, 0))^* y_a(t_a)] = 0. \tag{21}$$

We can solve this last equation for the optimum filter function $g_o(\cdot)$ by using the definition of $\hat{x}_a^*(t_a, 0)$ and by taking the complex conjugate of both sides of the above equation

$$g_o(t_a) = \frac{R_{x_a y_a}(t_a, t_a)}{R_{y_a y_a}(t_a, t_a)} \tag{22}$$

where the above correlation functions can be obtained from the correlation functions $R_{xy}(t, t')$ and $R_{yy}(t, t')$ by

$$\begin{aligned}R_{x_a y_a}(t_a, t_a) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_a(t_a, t) B_{-a}(t_a, t')\\ &\qquad \cdot R_{xy}(t, t')\, dt'\, dt\\ R_{y_a y_a}(t_a, t_a) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_a(t_a, t) B_{-a}(t_a, t')\\ &\qquad \cdot R_{yy}(t, t')\, dt'\, dt.\end{aligned} \tag{23}$$

Thus, the optimal filter function is

$$g_o(t_a) = \frac{\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_a(t_a, t) B_{-a}(t_a, t') R_{xy}(t, t')\, dt'\, dt}{\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_a(t_a, t) B_{-a}(t_a, t') R_{yy}(t, t')\, dt'\, dt}. \tag{24}$$

The last equation provides us the optimal multiplicative filter function in the ath fractional domain. To find the optimal value of a, that is, the domain in which the smallest error is obtained, we plug the optimum filter function into the MSE expression

$$\begin{aligned}\sigma_{e,o}^2 &= E\bigg[\int_{-\infty}^{\infty} (x_a(t_a) - \hat{x}_{a,o}(t_a))(x_a(t_a) - \hat{x}_{a,o}(t_a))^*\, dt_a\bigg]\\ &= \int_{-\infty}^{\infty} (R_{x_a x_a}(t_a, t_a) - 2\,\mathrm{Re}(g_o^*(t_a) R_{x_a y_a}(t_a, t_a))\\ &\qquad + g_o(t_a) g_o^*(t_a) R_{y_a y_a}(t_a, t_a))\, dt_a\end{aligned} \tag{25}$$

and then choose a as the minimizer of $\sigma_{e,o}^2$ in the range $[-1, 1]$. This minimizer can be found analytically in certain cases but

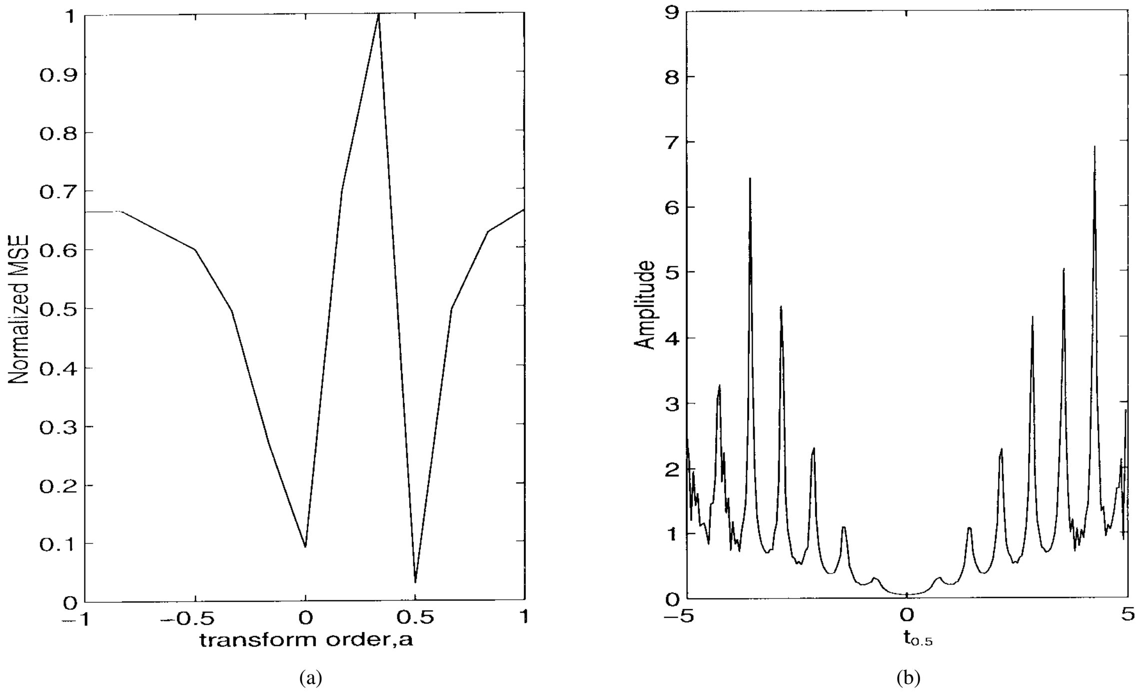


Fig. 2. [**Example 1**] (a) Normalized MSE for different values of transform order a. (b) Optimum filter function $g_o(t_{0.5})$ in the $a_{\mathrm{opt}} = 0.5$th domain (absolute value).

not in general. It can also be found by simply calculating the MSE for sufficiently closely spaced discrete values of a (for example, by step size of 0.1) and choosing the one that minimizes the MSE. If greater accuracy is needed, we can refine this process around the neighborhood of the initially obtained value of a. Other more sophisticated and efficient minimization routines with the initial estimate found by coarse discretization can also be employed [45]. The key point is that given the noise and signal statistics, the optimum value of a is calculated only once. After this, the filtering process can be implemented in $O(N \log N)$ time for arbitrary many realizations of that statistics.

Overall, the procedure can be outlined as follows: Given the autocorrelation functions of the input ($\boldsymbol{x}$) and noise ($\boldsymbol{n}$) processes and the degradation ($\mathcal{H}$), we can find the correlation function between the input and output ($\boldsymbol{y}$) processes and the autocorrelation function of the output process. Then, using these, we can find the optimal filter function in the ath domain by using (24). The optimal value of a is then chosen as that which minimizes (25).

V. Simulations

In this section, computer simulations that illustrate the applications and performance of fractional Fourier domain filtering are presented. Notice that the integrals appearing in (24) are simply fractional Fourier transformations and can be simulated using the procedure given in [5]. Then, the MSE can be computed by using (25).

Example 1: As a first example, we consider a degradation model that corresponds to a time-varying bandpass filter whose center frequency changes linearly with time $h(t, t') = e^{-j2\pi t(t-t')}\,\mathrm{sinc}(t - t')$ so that the system function [39] is given by

$$\overline{H}(f, t) = \int_{-\infty}^{\infty} h(t, t') e^{-j2\pi f(t-t')}\, dt' = \mathrm{rect}(f + t). \qquad (26)$$

The input process $\boldsymbol{x}$ is a sequence of rectangular pulses whose amplitudes takes the value of 1 or 0 with equal probability. There is no noise process.

The normalized MSE is plotted for different values of a in Fig. 2(a). (The normalization is obtained by dividing the MSE values by the maximum value of MSE.) The minimum MSE is obtained in the $a = a_{\mathrm{opt}} = 0.5$th domain. The absolute value of the optimal filter function in the a_{opt}th domain is plotted in Fig. 2(b).

Fig. 3(a) and (b) show a realization of the input process $\boldsymbol{x}$ and the corresponding output process $\boldsymbol{y}$, respectively. The input process is totally unrecognizable. The estimates obtained by filtering the distorted signal realization in the optimum domain ($a = 0.5$) by the filter function plotted in Fig. 2(b) and in the conventional Fourier domain ($a = 1$) are plotted together with the desired signal (input) realization in Fig. 3(c) and (d). Notice that filtering in the optimal fractional domain is significantly better than filtering in the conventional frequency domain.

The Wigner distributions of the above realizations are plotted in Fig. 4 to show the TF content of the signals and the optimal filtering domain ($a = 0.5$). The optimal domain is the one that is perpendicular to the one defined by (26) ($\mathrm{rect}(f + t)$). Since the degradation roughly corresponds to multiplication with a window along the line $f + t = 0$, the effect of the degradation can be reduced in the domain where this window is localized. This is the domain perpendicular to the line $f + t = 0$. (Consider projections onto the filtering domain.)

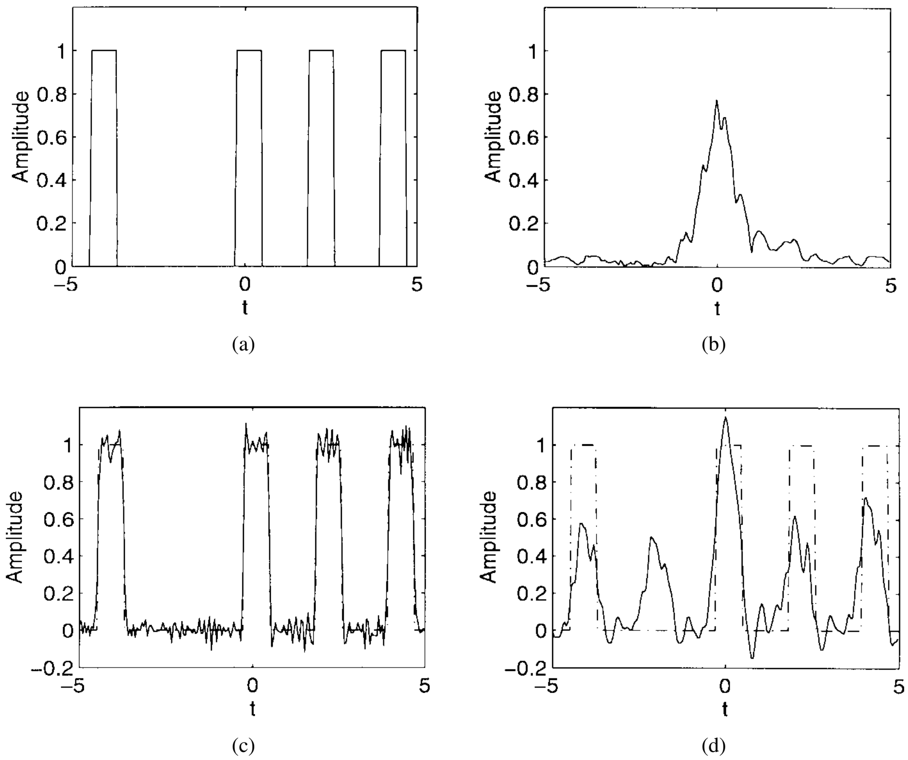


Fig. 3. [**Example 1**] (a) Realization of the input process $\boldsymbol{x}$. (b) Corresponding output process $\boldsymbol{y}$. (c) Estimate $\hat{\boldsymbol{x}}$ obtained by filtering in the $a = 0.5$th domain (solid) and the desired undistorted signal $\boldsymbol{x}$ (dashed). (d) Estimate obtained by filtering in the $a = 1$st domain (solid) and the desired signal (dashed).

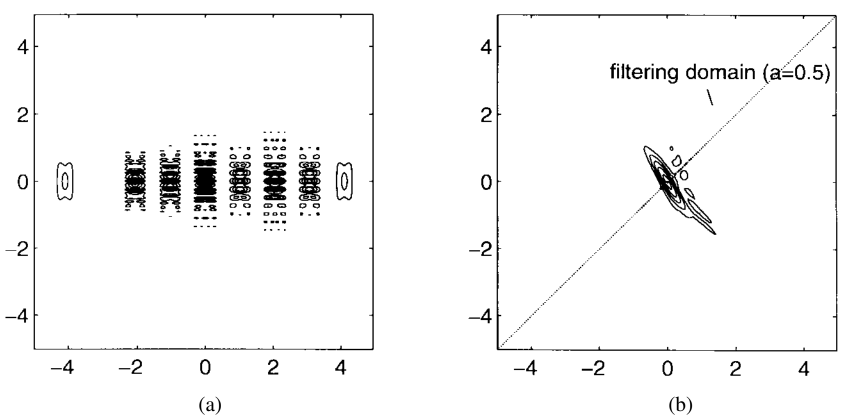
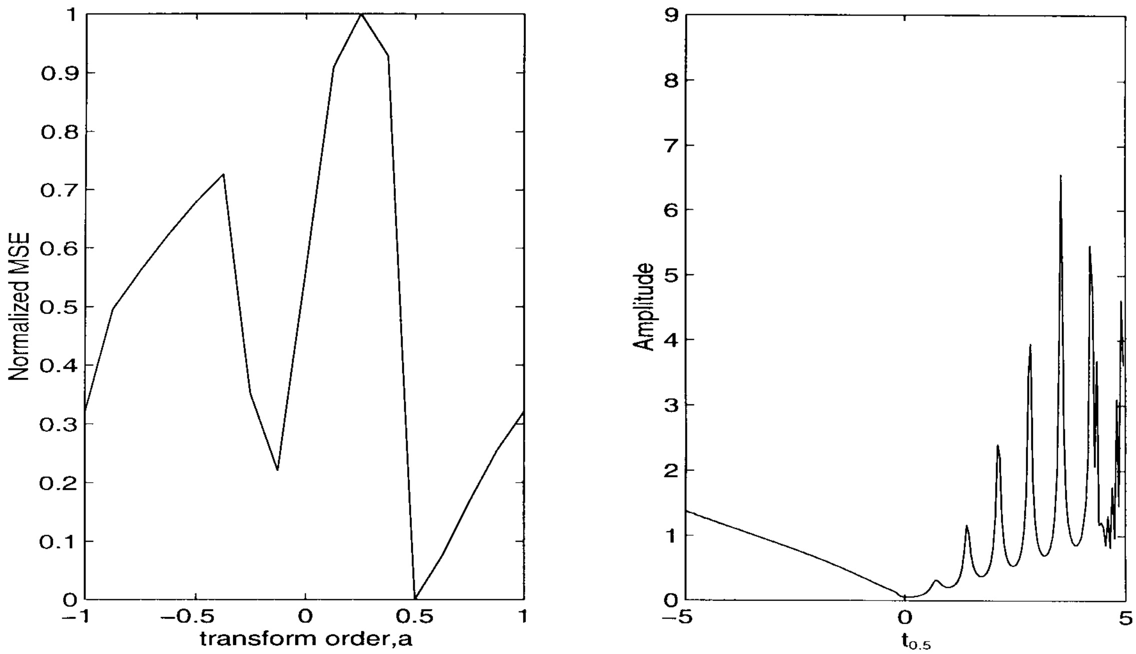


Fig. 4. [**Example 1**] (a) Wigner distribution of the input process realization plotted in Fig. 3(a). (b) Wigner distribution of the corresponding output realization. Tilted solid line shows the optimal fractional domain.

Example 2: In the second example, we consider the same degradation model; however, in this case, the input process is a shifted Gaussian function that is deterministic except for a random amplitude factor and random shift. That is, the input process is given by $x(t) = \boldsymbol{A}\exp(-\pi(t - \boldsymbol{s})^2)$, where $\boldsymbol{A}$ and $\boldsymbol{s}$ are random variables uniformly distributed on interval [1 3]. Again, there is no noise process. Fig. 5(a) shows the normalized MSE plot for different values of a, and in Fig. 5(b), the optimal filter function for the $a_{\text{opt}} = 0.5$th domain can be seen. Again, the minimum MSE is obtained in the $a = a_{\text{opt}} = 0.5$ domain with nearly perfect reconstruction.

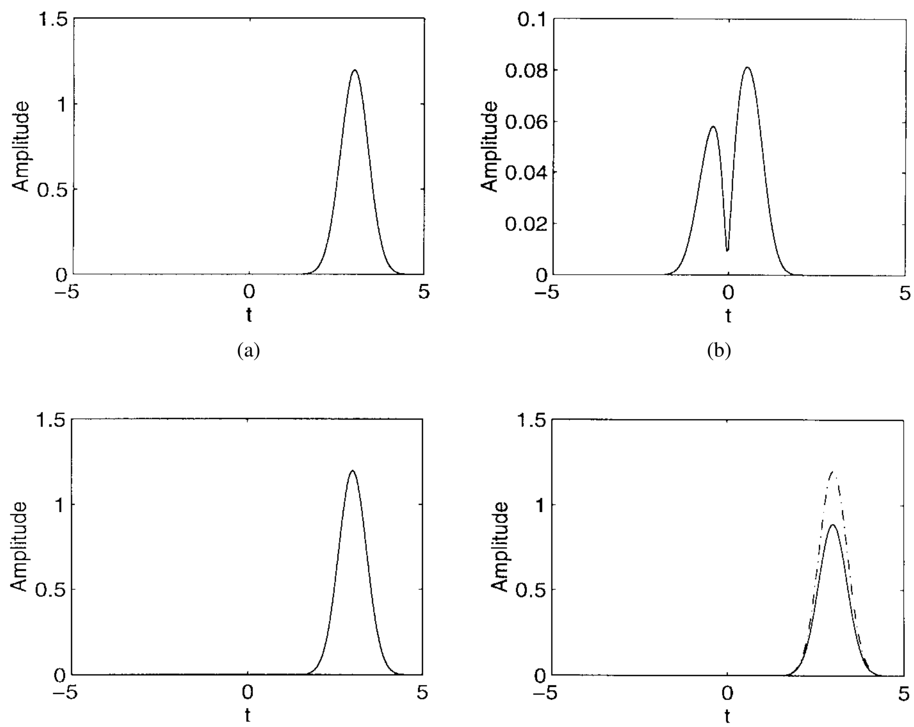
We have shown a realization of the input process $\boldsymbol{x}$, the corresponding output process, and the estimates obtained by filtering the output realization in the optimum domain $(a = 0.5)$ and in the conventional Fourier domain $(a = 1)$ in Fig. 6(a)–(d), respectively. Almost perfect reconstruction is obtained as a result of filtering in $a = 0.5$th domain.

Example 3: The third example deals with a noise separation problem. The input process $\boldsymbol{x}$ is the process of the second example. The noise process is a finite duration bandpass noise that is modulated with a quadratic exponential (chirp) function $(\exp(-j1.73\pi t^2))$ so that its center frequency changes linearly with time. We assume there is no degradation so we take the degradation operator as the identity operator. Fig. 7(a) shows the normalized MSE for different values of a, and the optimal filter function for $a_{\text{opt}} = 0.33$ is plotted in Fig. 7(b). In this case, the minimum MSE is obtained in the $a = a_{\text{opt}} = 0.33$rd domain. The realizations of the input



(a) (b)

Fig. 5. [**Example 2**] (a) Normalized MSE for different values of a. (b) Optimum filter function $g_o(t_{0.5})$ in the $a_{\text{opt}} = 0.5$th domain (absolute value).



(c) (d)

Fig. 6. [**Example 2**] (a) Realization of the input process $\boldsymbol{x}$. (b) Corresponding output process $\boldsymbol{y}$. (c) Estimate obtained by filtering in the $a = 0.5$th domain (solid) and the desired signal (dashed). (d) Estimate obtained by filtering in the $a = 1$st domain (solid) and the desired signal (dashed).

and output processes together with the estimates obtained by filtering the output realization in the optimum domain $(a = 0.33)$ and in the conventional Fourier domain $(a = 1)$ are plotted in Fig. 8(a)–(d), respectively.

Fig. 9 shows the Wigner distributions of the realizations. The optimal filter domain is also shown in Fig. 9(b). It is intuitively clear from this figure that $a = 0.33$ is the optimal domain. (Consider the projections onto the filtering domain.)

Up to this point, we have given examples for which only degradation or only noise is present. For the following examples, degradation and noise are present together. We consider two cases: i) The optimum values of a for degradation alone

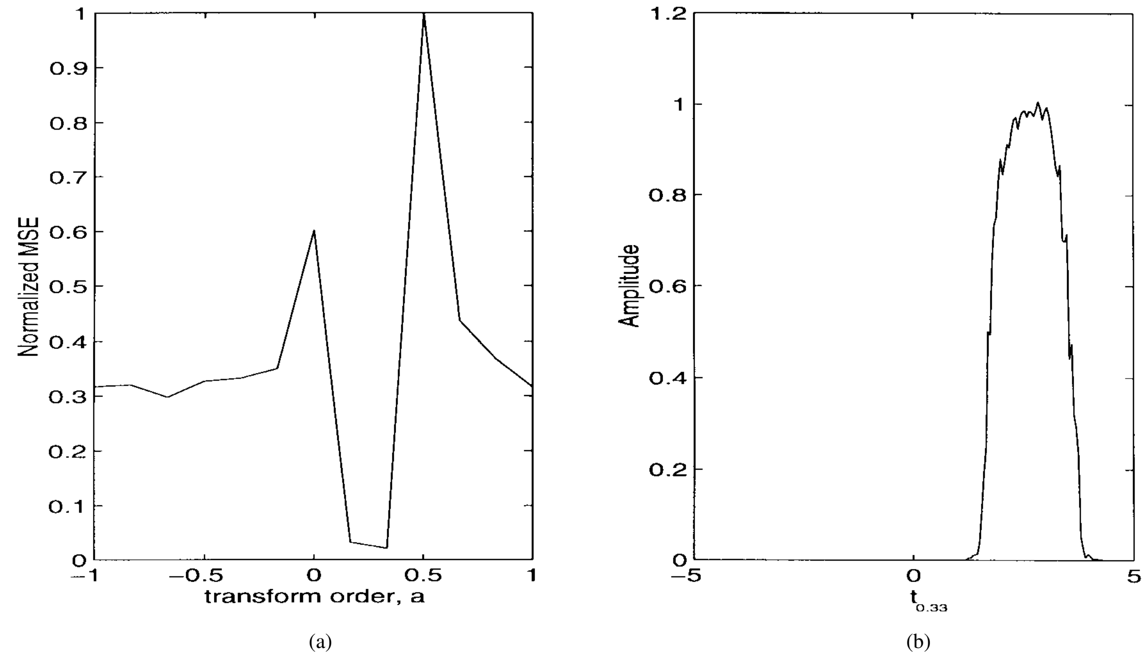


Fig. 7. [**Example 3**] (a) Normalized MSE for different values of transform order a. (b) Optimum filter function $g_o(t_{0.33})$ in the $a_{\rm opt} = 0.33$th domain (absolute value).

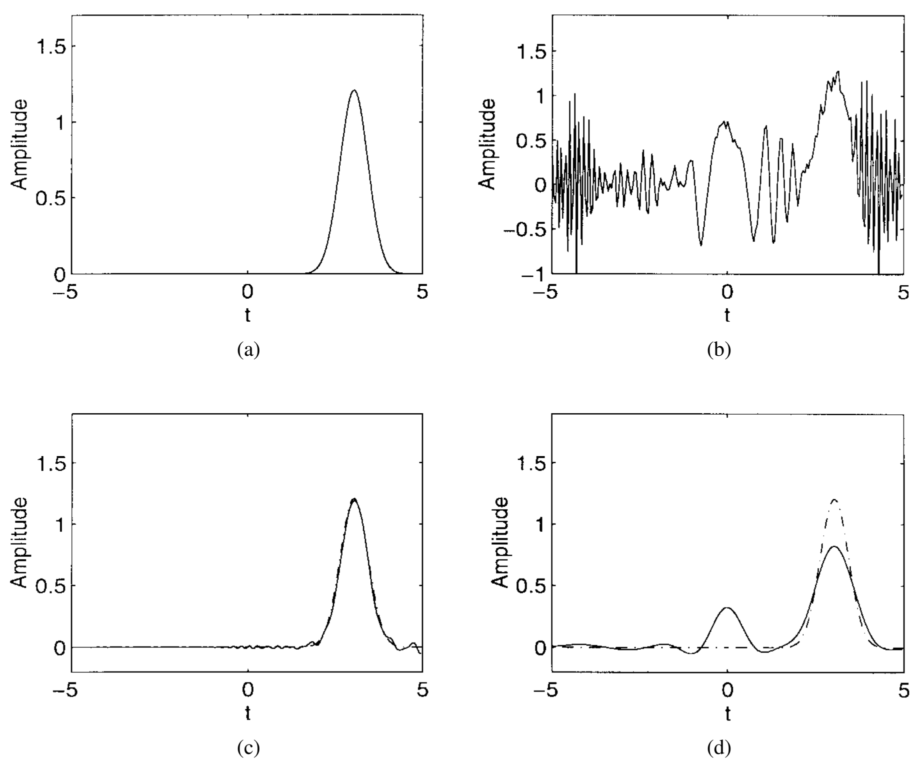
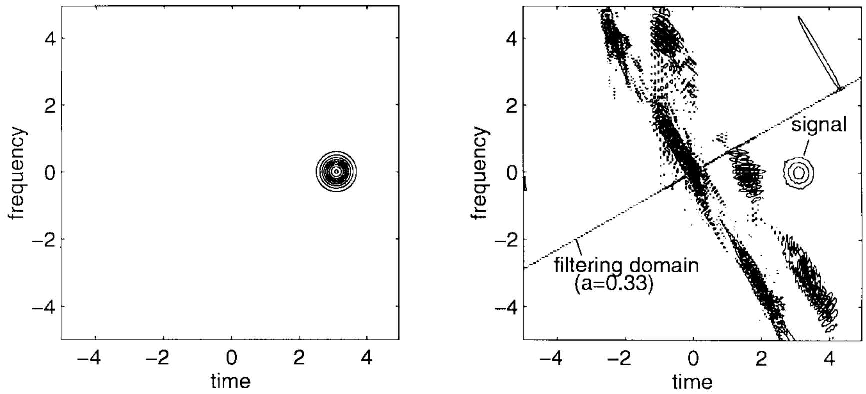


Fig. 8. [**Example 3**] (a) Realization of the input process $\boldsymbol{x}$. (b) Corresponding output process $\boldsymbol{y}$. (c) Estimate $\hat{\boldsymbol{x}}$ obtained by filtering in the $a = 0.33$th domain (solid) and the desired undistorted signal $\boldsymbol{x}$ (dashed). (d) Estimate obtained by filtering in the $a = 1$st domain (solid) and the desired signal (dashed).

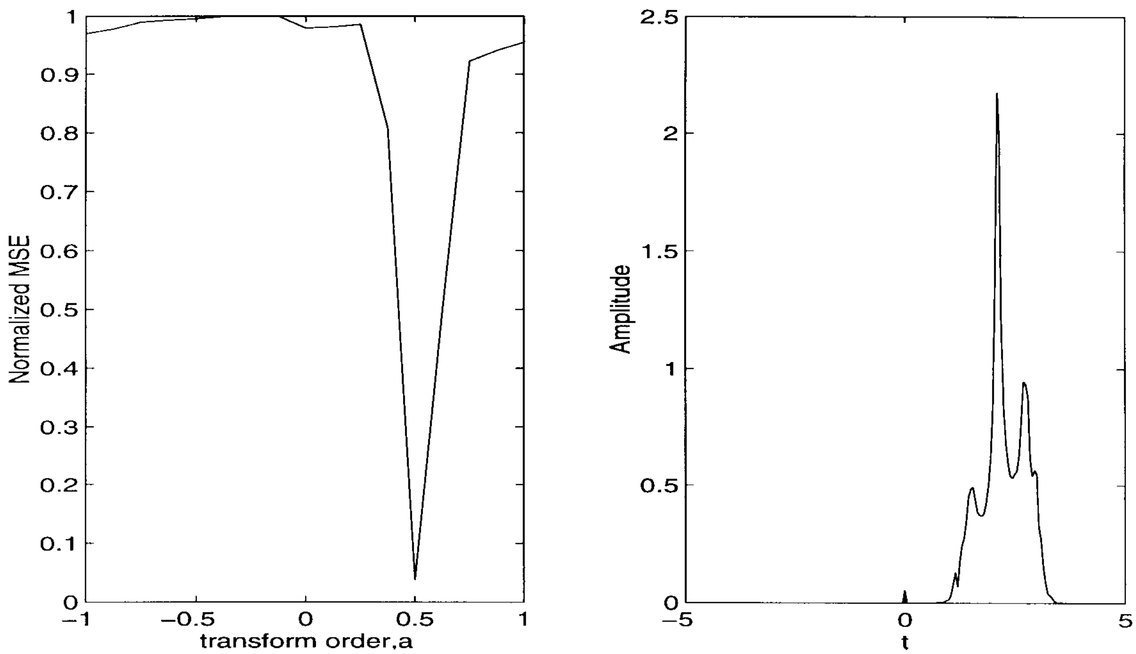
and noise alone are the same or nearly same, and ii) they are significantly different.

Example 4: As an example for the first case, we assume the same input process as in the third example. We assume the noise is finite duration bandpass noise that is modulated with a quadratic exponential chirp function $(\exp(-j\pi t^2)$. Degradation is as given in (26). Figs. 10–12 show this example. The normalized MSE plot for different values of a and the optimal filter function for $a_{\rm opt} = 0.5$ are plotted in parts Fig. 10(a) and (b), respectively. Significant reduction in the MSE is obtained in this case since the optimum values of a for the degradation and noise considered separately coincide; therefore, there is no tradeoff involved in choosing the optimum domain. Fig. 11 shows the filtering result. Fig. 12 shows the Wigner



(a) (b)

Fig. 9. [**Example 3**] (a) Wigner distribution of the input process realization plotted in Fig. 8(a). (b) Wigner distribution of the corresponding output realization. Tilted solid line shows the optimal fractional domain. Signal can be recovered by a simple mask in this domain (consider projections on to this domain).



(a) (b)

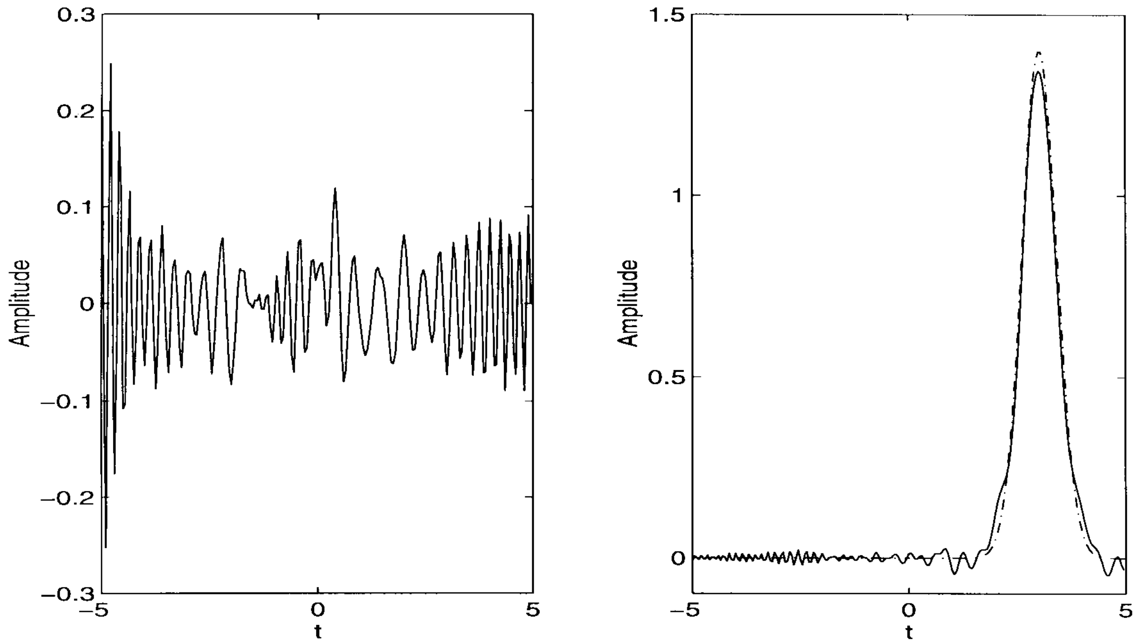
Fig. 10. [**Example 4**] (a) Normalized MSE for different values of transform order a. (b) Optimum filter function $g_o(t_{0.5})$ in the $a_{\rm opt} = 0.5$th domain (absolute value).

distributions of the desired signal and distorted signal together with the domain in which optimal filtering is performed.

Example 5: Now, we consider the case where the optimum values of a for degradation alone and noise alone are different and thus impose conflicting requirements in choosing the filtering domain. We combine the second and third examples so that our input process $\boldsymbol{x}$ is the input process of Examples 2 and 3, and the noise is a finite duration bandpass noise that is modulated with a quadratic exponential chirp function $(\exp(-j1.73\pi t^2))$. The degradation is as given in the first and second examples. The normalized MSE plot for different values of a can be seen in Fig. 13(a). The optimum value of MSE is obtained for $a = 0.5$, but this value is much larger compared with the cases where we consider degradation alone or noise alone. We also plot a realization of the output $\boldsymbol{y}$ and the estimate obtained by filtering in the $a = 0.5$th domain in Fig. 14(a) and (b), respectively. We see that this estimate obtained by filtering in a single domain is not satisfactory.

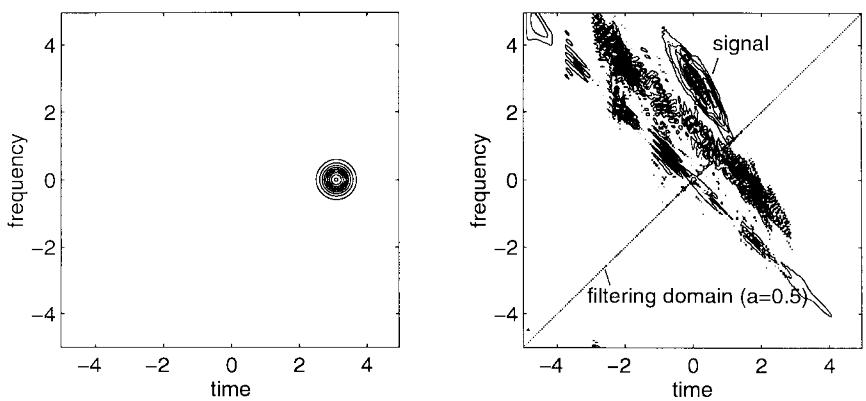
We might try a heuristic two-step filtering procedure to obtain a smaller MSE. Fig. 14(c) and (d) show the results of this two-step filtering procedure. We first go to the $a = 0.5$th domain and eliminate the degradation by using the optimum filter function obtained in the second example. We then go to the $a = 0.33$th domain and eliminate the noise by using the optimum filter function obtained in the third example. The resulting estimate is much better than that obtained by filtering in a single domain.

Fig. 15 shows this two-step filtering process in the TF plane. Fig. 15(a) is the Wigner distribution of the output realization of Fig. 14(a). The domain in which the degradation is eliminated is also shown. Fig. 15(b) shows the Wigner distribution of the intermediate estimate in which the degradation has been



(a) (b)

Fig. 11. [**Example 4**] (a) Realization of the output process $\boldsymbol{y}$. (b) Estimate $\hat{\boldsymbol{x}}$ obtained by filtering in the $a = 0.5$th domain (solid) and the desired undistorted signal $\boldsymbol{x}$ (dashed).



(a) (b)

Fig. 12. [**Example 4**] (a) Wigner distribution of the input process realization plotted in Fig. 11(a). (b) Wigner distribution of the corresponding output realization. Tilted line shows the optimal fractional domain.

eliminated. The second filtering domain in which the noise will be eliminated is also shown in Fig. 15(b).

The orders in the above procedure have been chosen heuristically by looking at the optimal order resulting from a consideration of each effect by itself. (In this case, they correspond to the optimal orders of Examples 2 and 3.) This procedure does not determine which of the two domains should be visited first. We have simply tried both ways to determine the one that results in the smaller MSE.

We should stress that this heuristic procedure does not necessarily yield the smallest possible MSE that can be obtained by filtering in two domains. An exact solution for the optimal filtering problem in two or more domains has yet to be found.

The above examples represent situations in which filtering in fractional Fourier domains yield substantially smaller MSE's as compared with conventional Fourier domain filtering. Fractional Fourier domain filtering in a single domain is particularly advantageous when the distortion or noise is of a chirped nature. Such situations are encountered in several real-life applications, some of which we will mention briefly. One application arises in synthetic aperture radar (SAR), which employs chirps as transmitted pulses so that the measurements are related to the terrain reflectivity function through a chirp convolution. This process results in chirp-type disturbances caused by moving objects in the terrain, which should be removed if high-resolution imaging is to be achieved [41]. Other applications arise in holography where different chirp rates involved in in-line holograms can be used for extraction of 3-D object-location information. The separation of these chirps, which is similar to Example 3, directly yields location information [42], [43]. A major problem in the reconstruction from holograms is the elimination of twin-image noise. Since

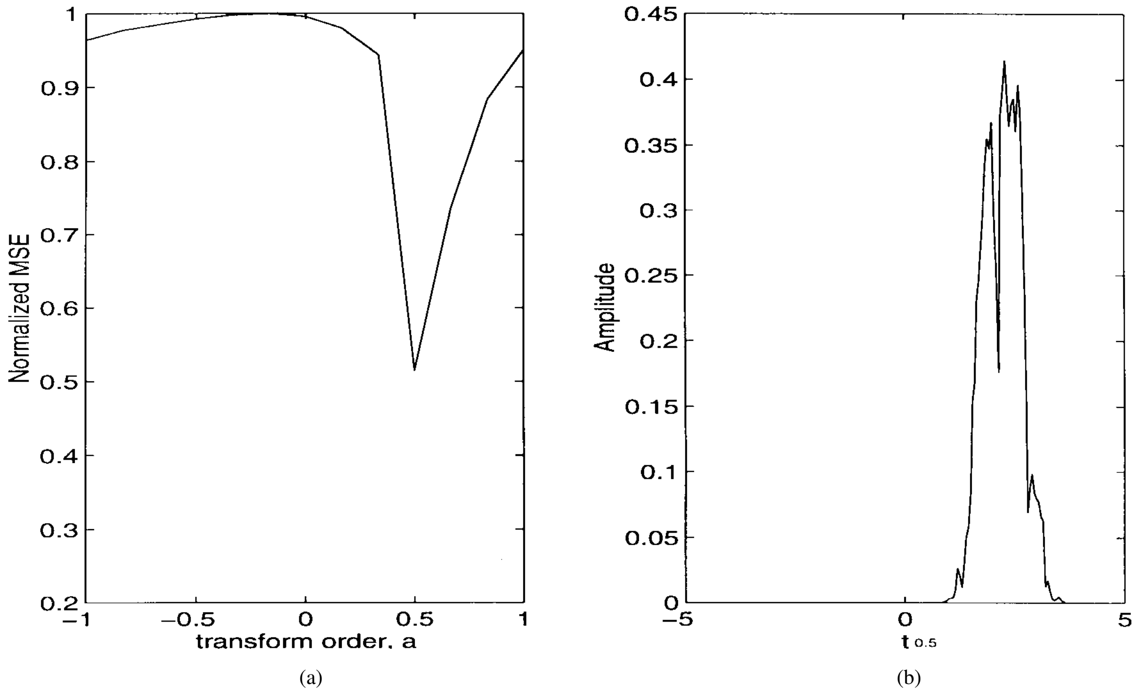


Fig. 13. [**Example 5**] (a) Normalized MSE for different values of transform order a. (b) Optimum filter function $g_o(t_{0.5})$ in the $a_{\mathrm{opt}} = 0.5$th domain (absolute value).

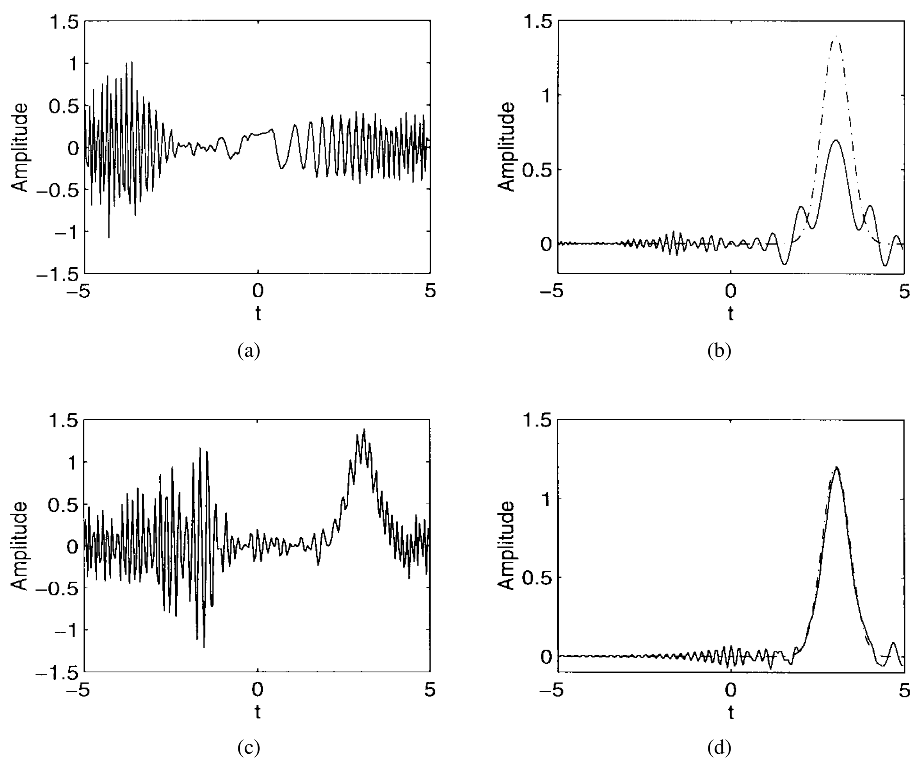


Fig. 14. [**Example 5**] (a) Realization of the output process $\boldsymbol{y}$. (b) Estimate $\hat{\boldsymbol{x}}$ obtained by filtering in the $a = 0.5$th domain (solid) and the desired undistorted signal $\boldsymbol{x}$ (dashed). Successive filtering of (a) in different domains. (c) Effect of degradation is removed by filtering in the $a = 0.5$th domain. (d) Noise is eliminated by filtering in the $a = 0.33$th domain.

this noise is essentially a modulated chirp signal [42], it can be removed with the filtering procedure presented in this paper. Another application arises in the correction of the effects of point or line defects found on lenses or filters in optical systems, which appear at the output plane in the form of chirp artifacts that are essentially similar to the examples given in this paper [44].

Furthermore, the class of noise and distortions that can be treated effectively with filtering in multiple fractional domains will be much larger; therefore, the extension of the method

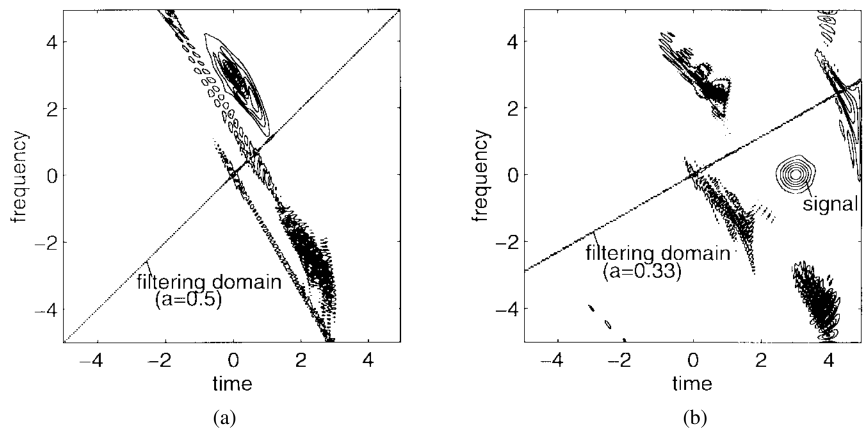


Fig. 15. [**Example 5**] (a) Wigner distribution of the output process realization plotted in Fig. 14(a). Tilted line shows the domain where degradation is removed. (b) Wigner distribution of Fig. 14(c). Tilted line shows the optimal fractional domain where noise is removed.

to multiple domains will provide a powerful and flexible tool for signal restoration. The method presented here should prove useful not only in itself but also in providing a foundation for the multiple domain filtering problem.

VI. Discrete Time Formulation

A definition of the discrete fractional Fourier transform is suggested in [5]. The precise definition is not important for the purpose of this paper. We only note that the essential property of the discrete fractional Fourier transform is that it maps the samples of a function into the samples of its fractional Fourier transform to some sufficient degree of accuracy.

In this section, the discrete time counterpart of the problem will be formulated, and its solution based on the discrete fractional Fourier transform will be presented. At the end, we show that the solution is analogous to its continuous time counterpart.

Problem Statement in Discrete Time: The signal observation model is given by

$$\underline{y} = \underline{\underline{H}}\,\underline{x} + \underline{n} \tag{27}$$

where $\underline{y}, \underline{x}$, and $\underline{n}$ are column vectors, and $\underline{\underline{H}}$ is the matrix characterizing the degradation process. We assume that input and output processes and noise are finite length random processes and that we know the correlation matrix of the input process $\underline{x}$ and noise $\underline{n}$. We will further assume that the noise is independent of the input process and is zero mean.

We consider an estimate of the form

$$\hat{\underline{x}} = \underline{\underline{F}}^{-a}\underline{\underline{\Lambda}}_g\underline{\underline{F}}^{a}\underline{y} \tag{28}$$

where $\underline{\underline{F}}^{-a}$ and $\underline{\underline{F}}^{a}$ are discrete fractional Fourier transform matrices of order $-a$ and a, respectively, and $\underline{\underline{\Lambda}}_g$ is a diagonal matrix whose diagonal consists of the elements of the vector $\underline{g}$. We note that since the fractional Fourier transformation is a unitary transformation, the fractional Fourier transformation matrices are related by $\underline{\underline{F}}^{-a} = (\underline{\underline{F}}^{a})^H$, where $(\cdot)^H$ denotes the conjugate transpose operation. This estimate corresponds to a multiplicative filter in the ath fractional Fourier domain. As in the continuous-time case, if $a = 1, \underline{\underline{F}}^{a}$ corresponds to the DFT matrix, and our estimation corresponds to that obtained by conventional Fourier domain filtering.

Our filter design criteria is the mean square error (MSE), which is defined as

$$\sigma_e^2 = \frac{1}{N}E[(\underline{x} - \hat{\underline{x}})^H(\underline{x} - \hat{\underline{x}})] \tag{29}$$

where N is the size of the input vector $\underline{x}$. The problem is then to find the vector $\underline{g}$, which minimizes σ_e^2.

In order to solve this discrete time problem, we first define the cost function J_d to be equal to the MSE defined in (29), which is also equal to the error in the ath domain:

$$\begin{aligned} J_d &= \frac{1}{N}E[(\underline{x} - \hat{\underline{x}})^H(\underline{x} - \hat{\underline{x}})] \\ &= \frac{1}{N}E[(\underline{x}_a - \hat{\underline{x}}_a)^H(\underline{x}_a - \hat{\underline{x}}_a)]. \end{aligned} \tag{30}$$

where

$$\underline{x}_a = \underline{\underline{F}}^{a}\underline{x}, \quad \text{and} \quad \hat{\underline{x}}_a = \underline{\underline{\Lambda}}_g\underline{\underline{F}}^{a}\underline{y} = \underline{\underline{\Lambda}}_g\underline{y}_a.$$

We can then follow the discrete analogous of the steps in Section IV and easily find the components of the optimal vector to be

$$g_{\text{opt},j} = \frac{\underline{\underline{R}}_{x_a y_a}(j,j)}{\underline{\underline{R}}_{y_a y_a}(j,j)} \qquad j = 1, \cdots, N. \tag{31}$$

The above correlation matrices can be obtained from the input and noise correlation matrices as

$$\begin{aligned} \underline{\underline{R}}_{x_a y_a} &= \underline{\underline{F}}^{a}\underline{\underline{R}}_{xx}\underline{\underline{H}}^H\underline{\underline{F}}^{-a} \\ \underline{\underline{R}}_{y_a y_a} &= \underline{\underline{F}}^{a}(\underline{\underline{H}}\,\underline{\underline{R}}_{xx}\underline{\underline{H}}^H + \underline{\underline{R}}_{nn})\underline{\underline{F}}^{-a}. \end{aligned} \tag{32}$$

Equation (31) provides the solution to our minimization problem in the discrete time setting. We note that this result is fully analogous to the solution obtained in (24) for the continuous time case. In fact, we will now show that the discrete time implementation of (24) yields a result very similar to (31).

Equation (24) can be written in the form

$$g_o(t') = \frac{E\left[\int_{-\infty}^{\infty} B_a(t,t')x(t)\,dt \int_{-\infty}^{\infty} B_a^*(t',t'')y^*(t'')\,dt''\right]}{E\left[\int_{-\infty}^{\infty} B_a^*(t',t'')y^*(t'')\,dt'' \int_{-\infty}^{\infty} B_a(t',x'')y(x'')\,dx''\right]}. \tag{33}$$

The processes inside the expectation operators are simply the ath fractional Fourier transform of the input and output processes and their complex conjugates. We can find the samples of the ath fractional Fourier transform of the input and output processes from the following equations [5]:

$$\begin{aligned}\{\mathcal{F}^a x\}\left(\frac{m}{2\Delta R}\right) &\simeq \frac{A_\phi}{2\Delta R}\exp\left[j\pi(\alpha-\beta)\left(\frac{m}{2\Delta R}\right)^2\right] \\ &\quad\cdot \sum_{n=-N}^{N}\exp\left[j\pi\beta\left(\frac{m-n}{2\Delta R}\right)^2\right] \\ &\quad\cdot \exp\left[j\pi(\alpha-\beta)\left(\frac{n}{2\Delta R}\right)^2\right] \\ &\quad\cdot x\left(\frac{n}{2\Delta R}\right), \\ \{\mathcal{F}^a y\}\left(\frac{m}{2\Delta R}\right) &\simeq \frac{A_\phi}{2\Delta R}\exp\left[j\pi(\alpha-\beta)\left(\frac{m}{2\Delta R}\right)^2\right] \\ &\quad\cdot \sum_{n=-N}^{N}\exp\left[j\pi\beta\left(\frac{m-n}{2\Delta R}\right)^2\right] \\ &\quad\cdot \exp\left[j\pi(\alpha-\beta)\left(\frac{n}{2\Delta R}\right)^2\right] \\ &\quad\cdot y\left(\frac{n}{2\Delta R}\right)\end{aligned} \tag{34}$$

where $2N$ is the number of samples of the input and output processes, and $\alpha = \cot\phi$ and $\beta = \csc\phi$. Equation (33) can then be written as

$$\begin{aligned} g_o\left(\frac{m}{2\Delta R}\right) &= \frac{E\left[\{\mathcal{F}^a x\}\left(\frac{m}{2\Delta R}\right)\{\mathcal{F}^a y\}^*\left(\frac{m}{2\Delta R}\right)\right]}{E\left[(\mathcal{F}^a[y])\left(\frac{m}{2\Delta R}\right)(\mathcal{F}^a[y])^*\left(\frac{m}{2\Delta R}\right)\right]} \\ &= \frac{\underline{\underline{R}}_{x_a y_a}(m,m)}{\underline{\underline{R}}_{y_a y_a}(m,m)}\end{aligned} \tag{35}$$

with

$$\underline{\underline{R}}_{x_a y_a} = (\underline{\underline{R}}_{x_a y_a}(k,l))$$

where

$$\underline{\underline{R}}_{x_a y_a}(k,l) = E\left[x_a\left(\frac{k}{2\Delta R}\right)y_a^*\left(\frac{l}{2\Delta R}\right)\right]$$

is the correlation matrix of the samples of the processes $\boldsymbol{x_a}$ and $\boldsymbol{y_a}$, and $\underline{\underline{R}}_{y_a y_a}$ is similarly defined. Equation (35) is very similar to (31), which shows that our discrete time formulation is analogous to the continuous case.

VII. Discussions and Conclusions

In the previous sections, we have shown how the fractional Fourier transform can be applied to the problem of time-varying filtering of nonstationary, finite energy processes both in continuous-time and discrete-time frameworks. We derived the optimum multiplicative filter function (vector) that minimizes the mean square error in the ath fractional Fourier domain.

Our simulation examples show that filtering in fractional domains will work better for certain kinds of distortions and signal and noise statistics in comparison with others. The presence of time-varying distortion and nonstationary statistics suggests that the fractional transform may be of use but may not guarantee significant improvements in every case.

The solutions obtained in (24) and (31) are not the most optimal linear estimators in the sense that we restrict the general linear form [see (2)] so that it corresponds to a multiplicative filter in the ath domain [cf. (6)]. However, in many cases, we can expect to reduce significantly the MSE in comparison with ordinary Fourier domain filtering [46]. These reductions come essentially for free since filtering in a fractional domain can be implemented with similar computational cost as filtering in the ordinary Fourier domain.

In this paper, we have analytically formulated filtering in a single fractional domain. As discussed in the fifth example of Section V, the method can sometimes be greatly improved by filtering in not one, but several consecutive fractional Fourier domains. This will not only allow one to handle a much wider variety of signals but may also make possible a fairly good approximation of the most general optimal linear recovery operator with a reasonable number of stages [2], [10]. A rigorous analytical solution of the multistage filtering problem, or at least a satisfactory numerical algorithm for its solution, has not yet been found and requires further work.

As a conclusion, filtering in fractional Fourier domains may enable significant reduction of the MSE compared with ordinary Fourier domain filtering. This reduction comes at essentially no additional cost since the fractional Fourier transform has an $O(N\log N)$ algorithm. We have presented a mathematical formulation and solution of this problem that is analogous to the formulation and solution of the classical optimal Wiener filtering problem.

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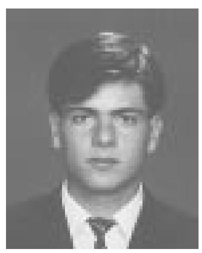

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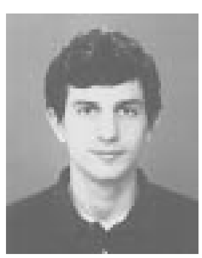

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