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Optimal fleet utilization and replacement

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Abstract

The fleet replacement problem of a profit-maximizing manager is examined using an optimal control model that captures both utilization and replacement decisions. Conditions for optimal utilization of each vessel in the fleet and optimal vessel acquisition and retirement strategies are discussed. The results indicate that the optimal replacement schedule and fleet size are influenced by utilization schedules, and vice versa. Thus, replacement and utilization strategies should be determined jointly. We develop a numerical example to illustrate the model's potential as a practical management decision tool and the procedures to solve it. © 1999 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Replacement theory deals with the optimal life of capital equipment. “Optimal life” can be defined as the period between the time the equipment enters service and the time when it should be replaced for economic reasons. Optimal life and replacement policy are important topics in the management of capital equipment, and have been studied by many economists (Williamson, 1971; Denslow and Schulze, 1974; Malcomson, 1975, 1979; Evans, 1988; van Hilten, 1991). Examples of equipment to which replacement analysis applies include ships (Evans, 1989), trucks (Ahmed, 1973), farm tractors (Chisholm, 1974; Reid and Bradford, 1983), cars (Smith, 1974), bus engines (Rust, 1987), and textile machines (Williamson, 1971).

Generally, the operating cost of a piece of capital equipment rises as its condition deteriorates over time. When the cost reaches a certain level, the long-run cost associated with investing in a new piece of equipment becomes less than that of keeping the old equipment. At that point, replacement

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is called for. Thus, a basic replacement analysis usually examines both the trend in operating cost and the net cost of replacement, which is defined as the difference between the cost of the new equipment and the salvage value of the old (Li et al., 1982; Rust, 1987; McClelland et al., 1989). In some cases, replacement analysis also considers the resale value of equipment at various stages of its service life (Ahmed, 1973; Chisholm, 1974; Kay and Rister, 1976; Reid and Bradford, 1983).

Several other factors affect the replacement decision. For example, new technology can lower cost or improve efficiency. Thus, to develop correct estimates of future capital and operating costs, a manager should consider changes in equipment design, efficiency, and capital and labor requirements. Replacement policy under conditions of rapid technical change has been examined by Williamson (1971) and Denslow and Schulze (1974). Some replacement problems are affected by stochastic factors, such as equipment failure rate (Devanney, 1971; Hillier and Lieberman, 1974; Rust, 1987) and the stream of revenues generated by a productive asset (Devanney, 1971; Smith and Wetzstein, 1992).

While most previous studies in the economic literature have examined replacement policy for a single piece of capital equipment (for example, see Ahmed, 1973; Chisholm, 1974; Evans, 1989), a small number of papers has addressed the overall replacement strategy for a firm operating multiple pieces of equipment. For example, in the vintage model of capital equipment developed by Malcomson (1975) and extended by van Hilten (1991), all equipment purchased at a given time is subject to the same utilization and retirement schedule.

The replacement decision is also a classical research topic in the industrial engineering and operations research literature. Most studies utilize dynamic programming (DP) techniques to consider a replacement problem in which there is one defender (one piece of equipment), multiple challengers (replacement options), and several cash flow components reflecting factors like technology change and inflation. The planning horizon may be finite (Oakford et al., 1981, 1984) or infinite (Bean et al., 1985).

The classical type of analysis has been extended to include various factors such as salvage value (Bean et al., 1991) and capital rationing (Karabakal et al., 1994). Another extension is the consideration of uncertainty using stochastic DP. For example, Lohmann (1986) examined uncertainty about the cash flows of current and future challengers. Hopp and Nair (1991) developed a decision model in which the time of appearance of future technologies is uncertain. The model was further improved by including a forecast horizon for technology, revenue and cost (Nair and Hopp, 1992).

There are several studies that examine replacement decisions for a group of equipment. Jones et al. (1991) considered the optimal replacement schedule for clusters of like-aged machines, keeping the total number of machines constant over time. Vander Veen and Jordan (1989) developed a model to optimize the investment decision for a number of new machines depending on their utilization (e.g., product types and output levels).

In this paper, we examine the replacement problem for a fleet of ships. (Our treatment can be applied readily to vehicles used in land or air transportation as well.) Generalizing “replacement” as a combination of scrapping and newbuilding, we develop an optimal control model that provides guidance on (a) utilization of individual ships in a fleet, (b) ordering new ships, and (c) scrapping old ships. By exploring several propositions about optimal replacement and utilization schedules, we show that the determination of replacement schedules and of utilization rates are related problems that should be solved jointly. Our model extends the existing literature by con-

sidering a fleet with a variable number of vessels over time, and breaking the replacement decision into two separate parts. The investment decision is only related to the newest vessel in the fleet and its asset value, while the scrapping decision depends on the oldest vessel in the fleet and its asset value. Thus, a new ship is not necessarily acquired to replace an old one. By contrast, in a case of a single machine (or a fixed number of machines), a new one is only purchased to replace the old.

The paper is organized as follows. The model is described in Section 2, along with a discussion of general implications through four propositions. In Section 3, the model is further explored using a numerical example, and Section 4 presents the conclusions.

2. The model

In this section, we describe a dynamic model that optimizes vessel utilization and replacement schedules. The decision maker is the manager of a water carrier such as a shipping line. The manager is to choose an optimal strategy of vessel utilization, newbuilding, and scrapping, so as to maximize the discounted sum of net revenues over the planning period. For simplicity, we consider only the deterministic case. To focus on replacement and utilization, we assume that the fleet is homogeneous and all ships are identical in design and size. Also, our model does not consider the effects of technical changes in ships, management alternatives, ship underutilization due to insufficient demand, and network characteristics (length of voyage, number of ports of call, etc.).¹

The unit operating cost of capital equipment can be modeled in a number of ways. For example, it can be specified as a function of age (Malcomson, 1975), as a function of cumulative usage such as milage (Rust, 1987), or as a function of age, cumulative usage and mechanical condition (Ahmed, 1973). To capture the effects of technical change, operating cost may also be specified as a function of time (Williamson, 1971).² In our model, we consider a fleet of ships, each identified by index variable k with lower bound N_L and upper bound N_H (i.e., the fleet consists of $N_H - N_L$ ships). The operating cost of vessel k is specified as a function of its usage at time t , $q(k, t)$, and its cumulative usage as of t , $G(k, t)$: $c[q(k, t), G(k, t)]$ with

$$\frac{\partial c}{\partial q} > 0, \quad \frac{\partial^2 c}{\partial q^2} > 0, \quad \frac{\partial c}{\partial G} > 0, \quad \frac{\partial^2 c}{\partial q \partial G} > 0. \quad (1)$$

The salvage value of the fleet of ships of the carrier (X) is assumed to be a function of the number of ships sold for scrapping in time period t by the carrier, $s(t)$. Since more ships retiring will depress the scrapping market, we assume

$$\frac{dX}{ds} > 0, \quad \frac{dX^2}{ds^2} < 0. \quad (2)$$

¹ All these aspects may be modeled explicitly by including additional control (decision) variables, such as vessel size. However, this could make the model very complex and impossible to solve analytically. In practice, when several control variables are involved, researchers often use discrete models and DP techniques to simulate specific management alternatives.

² Frankel (1991) discussed the economics of technological change in shipping. For a discussion about long-term technological change in transportation, see Nakicenovic (1988).

Similarly, the cost of purchasing new ships (W) is a function of the number of ships on order, $n(t)$. Since newbuilding price will rise with an increase in demand, we assume

$$\frac{dW}{dn} > 0, \quad \frac{d^2W}{dn^2} > 0. \quad (3)$$

The dynamic model encompasses the planning period from 0 to T , the number of vessels in the fleet ($N_H - N_L$), and the cumulative usage of each vessel. Thus, total net revenues are estimated by integrating over all vessels in the fleet and by integrating revenues and costs over time. Using the concept developed by Livernois and Uhler (1987) in their resource management model, we formulate the problem as follows. The fleet manager is to

$$\max \int_0^T \left\{ \int_{N_L(t)}^{N_H(t)} \{p(t)q(k, t) - c[q(k, t), G(k, t)]\} dk + X[s(t)] - W[n(t)] \right\} e^{-\delta t} dt. \quad (4)$$

Subject to

$$\dot{G}(k, t) = q(k, t), \quad k \in (N_L(t), N_H(t)), \quad (5)$$

$$\dot{N}_L(t) = s(t), \quad (6)$$

$$\dot{N}_H(t) = n(t), \quad (7)$$

where t is time; k is the indexing variable; $p(t)$ is the freight rate in dollars per ton-mile at t ; $q(k, t)$ the usage (output) of the k th vessel in ton-miles at t ; $G(k, t)$ the cumulative usage of the k th vessel at t ; $c[q, G]$ the operating cost of the k th vessel at t ; $X[s(t)]$ the total salvage value of vessels retired at t ; $s(t)$ the number of vessels retired at t ; $W[n(t)]$ the total investment in newbuildings at t ; $n(t)$ the number of new ships purchased at t ; $N_L(t)$ the lower bound on k at t ; $N_H(t)$ the upper bound on k at t ; $e^{-\delta t}$ the continuous-time discount factor; and δ is the discount rate.

We assume the shipping market to be competitive, and thus $p(t)$ is exogenously determined. The revenue generated by the k th vessel is captured by $p(t)q(k, t)$, while the operating cost of the ship is represented by $c[q(k, t), G(k, t)]$. As noted, the total net revenue from fleet operation is the sum of the net revenues of individual ships. At any time t , $N_L \leq k \leq N_H$, and thus, the actual fleet size is given by $N_H - N_L$.

For each t , the manager determines the utilization of each individual ship, as well as the replacement decisions. The total value recovered from ship retirement at t is X , while the total investment in new ships at t is W .

For each ship, the relationship between current usage and cumulative usage is described by (5). As ships retire from the fleet, the lower bound on the vessel indexing variable (k) rises as defined in (6). Eq. (7) shows the impact of newbuildings on the upper bound on k : as newbuildings are added to the fleet, the upper bound rises.

Our specification of operating cost as a function of current usage³ (q) and its cumulative usage (G) is a simplification. In fact, operating cost may increase with the age of the ship regardless of cumulative usage, such as when a ship has been laid-up. Also, the model does not consider the

³ For a discussion of marginal cost in ship operation, see Evans (1988).

second-hand vessel market, the possibility of a surplus in the shipping market, or the option of acquiring ships through chartering. If there is a surplus of vessels in the market, the cost of acquiring vessel services may be much lower than the cost of newbuildings.

The system has three control variables: $q(k)$, s and n , and three state variables: $G(k)$, N_L and N_H . The current value Hamiltonian is

$$\begin{aligned}
 H = & \int_{N_L(t)}^{N_H(t)} \{p(t)q(k, t) - c(k, t)\}dk + X[s(t)] - W[n(t)] \\
 & + \int_{N_L(t)}^{N_H(t)} \alpha(k, t)q(k, t)dk + \beta(t)s(t) + \gamma(t)n(t)
 \end{aligned} \tag{8}$$

with

$$c(k, t) = c[q(k, t), G(k, t)]. \tag{9}$$

The optimality condition requires ⁴

$$\frac{\partial H}{\partial q(k, t)} = 0, \quad \frac{\partial H}{\partial s(t)} = 0, \quad \frac{\partial H}{\partial n(t)} = 0. \tag{10}$$

These lead to

$$\alpha(k, t) = \frac{\partial c(k, t)}{\partial q(k, t)} - p(t), \quad k \in (N_L(t), N_H(t)), \tag{11}$$

$$\beta(t) = -\frac{dX[s(t)]}{ds(t)}, \tag{12}$$

$$\gamma(t) = \frac{dW[n(t)]}{dn(t)}. \tag{13}$$

The costate (adjoint) equations are

$$\dot{\alpha}(k, t) - \delta\alpha(k, t) = -\frac{\partial H}{\partial G(k, t)}, \quad \dot{\beta}(t) - \delta\beta(t) = -\frac{\partial H}{\partial N_L(t)}, \quad \dot{\gamma}(t) - \delta\gamma(t) = -\frac{\partial H}{\partial N_H(t)}. \tag{14}$$

These lead to

$$\dot{\alpha}(k, t) - \delta\alpha(k, t) - \frac{\partial c(k, t)}{\partial G(k, t)} = 0, \quad k \in (N_L(t), N_H(t)), \tag{15}$$

$$\dot{\beta}(t) - \delta\beta(t) - p(t)q(N_L, t) + c(N_L, t) - \alpha(N_L, t)q(N_L, t) = 0, \tag{16}$$

$$\dot{\gamma}(t) - \delta\gamma(t) + p(t)q(N_H, t) - c(N_H, t) + \alpha(N_H, t)q(N_H, t) = 0. \tag{17}$$

⁴ For an introduction to optimal control theory, see Kamien and Schwartz (1981) and Conrad and Clark (1987).

The change in the utilization rate of vessel $k(q(k, t))$ with respect to time can be solved from (11) and (15). To simplify the notation, we drop k and t from the expression

$$\dot{q} = \left(\dot{p} - \delta \left(p - \frac{\partial c}{\partial q} \right) - \frac{\partial^2 c}{\partial q \partial G} q + \frac{\partial c}{\partial G} \right) / \left(\frac{\partial^2 c}{\partial q^2} \right), \quad k \in (N_L(t), N_H(t)) \quad (18)$$

Eq. (18) leads to the first of four propositions regarding optimal utilization and replacement:

Proposition 1. *The utilization rate of each vessel in the fleet, $q(k, t)$, will decline over time, unless the freight rate is expected to increase substantially over time and/or the impact of cumulative usage on the operating cost of the vessel is very large.*

Proof. From (1), we have $\partial c / \partial q > 0$, $\partial^2 c / \partial q^2 > 0$, $\partial c / \partial G > 0$, and $\partial^2 c / \partial q \partial G > 0$. Thus, the denominator of (18) is positive. There are four terms in the numerator of (18). For profitable operation, $p > \partial c / \partial q$. Thus, the second $(-\delta(p - \partial c / \partial q))$ and the third $(-q \partial^2 c / \partial q \partial G)$ terms are negative. p -dot can be either positive or negative, depending on the trend in expected future prices. The last term $(\partial c / \partial G)$ is positive. $\partial c / \partial G$ is usually relatively small. If p -dot is zero or negative, q -dot is negative. However, if p -dot is positive and relatively large and/or $\partial c / \partial G$ is relatively large, q -dot may be positive. \square

As noted, p is greater than $\partial c / \partial q$ in (11), and therefore $\partial c / \partial q - p$ is negative. We define $\alpha = \partial c / \partial q - p$ as the “usage cost,” which reflects the influence of current usage (q) on the change in the state variable (G). As an increase in q today leads to an increase in its cumulative usage (G) and all future operating costs over the remainder of the planning period, α reflects an intertemporal cost. The usage cost can also be considered as “marginal losses” associated with increased operating costs over the remaining future, resulting from undertaking an incremental action in q today. The manager must balance current usage (q) and future usage (affected by G), for each ship.

Proposition 2. *The utilization rates for the fleet, $q(k, t)$, should be chosen so that the sum of marginal operating cost and usage cost is the same for all vessels.*

Proof. From (11) we have

$$\frac{\partial c(i, t)}{\partial q(i, t)} - \alpha(i, t) = \frac{\partial c(j, t)}{\partial q(j, t)} - \alpha(j, t) \quad (19)$$

for all $i \neq j \in (N_L(t), N_H(t))$. \square

Propositions 1 and 2 imply that since older ships are more costly to operate, fleet operation can be optimized by using new ships more and old ships less.

Eqs. (13) and (17) govern the investment in new ships. The costate variable γ represents the marginal benefit of adding ships to the fleet. Thus, Eq. (13) shows that newbuildings will be ordered when the marginal benefit is equal to the marginal investment cost (dW/dn). From (3), $dW/dn > 0$, so $\gamma > 0$. Eq. (17) indicates that at any t , the marginal benefit of adding ships to the fleet (γ) is influenced only by the net benefit of the newest vessel (N_H) in the fleet.

The change in newbuilding rate with respect to time (\dot{n}) can be obtained from (11), (13) and (17):

$$\dot{n} = \left(\delta \frac{dW}{dn} + c(N_H) - \frac{\partial c(N_H)}{\partial q(N_H)} q(N_H) \right) / \left(\frac{d^2W}{dn^2} \right). \quad (20)$$

Proposition 3. *The addition of new ships is influenced by the operating cost of the newest vessel (N_H) in the fleet. Newbuilding will increase over time when the marginal operating cost of vessel N_H ($\partial c/\partial q$) is smaller than its average cost (c/q). Newbuilding will decrease over time when the marginal operating cost of vessel N_H is much greater than its average cost ($q\partial c/\partial q > \delta dW/dn + c$).*

Proof. From (3), $d^2W/dn^2 > 0$. The denominator of (20) is positive. There are three terms in the numerator of (20). The first two terms ($\delta dW/dn$ and c) are positive, and the last term ($-q\partial c/\partial q$) is negative. \dot{n} is positive if $c/q > \partial c/\partial q$ ($c > q\partial c/\partial q$). When $c/q < \partial c/\partial q$, \dot{n} may still be positive, if $\delta dW/dn + c > q\partial c/\partial q$. However, if $\partial c/\partial q$ is much greater than c/q , $\delta dW/dn + c < q\partial c/\partial q$, then \dot{n} is negative. \square

Finally, Eqs. (12) and (16) govern the scrapping of old ships. Since dX/ds is the marginal benefit from ships sold for scrapping, Eq. (12) suggests that β is the marginal cost of taking ships out of the fleet. Eq. (16) shows that the marginal cost of taking ships out of the fleet (β) is determined only by the net benefit of the oldest ship (N_L) in the fleet. Eq. (12) states that ships should retire when their marginal salvage value is equal to the marginal cost of retiring these ships. From (2), $dX/ds > 0$. Thus, $\beta < 0$.

The change in scrapping rate with respect to time (\dot{s}) can be solved using (11), (12) and (16):

$$\dot{s} = \left(\delta \frac{dX}{ds} + c(N_L) - \frac{\partial c(N_L)}{\partial q(N_L)} q(N_L) \right) / \left(\frac{d^2X}{ds^2} \right). \quad (21)$$

Proposition 4. *The rate of scrapping is influenced by the operating cost of the oldest vessel (N_L) in the fleet. Scrapping will decrease over time when the average cost of vessel N_L (c/q) is greater than its marginal cost ($\partial c/\partial q$). Scrapping will increase over time when the marginal cost of vessel N_L is much greater than its average cost ($q\partial c/\partial q > \delta dX/ds + c$).*

Proof. From (2), $d^2X/ds^2 < 0$, so the denominator of (21) is negative. Among the three terms in the numerator of (21), the first ($\delta dX/ds$) and the second term (c) are positive, while the third ($-q\partial c/\partial q$) is negative. \dot{s} is negative if $c/q > \partial c/\partial q$ ($c > q\partial c/\partial q$). \dot{s} is positive when $\partial c/\partial q$ is much greater than c/q , so that $q\partial c/\partial q > \delta dX/ds + c$. \square

Our model provides an analytical framework in which the fleet manager determines (a) fleet utilization strategy, (b) vessel retirement rate, (c) newbuilding rate, and as a result, (d) fleet size. The model explicitly shows that replacement decisions (the rates of building and scrapping ships) are affected by vessel utilization rates, and that the replacement and utilization problems must be considered jointly to obtain an efficient solution. The size of the fleet is determined by the resulting scrapping and newbuilding decisions.

3. A numerical example

As noted, the model integrates three aspects of fleet management: vessel utilization, new-building, and scrapping. Although the continuous-time model is useful in discussing the general trends (rules of thumb) of utilization and replacement, the model is not generally solvable for analytical solutions (e.g., the path of $q(t), s(t)$ and $n(t)$). Thus, it is not useful as a practical management decision tool. In this section, we develop a numerical example using a set of specific functional forms for operating cost (c), investment cost (W), and salvage value (X) to illustrate optimal strategies for utilization and replacement, and the linkages among them.

To develop a numerical example, we first present the discrete-time⁵ version of the model in Eqs. (4)–(7). The fleet manager is to

$$\max \sum_{t=0}^T \rho^t \left\{ \sum_{k=N_L(t)}^{N_H(t)} \{p(t)q(k, t) - c[q(k, t), G(k, t)]\} + X[s(t)] - W[n(t)] \right\} \quad (22)$$

subject to

$$G(k, t+1) - G(k, t) = q(k, t), \quad k \in (N_L(t), N_H(t)), \quad (23)$$

$$N_L(t+1) - N_L(t) = s(t), \quad (24)$$

$$N_H(t+1) - N_H(t) = n(t), \quad (25)$$

where ρ is the discrete-time discount factor ($\rho = 1/(1 + \delta)$).

The discrete-time current value Hamiltonian is

$$H = \sum_{k=N_L(t)}^{N_H(t)} \{p(t)q(k, t) - c(k, t)\} + X[s(t)] - W[n(t)] + \rho \sum_{k=N_L(t)}^{N_H(t)} \alpha(k, t+1)q(k, t) + \rho\beta(t+1)s(t) + \rho\gamma(t+1)n(t) \quad (26)$$

with

$$c(k, t) = c[q(k, t), G(k, t)]. \quad (27)$$

The optimality condition requires

$$\frac{\partial H}{\partial q(k, t)} = 0, \quad \frac{\partial H}{\partial s(t)} = 0, \quad \frac{\partial H}{\partial n(t)} = 0. \quad (28)$$

Eqs. (11)–(13) become

$$\rho\alpha(k, t+1) = \frac{\partial c(k, t)}{\partial q(k, t)} - p(t), \quad k \in (N_L(t), N_H(t)) \quad (29)$$

$$\rho\beta(t+1) = -\frac{dX[s(t)]}{ds(t)}, \quad (30)$$

⁵ For a general discussion of discrete dynamic modeling, see Sandefur (1993).

$$\rho\gamma(t+1) = \frac{dW[n(t)]}{dn(t)}. \quad (31)$$

The costate equations are

$$\begin{aligned} \rho\alpha(k, t+1) - \alpha(k, t) &= -\frac{\partial H}{\partial G(k, t)}, & \rho\beta(t+1) - \beta(t) &= -\frac{\partial H}{\partial N_L(t)}, \\ \rho\gamma(t+1) - \gamma(t) &= -\frac{\partial H}{\partial N_H(t)}. \end{aligned} \quad (32)$$

Eqs. (15)–(17) become

$$\rho\alpha(k, t+1) - \alpha(k, t) - \frac{\partial c(k, t)}{\partial G(k, t)} = 0, \quad k \in (N_L(t), N_H(t)), \quad (33)$$

$$\rho\beta(t+1) - \beta(t) - p(t)q(N_L, t) + c(N_L, t) - \rho\alpha(N_L, t+1)q(N_L, t) = 0, \quad (34)$$

$$\rho\gamma(t+1) - \gamma(t) + p(t)q(N_H, t) - c(N_H, t) + \rho\alpha(N_H, t+1)q(N_H, t) = 0. \quad (35)$$

Now, the discrete model can be used to generate numerical results if specific functional forms are defined for c , W and X . In this example, we solve a simple case of this model with additional assumptions in three steps. First, we examine the optimal utilization schedule for individual vessels. We then solve the optimal schedule for new ship acquired, given the utilization schedule. Finally, we analyze the optimal schedule for scrapping old vessels, given the optimized utilization and purchasing schedules. Unfortunately, as we will show, a more general joint solution for utilization and replacement is difficult to achieve even in this simple case.⁶

3.1. Utilization

Whether the optimal utilization schedule for each vessel k can be solved depends on the cost structure and terminal conditions. Here, we use a simple example⁷ which is easy to solve. We specify cost function $c(k, t)$ as

$$c(k, t) = \frac{q^2(k, t)}{G(k, t)}, \quad (36)$$

where G is defined differently from the above to obtain easy numerical solutions. Suppose that each new vessel has a designed life in terms of cumulative usage (e.g., ton-miles). When life-time cumulative usage is reached, the vessel's life ends (this is analogous to a resource stock that has been exhausted). Here, $G(t)$ is the *unused* part of the life-time cumulative usage at time t , which equals the life-time cumulative usage ($G(0)$) minus the cumulative usage up to $t-1$ ($\sum_{i=0}^{t-1} q(t)$). Thus, G decreases over time, contrary to the previous definition. Considering the new notion of G , the cost function specified in (36) satisfies the properties of c described in (1).

⁶ Our three step approach assumes that all vessels follow a specific utilization schedule which is independent of replacement schedules.

⁷ The example is from Conrad and Clark (1987).

Substituting (36) into (29) and (33), we have

$$-\rho\alpha(k, t+1) = \frac{2q(k, t)}{G(k, t)} - p, \quad k \in (N_L(t), N_H(t)), \quad (37)$$

$$\rho\alpha(k, t+1) - \alpha(k, t) + \frac{q^2(k, t)}{G^2(k, t)} = 0, \quad k \in (N_L(t), N_H(t)). \quad (38)$$

The negative sign on the left-hand side in (37) is due to the new notion of G , as (23) becomes

$$G(k, t+1) - G(k, t) = -q(k, t). \quad (39)$$

Also, we assume that price is a constant.

For vessel k , $q(k, t)$ is defined only during the life of k . As shown in Fig. 1, the upper bound of the indexing variable $k = N_H(t)$ describes the last new ship added to the fleet as of t . Similarly, the lower bound $k = N_L(t)$ describes the last ship retired from the fleet as of t . We define the inverse functions as $t = H(k) = N_H^{-1}(k)$ for the upper bound (the time when vessel k enters the fleet) and $t = L(k) = N_L^{-1}(k)$ for the lower bound (the time when vessel k is scrapped). Then, for any k , $q(t)$ is defined for $H(k) \leq t \leq L(k)$, i.e. from the time vessel k enters the fleet to its retirement (see Fig. 1).

Since k is the vessel index, for a given vessel k is fixed. t is the only variable affecting the utilization dynamics of vessel k . Since it is assumed that all vessels are identical, we can ignore the vessel index k . t is defined between 0 and T (the planning horizon). $q(t)$ is defined between $t = H(k)$ and $t = L(k)$ for vessel k . Thus, the general expressions $q(k, t)$ and $G(k, t)$ can be rewritten as $q(\tau)$ and $G(\tau)$, with $\tau = t - H(k) > 0$. τ is the time variable which starts counting when a vessel enters the fleet. In other words, for each vessel k , the utilization dynamics follow $q(t)$, but the variable t starts at $H(k)$ (when k enters the fleet).

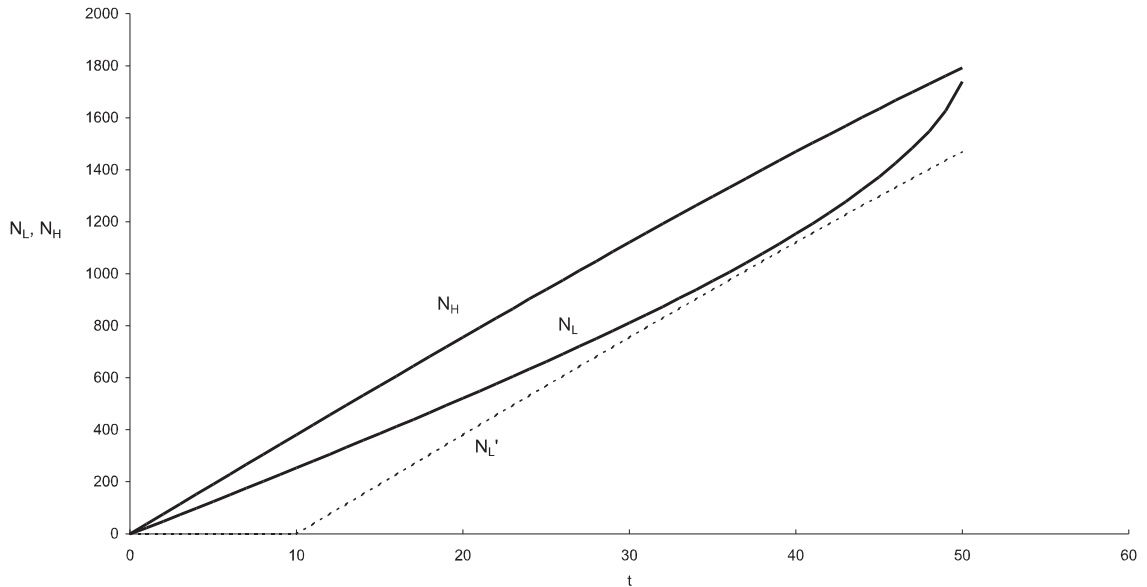


Fig. 1. Vessel acquisition and retirement schedule.

Table 1
Parameter values

Parameter	Description	Value
δ	Discount rate	0.05
f	Maximum vessel life	10
p	Freight rate	1
$G(0)$	Initial condition for G	1000
$\alpha(f)$	Terminal condition for α	0
T	Planning horizon	50
μ	New vessel cost coefficient	10
$N_H(0)$	Initial condition for N_H	0
$\gamma(T)$	Terminal condition for γ	$v(0)$
φ	Vessel salvage value coefficient	200,000
θ	Vessel salvage value coefficient	200,000
$\beta(T)$	Terminal condition for β	$-v(f-1)$
$N_L(0)$	Initial condition for N_L	0

To solve for $q(\tau)$, we need initial and terminal conditions. The conditions describe α , q and G when a ship k enters or leaves the fleet. For simplicity, we assume that besides the life-time cumulative usage, we also have a life (f) in time periods. For example, a vessel must retire after f periods due to regulatory requirements. Then, this becomes a fixed-time free state problem: we need to solve for the optimal utilization for $\tau = 0, 1, 2, \dots, f$. Clearly, any units of G remaining in f must be worthless.⁸ Thus, $\alpha(f) = 0$. In our numerical example, $f = 10$. Also, the life-time cumulative usage $G(0) = 1000$. All input parameter values for our numerical example are summarized in Table 1.

Now $q(\tau)$ can be solved using the terminal conditions and Eqs. (37)–(39). Specifically, we first solve for α and the ratio q/G using (37) and (38) from $\tau = 10$ to $\tau = 0$; and then solve q and G from $\tau = 0$ to $\tau = 10$. The results are shown in Fig. 2 and Table 2. As shown in Fig. 2, with a constant freight rate, utilization rate (q) declines throughout the life of a vessel, as suggested by Proposition 1 in the previous section. Table 2 also includes the asset value (v), the discounted sum of net revenues from all future periods, and net revenue (π) for each period.

We emphasize that fixed vessel life (f) and identical utilization schedule for all vessels are important assumptions for solving the entire model. If f varies, the optimal path $q(\tau)$ changes accordingly. The system will be difficult to solve, since $H(k)$ and $L(k)$ are affected by investment and scrapping decisions, and $q(k, t)$ is determined jointly by the initial/terminal conditions associated with these replacement decisions. With $f = 10$, we have a unique optimal path for q , which makes the new building and retirement schedules solvable.

3.2. Investment in new vessels

For investment in new ships, we specify

$$W[n(t)] = \mu n^2(t), \quad (40)$$

⁸ α is now positive due to the new definition of G .

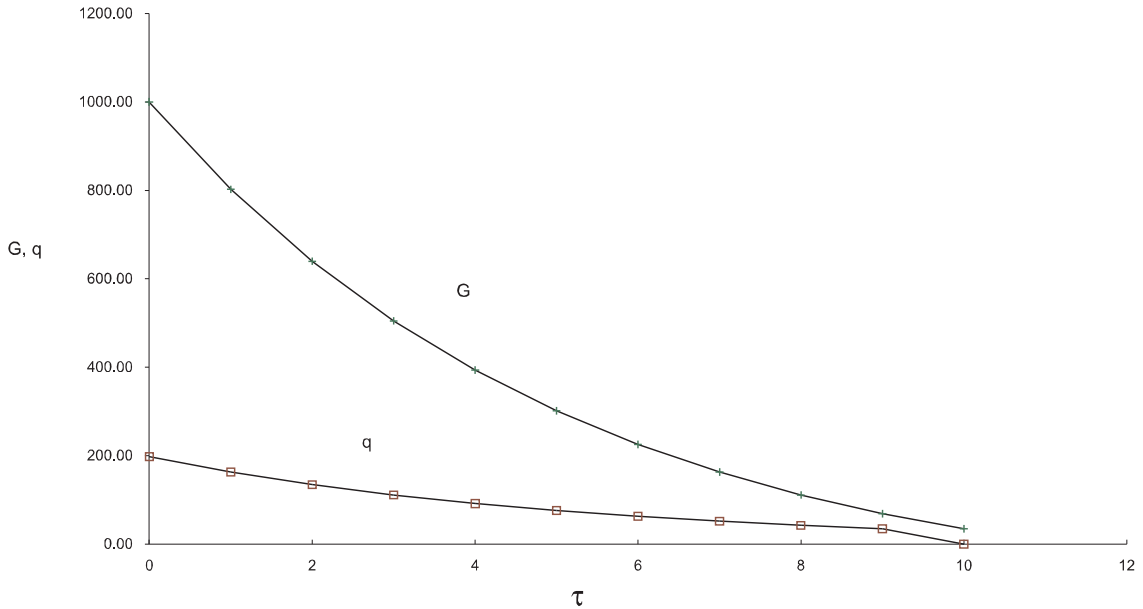


Fig. 2. Utilization schedule.

Table 2
Utilization schedule

τ	$G(\tau)$	$q(\tau)$	$\alpha(\tau)$	$v(\tau)$	$\pi(\tau)$
0	1000.00	197.64	0.64	643.79	158.58
1	802.36	163.00	0.64	509.47	129.89
2	639.36	134.56	0.62	398.57	106.24
3	504.81	111.17	0.61	306.94	86.69
4	393.63	91.91	0.59	231.27	70.45
5	301.72	76.00	0.56	168.86	56.86
6	225.71	62.79	0.52	117.60	45.32
7	162.92	51.73	0.47	75.89	35.30
8	111.19	42.36	0.38	42.61	26.22
9	68.83	34.42	0.25	17.21	17.21
10	34.42	0.00	0.00	0.00	0.00

where μ is a positive constant. With this specification, we have $W > 0, dW/dn > 0$, and $d^2W/dn^2 > 0$ (see (3)). Substituting (40) and (36) into (31) and (35), and also considering (39), we have

$$\rho\gamma(t + 1) = 2\mu n(t), \tag{41}$$

$$\rho\gamma(t + 1) - \gamma(t) + p(t)q(N_H, t) - \frac{q^2(N_H, t)}{G(N_H, t)} - \rho\alpha(N_H, t + 1)q(N_H, t) = 0. \tag{42}$$

The number of new ships purchased at $t, n(t)$, is affected by the utilization rate of the newest ship and its asset value. On the path of $N_H(t)$ (see Fig. 1), $t = H(k)$ and $\tau = 0$. Thus,

$q(N_H, t) = q(\tau = 0)$, $G(N_H, t) = G(\tau = 0)$, and $\alpha(N_H, t + 1) = \alpha(\tau = 1)$, which have been solved in the previous subsection (Table 2).

To solve for $n(t)$, we also need to define initial and terminal conditions. For our numerical example, we assume a planning horizon (T) of 50 time periods ($t = 0, 1, 2, \dots, T = 50$), $N_H(0) = 0$, and $\gamma(50)$ equals the asset value of a new vessel ($v(0) = 643.8$, see Table 2).⁹ Under these assumptions, we actually optimize replacement during the 50 periods and assume that the operation will continue for another 10 periods after T (hence $\gamma(50) = v(0)$). Now, $n(t)$ and N_H can be solved using (25), (41) and (42) and similar procedures as in the utilization case.

For our cost function (36), marginal cost ($2q/G$) is always greater than average cost (q/G) and $q\partial c/\partial q > \delta dW/dn + c$. Thus, the newbuilding rate ($n(t)$) is decreasing over time from $n(0) = 38$ to $n(49) = 30$, which is consistent with Proposition 3. The path of N_H is depicted in Fig. 1.

3.3. Scrapping of old vessels

To examine the vessel scrapping schedule, we specify

$$X[s(t)] = \varphi - \frac{\theta}{s(t)}, \tag{43}$$

where φ and θ are positive constants ($\varphi \geq \theta$). This specification satisfies that $X > 0$, $dX/ds > 0$, and $d^2X/ds^2 < 0$ (see (2)). As noted, this is a simplified case since the salvage value of old vessels is not related to their usage condition (G). If the salvage value (X) is affected by G as well, the utilization schedule (q) will not be unique and will be interrelated with the scrapping schedule.

Substituting (43) and (36) into (30) and (34), and again considering (39), we have

$$\rho\beta(t + 1) = -\frac{\theta}{s^2(t)}, \tag{44}$$

$$\rho\beta(t + 1) - \beta(t) - p(t)q(N_L, t) + \frac{q^2(N_L, t)}{G(N_L, t)} + \rho\alpha(N_L, t + 1)q(N_L, t) = 0, \tag{45}$$

$s(t)$ is affected by the utilization rate of the oldest ship, $q(N_L)$, and its asset value. Unlike the investment case, $q(N_L)$ varies in each t , depending on the scrapping rate $s(t)$. Thus, a procedure is required to determine the relationship between $s(t)$ and $q(N_L)$.

We develop the following procedure to determine $q(N_L)$ at t . Suppose that we have no market for ships of 10 periods old, each vessel exits when it is 10 periods old, as shown by N'_L (the dashed line) in Fig. 1. Then, fleet size (g) is

$$g(t + 1) = \sum_{i=0}^{f-1} n(t - i). \tag{46}$$

Note that in our numerical example, $f = 10$, that is the fleet consists of vessels of age 0–9. Thus, we know the age structure of the fleet. Furthermore, in our example, the utilization path $q(0)$ – $q(9)$

⁹ For a discussion of this relationship see Livernois and Uhler (1984).

is determined (see Table 2 and Fig. 2), thus $q(N_L, t)$, $G(N_L, t)$, $\alpha(N_L, t + 1)$ can be determined by $s(t)$ and the age structure.

We construct a function

$$m(t, f - i) = \sum_{j=f-i}^{f-1} n(t - j) \quad \text{for } i = 1, \dots, f. \quad (47)$$

where m is the number of vessels in the fleet purchased more than $f - i$ periods ago. For example, for $f = 10$ and $i = 2$, $m(t, 8) = n(t - 9) + n(t - 8)$; that is $m(t, 8)$ is the sum of the number of vessels purchased eight periods ago and nine periods ago. Since each vessel follows the optimal utilization path in Fig. 2, the corresponding q and G are known.

Thus, for

$$m(t, f - j + 1) < s(t) \leq m(t, f - j) \quad \text{for } j = 1, \dots, f, \quad (48)$$

we have

$$q(N_L, t) = q(f - j), \quad G(N_L, t) = G(f - j), \quad \alpha(N_L, t + 1) = \alpha(f - j + 1). \quad (49)$$

For example, for $j = 2$, $m(t, 9) < s(t) \leq m(t, 8)$, and $q(N_L, t) = q(8)$, $G(N_L, t) = G(8)$, and $\alpha(N_L, t + 1) = \alpha(9)$. That is: if s is greater than the number of vessels acquired nine periods ago but smaller than (or equal to) the sum of the number of vessels acquired nine periods and eight periods ago, $q(t)$ and $G(t)$ equal their value at period eight and $\alpha(t + 1)$ equals its value at period nine (all in Table 2).

Again, to solve for $s(t)$, we need to define initial and terminal conditions. For our numerical example, we assume that $N_L(0) = 0$, and $\gamma(50) = v(9) = 17.2$ (see Table 2). Now, $s(t)$ and N_L can be solved using (24), (44) and (45) and similar procedures as in the newbuilding case.

As noted, for our cost function (36), marginal cost ($2q/G$) is always greater than average cost (q/G) and $q\partial c/\partial q > \delta dX/ds + c$. Thus, the scrapping rate ($s(t)$) is increasing over time from $s(0) = 24$ to $s(49) = 110$, as suggested by Proposition 4. The path of N_L is also depicted in Fig. 1.

An interesting result from this model is that, if the salvage value (X) is influenced by the number of vessels scrapped, vessels acquired together will not necessarily all be replaced at the same time. Jones et al. (1991) developed a no-splitting rule for parallel machine replacement where the total number of machines remains constant over time. The rule suggests that machines of the same age are either all kept or all replaced. In our example, the no-splitting rule does not hold in general.

As noted, in our example, the model is simplified in that utilization schedule is first solved independently. However, if the utilization rates affect the newbuilding and scrapping schedules, and vice versa, the model will be more difficult to solve. In practice, those cases may be solved numerically using iterative procedures (Malcomson, 1979).

4. Conclusions

We have developed a model of fleet management that optimizes both utilization and replacement schedules. Under the assumption of fixed output, the replacement decision is simply a question of whether the operating cost of an existing vessel is greater than the annualized total cost of a replacement vessel (Evans, 1989; see also Malcomson, 1979). More generally, our fleet

operation and replacement model suggests how replacement decisions are interrelated with vessel operating strategies. The economic efficiency of a fleet can be improved by optimizing its vessel utilization plan. New ships will be acquired only if such action improves the economic efficiency of the fleet. Under certain conditions, it may be optimal to scrap a ship without replacing it. We developed a simple numerical example to illustrate how this model may be solved, and its potential as a practical management decision tool.

As noted, our model extends the existing literature in that it considers a fleet with varying number of vessels over time, where the replacement decision consists of two separate parts: the investment decision is only related to the newest vessel in the fleet and its asset value, while the scrapping decision depends on the oldest vessel in the fleet and its asset value. Thus, a new ship is not necessarily acquired to replace an old one. By contrast, in the case of a single machine (or a fixed number of machines), a new one is only (and always) purchased to replace the old.

In summary, our model illustrates several important points:

(a) Fleet replacement and operation are joint decisions. To develop an efficient replacement schedule, a fleet manager must first optimize fleet utilization. In general, this implies high utilization rates for new ships and (relatively) low utilization of old ships.

(b) Replacement strategy is considered only after fleet utilization has been optimized. The economic efficiency of the fleet can be improved by adding new vessels when the marginal benefit of adding ships is greater than the marginal cost of these newbuildings.

(c) Optimal replacement does not necessarily imply maintaining a constant fleet size. Depending on the vessel cost function, newbuilding and scrapping rates may increase or decrease over time. A vessel is retired only if its salvage value is greater than or equal to the net benefit it can generate in the fleet.

The analytical framework described in this paper provides a solid basis for the development of practical decision models. Such models should explicitly consider technological and management alternatives, and different sets of financial and physical constraints.¹⁰ In its present formulation, the model does not consider the effects of technical change or of management alternatives, such as chartering, on utilization and replacement decisions. It also ignores the uncertainty associated with under utilization of an asset due to insufficient demand, exogenous cost escalation, equipment failure, and asset loss or unavailability. These factors are leading candidates for future extensions of the model.

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¹⁰ For a discussion of various issues related to shipping fleet utilization and replacement, see Devanney (1971) and Frankel and Marcus (1973). Examples of fleet-operation optimization models can be found in Perakis and Papadakis (1987) and Perakis and Bremer (1992).

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Appendix A

Glossary of symbols

T	planning horizon.
t	time between 0 and T .
k	vessel indexing variable.
$p(t)$	freight in dollars per ton-mile at t .
$q(k, t)$	usage (output) of the k th vessel in ton-miles at t (1st control variable).
$G(k, t)$	cumulative usage of the k th vessel at t (G is the unused ton-mile capacity in the numerical example) (1st state variable).
$c[q, G]$	operating cost of the k th vessel at t .
$s(t)$	number of vessels sold for scrapping by the carrier at t (2nd control variable).
$X(s)$	salvage value of vessels retired from the fleet at t .
$n(t)$	number of new vessels acquired at t (3rd control variable).
$W(n)$	total investment in newbuildings by the carrier at t .
$N_L(t)$	lower bound on k (the oldest vessel in the fleet) at t (2nd state variable).
$N_H(t)$	upper bound on k (the newest vessel in the fleet) at t (3rd state variable).
$e^{-\delta t}$	continuous-time discount factor.
δ	discount rate.
α	1st costate variable.
β	2nd costate variable.
γ	3rd costate variable.
ρ	discrete-time discount factor.
$L(k)$	time when vessel k exits the fleet (the inverse function of $N_L(k)$).
$H(k)$	time when vessel k enters the fleet (the inverse function of $N_H(k)$).
f	vessel life.
τ	relative time for each vessel (between 0 and f).
μ	new vessel cost coefficient.
φ	vessel salvage value coefficient.
θ	vessel salvage value coefficient.
$g(t)$	fleet size without scrapping at t .
$m(t, f - i)$	number of vessels in the fleet purchased more than $f - i$ periods ago ($i = 1, \dots, f$) at t .

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