Optimal Harvesting of Structured Populations*†

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ABSTRACT

A general harvesting model is presented that allows the dynamics of the population to be divided into two distinct phases, viz, the harvesting-season dynamics, modeled by an n-dimensional system of ordinary nonlinear differential equations, and the spawning season, modeled by n difference equations. A maximum principle for this type of system is presented. The concept of "maximum sustainable yield" for periodic forms of such systems is introduced and discussed. The model is then simplified to exhibit linear-bilinear dynamics and in this form is shown to be a natural extension of the Beverton-Holt model in fisheries management. A method for deriving maximum-sustainable-yield solutions is presented, using this formulation and its corresponding maximum principle. By considering the solution in the limit as the control constraint set $[\mathbf{0},\mathbf{b}]$ becomes unbounded above, the concept of the "ultimate" sustainable yield is introduced. Finally, models including only scalar harvesting are introduced as being of practical value. The question is explored as to how maximum-sustainable-yield solutions are to be constructed from the maximum principle.

1. INTRODUCTION

While the need for sound management policies in the exploitation of renewable resource stocks is self-evident, the development of such policies by the use of analytical techniques (mathematical modeling and systems theory) presents many problems. It is impossible to take into account all factors that influence the spatial and temporal dynamics of a biological population integrated into its natural environment. We can hope to estimate only certain aspects of the behavior of such populations: those aspects which are subjectively deemed to be the most essential in the formulation of sound management policies.

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In this paper the problem considered is that of optimally harvesting a structured population. The application of results to problems in fisheries management is emphasized. In particular, consideration is given to the problem of estimating maximum sustainable economic returns and biomass yields for certain types of fisheries, and finding the corresponding harvesting strategies.

The fundamental work on quantitative fisheries management was done mainly by Beverton and Holt [1], Ricker [2, 3] and Schaefer [4, 5]. Clark [6] provides a recent comprehensive account of the field, while subsequent results may be found in Goh [7] and Silvert [8].

Various approaches have been taken in analyzing the problem of optimally exploiting a renewable resource. In one of these approaches it is assumed that the biomass density of the population [denoted by $x(t)$] satisfies a nonlinear ordinary differential equation with forcing terms relating to the harvesting rate [$u(t)$ will be used to denote harvesting intensity]. The well-known "Schaefer model" is of this type: a logistic differential equation plus a harvesting term, bilinear in x and u. Maximum-sustainable-yield solutions can be derived on assuming that the population is in equilibrium (i.e., that the growth rate and harvest rates in the differential equation balançe each other) as was done by Fox [9] and Clark [6]. On the other hand Cliff and Vincent [10], Clark and Munro [11] and Vincent et al. [12] obtained more general results by considering the population model in an optimal control setting, while Palm [13] specifically examined problems formulated in terms of linear-quadratic optimal control.

Two important drawbacks of the differential equation models discussed above are the following: the class structure within the population (e.g. age) is ignored; also ignored is the occurrence of certain discrete events (such as seasonal spawning). In the management of certain types of fisheries it is essential that both these features should be considered.

Population class structures can be handled by constructing a separate equation for each class [14] or allowing each class to satisfy the same differential equation but start off at a different initial point in time. An example of the latter approach is the multiple-cohort model of Clark et al. [15]. Both approaches fail, however, to take cognizance of periodic discrete events such as spawning, where the number of new recruits entering the fishery is estimated by a suitable stock-recruitment relationship.

Discrete events can be modeled by systems of difference equations, but such models, in contrast to differential-equation models, ignore effects related to continuous processes. For example, the Leslie matrix population model [16] has been used by Doubleday [17] and Rorres and Fair [18] to analyze the problem of harvesting a structured population, but the following essential information was not forthcoming: at what points in time between the iteration intervals of the model should harvesting take place?

This information is especially relevant to the problem of maximizing biomass yield in a population continuously gaining mass according to some prescribed growth function $w(t)$.

In the next section a model is presented that includes both a discrete periodic spawning event and a continuous harvesting rate. The model takes the form of a system of nonlinear ordinary differential equations with state-variable jump discontinuities at specified points. A maximum principle for such systems in an optimal-control setting has been derived by Getz and Martin [19] and applied to the problem of constructing maximum-sustainable-rent solutions for certain management problems. A more detailed discussion then follows on systems to which Beverton-Holt type mortality and harvesting dynamics are assumed to be applicable. In particular, a method is presented by means of which maximum-sustainable-yield strategies for such systems can be evaluated, and which also indicates the maximum-sustainable-yield strategy when constraints on the harvesting rates are removed. The latter strategy thus provides an estimate of the ultimate biomass yield of the population under an "ideal" harvesting policy.

The paper concludes with a discussion on the evaluation of maximum sustainable yields subject to the practical constraints of scalar control and continuous harvesting over a given harvesting season.

2. GENERAL FORMULATION

Let time, denoted by t, be measured in years. Let $\mathbf{x} \in \mathbb{R}^n$ be a vector whose ith component $x_i(t)$, $i=1,\ldots,n-1$, represents the number of individuals in the age group of i-year-olds and let $x_n(t)$ represent the number of individuals of age n years or more. Let[1] $\mathbf{u}(t) \in [\mathbf{0},\mathbf{b}] \subseteq \mathbb{R}^m$, for all t, represent the intensities of m different ways of harvesting the population (e.g. by using nets with different mesh sizes or fishing in spatially distinct regions that have dissimilar population age distributions), where $u_j(t) \in [0,b_j]$ for all t, $b_j > 0$, $j=1,\ldots,m$. Let $\nu^{(k)} \in D \subseteq \mathbb{R}^q$ be vector of q parameters, each influencing in some way, at time $t=k$, $k=1,2,3,\ldots$, the stock-recruitment relationship and/or any of the functions associated with discrete periodic events occurring during the time interval $[k,k+1]$ as included in the model below.

Suppose that each year can be divided into a spawning and harvesting season, where the spawning season is closed to harvesting. A continuous-time system of differential equations will be used to model the dynamics of the population during the harvesting season, while a discrete-time system of difference equations will be used to model the transition of the population from the beginning of the spawning season to the beginning of the subse-

[1]Here $[\mathbf{0},\mathbf{b}]$ is the m-rectangle $[0,b_1]\times[0,b_2]\times\cdots\times[0,b_n]$.

quent harvesting season. Thus for some $\bar{t}\in(0,1]$ our model assumes the following form:

Harvesting season, $t\in[k,k+\bar{t})$, $k\in I \triangleq \{\text{non-negative integers}\}$.

$$\dot{x}_i = f_i(\mathbf{x},t) - h_i(\mathbf{x},\mathbf{u},t), \qquad i=1,\ldots,n, \tag{2.1}$$

where conditions are imposed on $f_i(\cdot,t)$ and $h_i(\cdot,\cdot,t)$ to ensure that for all t, $\dot{x}_i=0$ when $x_i=0$ [i.e., that the non-negative orthant of $\mathbb{R}^n$ is invariant with respect to (2.1)], that $f_i(\cdot,t)$ is a negative monotonically decreasing function of x_i—since it reflects mortality—and that $h_i(\cdot,\cdot,t)$ is monotonically increasing in x_i and u_j, $j=1,\ldots,m$—since it reflects harvesting. We shall also assume that f_i and h_i are C^1 in $\mathbf{x}$ and $\mathbf{x},\mathbf{u}$ respectively.

Spawning season.

$$\mathbf{x}(k+1) = \mathbf{p}^{(k+1)}\big(\mathbf{x}(k+\bar{t}),\boldsymbol{\nu}^{(k+1)}\big), \qquad k\in I, \tag{2.2}$$

where $\mathbf{p}^{k+1}:\mathbb{R}^n\times D\to\mathbb{R}$, $k\in I$, is assumed to be C^1.

Note that for the case $\bar{t}=1$ a spawning season of zero duration is implied, whence (using a superscripted minus sign to denote the value of the limit "from the left") $\mathbf{x}(k+\bar{t})$ in (2.2) will be replaced by $\mathbf{x}(k+1^-)$.

The problem of maximizing economic rent from a fishery over an N-year period can be stated as follows: Given $\mathbf{x}(0)=\mathbf{x}_0$, maximize the functional

$$J(\mathbf{u};\boldsymbol{\nu}^{(1)},\ldots,\boldsymbol{\nu}^{(N-1)}) = \sum_{k=0}^{N-1}\left[\int_k^{k+\bar{t}}\left[\sum_{i=1}^{n} g_i(t)h_i(\mathbf{x},\mathbf{u},t) - c(\mathbf{u},t)\right]\alpha(t)\,dt\right] + \sum_{k=0}^{N-2} F^{(k+1)}\big(\mathbf{x}(k+\bar{t}),\boldsymbol{\nu}^{(k+1)}\big) + F^N\big(\mathbf{x}(N-1+\bar{t})\big) \tag{2.3}$$

over all choices of piecewise continuous, and continuous from the left, controls $\mathbf{u}(t)\in[\mathbf{0},\mathbf{b}]$ for all t, and[2] $\boldsymbol{\nu}^{(k+1)}\in D$, $k=0,\ldots,N-1$, subject to the constraint equation (2.1) for $k=0,\ldots,N-1$ and interior jump conditions (2.2), $k=0,\ldots,N-2$. The final state $\mathbf{x}(N-1+\bar{t})$ will be regarded as free, although the case is easily dealt with for which $\mathbf{x}(0)$ and $\mathbf{x}(N-1+\bar{t})$ are constrained to initial and terminal manifolds.

The function $g_i(t)$ weights the yield $h_i(\mathbf{x},\mathbf{u},t)$ according to the commercial value of the individuals in the ith age class. The functions $c(\mathbf{u},t)$ and $\alpha(t)$ respectively account for harvesting costs and the present-value discount rate

[2]The greater generality of assuming $u(t)\in\Omega^{(k+1)}$ (a suitable constraint set) for $t\in[k,k+\bar{t}]$ and $\nu^{(k+1)}\in D^{(k+1)}$ can be included without essential modification of later results.

on future returns. Since fixed capital costs do not enter into the minimization problem, $c(\mathbf{u},t)$ is normalized to satisfy $c(\mathbf{0},t)=0$. Further, $c(u,t)$ is assumed to be (as it should be) a monotonically increasing function of u_j, $j=1,\ldots,m$. The functions $F^{(k+1)}(\mathbf{x}(k+\bar{t}),\boldsymbol{\nu}^{(k+1)})$, $k=0,\ldots,N-2$, reflect costs associated with the state of the system and the application of parameters during the kth spawning season, while the function $F^N(\mathbf{x}(N-1+\bar{t}))$ reflects the cost associated with the final state of the fisheries.

For convenience define $T=\bigcup_{k=0}^{N-1}[k,k+\bar{t})$. Then we define an admissible arc $\mathbf{x}(\cdot),\mathbf{u}(\cdot),\boldsymbol{\nu}^{(1)},\ldots,\boldsymbol{\nu}^{(N-1)}$, for the above control problem to be the following: a piecewise continuous control $\mathbf{u}(\cdot)$, with $\mathbf{u}(t)\in[\mathbf{0},\mathbf{b}]$, $t\in T$; a sequence of vectors $\boldsymbol{\nu}^{(k)}\in D$, $k=1,\ldots,N-1$; and a function $\mathbf{x}(\cdot)$ that is absolutely continuous on each subinterval $[k,k+\bar{t})$, that satisfies (2.1) for all $t\in T$ with finite limits from the left at $k+\bar{t}$, $k=0,\ldots,N-1$, and that satisfies (2.2) for $k=0,\ldots,N-2$.

We are now in a position to state a maximum principle (necessary conditions) for the problem of maximizing (2.3). Since the problem under consideration refers to a special case of a more general class of optimal control systems with state-variable jump discontinuities considered in Getz and Martin [19], we adapt without proof the necessary conditions derived in this latter paper[3].

THEOREM 2.1

Suppose the arc $\hat{\mathbf{x}}(\cdot),\hat{\mathbf{u}}(\cdot),\hat{\boldsymbol{\nu}}^{(1)},\ldots,\hat{\boldsymbol{\nu}}^{(N-1)}$ *is optimal for the problem formulated above. Then there exist costate variables* $\lambda_1(\cdot),\ldots,\lambda_n(\cdot)$, *not all zero, such that:*[4]

(i) *The costate variables* $\lambda_i(\cdot)$ *satisfy, for* $i=1,\ldots,n$, *the differential equations*

$$\frac{d\lambda_i}{dt}=-\frac{\partial H(\boldsymbol{\lambda}(t),\hat{\mathbf{x}}(t),\hat{\mathbf{u}}(t),t)}{\partial x_i},\qquad t\in T, \tag{2.4}$$

where

$$H(\boldsymbol{\lambda},\mathbf{x},\mathbf{u},t)\triangleq\left[\sum_{i=1}^{n}g_i(t)h_i(\mathbf{x},\mathbf{u},t)-c(\mathbf{u},t)\right]\alpha(t)$$
$$+\sum_{i=1}^{n}\lambda_i[f_i(\mathbf{x},t)-h_i(\mathbf{x},\mathbf{u},t)] \tag{2.5}$$

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[3]Certain sign changes have been made, bearing in mind that the problem under consideration is a maximization rather than a minimization problem.

[4]The multiplier λ_0 is positive for the problem under consideration and has been normalized to unity.

satisfy the jump conditions

$$\boldsymbol{\lambda}(k+\bar{t})=\left[\frac{\partial \mathbf{p}^{(k+1)}\big(\hat{\mathbf{x}}(k+\bar{t}),\hat{\boldsymbol{\nu}}^{(k+1)}\big)}{\partial \mathbf{x}(k+\bar{t})}\right]^T \boldsymbol{\lambda}(k+1)$$

$$+\frac{\partial F^{(k+1)}\big(\hat{\mathbf{x}}(k+\bar{t}),\hat{\boldsymbol{\nu}}^{(k+1)}\big)^T}{\partial \mathbf{x}(k+\bar{t})}, \qquad k=0,\ldots,N-2, \tag{2.6}$$

and the final time condition[5]

$$\boldsymbol{\lambda}(N-1+\bar{t})=\frac{\partial F^{(N)}\big(\mathbf{x}(N-1+\bar{t})\big)^T}{\partial \mathbf{x}(N-1+\bar{t})}; \tag{2.7}$$

(ii) *the parameters* $\hat{\boldsymbol{\nu}}^{(k)}$ *satisfy*

$$\left[\frac{\partial \mathbf{p}^{(k+1)}\big(\hat{\mathbf{x}}(k+\bar{t}),\hat{\boldsymbol{\nu}}^{(k+1)}\big)}{\partial \boldsymbol{\nu}^{(k+1)}}\right]^T \boldsymbol{\lambda}(k+1)+\frac{\partial F^{(k+1)}\big(\hat{\mathbf{x}}(k+\bar{t}),\hat{\boldsymbol{\nu}}^{(k+1)}\big)^T}{\partial \boldsymbol{\nu}^{(k+1)}}=0,$$

$$k=0,\ldots,N-2, \tag{2.8}$$

$$\frac{\partial F^{(N)}\big(\hat{\mathbf{x}}(N-1+\bar{t}),\hat{\boldsymbol{\nu}}^{(N)}\big)}{\partial \boldsymbol{\nu}^{(N)}}=\mathbf{0}; \tag{2.9}$$

(iii) *the inequality*

$$\hat{H}(t)\triangleq H(\boldsymbol{\lambda}(t),\hat{\mathbf{x}}(t),\hat{\mathbf{u}}(t),t)\geqslant H(\boldsymbol{\lambda}(t),\hat{\mathbf{x}}(t),\mathbf{u},t) \tag{2.10}$$

holds for any $\mathbf{u}\in[\mathbf{0},\mathbf{b}]$ *and all* $t\in T$. *Furthermore,* $\hat{H}(t)$ *is continuous for* $t\in T$, *and the limits* $\hat{H}(k+\bar{t})$, $k=1,\ldots,N-1$, *exist.*

Theorem 2.1 can be utilized in the same manner as the standard maximum principle is utilized to construct candidate solutions to the maximization problem under consideration: given $\mathbf{x}_0,\bar{\mathbf{u}}(\cdot),\bar{\boldsymbol{\nu}}^{(1)},\ldots,\bar{\boldsymbol{\nu}}^{(N-1)}$, (2.1) and (2.2) can be used to obtain $\bar{\mathbf{x}}(t)$, $t\in T$, from which—using (2.4) and (2.6)—$\bar{\boldsymbol{\lambda}}(t)$ can be solved backwards on T from the final-time condition (2.9), whence the nominal controls $\bar{\mathbf{u}}(\cdot)$ and nominal parameters $\bar{\boldsymbol{\nu}}^{(k)}$, $k=1,\ldots,N-1$, can be updated according to some algorithm (e.g. [20]).

Owing to the seasonal structure imposed upon biological systems, it may often be appropriate to model the dynamics of such systems using periodic functions of time.

Assume that the functions $f_i(\mathbf{x},\mathbf{u},t)$, $h_i(\mathbf{x},\mathbf{u},t)$, $g_i(t)$ $(i=1,\ldots,n)$ and $c(\mathbf{u},t)$ are periodic with period unity (i.e., one year), and that the functions $\mathbf{p}^{(k+1)}$ and similarly $F^{(k+1)}$ are identical for each k. Since the discount factor $\alpha(t)$ is by nature a monotonically decreasing function of time, it is always

[5]Note that from (2.3) the optimization problem clearly terminates at $N-1+\bar{t}$ and not at N.

aperiodic, and thus for the purpose of discussing sustainable yields for periodic systems, $\alpha(t)$ will be ignored [by setting $\alpha(t)=1$ in (2.3)]. Now if, for some N, a particular management policy applied to (2.1) and (2.2) (not necessarily the same policy on each subinterval $[k,k+\bar{t})$) results in

$$\mathbf{x}(N)=\mathbf{x}(0), \tag{2.11}$$

then, because of the periodicity of the system, a sustainable yield policy over an infinite time horizon is attainable by means of repeated application of that management policy over N-year intervals.

Consider firstly the following one-year horizon problem: Maximize

$$J_1(\mathbf{u}(\cdot),\boldsymbol{\nu})=\int_0^{\bar{t}}\left[\sum_{i=1}^{n} g_i(t)h_i(\mathbf{x},\mathbf{u},t)-c(\mathbf{u},t)\right]dt+F(\mathbf{x}(\bar{t}),\boldsymbol{\nu}) \tag{2.12}$$

subject to (2.1) and the constraints

$$\mathbf{x}(1)=\mathbf{p}(\mathbf{x}(\bar{t}),\boldsymbol{\nu}) \tag{2.13}$$

and

$$\mathbf{x}(1)=\mathbf{x}(0). \tag{2.14}$$

Since (2.13) and (2.14) provide the boundary condition

$$\mathbf{x}(0)=\mathbf{p}(\mathbf{x}(\bar{t}),\boldsymbol{\nu}), \tag{2.15}$$

the maximization of (2.12) subject to (2.1) is in a standard form (i.e., no jump discontinuities are evident on $[0,\bar{t})$). The standard maximum principle of Pontryagin [21, Chapter 7.9] asserts that if $\mathbf{x}^s(\cdot),\mathbf{u}^s(\cdot),\boldsymbol{\nu}^s$ maximizes (2.12) subject to (2.1) and (2.15), then there exists $\boldsymbol{\lambda}^s(\cdot)$ satisfying (2.4) on $[0,\bar{t})$ such that

$$\boldsymbol{\lambda}^s(\bar{t})=\frac{\partial \mathbf{p}^T(\mathbf{x}^s(\bar{t}),\boldsymbol{\nu}^s)}{\partial \mathbf{x}(\bar{t})}\boldsymbol{\lambda}^s(0)+\frac{\partial F^T(\mathbf{x}^s(\bar{t}),\boldsymbol{\nu}^s)}{\partial \mathbf{x}(\bar{t})}, \tag{2.16}$$

$$\frac{\partial \mathbf{p}^T(\mathbf{x}^s(\bar{t}),\boldsymbol{\nu}^s)}{\partial \boldsymbol{\nu}}\boldsymbol{\lambda}^s(0)+\frac{\partial F^T(\mathbf{x}^s(\bar{t}),\boldsymbol{\nu}^s)}{\partial \boldsymbol{\nu}}=0, \tag{2.17}$$

and condition (iii) of Theorem 2.1 holds on $[0,\bar{t})$.

Now consider the problem of maximizing

$$\begin{aligned}J_N(\mathbf{u}(\cdot),\boldsymbol{\nu}^{(1)},\ldots,\boldsymbol{\nu}^{(N)})=\sum_{k=0}^{N-1}\Bigg\{\int_k^{k+\bar{t}}&\left[\sum_{i=1}^{n} g_i(t)h_i(\mathbf{x},\mathbf{u},t)-c(\mathbf{u},t)\right]dt\\&+F(\mathbf{x}(k+\bar{t}),\boldsymbol{\nu}^{(k+1)})\Bigg\}\end{aligned} \tag{2.18}$$

subject to (2.1) for $t \in T$, (2.2) (bearing in mind that $\mathbf{p}$ is independent of k) for $k=0,\ldots,N-1$, and (2.11). Define for $k=0,\ldots,N-1$ and $t \in [k,k+\bar{t})$:

$$\hat{\mathbf{x}}(t)=\mathbf{x}^s(t-k); \qquad \hat{\mathbf{u}}(t)=\mathbf{u}^s(t-k); \qquad \boldsymbol{\lambda}(t)=\boldsymbol{\lambda}^s(t-k); \qquad \hat{\boldsymbol{\nu}}^{(k+1)}=\boldsymbol{\nu}^s. \tag{2.19}$$

Then it is clear (recall the unit periodicity of all functions) that $\boldsymbol{\lambda}(t)$ satisfies Theorem 2.1 for $\hat{\mathbf{x}}(\cdot),\hat{\mathbf{u}}(\cdot),\hat{\boldsymbol{\nu}}^{(1)},\ldots,\hat{\boldsymbol{\nu}}^{(N)}$ defined in (2.19) when the boundary conditions [(2.7) and (2.9) are replaced by conditions of the type (2.16) and (2.17), since the problem under consideration is subject to (2.11) rather than to a fixed initial point and a free final point specifications] are

$$\boldsymbol{\lambda}(N-1+\bar{t})=\frac{\partial \mathbf{p}^T(\hat{\mathbf{x}}(N-1+\bar{t}),\boldsymbol{\nu}^s)}{\partial \mathbf{x}(N-1+\bar{t})}\boldsymbol{\lambda}(0)+\frac{\partial F^T(\hat{\mathbf{x}}(N-1+\bar{t}),\boldsymbol{\nu}^s)}{\partial \mathbf{x}(N-1+\bar{t})}, \tag{2.20}$$

$$\frac{\partial \mathbf{p}^T(\hat{\mathbf{x}}(N-1+\bar{t}),\boldsymbol{\nu}^s)}{\partial \boldsymbol{\nu}^s}\boldsymbol{\lambda}(0)+\frac{\partial F^T(\hat{\mathbf{x}}(N-1+\bar{t}),\boldsymbol{\nu}^s)}{\partial \boldsymbol{\nu}^s}=0. \tag{2.21}$$

Note that since $\boldsymbol{\lambda}(t)=\boldsymbol{\lambda}(k+t)$ and $\hat{\mathbf{x}}(t)=\hat{\mathbf{x}}(k+t)$ $(k=0,\ldots,N-1)$, (2.20) and (2.21) are really the only conditions on $\boldsymbol{\lambda}(k+t)$ $(k=0,\ldots,N-1)$ and $\boldsymbol{\nu}^s$ [cf. (2.6) and (2.8)].

Hence, from the above, $\hat{\mathbf{x}}(\cdot),\hat{\mathbf{u}}(\cdot),\hat{\boldsymbol{\nu}}^{(k)}=\boldsymbol{\nu}^s$ $(k=1,\ldots,N)$ is extremal for the N-horizon problem. For obvious reasons the arc $\mathbf{x}^s(\cdot),\mathbf{u}^s(\cdot),\boldsymbol{\nu}^s$ will be referred to as the maximum-sustainable-rent solution. Although for given but arbitrary $N>1$ it does not immediately follow that the arc $\hat{\mathbf{x}}(\cdot),\hat{\mathbf{u}}(\cdot),\boldsymbol{\nu}^{(1)},\ldots,\boldsymbol{\nu}^{(N)}$ maximizes (2.18). [Since only necessary conditions are satisfied, it does follow that if the maximum sustainable rent solution satisfies sufficiency conditions involving the existence of a suitable function $\boldsymbol{\lambda}(t)$ (e.g. [22]) for the problem associated with (2.12), then the above arc satisfies the same sufficiency conditions for the problem associated with (2.18).]

3. EXTENDED BEVERTON-HOLT THEORY

Beverton and Holt [1] considered the problem of harvesting a single cohort recruited at time zero, using "knife-edge" selective fishing gear, of mesh size μ, at intensity b, on a population of size $x(t)$. They assumed the dynamics of the population to be modeled by

$$\begin{aligned} \dot{x}&=-\alpha x \qquad &&\text{for} \quad 0<t<t_\mu, \\ \dot{x}&=-(\alpha+b)x \qquad &&\text{for} \quad t\geqslant t_\mu, \end{aligned} \tag{3.1}$$

where t_μ is the age at which the fish, when encountering the fishing gear, are first captured, and $\alpha > 0$ is a constant natural mortality rate. They also assumed that the biomass increase in the population could be modeled by the three-parameter von Bertalanffy growth function

$$w(t) = \omega(1 - \kappa e^{-\rho t})^3, \quad t \geqslant 0 \qquad \text{with constants} \quad \omega > 0,\ \kappa > 0,\ \rho > 0, \tag{3.2}$$

where $w(t)$ is the average biomass of an individual (recruited to the population at time zero) at time t. Beverton and Holt then considered the problem of maximizing the biomass yield (for given b) over all choices of t_μ at which harvesting should commence. Viewing the optimal yield as a function of b, they were led to introduce the concept of the "eumetric yield curve."

The problem of optimally harvesting a multiple-cohort fishery is significantly more difficult to analyze. Clark [6, Sec. 8.6] extended the Beverton-Holt approach to multiple cohort fisheries by considering a fishery to be a conglomerate of single cohorts $x_k(t)$ each satisfying equations of the type (3.1), but with time intervals shifted to the right by the integer k and subject to the initial condition

$$x_k(k) = R, \qquad k = 0, 1, 2, \ldots, \tag{3.3}$$

for some given constant R. The relation (3.3) ignores an essential aspect of multiple-cohort fisheries: the existence of a stock-recruitment relationship. A multiple-cohort fishery including a stock-recruitment relationship can be modeled following the formulation of state-variable jump-discontinuous systems considered in the previous section.

A multiple-cohort model where the dynamics of each cohort is given by (3.1) can be translated into the following age-structure model: Suppose $t_\mu \in [r, r+1]$ and $n \geqslant r+1$; then for $t \in [k, k+\bar{t})$ and $k = 0, 1, 2, \ldots,$

$$\begin{aligned} \dot{x}_i &= -\alpha x_i, && i = 1, \ldots, r-1, \\ \dot{x}_r &= -\alpha x_r, && t \in [0, t'), \\ \dot{x}_r &= -(\alpha + b) x_r, && t \in [t', \bar{t}), \\ \dot{x}_i &= -(\alpha + b) x_i, && i = r+1, \ldots, n, \end{aligned} \tag{3.4}$$

where $t' = t_\mu - r$ if $t_\mu > r + \bar{t}$, and $t' = \bar{t}$ otherwise. Furthermore, it follows for $k = 0, 1, 2, \ldots$ that

$$\begin{aligned} x_{i+1}(k+1) &= a_i x_i(k+\bar{t}), \qquad i = 1, \ldots, n-2, \\ x_n(k+1) &= a_{n-1} x_{n-1}(k+\bar{t}) + a_n x_n(k+\bar{t}), \end{aligned} \tag{3.5}$$

where $a_i = e^{-\alpha(1-\bar{t})}$ $(i=1,\ldots,r-1)$, $a_i = e^{-(\alpha+b)(1-\bar{t})}$ $(i=r+1,\ldots,n)$ and $a_r = e^{-(\alpha+b)(1-\bar{t})}$ if $t_\mu \leqslant r+\bar{t}$ or $a_r = e^{-\alpha(1-\bar{t})-b(1-t_\mu)}$ if $t_\mu > r+\bar{t}$.

In the Beverton-Holt model, harvesting is assumed to continue over the whole period $[k,k+1)$, $k \in I$. We shall revert, however, to our assumption of distinct harvesting and spawning seasons, i.e., we shall assume in case $\bar{t}<1$ that (3.5) holds with

$$a_i = e^{-\alpha_i(1-\bar{t})}, \qquad i=1,\ldots,n, \tag{3.6}$$

where, for generality, distinct α_i's are assumed to hold for each age group and (3.4) is replaced on $[k,k+\bar{t})$, $k \in I$, by the more general model

$$\dot{x}_i = -[\alpha_i + u_i(t)]x_i, \quad \alpha_i > 0, \qquad i=1,\ldots,n, \tag{3.7}$$

and $u_i(t) \in [0,b_i]$.

We notice that (3.5) contains only $n-1$ equations: an expression for $x_1(k+1)$ is absent. This expression is in fact the stock-recruitment relationship, which we assume has the form

$$x_1(k+1) = p_1\big(\mathbf{x}(k+\bar{t})\big), \qquad k \in I, \tag{3.8}$$

where $p_1 : \mathbb{R}^n \to \mathbb{R}$ is C^1.

If p_1 is linear, i.e., if

$$x_1(k+1) = \mathbf{c}^T\mathbf{x}(k+\bar{t}) \triangleq \sum_{i=1}^{n} c_i x_i(k+\bar{t}), \quad c_i \geqslant 0, \qquad i=1,\ldots,n,$$

then (3.5) and (3.8) are the well-known Leslie matrix transformation [16]. For certain demersal populations, Beverton and Holt [1] found a saturating stock-recruitment relationship, for which a general expression is

$$x_1(k+1) = \frac{\beta\mathbf{c}^T\mathbf{x}(k+\bar{t})}{\gamma + \mathbf{c}^T\mathbf{x}(k+\bar{t})}, \qquad \beta>0, \quad \gamma>0, \quad c_i \geqslant 0, \; i=1,\ldots,n, \tag{3.9}$$

more suitable then a linear relationship, while Ricker [2] found an "over-compensating" curve, for which a general expression is

$$x_1(k+1) = \mathbf{c}^T\mathbf{x}(k+\bar{t})\exp\left\{\beta\left(1 - \frac{\mathbf{c}^T\mathbf{x}(k+\bar{t})}{\gamma}\right)\right\},$$
$$\beta>0, \quad \gamma>0, \quad c_i \geqslant 0, \; i=1,\ldots,n,$$

suitable for salmon populations.

Finally, for any given growth function $w(t)$, of which (3.2) is a specific example, it is clear in terms of age structure that the average biomass of an individual in age class i at time t will, for any $k \in I$ and $t \in [k, k+1)$, be

$$w_i(t) = w(t-k+i-1), \qquad i = 1, \ldots, n. \tag{3.10}$$

We note that an error arises in assuming that $w_n(t)$ is the average weight of an individual in class n at time t, since $x_n(t)$ includes not only n-year-old individuals but older individuals as well [see (3.5)]. This error is small, however, if the mortality rate α_n of the nth class is high or if n is large [since $w(t)$ saturates as a function of increasing t, and hence, from (3.10), $w_i(t)$ saturates as a function of increasing i]. Alternatively $w_n(t)$ can be modified to account for this truncation.

We also note that, by nature, biomass growth functions for fish should have the following intrinsic properties for all $t \geqslant 0$ and some $\bar{w} > 0$:

$$0 < w(t) < \bar{w}, \qquad \dot{w}(t) \geqslant 0 \quad \text{and} \quad \frac{d}{dt}\left(\frac{\dot{w}(t)}{w(t)}\right) \leqslant 0. \tag{3.11}$$

The von Bertalanffy growth function (3.2) satisfies inequalities (3.11) strictly. The last inequality in (3.11) is interperated as follows: the proportional rate of increase of biomass is a decreasing function of time. Assuming that the last inequality in (3.11) holds strictly, it follows from (3.10) that

$$\frac{\dot{w}_i(t_1)}{w_i(t_1)} > \frac{\dot{w}_i(t_2)}{w_i(t_2)} \qquad \text{for} \quad 0 \leqslant t_1 \leqslant t_2 < 1, \quad i = 1, \ldots, n, \tag{3.12}$$

and

$$\frac{\dot{w}_i(t)}{w_i(t)} > \frac{\dot{w}_{i+1}(t)}{w_{i+1}(t)} \qquad \text{for all } t \text{ and } i = 1, \ldots, n-1. \tag{3.13}$$

Theorem 2.1 can now be applied to the problem of constructing optimal harvesting strategies for systems modeled by the above "extended Beverton-Holt model." Since, however, (3.7) is a linear system [for given $\mathbf{u}(\cdot)$], analytical techniques can be used to gain further insight into solutions to such problems. This is done in the next section with respect to the problem of obtaining maximum-sustainable-yield strategies.

4. MAXIMUM SUSTAINABLE YIELDS

One of the long-term management problems to consider, and one that is essentially of sociological rather than economic importance, is that of estimating the maximum sustainable yield (disregarding control and other costs) that can be obtained from a fishery. Thus we shall consider the problem of maximizing

$$J(\mathbf{u}(\cdot)) = \int_0^{\bar{t}} \sum_{i=1}^{n} w_i(t) u_i(t) x_i(t)\, dt \tag{4.1}$$

over all piecewise continuous $\mathbf{u}(\cdot)$ taking values in $[\mathbf{0}, \mathbf{b}]$, subject to (3.7) and the following boundary conditions [recalling (2.14), (3.5) and (3.6), and assuming (3.8) has the form (3.9)]:

$$\begin{aligned} x_{i+1}(0) &= e^{-\alpha_i(1-\bar{t})} x_i(\bar{t}), \qquad i=1,\ldots,n-2, \\ x_n(0) &= e^{-\alpha_{n-1}(1-\bar{t})} x_{n-1}(\bar{t}) + e^{-\alpha_n(1-\bar{t})} x_n(\bar{t}), \\ x_1(0) &= \frac{\beta \mathbf{c}^T \mathbf{x}(\bar{t})}{\gamma + \mathbf{c}^T \mathbf{x}(\bar{t})}. \end{aligned} \tag{4.2}$$

The Hamiltonian for the problem under consideration [cf. (2.5)] is

$$H(\boldsymbol{\lambda}, \mathbf{x}, \mathbf{u}, t) = \sum_{i=1}^{n} [(w_i(t) - \lambda_i) u_i x_i - \alpha_i \lambda_i x_i], \tag{4.3}$$

which we note from (3.10) is a periodic function of t.

Hence as discussed in Sec. 2, if $\mathbf{x}^s(\cdot), \mathbf{u}^s(\cdot)$ maximizes (4.1), then it provides for a maximum-sustainable-yield solution [see Eq. (2.19)] and there exist $\boldsymbol{\lambda}(\cdot)$ satisfying the following differential equation [see (2.4), (2.16), (4.3), and replace (2.15) with (4.2)]:

$$-\dot{\lambda}_i = -(\alpha_i + u_i^s)\lambda_i + w_i(t) u_i^s, \qquad i=1,\ldots,n, \tag{4.4}$$

and boundary conditions

$$\begin{aligned} \lambda_i(\bar{t}) &= e^{-\alpha_i(1-\bar{t})} \lambda_{i+1}(0) + \left[\frac{\beta \gamma c_i}{[\gamma + \mathbf{c}^T \mathbf{x}^s(\bar{t})]^2} \right] \lambda_1(0), \qquad i=1,\ldots,n-1 \\ \lambda_n(\bar{t}) &= e^{-\alpha_n(1-\bar{t})} \lambda_n(0) + \left[\frac{\beta \gamma c_n}{[\gamma + \mathbf{c}^T \mathbf{x}^s(\bar{t})]^2} \right] \lambda_1(0). \end{aligned} \tag{4.5}$$

In order that $\boldsymbol{\lambda}(\cdot)$ should satisfy (2.10), it is necessary that

$$u_i^s(t) = 0 \quad \text{whenever } H_{u_i}^s(t) < 0, \qquad i=1,\ldots,n, \tag{4.6}$$

and

$$u_i^s(t) = b_i \quad \text{whenever } H_{u_i}^s(t) > 0, \qquad i=1,\ldots,n, \tag{4.7}$$

where

$$H^s_{u_i}(t) \triangleq \frac{\partial H(\boldsymbol{\lambda}(t),\mathbf{x}^s(t),\mathbf{u}^s(t),t)}{\partial u_i} = [w_i(t) - \lambda_i(t)]x_i^s(t). \tag{4.8}$$

Using (3.7) and (4.4) to differentiate (4.8) with respect to time, it is possible to show that

$$\dot{H}^s_{u_i} = w_i(t)x_i^s(t)\left(\frac{\dot{w}_i(t)}{w_i(t)} - \alpha_i\right), \qquad i=1,\ldots,n. \tag{4.9}$$

From (3.12) it then follows that $\dot{H}_{u_i}$ can change sign at most once on $[0,\bar{t}]$ and cannot be identically zero on any open subinterval of $[0,\bar{t}]$. Hence it is easily deduced that $u_i^s(\cdot)$ must be nonsingular [i.e., $u_i^s(\cdot)$ is characterized by (4.6) and/or (4.7) for almost all[6] $t\in[0,\bar{t})$] and of the following form. For some τ_i, σ_i satisfying $0 \leqslant \tau_i \leqslant \sigma_i \leqslant \bar{t}$, $\mathbf{u}(\cdot)$ is specified by

$$u_i(t) = b_i, \quad t\in[\tau_i,\sigma_i), \quad \text{and} \quad u_i(t) = 0, \quad t\in\left[0,\bar{t}\right)\backslash[\tau_i,\sigma_i). \tag{4.10}$$

Note that there are five distinct cases in (4.10):

$\tau_i = \sigma_i$ [i.e., $u_i(\cdot) \equiv 0$];
$\tau_i = 0$, $\sigma_i = \bar{t}$ [i.e. $u_i(\cdot) \equiv b_i$];
$\tau_i = 0$, $\sigma_i < \bar{t}$ [i.e. $u_i(\cdot)$ is "switched off" at σ_i];
$\tau_i > 0$, $\sigma_i = \bar{t}$ [i.e., $u_i(\cdot)$ is "switched on" at τ_i];
$\tau_i > 0$, $\sigma_i < \bar{t}$ [i.e., $u_i(\cdot)$ is "switched on" at τ_i and is "switched off" at σ_i].

Furthermore, if it happens that $\alpha_i = \alpha$ $(i=1,\ldots,n)$, as is often assumed to be the case (e.g. [6, Sec. 8.6]), then using (3.13) it follows from (4.6)–(4.9) that for some $r\in\{1,\ldots,n\}$, $u_i(\cdot)$ $(i=1,\ldots,r-1)$ can at most be "switched on," $u_r(\cdot)$ can at most be "switched on and then off," and $u_i(\cdot)$ $(i=r+1,\ldots,n)$ can at most be "switched off." The interpretation of this is simply that in the younger age groups, where biomass gains "outweigh" losses to natural mortality, late-season harvesting is preferable, while in older age groups, where losses to natural mortality "outweigh" biomass gains, early-season harvesting is preferable.

The problem of finding $u_i^s(\cdot)$, $i=1,\ldots,n$, has now been reduced to one of finding the corresponding τ_i^s and σ_i^s [see (4.10)]. Integrating (3.7) by using (4.10), we obtain

$$x_i(\bar{t}) = x_i(0)e^{-\alpha_i\bar{t} - b_i(\sigma_i - \tau_i)}, \qquad i=1,\ldots,n. \tag{4.11}$$

[6]In the measure-theoretic sense.

Equation (4.11) can be substituted in (4.2) to eliminate $\mathbf{x}(\bar{t})$ and subsequently solved to yield

$$x_{i+1}(0)=\prod_{j=1}^{i} e^{-\alpha_j-b_j(\sigma_j-\tau_j)}x_1(0), \qquad i=1,\dots,n-2,$$

$$x_n(0)=\frac{\prod_{j=1}^{n-1} e^{-\alpha_j-b_j(\sigma_j-\tau_j)}}{1-e^{-\alpha_n-b_n(\sigma_n-\tau_n)}}x_1(0), \tag{4.12}$$

$$x_1(0)=\frac{\beta\phi(\mathbf{u})-\gamma}{\phi(\mathbf{u})},$$

where

$$\phi(\mathbf{u})=\sum_{i=1}^{n-1} c_i e^{\alpha_i(1-\bar{t})}\prod_{j=1}^{i} e^{-\alpha_j-b_j(\sigma_j-\tau_j)}$$

$$+c_n e^{\alpha_n(1-\bar{t})}\frac{\prod_{j=1}^{n} e^{-\alpha_j-b_j(\sigma_j-\tau_j)}}{1-e^{-\alpha_n-b_n(\sigma_n-\tau_n)}}. \tag{4.13}$$

Note that ϕ depends on $\mathbf{u}$ through the specification of $\mathbf{u}$ by $\mathbf{b}$, $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ using (4.10). Since $\mathbf{u}^s$ is of the form (4.10), $\mathbf{x}^s(0)$ and hence $\mathbf{x}^s(t)$ are obtained by substituting the corresponding $\boldsymbol{\tau}^s$ and $\boldsymbol{\sigma}^s$ and the given $\mathbf{b}$ in (4.12) and (4.13) and integrating (3.7) on $[0,\bar{t})$. From (4.12), however, a solution $\mathbf{x}(0)>\mathbf{0}$ exists only if for some $\mathbf{u}$

$$\phi(\mathbf{u})>\gamma/\beta. \tag{4.14}$$

From (4.13) it is clear that $\phi(\mathbf{u})$ is a maximum at $\mathbf{u}=\mathbf{0}$ [i.e., for given $b_i>0$ we have $\sigma_i-\tau_i=0$, $i=1,\dots,n$ in (4.10)] and that $\phi(\mathbf{u})\to 0$ as $b_1\to\infty$ for $\sigma_1-\tau_1>0$. Hence if $\phi(\mathbf{0})>\gamma/\beta$, it is possible, since ϕ is a continuous function of $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$, to find $\mathbf{u}$ such that (4.14) holds. If, however, $\phi(\mathbf{0})\leqslant\alpha/\beta$, then it is clear that no sustainable-yield solution exists [i.e., no solution for which (2.14) is satisfied]. We shall thus assume that (4.14) holds with $\mathbf{u}=\mathbf{0}$ $(\sigma_i-\tau_i=0)$.

Using (4.10), we can integrate (4.4) to obtain

$$\lambda_i(\bar{t})=e^{\alpha_i\bar{t}+b_i(\sigma_i-\tau_i)}\left(\lambda_i(0)-b_i\int_{\tau_i}^{\sigma_i} w_i(t)e^{-\alpha_i t-b_i(t-\tau_i)}dt\right). \tag{4.15}$$

Substitution of (4.15) in (4.5) to eliminate $\boldsymbol{\lambda}(\bar{t})$ yields the following linear system in $\boldsymbol{\lambda}(0)$ [where $\mathbf{x}^s(\bar{t})$ is replaced by a candidate $\mathbf{x}(\bar{t})$ derived in (4.11)–(4.13) from a candidate $\mathbf{u}$ specified in terms of $\mathbf{b},\boldsymbol{\tau},\boldsymbol{\sigma}$]:

$$
\begin{aligned}
&-e^{-\alpha_i(1-\bar{t})}\lambda_{i+1}(0)+e^{\alpha_i\bar{t}+b_i(\sigma_i-\tau_i)}\lambda_i(0)-\frac{\gamma c_i}{\beta\phi^2(\mathbf{u})}\lambda_1(0)\\
&\qquad=b_ie^{\alpha_i\bar{t}+b_i(\sigma_i-\tau_i)}\int_{\tau_i}^{\sigma_i}w_i(t)^{-\alpha_it-b_i(t-\tau_i)}\,dt,\qquad i=1,\dots,n-1,\\
&e^{\alpha_n\bar{t}}\left[e^{b_i(\sigma_i-\tau_i)}-e^{-\alpha_n}\right]\lambda_n(0)-\frac{\gamma c_n}{\beta\phi^2(\mathbf{u})}\lambda_1(0)\\
&\qquad=b_ne^{\alpha_n\bar{t}+b_n(\sigma_n-\tau_n)}\int_{\tau_n}^{\sigma_n}w_n(t)e^{-\alpha_nt-b_n(t-\tau_n)}\,dt,
\end{aligned}
\tag{4.16}
$$

where $\phi(\mathbf{u})$ is given by (4.13).

Thus given $\bar{\mathbf{u}}$ defined in (4.10) in terms of $\bar{\boldsymbol{\tau}}$, $\bar{\boldsymbol{\sigma}}$ for given $\mathbf{b}$, we can calculate $\bar{\mathbf{x}}(0)$ and $\bar{\boldsymbol{\lambda}}(0)$ respectively from (4.12), (4.13) and from (4.16), whence $\bar{\mathbf{x}}(\cdot)$ and $\bar{\boldsymbol{\lambda}}(\cdot)$ can respectively be determined by integrating (3.7) and (4.4) [replacing $\mathbf{u}^s(\cdot)$ and $\mathbf{x}^s(\cdot)$ by $\bar{\mathbf{u}}(\cdot)$ and $\bar{\mathbf{x}}(\cdot)$ in (4.4)]. The nominal control $\bar{\mathbf{u}}$ can then be updated using (4.6), (4.7) and a suitable algorithmic procedure. This aspect of the problem will not be pursued here. It should be borne in mind that even if a solution $\mathbf{u}^s(\cdot)$ is obtained in terms of $\boldsymbol{\tau}^s$ and $\boldsymbol{\sigma}^s$, this solution may be impractical to implement, since it is very difficult, in general, to single out a specific age group for application of a given harvesting strategy u_i.

It is, however, worth considering the case for which the controls u_i are allowed to become unbounded (i.e., when $\mathbf{u}$ is unconstrained), since this, as discussed in the next section, will lead to an estimate of the "ultimate" sustainable biomass yield potential of the population. Such an estimate will provide a useful standard against which the performance of any sustainable yield policy can be measured.

5. THE ULTIMATE SUSTAINABLE YIELD

Consider the problem of maximizing (4.1) subject to (3.7) and the boundary condition (4.2), where $\mathbf{u}(\cdot)$ is now taken to be unconstrained. It follows from (3.7) that if $u_i(\cdot)$ assumes the value b_i on any fixed open interval $(\tau_i,\sigma_i)\subset[0,\bar{t})$ (no matter how small) and $x_i(\tau_i)>0$, then $x_i(\sigma_i)\to 0$ as $b_i\to\infty$. Hence the optimal controller, in the unconstrained case, either removes all members of the ith age class at time τ_i or acts impulsively,[7] i.e.

[7]We shall admit impulse controls, and the ensuing discussion will be on a heuristic rather than a rigorous level.

$\tau_i \to \sigma_i$ as $b_i \to \infty$. In both cases, the application of the optimal controller is equivalent to removing a finite number of individuals, say z_i, from the ith class at time $\hat{t}_i$, say. It is thus clear that $x_i(t)$ will satisfy

$$\begin{aligned} x_i(t) &= x_i(0)e^{-\alpha_i t}, \qquad t < \hat{t}_i, \\ x_i(t) &= \left(x_i(0)e^{-\alpha_i \hat{t}_i} - z_i\right)e^{-\alpha_i\left(t - \hat{t}_i\right)}, \qquad \hat{t}_i < t < \bar{t}, \end{aligned} \tag{5.1}$$

where the latter follows because the optimal controller either is an impulse at $\hat{t}_i$ or removes all individuals in the ith age class at time $\hat{t}_i$, whence

$$z_i = x_i(0)e^{-\alpha_i \hat{t}_i} \quad \text{and} \quad x_i(t) = 0, \quad t \geqslant \hat{t}_i. \tag{5.2}$$

For finite **b** we know from (4.10) that $x_i(t)$ will satisfy

$$\begin{aligned} x_i(t) &= x_i(0)e^{-\alpha_i t}, \qquad t < \tau_i \\ x_i(t) &= x_i(0)e^{-\alpha_i t - b_i(t - \tau_i)}, \qquad \tau_i \leqslant t < \sigma_i \\ x_i(t) &= \left(x_i(0)e^{-\alpha_i \sigma_i - b_i(\sigma_i - \tau_i)}\right)e^{-\alpha_i(t - \sigma_i)} \qquad \sigma_i \leqslant t \leqslant \bar{t}. \end{aligned} \tag{5.3}$$

Hence in the limit as $b_i \to \infty$, we have, comparing (5.1) and (5.3), that

$$\lim_{b_i \to \infty} e^{-b_i(\sigma_i - \tau_i)} = \frac{x_i(0) - e^{\alpha_i \hat{t}_i} z_i}{x_i(0)}. \tag{5.4}$$

Furthermore, for finite **b** we can use a mean-value theorem [to take $w_i(t)$ outside the integral sign] and integrate (4.1) to obtain

$$J(u(\cdot)) = \sum_{i=1}^{n} w_i(\hat{t}_i)x_i(0)e^{-\alpha_i \tau_i}\left[1 - e^{-(\alpha_i + b_i)(\sigma_i - \tau_i)}\right]\frac{b_i}{\alpha_i + b_i}, \tag{5.5}$$

where $\hat{t}_i \in [\tau_i, \sigma_i]$. On letting $b_i \to \infty$, $i = 1, \ldots, n$, and recalling that either $\tau_i = \sigma_i$ or (5.2) holds, it is clear that (5.5) reduces to

$$J(u(\cdot)) = \sum_{i=1}^{n} w_i(\hat{t}_i) z_i. \tag{5.6}$$

In Sec. 4 it was shown that the problem of finding the $\mathbf{u}(\cdot)$ that minimizes (4.1) reduces to the problem of finding three vectors $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$ and $\mathbf{x}(0)$ related by n equations [see (4.12)]. From (5.1) and (5.6) we see that in the unconstrained case the problem now reduces to one of finding three vectors $\hat{\mathbf{t}}$, $\mathbf{z}$ and $\mathbf{x}(0)$ also related by n equations [see (5.8) below]. This latter problem can be viewed as the following programming problem: Firstly, the constraints

$$x_i(0) \geqslant 0 \text{ and } z_i \geqslant 0, \qquad i = 1, \ldots, n, \tag{5.7}$$

follow from the original non-negativity constraints on $\mathbf{x}(t)$ and $\mathbf{u}(t)$. Secondly, using (4.2) and (5.1) we obtain

$$\begin{aligned} x_{i+1}(0) &= x_i(0)e^{-\alpha_i} - z_i e^{-\alpha_i(1-\hat{t}_i)} \\ x_n(0) &= x_{n-1}(0)e^{-\alpha_{n-1}} - z_{n-1}e^{-\alpha_{n-1}(1-\hat{t}_{n-1})} + x_n(0)e^{-\alpha_n} - z_n e^{-\alpha_n(1-\hat{t}_n)} \\ x_1(0) &= \frac{\beta \sum_{i=1}^{n} c_i\left[x_i(0)e^{-\alpha_i \bar{t}} - z_i e^{-\alpha_i(\bar{t}-\hat{t}_i)}\right]}{\gamma + \sum_{i=1}^{n} c_i\left[x_i(0)e^{-\alpha_i \bar{t}} - z_i e^{-\alpha_i(\bar{t}-\hat{t}_i)}\right]}. \end{aligned} \tag{5.8}$$

Suppose $\hat{\mathbf{t}}$ is given (a simple means of determining $\hat{\mathbf{t}}$ is discussed below). Then there are various techniques and algorithmic procedures available for solving the nonlinear programming problem of maximizing (5.6) subject to (5.7) and (5.8). This aspect of the problem will not be pursued here.

Consider $\hat{\mathbf{t}}$ selected in the following way. Let $b_i(t)$ denote the biomass of the ith class at time t. Then

$$b_i(t) = w_i(t)x_i(t), \qquad t \in [0, \bar{t}), \quad i = 1, \ldots, n. \tag{5.9}$$

In a naturally evolving system [(3.7) without controls] it follows that

$$\dot{b}_i = w_i(t)x_i(t)\left(\frac{\dot{w}_i(t)}{w_i(t)} - \alpha_i\right). \tag{5.10}$$

Then from (3.12), $b_i(t)$ is a maximum on $[0, \bar{t})$ at: $\bar{t}$ if $\dot{w}_i(\bar{t})/w_i(\bar{t}) \geqslant \alpha_i$; 0 if $\dot{w}_i(0)/w_i(0) \leqslant \alpha_i$; $t_i^s \in (0, \bar{t})$ if $\dot{w}_i(t_i^s)/w_i(t_i^s) = \alpha_i$. If $\hat{t}_i$ is chosen to be the value of t at which $b_i(t)$ assumes its maximum on $[0, \bar{t})$, this will ensure that for a given set $\{\mathbf{x}(0), \mathbf{x}(\bar{t})\}$ such that $\mathbf{x}(\bar{t})$ is obtainable from $\mathbf{x}(0)$ for some $\mathbf{z}$ for which (5.7) is satisfied, the greatest biomass yield is obtained from the population. Furthermore, since (4.9) and (5.10) are identical functions in $\mathbf{x}$ and t, it can be seen that $\hat{t}_i$ is the point at which one would expect the ith impulse controller to act when this impulse controller is taken as the limit of solutions to the constrained problem (i.e., letting $b_i \to \infty$).

Note that from (5.1), (5.7) and (5.8) it follows that if (5.2) holds for $i = k$, then $x_i(0) = z_i(0) = 0$, $i = k+1, \ldots, n$. This fact may lead in some cases to the simplification of the solution procedure to the programming problem {(5.6), (5.7), (5.8)} as it stands.

Now let $J_{\mathbf{b}}$ denote the maximum value of (4.1) where $u(\cdot)$ satisfies the constraint $\mathbf{u}(t) \in [\mathbf{0}, \mathbf{b}]$ for $t \in [0, \bar{t}]$. Then it follows from the maximum principle that[8]

$$\mathbf{b}^2 \geqslant \mathbf{b}^1 \quad \Rightarrow \quad J_{\mathbf{b}^2} \geqslant J_{\mathbf{b}^1},$$

[8]Vector inequalities are taken to hold componentwise.

since $[\mathbf{0},\mathbf{b}^2]\supset[\mathbf{0},\mathbf{b}^1]$. Let J_∞ denote the maximum value in the case of the unconstrained problem, i.e., let J_∞ be the maximum value of (5.6). Then it is easily shown as follows that

$$J_\infty = \sup_{\mathbf{b}>0} J_{\mathbf{b}}. \tag{5.11}$$

Let $K=\sup_{\mathbf{b}>0} J_{\mathbf{b}}$. Then since $[\mathbf{0},\mathbf{b}]\supset[\mathbf{0},\infty)$ for all $\mathbf{b}\geqslant 0$, we have $J_\infty \geqslant K$. However, by representing the optimal impulse controller as the limit of a sequence of controllers of the form $u_i^{(N)}(\cdot)=N$ on $[\tau_i^{(N)},\sigma_i^{(N)}]$, $u_i^{(N)}(\cdot)=0$ otherwise, where $\tau_i^{(N)},\sigma_i^{(N)}$ are chosen so that the optimal values for z_i and $\hat{t}_i$ are obtained in the limit as $N\to\infty$, one shows that $J(\mathbf{u}^{(N)})\leqslant J_{\mathbf{N}}\leqslant K$ and $J(\mathbf{u}^{(N)})\to J_\infty$ as $N\to\infty$, whence $J_\infty\leqslant K$, i.e., (5.11) holds.

Thus J_∞ can be regarded as the "ultimate" yield potential of the population under sustained exploitation. Since, however, J_∞ can be attained only by using impulse controls which are not physically implementable, J_∞ should be regarded as an idealized standard with which the performance of the physically constrained controls can be compared.

6. SCALAR HARVESTING

In many fisheries, for example purse-seine fisheries, harvesting proceeds by removing (according to mesh size of nets) individuals in a given age class and above, at a given intensity $u(t)$. In this case u is a scalar and (3.7) becomes [cf. (3.4)], for[9] $r\in\{0,\ldots,n-1\}$,

$$\begin{aligned} \dot{x}_i &= -\alpha_i x_i, \qquad i=1,\ldots,r,\\ \dot{x}_i &= -[\alpha_i+u(t)]x_i, \qquad i=r+1,\ldots,n. \end{aligned} \tag{6.1}$$

The problem now under consideration is to maximize [cf. (4.1)]

$$J_r(u(\cdot)) = \int_0^{\bar{t}} u(t) \sum_{i=r+1}^{n} w_i(t)x_i(t)\,dt \tag{6.2}$$

over all piecewise continuous $u(\cdot)$ taking values in $[0,b]$ and subject to (6.1).

The Hamiltonian for this problem is [cf. (4.3)]

$$H(\lambda,\mathbf{x},u,t) = u\sum_{i=r+1}^{n}[w_i(t)-\lambda_i]x_i - \sum_{i=1}^{n}\alpha_i\lambda_i x_i,$$

whence [cf. (4.8)]

$$H_u(t) = \sum_{i=r+1}^{n}[w_i(t)-\lambda_i(t)]x_i(t), \tag{6.3}$$

[9] $r=0$ is given the obvious interpretation of u included in all n equations.

and $\lambda(\cdot)$ should satisfy [cf. (4.4)]

$$\begin{aligned} -\dot{\lambda}_i &= -\alpha_i\lambda_i, \qquad i=1,\ldots,r, \\ -\dot{\lambda}_i &= -(\alpha_i+u)\lambda_i + w_i(t)u, \qquad i=r+1,\ldots,n. \end{aligned} \tag{6.4}$$

As in (4.6) and (4.7),

$$u^s(t)=0 \qquad \text{whenever} \quad H_u^s(t)<0, \tag{6.5}$$

$$u^s(t)=b \qquad \text{whenever} \quad H_u^s(t)>0. \tag{6.6}$$

Given r and b, the problem is to find the $u(\cdot)$ that maximizes (6.2) and hence results in a maximum sustainable yield. Let $u^r(\cdot)$ denote the controller that maximizes $J_r(u(\cdot))$. Suppose $u^r(\cdot)$ is found for $r=1,\ldots,n$; then the value of r that maximizes $J_r(u^r(\cdot))$ indicates the optimal choice of mesh size [in terms of the youngest age class which is harvested in (6.1)].

We note, however, that the optimal controller no longer has the simple form of $u_i(t)$ given in (4.10), since $H_u(t)$, given by (6.3), may pass through zero several times on $[0,\bar{t})$, and may in fact be singular on subintervals of $[0,\bar{t})$. In this case, systems of equations in $\mathbf{x}(0)$ and $\boldsymbol{\lambda}(0)$, similar to (4.12) and (4.16), can be obtained, but a general expression for $\mathbf{u}(\cdot)$ will have to be retained, i.e., we cannot *a priori* simplify the expressions [cf. (4.11) and (4.15)]

$$x_i(\bar{t}) = x_i(0)e^{-\alpha_i(t) - \int_0^{\bar{t}} u(t)\,dt}$$

and

$$\lambda_i(\bar{t}) = e^{\alpha_i\bar{t} + \int_0^{\bar{t}} u(t)\,dt}\left(\lambda_i(0) - \int_0^{\bar{t}} u(t)w_i(t)e^{-\alpha_i t - \int_0^{t} u(s)\,ds}\,dt\right),$$

$i=r+1,\ldots,n$. This then would seem to offer no substantial progress in solving the two-point boundary-value problem associated with the maximum principle for the problem under consideration.

Consider, however, the following problem relating to (6.1) and (6.2). Let r be given, and suppose $u(\cdot)$ is constrained, not to be piecewise continuous and taking values in $[0,b]$, but rather to be constant on $[0,\bar{t})$. This problem is of practical importance, since the constraint of harvesting at a given level over the whole season is often desirable: it avoids having to lay off fishermen during the harvesting season. The fact that there is no upper bound to u does not (in this case) necessarily imply that the maximum sustainable yield is obtained by letting $u\to\infty$, since, depending on the value of r, this may cause the breeding-age classes to become depleted, in which case the population would not be able to reproduce.

Let $u(\cdot)=v$ for $t\in[0,\bar{t})$. Then, as in deriving (4.12) and (4.13), we obtain (assuming $r\leqslant n-2$)[10]

$$\begin{aligned}
x_{i+1}(0)&=\prod_{j=1}^{i} e^{-\alpha_j}x_1(0), \qquad i=1,\ldots,r,\\
x_{i+1}(0)&=\prod_{j=1}^{r} e^{-\alpha_j}\prod_{j=r+1}^{i} e^{-\alpha_j-v\bar{t}}x_1(0), \qquad i=r+1,\ldots,n-2,\\
x_n(0)&=\left[\prod_{j=1}^{r} e^{-\alpha_j}\prod_{j=r+1}^{n-1}\frac{e^{-\alpha_j-v\bar{t}}}{1-e^{-\alpha_n-v\bar{t}}}\right]x_1(0),\\
x_1(0)&=\frac{\beta\phi_r(v)-\gamma}{\phi_r(v)},
\end{aligned} \tag{6.7}$$

where

$$\begin{aligned}
\phi_r(v)=&\sum_{i=1}^{r} c_i e^{\alpha_i(1-\bar{t})}\prod_{j=1}^{i} e^{-\alpha_j}\\
&+\sum_{i=r+1}^{n-1} c_i e^{\alpha_i(1-\bar{t})}\prod_{j=1}^{r} e^{-\alpha_j}\prod_{j=r+1}^{i} e^{-\alpha_j-v\bar{t}}\\
&+c_n e^{\alpha_n(1-\bar{t})}\prod_{j=1}^{r} e^{-\alpha_j}\prod_{j=r+1}^{n}\frac{e^{-\alpha_j-v\bar{t}}}{\left(1-e^{-\alpha_n-v\bar{t}}\right)}.
\end{aligned} \tag{6.8}$$

Furthermore, (6.2) becomes

$$J_r(u(\cdot)=v)=v\sum_{i=r+1}^{n} x_i(0)\int_0^{\bar{t}} w_i(t)e^{-(\alpha_i+v)t}\,dt. \tag{6.9}$$

Substituting (6.7) and (6.8) in (6.9), the problem under consideration reduces to that of finding the maximum of a scalar function $J(v)$ with respect to the parameter $v\geqslant 0$. $J(v)$ may, however, be an increasing function of v, in which case no maximum exists and $J(\infty)\geqslant J(v)$ for all $v\geqslant 0$. Again, $J(\infty)$ is finite and can be used as an idealized standard with which yields obtained for finite v can be compared. Also, it is clear from (6.1) that if $v\to\infty$, then in the limit all members in the age classes $r+1$ to n will be removed at the instant $t=0$.

Let the optimal value of b, for given r, be denoted by v^r, where possibly $v^r=\infty$. Then the corresponding yield $J_r(v^r)$ will satisfy

$$J_r(v^r)\geqslant J_r(v) \qquad \text{for all} \quad v\geqslant 0.$$

[10]The equations can be suitably adjusted for the case $r=n-1$. If $r=n-2$, then $i=n-1,\ldots,n-2$ is taken to imply that no equations are present in the second subsystem of equations (6.7).

As before, if $J_r(v^r)$ is calculated for $r=1,\dots,n$, then the optimal value of r, say r^*, is chosen to satisfy

$$J_{r^*}(v^{r^*}) \geqslant J_r(v^r) \qquad \text{for all} \quad r\in\{1,\dots,n\}.$$

7. CONCLUSION

In the Introduction (Sec. 1), the history and present "state of the art" in fisheries management models was briefly reviewed. It was pointed out that an outstanding problem in this area is to link up into a unified setting the modeling of discrete events (using systems of difference equations) with the modeling of continuous ongoing processes (using systems of differential equations). This is done in Sec. 2 in the general setting of systems of differential equations with state-variable jump discontinuities. A general harvesting model is presented that divides the population into two distinct phases, viz. the harvesting-season dynamics and the spawning-season transformation. A maximum principle is presented for this type of system, and it is shown that in the case of periodic systems the maximum-sustainable-rent solution for the one-year harvesting problem satisfies the maximum principle for the N-year harvesting problem. This maximum principle can be used in the same manner as the standard maximum principle is used to construct candidate optimal solutions from the associated two-point boundary-value problem.

In Sec. 3 it is shown that the Beverton-Holt theory in fisheries management naturally extends into the state-variable jump-discontinuity setting: the harvesting-season dynamics are modeled by a system of differential equations that are linear in state but bilinear in state and control, while a stock-recruitment relationship is included in the spawing-season transformation. This model is used in Sec. 4 in the presentation and subsequent analysis of a maximum-sustainable-yield problem. Application of the maximum principle makes it possible to deduce the form of the optimal solution. This facilitates the reduction of the associated two-point boundary value to a parameter-selection problem. Numerical algorithms for solving the latter problem are not discussed in this paper, and they present an area for future research.

In Sec. 5 the unconstrained version of the above maximum-sustained-yield problem is introduced as the limit of associated constrained problems as the constraints increase without bound. The solution to the unconstrained problem is designated as the "ultimate"-sustainable-yield solution, as it provides the least upper bound to sustainable yields that can be realized by any bounded harvesting strategy. The unconstrained problem is shown to be equivalent to a programming problem which, although nonlinear, contains a number of linear structures. In a companion to this paper

[25], it has been shown that the above programming problem reduces to a linear parametric programming problem with two constraints. Furthermore, a solution to this problem is provided in [25] where the ultimate sustainable yield is evaluated for a real fishery.

Finally, in Sec. 6. models including only scalar control (harvesting) are introduced, as these are particularly important in practical applications. These models, apart from being discussed in the usual maximum-sustainable-yield formulation, are also discussed in the formulation where the additional constraint is imposed of constant harvesting intensity over the whole harvesting season. It is shown that under this constraint the problem reduces to a classical single-parameter optimization problem, which can be solved by applying standard techniques. More importantly, however, for given r and v the corresponding yield can be easily and directly computed from (6.9). If this is done over a grid of (r, v)-values, a comprehensive knowledge of the yield characteristics of the fishery as a function of r and v can be obtained. A full numerical example along these lines is presented in [26], where a refinement of the model presented here in Sec. 6 is also considered.

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