Table 3 Estimates of state variable speed for F-4C aircraft

| Variable | Speed | |
|---|---|---|
| | Method 1 | Method 2 |
| $x,y$ | 0.00049 | 0.00013 |
| $E$ | 0.0055 | 0.0027 |
| $h$ | 0.023 | 0.0029 |
| V | 0.035 | 0.0033 |
| $\gamma$ | 0.090 | 0.044 |
| $\chi$ | $\infty$ | 0.059 |

magnitude of the time-scale separations. For example, method 1 predicts that $E$ is much slower than $h$ or $V$, which are themselves of about the same speed, whereas method 2 indicates that $E$, $h$, and $V$ all have very nearly the same speed.

Comparing the variable ordering shown in Table 3 with the ordering assumed in past analyses (Table 1) shows general agreement. The only exception is that $\chi$ has been treated as a variable of intermediate speed, whereas the present analysis shows it to be the fastest variable of all. This discrepancy has been recognized,[8-19] but $\chi$ has been retained as an intermediate variable for two main reasons in spite of this recognition. First, treating $\chi$ as slower than $h$ and $\gamma$ gives singular perturbation solutions that model maneuvers such as "high- and low-speed yo-yos," which are known to be important in optimal turning of high-performance aircraft, whereas treating $\chi$ as faster than $h$ and $\gamma$ does not.[12] Second, the adjoint equation associated with $\chi$ can be analytically integrated, making the inclusion of $\chi$ in slower subsystems relatively easy.

## Conclusion

Two methods for time-scale separation analysis of dynamic systems have been proposed. These methods are based on the concept of state variable speed and require knowledge only of the dynamical equations and bounds on state and control variables. They are not as rigorous as other proposed methods, but they do not require a priori knowledge of an optimal trajectory, are relatively easy to apply, and are an improvement over the ad hoc methods currently in use.

The two methods were applied to a typical class of aircraft flight dynamics problems and equations were derived for state variable speed estimation. A numerical example showed that the time-scale separations as computed by the two methods proposed here generally agree with previous ad hoc time-scale separation assumptions.

## References


[1]Kelley, H.J. and Edelbaum, T.N., "Energy Climbs, Energy Turns, and Asymptotic Expansions," *Journal of Aircraft,* Vol. 7, Jan.-Feb. 1970, pp. 93-94.

[2]Kelley, H.J., "Reduced-Order Modeling in Aircraft Mission Analysis," *AIAA Journal,* Vol. 9, Feb. 1971, pp. 349-350.

[3]Kelley, H.J., "Flight Path Optimization with Multiple Time Scales," *Journal of Aircraft,* Vol. 8, April 1971, pp. 238-240.

[4]Kelley, H.J., "Aircraft Maneuver Optimization by Reduced-Order Approximation," *Control and Dynamic Systems,* Vol. 10, edited by C.T. Leondes, Academic Press, New York, 1973.

[5]Ardema, M.D., "Singular Perturbations in Flight Mechanics," NASA TM X-62, 380, 1974 (revised 1977).

[6]Ardema, M.D., "Solution of the Minimum Time-to-Climb Problem by Matched Asymptotic Expansions," *AIAA Journal,* Vol. 14, July 1976.

[7]Ardema, M.D., "Linearization of the Boundary-Layer Equations of the Minimum Time-to-Climb Problem," *Journal of Guidance and Control,* Vol. 2, Sept.-Oct. 1979, pp. 434-435.

[8]Calise, A.J., "Singular Perturbation Methods for Variational Problems in Aircraft Flight," *IEEE Transactions Automatic Control,* Vol. AC-21, No. 3, June 1976.

[9]Calise, A.J., "Extended Energy Management Methods for Flight Performance Optimization," *AIAA Journal,* Vol. 15, 1977.

[10]Calise, A.J., "A New Boundary Layer Matching Procedure for Singularly Perturbed Systems," *IEEE Transactions Automatic Control,* Vol. AC-23, No. 3, June 1978.

[11]Calise, A.J., Aggarwal, R., and Anderson, G.M., "Aircraft Optimal Weapon Delivery Maneuvers Based on Extended Energy Management," AIAA Paper 75-1076, 1975.

[12]Calise, A.J. and Moerder, D.D., "Singular Perturbation Techniques for Real Time Aircraft Trajectory Optimization and Control," NASA CR-3497, 1982.

[13]Calise, A.J., "A Singular Perturbation Analysis of Optimal Aerodynamic and Thrust Magnitude Control," *IEEE Transactions Automatic Control,* Vol. AC-24, No. 5, Oct. 1979.

[14]Calise, A.J., "Optimal Thrust Control with Proportional Navigation Guidance," *Journal of Guidance and Control,* Vol. 3, July-Aug. 1980, pp. 312-318.

[15]Mehra, R., Washburn, R., Sajan, S., and Carroll, J., "A Study of the Application of Singular Perturbation Theory," NASA CR-3167, 1979.

[16]Shinar, J., "Remarks on Singular Perturbation Technique Applied in Nonlinear Optimal Contol," *Proceedings of the 2nd IFAC Workshop on Control Applications on Nonlinear Programming and Optimization,* 1980.

[17]Shinar, J., "Validation of Zero-Order Feedback Strategies for Medium-Range Air-to-Air Interception in a Horizontal Plane," NASA TM-84237, 1982.

[18]Shinar, J. and Farber, N., "Nonlinear Zero-Sum Differential Game Analysis by Singular Perturbation Methods," NASA TM-84271, 1982.

[19]Shinar, J., Farber, N., and Negrin, M., "A Three-Dimensional Air Combat Game Analysis by Forced Singular Perturbations," AIAA Paper 82-1327, 1982.

[20]Breakwell, J.V., "Optimal Flight-Path-Angle Transitions in Minimum-Time Airplane Climbs," *Journal of Aircraft,* Vol. 14, 1977.

[21]Chow, J.H. and Kokotovic, P.V., "Eigenvalue Placement in Two Time Scale Systems," *Proceedings of the IFAC Workshop on Large Scale Systems,* 1976.

[22]Syrcos, G.P. and Sannuti, P., "Singular Perturbation Modeling of Continuous and Discrete Physical Systems," *IFAC Workshop on Singular Perturbations and Robustness in Control Systems,* July 1982.

[23]Sridar, B. and Gupta, N.K., "Missile Guidance Laws Based on Singular-Perturbation Methodology," *Proceedings of the Joint Automatic Control Conference,* June 1979.

[24]Kelley, H.J., "Comments on 'A New Boundary-Layer-Matching Procedure for Singularly Perturbed Systems," *IEEE Transactions on Automatic Control,* Vol. AC-23, No. 3, June 1978.


# Optimal Finite Horizon Approximation of Unstable Linear Systems


A.-M. Guillaume*
*Université Catholique de Louvain*
*Louvain-la-Neuve, Belgium*
and
P.T. Kabamba
*University of Michigan, Ann Arbor, Michigan*


## Introduction

THE general problem of approximating a high-order linear dynamical system by a low-order "reduced-order model" has received considerable attention in the literature during the last fifteen years. See Ref. (1) for an extensive bibliographical

Received July 13, 1983; revision received May 7, 1984. 

*Research Assistant, Unité de Mécanique Appliquée.

†Assistant Professor, Department of Aerospace Engineering.

list. However, as pointed out,[1] it is important to know a measure of the error introduced by the approximation. The more restricted problem of the optimal approximation of a linear system by a reduced-order model for a given error measure has also been treated in the literature.[2-10] The original solution by Wilson[2,3] is as follows: Consider a minimal, asymptotically stable, time-invariant, linear system with a candidate reduced-order model.

$$\dot{x}_s = A_s x_s + B_s u \tag{1}$$

$$y = C_s x_s \tag{2}$$

$$\dot{x}_r = A_r x_r + B_r u \tag{3}$$

$$y = C_r x_r \tag{4}$$

where $x_s \epsilon R^{n_s}$ is the system state, $x_r \epsilon R^{n_r}$ is the reduced-order model state $(n_r < n_s)$, $u \epsilon R^m$ is the input, $y \epsilon R^p$ is the output, and $A_s$, $B_s$, $C_s$, $A_r$, $B_r$, $C_r$ are matrices of appropriate dimensions. Also consider the corresponding "error system"

$$\dot{x}_e = \begin{bmatrix} A_s & O \\ O & A_r \end{bmatrix} x_e + \begin{bmatrix} B_s \\ B_r \end{bmatrix} u = A_e x_e + B_e u \tag{5}$$

$$e = [C_s, -C_r] x_e = C_e x_e$$

with impulse response

$$H_e(t) = C_e e^{A_e t} B_e \tag{6}$$

The reduced-order model will be chosen in such a way as to minimize the square of the norm of the error system impulse response

$$J = \mathrm{tr} \int_o^\infty H_e(t) H_e^T(t) \mathrm{d}t \tag{7}$$

The gradient of $J$ with respect to $A_r$, $B_r$ and $C_r$ is computed in the following way:

$$A_e W_c + W_c A_e^T + B_e B_e^T = 0 \tag{8}$$

$$A_e^T W_o + W_o A_e + C_e^T C_e = 0 \tag{9}$$

$$J = \mathrm{tr} B_e^T W_o B_e = \mathrm{tr} C_e W_c C_e^T \tag{10}$$

$$\frac{\partial J}{\partial A_e} = 2 W_c W_o \tag{11}$$

$$\frac{\partial J}{\partial B_e} = 2 W_o B_e \tag{12}$$

$$\frac{\partial J}{\partial C_e} = 2 C_e W_c \tag{13}$$

where superscript $T$ denotes matric transposition, and $W_c$ and $W_o$ are symmetric positive definite matrices of order $n_s + n_r$. The matrices $A_r$, $B_r$, $C_r$ being submatrices of $A_e$, $B_e$ and $C_e$, the gradients $\partial J / \partial A_r$, $\partial J / \partial B_r$, $\partial J / \partial C_r$ are the corresponding submatrices of $\partial J / \partial A_e$, $\partial J / \partial B_e$ and $\partial J / \partial C_e$. Therefore, once Eqs. (8-9) are solved for $W_c$ and $W_o$, Eqs. (11-13) can be used by a gradient algorithm to update $A_r$, $B_r$ and $C_r$ and converge to an optimal reduced-order model.

One fundamental restriction of this method and other kindred results found in the literature[4-10] is that the system, Eqs. (1-2) and its reduced-order model, Eqs. (3-4) must be asymptotically stable. If this were not the case, the cost $J$ of Eq. (7) would generally not exist and Eqs. (8-9) would become meaningless. Indeed, even if the unstable dynamics of Eqs. (1-2) were compensated by some unstable dynamics of Eqs. (3-4), numerical errors would cause the generalized integral in Eq. (7) to diverge.

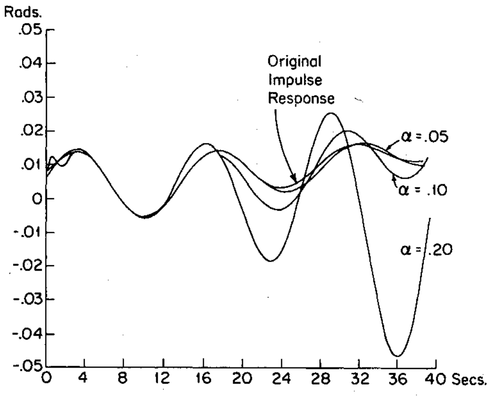


**Fig. 1 Impulse response of satellite and optimal reduced-order models.**

This restriction on the applicability of Wilson's method can be relaxed in the following way. Rather than using a uniform penalty on the impulse response in Eq. (7), we suggest using a degressive weight in such a way as to give more consideration to the recent past history of the system. Using an exponential weight yields a cost function of the form

$$J = \mathrm{tr} \int_o^\infty H_e(t) H_e^T(t) e^{-2\alpha t} \mathrm{d}t \tag{14}$$

where $\alpha$ is a real nonnegative constant. Note that for $\alpha$ positive and large enough, the cost $J$ of Eq. (14) is always guaranteed to exist. $1/\alpha$ can be viewed as a time constant over which we want the impulse response of the reduced-order model to approximate that of the system.

The idea of using exponential weighting functions is not original. In Refs. 12 and 13, such an idea is used to guarantee a stability margin for a time-invariant, linear-quadratic regulator. However, the present context is clearly different. Similarly to the results in Refs. 12 and 13 we have the following:

**Lemma:**

Suppose $\alpha$ is large enough to render the matrix $A_e - \alpha I$ stable, $I$ being the unit matrix. Then the cost function $J$ of Eq. (14) and its gradient are given by Eqs. (10)-(13), but where $W_o$ and $W_c$ satisfy

$$(A_e - \alpha I) W_c + W_c (A_e - \alpha I)^T + B_e B_e^T = 0$$

$$(A_e - \alpha I)^T W_o + W_o (A_e - \alpha I) + C_e^T C_e = 0$$

**Proof**

Equations (14) and (6) yield

$$J = \mathrm{tr} \int_o^\infty C_e e^{(A_e - \alpha I)t} B_e B_e^T e^{(A_e - \alpha I)^T t} C_e^T \mathrm{d}t$$

and the same derivation as in Ref. 2 can be made, with $A_e - \alpha I$ replacing $A_e$.

**Example**

The result of the lemma allows us to design optimal reduced order models for unstable and marginally stable systems by approximating their impulse responses over a finite horizon. This method has been applied to obtain a fourth-order-reduced-order model of the pitch motion of a satellite with flexible radial appendages. The original model is of order six. The input being an actuator torque along the pitch axis, the output is the pitch angle itself. The system has a rigid body mode, so that its impulse response is the sum of a ramp function and damped sinusoids (see Ref. 11 for details). Consequently, Wilson's method is not directly applicable. Figure 1 compares the impulse response of the original system with that of the optimal fourth-order model obtained for several values of $\alpha$. It appears, as expected, that the impulse response is approximated over an interval of duration proportional to $1/\alpha$.

## Conclusion

We have shown how methods now existing for optimal model reduction of asymptotically stable sytems can be extended to nonasymptotically-stable systems. This extension can be very simply implemented in software now existing, making it useful and convenient for the optimal model reduction of marginally stable and unstable systems.

## References

[1] Genesio, R. and Milanese, M., "A Note on the Derivation and Use of Reduced-Order Models," *IEEE Transactions on Automatic Control,* Vol. AC-21, Feb. 1976, pp. 118-112.

[2] Wilson, D.A., "Optimal Solution of Model Reduction Problem," *Proceedings of the IEEE,* Vol. 17, June 1970, pp. 1161-1165.

[3] Wilson, D.A., "Model Reduction for Multivariable Systems," *International Journal of Control,* Vol. 20 Jan. 1974, pp. 57-64.

[4] Galiana, F.D., "On the Approximation of Multiple Input-Multiple Output Constant Linear Systems," *International Journal of Control,* Vol. 17, 1973, pp. 1313-1324.

[5] Briggs, J.B. and Edgar, T.F., "Least Squares Reduction of Linear Systems Using Impulse Response," *International Journal of Control,* Vol. 20, 1974, pp. 213-223.

[6] Apelvich, J.D., "Approximation of Discrete Linear Systems," *International Journal of Control,* Vol. 17, 1973, pp. 565-575.

[7] Hirzinger, G. and Kreisselmeier, G., "On Optimal Approximation of High Order Linear Systems by Low-Order Models," *International Journal of Control,* Vol. 22, 1975, pp. 399-408.

[8] Wilson, D.A. and Mishra, R.N., "Optimal Reduction of Multivariable Systems," *Internal Journal of Control,* Vol. 29, 1979, pp. 267-278.

[9] Mishra, R.N. and Wilson, D.A., "A New Algorithm for Optimal Reduction of Multivariable Systems," *International Journal of Control,* Vol. 31, 1980, pp. 443-466.

[10] Wilson, D.A., and Mishra, R.N., "Design of Low-Order Estimators Using Reduced Order Models," *International Journal of Control,* Vol. 29, 1979, pp. 447-456.

[11] Guillaume, A.-M., "Réduction d'Ordre des Systémes Linéaires," Master's Thesis, Department of Applied Mathematics, The University of Louvain, Louvain-La-Neuve, Belgium, June 1983.

[12] Kailath, T., *Linear Systems,* Prentice-Hall, Englewood Cliffs, New Jersey, 1980.

[13] Anderson, B.D.O. and Moore, J.B., *Linear Optimal Control,* Prentice-Hall, Englewood Cliffs, New Jersey, 1971.

# Angular Motion Influence on Re-entry Vehicle Ablation or Erosion Asymmetry Formation

D.H. Platus*

*The Aerospace Corporation, El Segundo, California*

## Nomenclature

| | |
|---|---|
| $a$ | $=E/2\Omega^2$; $(\theta^2_{max}+\theta^2_{min})/2$ |
| $b$ | $=(E^2-4K^2\Omega^2)^{1/2}/2\Omega^2$; $(\theta^2_{max}-\theta^2_{min})/2$ |
| $E$ | = constant determined by initial conditions, Eq. (6) |
| $I$ | = pitch or yaw moment of inertia |
| $I_x$ | = roll moment of inertia |
| $K$ | = constant determined by initial conditions, Eq. (5) |
| $p$ | = roll rate |
| $p_r$ | = roll rate parameter, $\mu p/2$ |
| $t$ | = time |
| $\theta$ | = pitch angle (Euler angle) |
| $\theta_{max}$ | = maximum value of $\theta$ during epicyclic oscillation |
| $\theta_{min}$ | = minimum value of $\theta$ during epicyclic oscillation |
| $\dot{\theta}$ | = pitch rate |
| $\mu$ | $=I_x/I$ |
| $\tau$ | = phase angle, Eq. (8) |
| $\phi$ | = roll angle relative to wind (Euler angle) |
| $\Delta\phi_p$ | $=\pi p_r/\Omega$ |
| $\dot{\phi}$ | = windward-meridian rotation rate |
| $\dot{\phi}_{max}$ | = maximum value of $\dot{\phi}$ during epicyclic oscillation |
| $\dot{\phi}_{min}$ | = minimum value of $\dot{\phi}$ during epicyclic oscillation |
| $\dot{\psi}$ | = precession rate |
| $\dot{\Psi}$ | $=\dot{\psi}-p_r$ |
| $\omega$ | = undamped natural pitch frequency |
| $\Omega$ | $=(\omega^2+p_r^2)^{1/2}$ |

## Introduction

A BALLISTIC re-entry vehicle usually enters the atmosphere with some angular misalignment between the vehicle's axis of symmetry and its velocity vector which together, by definition, comprise the entry total angle of attack.[1,2] The angle of attack converges with increasing atmospheric density until it reaches a quasisteady trim value determined by the magnitude of mass and configurational asymmetries. During the period in which the angle of attack converges, the motion is generally epicyclic and is characterized by a highly transient windward-meridian rotation behavior, in contrast to trimmed motion in which both the angle of attack and the windward meridian tend to be quasisteady.[3] It has been postulated that the change in the vehicle's shape as a result of combined ablation and erosion should occur preferentially along surface meridians that spend the longest duration windward, i.e., where the windward-meridian rotation rate is minimum. Such points would be subjected to maximum cumulative pressures and heating and, in the case of erosive environments, to maximum cumulative particle impacts.

The coupling between angle of attack and windward-meridian rotation rate is derived for undamped epicyclic motion. The locus of meridians about the vehicle where the windward-meridian rotation rate is minimum and where incipient shape change would be expected to occur is calculated, and its influence on trim formation is discussed.

## Analysis

The undamped, small-angle equations of missile angular motion in classical Euler coordinates, for only a linear static moment and constant roll rate, are[1,3]

$$\ddot{\theta}+(\omega^2+\mu p\dot{\psi}-\dot{\psi}^2)\theta=0 \tag{1}$$

$$\frac{d}{dt}(\dot{\psi}\theta)+\dot{\theta}\dot{\psi}-\mu p\dot{\theta}=0 \tag{2}$$

$$p=\dot{\phi}+\dot{\psi}=\text{const} \tag{3}$$

in which $\omega^2$, the square of the aerodynamic pitch frequency, has been substituted for the ratio of the static moment derivative to the pitch moment of inertia. In the subsequent analysis, $\omega$ is assumed to be constant.

In terms of $\dot{\Psi}$ and $\Omega$, defined in the Nomenclature, Eqs. (1) and (2) can be written

$$\ddot{\theta}+(\Omega^2-\dot{\Psi}^2)\theta=0 \tag{4}$$

Received July 25, 1983; presented as Paper 83-2111 at the AIAA Atmospheric Flight Mechanics Conference, Gatlinburg, Tenn., Aug. 15-17, 1983; revision received May 15, 1984. 

*Senior Scientist, Aerophysics Laboratory. Associate Fellow, AIAA.