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Optimal Housing, Consumption, and Investment Decisions over the Life Cycle

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HOLGER KRAFT^{*} / CLAUS MUNK[†]

**OPTIMAL HOUSING, CONSUMPTION, AND INVESTMENT DECISIONS OVER THE
LIFE-CYCLE[‡]**

The paper contains graphs in color. Use color printer for best result.

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solely those of the authors*

Abstract

We provide explicit solutions to life-cycle utility maximization problems simultaneously involving dynamic decisions on investments in stocks and bonds, consumption of perishable goods, and the rental and the ownership of residential real estate. House prices, stock prices, interest rates, and the labor income of the decision-maker follow correlated stochastic processes. The preferences of the individual are of the Epstein-Zin recursive structure and depend on consumption of both perishable goods and housing services. The explicit consumption and investment strategies are simple and intuitive and are thoroughly discussed and illustrated in the paper. For a calibrated version of the model we find, among other things, that the fairly high correlation between labor income and house prices imply much larger life-cycle variations in the desired exposure to house price risks than in the exposure to the stock and bond markets. We demonstrate that the derived closed-form strategies are still very useful if the housing positions are only reset infrequently and if the investor is restricted from borrowing against future income. Our results suggest that markets for REITs or other financial contracts facilitating the hedging of house price risks will lead to non-negligible but moderate improvements of welfare.

Keywords: Housing, labor income, portfolio choice, life-cycle decisions, recursive utility, REITs

JEL-Classification: G11, D14, D91, C6

Optimal Housing, Consumption, and Investment Decisions over the Life-Cycle

1 Introduction

The two largest assets for many individuals are the human capital and the residential property owned and occupied by the individual. The financial decisions of individuals over the life-cycle are bound to be affected by the characteristics of these assets. While the early literature on dynamic consumption and portfolio decisions (Samuelson 1969; Merton 1969, 1971) ignored such non-financial assets, progress has recently been made with respect to incorporating and understanding housing decisions and labor income in a life-cycle framework of consumption and portfolio choice. Due to the complexity of such decision problems, almost all of these studies resort to rather coarse and computationally very intensive numerical solution techniques with an unknown precision. In contrast, this paper provides closed-form solutions for continuous-time problems involving both consumption, housing, and investment decisions when stock prices, interest rates, labor income, and house prices vary stochastically over time. Preferences are modeled by a two-good extension of Epstein-Zin recursive utility that allows for a separation of the risk aversion and the elasticity of intertemporal substitution, with exact closed-form solutions given for the two special cases of power utility and a unit elasticity of substitution and an approximate closed-form solution for the general case. These closed-form solutions lead to a deeper understanding of the economic forces driving individual decisions in such a complex setting. For a calibrated version of the model we show that the solutions from the model imply fairly realistic consumption and investment patterns over the life-cycle.

Our model has the following features. The individual derives utility from consumption of perishable goods and of housing services and maximizes life-time utility of the Epstein-Zin type. The individual receives an exogenous stochastic stream of labor income until a fixed retirement date after which the individual lives for another fixed period of time. Our specification of the income process encompasses life-cycle variations in the expected growth rate and volatility and also allows for variations in expected income growth related to the short-term interest rate in order to reflect dependence on the business cycle. The pure financial assets available are a stock, a bond, and short-term deposits (cash). The short-term interest rate and the returns on the bond are modeled by the classical Vasicek model, and for the stock price we assume a constant expected excess return, a constant volatility, and a constant correlation with the bond price. The individual can buy and sell houses² at a unit price that varies stochastically with a constant expected growth rate in excess of the short-term interest rate, a constant volatility, and constant correlations with labor income and financial asset prices. The purchase of a house serves a dual role by both generating consumption services and by constituting an investment affecting future wealth and consumption opportunities. We allow the individual to disentangle the two dimensions of housing by renting the house instead of owning it (the rent is proportional to the price

²In order to keep the terminology simple we use “house” instead of the more general term “residential property.”

of the house rented) and/or by investing in a financial asset linked to house prices. In current financial markets, shares in REITs (Real Estate Investment Trusts) and the S&P/Case-Shiller Home Price Indices (CSI) futures and options traded at the Chicago Mercantile Exchange offer such opportunities; more information on these contracts is provided in Section 2.

In order to derive closed-form solutions our main model exhibits market completeness (cf., e.g., [Liu 2007](#)) so, in particular, the labor income stream has to be spanned by the traded assets. The correlations between an individual's labor income and the returns on stocks and bonds are probably quite low.³ However, labor income tends to be highly correlated with house prices (e.g. [Cocco \(2005\)](#) reports a correlation of 0.55) so that the income spanning assumption is less unrealistic in our model with housing than in the models with labor income, but no housing, studied in the existing literature (references given below). Still it may not be possible to find a trading strategy in stocks, bonds, deposits, and houses that perfectly replicates the income risk. Without perfect spanning it seems impossible to derive the optimal investment strategy in closed-form or even with a precise, numerical solution technique. While the investment strategy we derive in this paper will then be sub-optimal, the results presented in [Bick, Kraft, and Munk \(2008\)](#) for a similar, though slightly simpler, model indicate that it will be near-optimal in the sense that the investor will at most suffer a loss corresponding to a few percent of his initial wealth by following the closed-form sub-optimal strategy instead of the unknown optimal strategy. The results we present below will therefore be highly relevant even without perfect spanning.

The high correlation between labor income and house prices implies the following distinct life-cycle pattern in the investment exposure to house price risk. When human wealth is big relative to financial wealth (e.g. early in life), the individual should invest very little in housing so that the desired housing consumption is mainly achieved by renting. When human wealth is low relative to financial wealth (e.g. late in life), the optimal housing investment is quite big due to its fairly attractive risk-return trade-off. We find that the optimal housing investment varies much more over the life-cycle than the optimal investments in bonds and stocks.

In our main model the individual can continuously and costlessly adjust both the housing consumption and the housing investment, but we also consider problems with limited flexibility in housing decisions. Changes in physical ownership of housing generate substantial transaction costs not included in our model, so continuous adjustments of housing investment must be implemented by rebalancing the position in the house-price linked financial asset. We have to assume a perfect correlation between the returns on that asset and house prices, which may be unattainable in actual markets but carefully selected REITs or CSI housing contracts will come close.⁴ The case where both housing consumption and housing investment are continuously adjustable can be seen as an upper bound on the life-time utility that the individual can realistically obtain. We investigate the importance of the frequency

³The correlation between average labor income and the general stock market is usually estimated to be close to zero (see, e.g., [Cocco, Gomes, and Maenhout 2005](#)), but it should be possible to find single stocks highly correlated with the labor income of a particular individual.

⁴[Tsai, Chen, and Sing \(2007\)](#) report that REITs behave more and more like real estate and less and less like ordinary stocks.

of adjustments of the housing consumption and housing investment in two ways. First, we derive an explicit solution to the problem where the individual consumes a constant level of housing services through life, adjusts the housing investment position continuously, and has time-additive power utility of consumption with no utility from terminal wealth. We find that the utility decrease due to the fixed housing consumption is equivalent to less than 1% of total wealth. Second, we implement a Monte Carlo simulation procedure to compute the expected utility of an individual restricted to infrequent adjustments of (a) housing consumption, (b) housing investment, or (c) both. Again, we find that the wealth-equivalent loss is fairly small even when the housing positions are only adjusted every five years. Our results indicate that it is more important to adjust the housing investment position frequently than the housing consumption position, suggesting that a well-functioning market for REITs or other financial contracts related to house prices can lead to a non-negligible improvement in the welfare of individual investors.

Another questionable feature of our main model is the possibility of the investor to borrow against future labor income. The optimal unconstrained consumption and investment strategy will lead to cases where the tangible wealth (the sum of financial wealth and the value of the housing stock owned) will be negative but the human wealth more than outweighs that so that total wealth is positive. Such a strategy may not be feasible in real life due to moral hazard and asymmetric information issues in the valuation of human wealth. We introduce a minor transformation of the optimal unconstrained consumption and investment strategy that ensures non-negative tangible wealth at all points in time. We evaluate the expected utility generated by this transformed strategy by Monte Carlo simulation and find that it corresponds to a wealth-equivalent loss of only about one percent compared to the maximum expected utility in the unconstrained case, using our benchmark parameter values. Hence, the transformation of the closed-form solution must be at least near-optimal in the borrowing constrained case, emphasizing the relevance of our closed-form solution in more realistic settings.

Next we briefly compare our setting and findings to some recent related papers. [Cocco \(2005\)](#) considers a model featuring stochastic house prices and labor income with an assumed perfect correlation between house prices and aggregate income shocks. Interest rates are assumed constant. Renting is not possible. The individual is allowed to borrow only up to a percentage of the current value of the house. There is a minimum choice of house size, and house transactions carry a proportional cost. The individual has to pay a one-time fixed fee to participate in the stock market. [Yao and Zhang \(2005a\)](#) add mortality risk and the possibility of renting to Cocco's framework and do not impose a perfect correlation between house prices and income. [Van Hemert \(2007\)](#) generalizes the setting further by allowing for stochastic variations in interest rates and thereby introducing a role for bonds, and he also addresses the choice between an adjustable-rate mortgage and a fixed-rate mortgage (ignoring the important prepayment option, however). The latter two papers disregard the stock market entry fee in Cocco's model.

All these three papers apply numerical solution techniques based on a discretization of time and the state space. [Yao and Zhang \(2005a\)](#) and [Cocco \(2005\)](#) solve the dynamic programming equation related to the problem by applying a very coarse discretization, e.g. using binomial processes and large time intervals between revisions of decisions. [Van Hemert \(2007\)](#) is able to handle a finer dis-

cretization by relying on 60 parallel computers. It is difficult to assess the precision of such numerical techniques and, in any case, the computational procedures are highly time-consuming and cumbersome. The closed-form solutions derived in this paper are much easier to analyze, interpret, and implement and thus facilitate an understanding and a quantification of the economic forces at play. Moreover, the three above-mentioned papers assume preferences of the time-additive Cobb-Douglas style. We build the instantaneous Cobb-Douglas utility of perishable consumption and housing services into an Epstein-Zin recursive utility formulation allowing us to disentangle the risk aversion γ and the elasticity of intertemporal substitution ψ , as has been shown to be valuable both for consumption-portfolio choice with one consumption good (see, e.g., [Campbell and Viceira \(1999\)](#), [Campbell, Cocco, Gomes, Maenhout, and Viceira \(2001\)](#), and [Chacko and Viceira \(2005\)](#)) and for equilibrium asset prices (see [Bansal and Yaron \(2004\)](#)). We provide exact closed-form solutions for the special case of time-additive Cobb-Douglas utility, corresponding to $\psi = 1/\gamma$, and for the more reasonable case where $\psi = 1$ and $\gamma > 1$. Extending the log-linearization technique of [Campbell \(1993\)](#) and [Chacko and Viceira \(2005\)](#), we derive an approximate closed-form solution for general combinations of ψ and γ .

[Damgaard, Fuglsbjerg, and Munk \(2003\)](#) do provide a closed-form solution for a related but much simpler problem of an individual maximizing time-additive Cobb-Douglas utility over consumption and owner-occupied housing, when the size of the house occupied can be continuously and costlessly rebalanced. They ignore the possibility of renting as well as labor income and variations in interest rates. They provide a mathematical and numerical analysis of the case with a proportional cost on house transactions.

Some more marginally related papers deserve to be mentioned. [Campbell and Cocco \(2003\)](#) study the mortgage choice in a life-cycle framework with stochastic house price, labor income, and interest rates. By fixing the house, however, they are not able to address the interaction between housing decisions and portfolio decisions. Moreover, their solution relies on a very coarse discretization of the model, e.g. with two year time intervals where decisions cannot be revised. [Munk and Sørensen \(2008\)](#) solve the life-cycle consumption and investment problem with stochastic labor income and interest rates, but do not incorporate houses in neither consumption nor investment decisions. They find a closed-form solution for a complete market version of their model, which is generalized to include housing decisions and recursive utility in our paper. They also report results from a numerical solution for the case where labor income risk is not spanned by traded financial assets.

While we investigate individual decision making in the presence of housing wealth and human capital on individual decisions, the role of these two factors in equilibrium asset pricing have also been subject to recent theoretical and empirical research. Papers on the impact of housing decisions and prices on financial asset prices include [Piazzesi, Schneider, and Tuzel \(2007\)](#), [Lustig and van Nieuwerburgh \(2005\)](#), and [Yogo \(2006\)](#), while papers such as [Constantinides, Donaldson, and Mehra \(2002\)](#), [Santos and Veronesi \(2006\)](#), and [Storesletten, Telmer, and Yaron \(2004, 2007\)](#) focus on the interaction of labor income risk and asset prices.

To summarize our contribution, we derive explicit expressions for the optimal life-cycle housing, consumption, and investment decisions of an investor having Epstein-Zin utility in a rich model taking into account variability in labor income, interest rates, and the prices of houses, stocks, and bonds.

We discuss the structure of the solution and show in a numerical example that our solution generates a life-cycle behavior with many realistic features. We also provide results suggesting that our solution will still be close to optimal if some of the unpleasant model assumptions are relaxed.

The remainder of the paper is organized as follows. Section 2 formulates and discusses the ingredients of our model and the utility maximization problem faced by the individual. Section 3 states, explains, and illustrates the optimal housing, consumption, and investment strategies in the case when housing decisions can be controlled continuously. Section 4 investigates the effect of limiting the flexibility in revising housing decisions and provides estimates of the value of being able to make continuous revisions for example via trade in financial contracts linked to house prices. Section 5 summarizes and concludes. All proofs are collected in the appendices at the end of the paper.

2 The problem

The main elements of our modeling framework are specified as follows.

Consumption goods. The individual can consume two goods: perishable consumption and housing. The perishable consumption good is taken as the numeraire so that the prices of the housing good and of all financial assets are measured in units of the perishable consumption good.

Financial assets. The individual can invest in three purely financial assets: a money market account (cash), a bond, and a stock (representing the stock market index). The return on the money market account equals the continuously compounded short-term real interest rate r_t , which is assumed to have Vasicek dynamics

$$dr_t = \kappa[\bar{r} - r_t] dt - \sigma_r dW_{rt}, \quad (2.1)$$

where $W_r = (W_{rt})$ is a standard Brownian motion. The price of any bond (or any other interest rate derivative) is then of the form $B_t = B(r_t, t)$ with dynamics

$$dB_t = B_t [(r_t + \lambda_B \sigma_B(r_t, t)) dt + \sigma_B(r_t, t) dW_{rt}], \quad (2.2)$$

where $\sigma_B(r, t) = -\sigma_r B_r(r, t)/B(r, t)$ is the volatility and λ_B the Sharpe ratio of the bond, which is identical to the market price of interest rate risk. In particular, if we introduce the notation

$$\mathcal{B}_m(\tau) = \frac{1}{m} (1 - e^{-m\tau})$$

for any positive constant m , the time t price of a real zero-coupon bond maturing at some date $T > t$ can be written as

$$B_t^T = e^{-a(T-t) - \mathcal{B}_\kappa(T-t)r_t}, \quad (2.3)$$

$$a(\tau) = \left(\bar{r} - \frac{\lambda_B \sigma_r}{\kappa} - \frac{\sigma_r^2}{2\kappa^2} \right) (\tau - \mathcal{B}_\kappa(\tau)) + \frac{\sigma_r^2}{4\kappa} \mathcal{B}_\kappa(\tau)^2. \quad (2.4)$$

An investor will not gain from trading in more than one bond in addition to the money market account. The stock price S_t has dynamics

$$dS_t = S_t \left[(r_t + \lambda_S \sigma_S) dt + \sigma_S \left(\rho_{SB} dW_{rt} + \sqrt{1 - \rho_{SB}^2} dW_{St} \right) \right], \quad (2.5)$$

where $W_S = (W_{St})$ is a standard Brownian motion independent of W_r , σ_S is the constant volatility and λ_S the constant Sharpe ratio of the stock, and ρ_{SB} is the constant correlation between the stock and the bond returns.

Houses. The individual can also buy or rent houses. A given house is assumed to be fully characterized by a number of housing units, where a “unit” is some one-dimensional representation of the size, quality, and location. Prices of all houses move in parallel. The purchase of a units of housing costs aH_t ; there are no transaction costs. The unit house price H_t is assumed to have dynamics

$$dH_t = H_t \left[(r_t + \lambda_H \sigma_H - r^{\text{imp}}) dt + \sigma_H (\rho_{HB} dW_{rt} + \hat{\rho}_{HS} dW_{St} + \hat{\rho}_H dW_{Ht}) \right], \quad (2.6)$$

where $W_H = (W_{Ht})$ is a standard Brownian motion independent of W_r and W_S , σ_H is the constant price volatility and λ_H the constant Sharpe ratio of houses, ρ_{HB} is the constant correlation between house and bond prices, and

$$\hat{\rho}_{HS} = \frac{\rho_{SH} - \rho_{SB}\rho_{HB}}{\sqrt{1 - \rho_{SB}^2}}, \quad \hat{\rho}_H = \sqrt{1 - \rho_{HB}^2 - \hat{\rho}_{HS}^2}$$

where ρ_{SH} is the constant correlation between house and stock prices. Finally, r^{imp} is the imputed rent, i.e. the market value associated with the net benefits offered by a house (similar to the convenience yield of commodities), which is assumed to be constant as, e.g., in [Van Hemert \(2007\)](#).

The unit rental cost of houses is assumed to be proportional to the current unit house price, i.e. νH_t for some constant ν . For later use, define $\lambda'_H = \lambda_H + (\nu - r^{\text{imp}})/\sigma_H$. By renting instead of owning the house, the individual can isolate the consumption role of housing.

We assume that the individual can invest in a financial asset with a price that follows the movements in house prices. In a number of countries, shares in REITs are publicly traded. A REIT (Real Estate Investment Trust) is an investment company that invests in (and often operates) real estate generating rental income and hopefully capital gains so, by construction, the prices of REIT shares will be closely related to real estate prices.⁵ While REITs in general may be interesting as an asset class improving the overall risk-return tradeoff, REITs specializing in residential real estate are particularly interesting for existing or prospective individual homeowners as a vehicle to manage exposure to house price risk without having to physically trade houses frequently.

⁵REITs were introduced in the U.S. in the 1960s and the REIT industry has experienced substantial growth since the early 1990s. According to the website of the National Association of Real Estate Investment Trusts on November 7, 2007 (see www.nareit.com) shares in 190 U.S. REITs are publicly traded with a total market capitalization of more than \$400 billion, and (as of September 28, 2007) 14 of the companies in S&P500 index are REITs. Well-established REIT markets also exist in countries such as Japan, Canada, France, and the Netherlands, and are under development in many other countries, e.g. in Germany.

If we let R_t denote the value of the REIT per unit of housing and assume that the REIT passes on rental income to shareholders as a dividend, we will have $R_t = H_t$ and the total instantaneous return from a REIT is

$$dH_t + \nu H_t dt = H_t [(r_t + \lambda'_H \sigma_H) dt + \sigma_H (\rho_{HB} dW_{rt} + \hat{\rho}_{HS} dW_{St} + \hat{\rho}_H dW_{Ht})]. \quad (2.7)$$

An alternative to REITs is the housing futures (and options on housing futures) traded since 2006 at the Chicago Mercantile Exchange. The payoff of such a contract is determined by either a U.S. national home price index or by a home price index for one of 10 major U.S. cities; the indices were developed by Case and Shiller, hence the contracts are also referred to as CSI futures and options. See [de Jong, Driessen, and Van Hemert \(2008\)](#) for a partial analysis of the economic benefits of having access to such housing futures.

Note that when an individual physically owns a house, a negative position in the money market account can be interpreted as an adjustable-rate mortgage, whereas a negative position in the long-term bond resembles a fixed-rate mortgage. In order to obtain closed-form solutions we do not limit borrowing to some fraction of the market value of the house owned.

Labor income. The individual is assumed to retire from working life at time \tilde{T} and live until time $T \geq \tilde{T}$. During working life the individual receives a continuous and exogenously given stream of income from non-financial sources (e.g. labor) at a rate of Y_t which has the dynamics

$$dY_t = Y_t [\mu_Y(r_t, t) dt + \sigma_Y(r_t, t) (\rho_{YB} dW_{rt} + \hat{\rho}_{YS} dW_{St} + \hat{\rho}_Y dW_{Ht})]. \quad (2.8)$$

For analytical tractability there is no idiosyncratic shock to the income process, hence the market is complete, but as discussed in the introduction this is not a crucial assumption for the relevance of our results. The expected percentage income growth μ_Y and volatility σ_Y are allowed to depend on time (age of the individual) and the interest rate level to reflect fluctuations of labor income over the life- and business cycle, cf., e.g., [Cocco, Gomes, and Maenhout \(2005\)](#) and [Munk and Sørensen \(2008\)](#). ρ_{YB} is the constant correlation between income growth and bond returns, and

$$\hat{\rho}_{YS} = \frac{\rho_{SY} - \rho_{SB}\rho_{YB}}{\sqrt{1 - \rho_{SB}^2}}, \quad \hat{\rho}_Y = \sqrt{1 - \rho_{YB}^2 - \hat{\rho}_{YS}^2}$$

where ρ_{SY} is the constant correlation between house and stock prices. Due to the completeness assumption the correlation between income growth and house prices follow from the other pairwise correlations,

$$\rho_{YH} = \rho_{HB}\rho_{YB} + \hat{\rho}_{HS}\hat{\rho}_{YS} + \hat{\rho}_H\hat{\rho}_Y.$$

In the retirement period $[\tilde{T}, T]$, the individual is assumed to have no income from non-financial sources.

The human capital of the individual is the present value of the entire future labor income stream. In a complete market with risk-neutral probability measure \mathbb{Q} , the human capital is unique and given by

$$L_t = L(t, r_t, y_t) = \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^{\tilde{T}} e^{-\int_t^s r_u du} y_s ds \right] = y_t \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^{\tilde{T}} e^{-\int_t^s r_u du} \frac{y_s}{y_t} ds \right] \equiv y_t F(t, r_t),$$

using the fact that the distribution of $\frac{y_s}{y_t}$ is independent of y_t . For general functions μ_Y and σ_Y , F can be found by solving a PDE. We specialize to the case

$$\mu_Y(r, t) = \bar{\mu}_Y(t) + br, \quad \sigma_Y(r, t) = \sigma_Y(t) \quad (2.9)$$

for deterministic functions $\bar{\mu}_Y$ and σ_Y , where the next theorem gives a closed-form solution for $F(t, r)$. This specification allows for the well-known life-cycle pattern in expected income growth and income volatility, see e.g. [Cocco, Gomes, and Maenhout \(2005\)](#), and also for a business-cycle variation in the expected income growth via the relation to the real interest rate, see e.g. [Munk and Sørensen \(2008\)](#).

Theorem 2.1 (Human capital) *When labor income is given by (2.8) and (2.9), the human capital is $L(t, r_t, y_t) = y_t F(t, r_t)$ with*

$$F(t, r) = 1_{\{t \leq \tilde{T}\}} \int_t^{\tilde{T}} e^{-\tilde{A}(t, s) - (1-b)\mathcal{B}_\kappa(s-t)r} ds, \quad (2.10)$$

where

$$\begin{aligned} \tilde{A}(t, s) = & (\kappa\bar{r} + \sigma_r\lambda_B)(1-b) \frac{s-t - \mathcal{B}_\kappa(s-t)}{\kappa} - \rho_{YB}\sigma_r(1-b) \int_t^s \sigma_Y(u)\mathcal{B}_\kappa(s-u) du \\ & - \frac{1}{2}\sigma_r^2(1-b)^2 \frac{1}{\kappa^2} [s-t - 2\mathcal{B}_\kappa(s-t) + \mathcal{B}_{2\kappa}(s-t)] - \int_t^s \bar{\mu}_Y(u) du + \lambda_Y \int_t^s \sigma_Y(u) du, \end{aligned}$$

and λ_Y is defined in (A.3) in Appendix A. The expected future income rate is

$$\begin{aligned} E_0[Y_t] = Y_0 \exp \left\{ \int_0^t \bar{\mu}_Y(u) du + br_0 t + b \left(\bar{r} - r_0 + \frac{b\sigma_r^2}{2\kappa^2} \right) (t - \mathcal{B}_\kappa(t)) \right. \\ \left. - \frac{b^2\sigma_r^2}{4\kappa} \mathcal{B}_\kappa(t)^2 - b\sigma_r\rho_{YB} \int_0^t \sigma_Y(u)\mathcal{B}_\kappa(t-u) du \right\}, \quad (2.11) \end{aligned}$$

and expected future human capital is $E_0[L(t, r_t, Y_t)] = \bar{F}(t)E_0[Y_t] \approx F(t, \bar{r})E_0[Y_t]$, where \bar{F} is given in (A.10).

For a proof, we refer the reader to Appendix A.

Wealth dynamics. The individual's tangible wealth at any time t is denoted by X_t and defined as the value of his current position in the money market account, the bond, the stock, and REITs, plus the value of the house owned by the individual. Let π_{St} and π_{Bt} denote the fraction of tangible wealth invested in the stock and the bond, respectively, at time t . Let φ_{ot} and φ_{rt} denote the units of housing owned and rented, respectively, at time t . Let φ_{Rt} denote the number of shares in REITs owned at time t . The wealth invested in the money market account is then $X_t(1 - \pi_{St} - \pi_{Bt}) - (\varphi_{ot} + \varphi_{Rt})H_t$. Finally, let c_t denote the rate at which the perishable good is consumed at time t . The dynamics of

tangible wealth is then

$$\begin{aligned}
dX_t &= \pi_{St} X_t \frac{dS_t}{S_t} + \pi_{Bt} X_t \frac{dB_t}{B_t} + [X_t(1 - \pi_{St} - \pi_{Bt}) - (\varphi_{ot} + \varphi_{Rt})H_t] r_t dt \\
&\quad + \varphi_{ot} dH_t + \varphi_{Rt} (dH_t + \nu H_t dt) - \varphi_{rt} \nu H_t dt - c_t dt + 1_{\{t \leq \tilde{T}\}} Y_t dt \\
&= \left[X_t (r_t + \pi_{St} \lambda_S \sigma_S + \pi_{Bt} \lambda_B \sigma_{Bt}) + \varphi_{It} \lambda'_H \sigma_H H_t - \varphi_{Ct} \nu H_t - c_t + 1_{\{t \leq \tilde{T}\}} Y_t \right] dt \\
&\quad + (\pi_{St} X_t \rho_{SB} \sigma_S + \pi_{Bt} X_t \sigma_{Bt} + \varphi_{It} H_t \rho_{HB} \sigma_H) dW_{rt} \\
&\quad + \left(\pi_{St} X_t \sigma_S \sqrt{1 - \rho_{SB}^2} + \varphi_{It} H_t \hat{\rho}_{HS} \sigma_H \right) dW_{St} + \varphi_{It} H_t \hat{\rho}_H \sigma_H dW_{Ht},
\end{aligned} \tag{2.12}$$

where

$$\varphi_{Ct} \equiv \varphi_{ot} + \varphi_{rt}, \quad \varphi_{It} \equiv \varphi_{ot} + \varphi_{Rt}, \tag{2.13}$$

so that φ_{Ct} is the total units of housing occupied by (and thus providing housing services to) the individual and φ_{It} is the total units of housing invested in either physically or indirectly through REITs. The wealth dynamics and the welfare of the individual are thus only affected by φ_{Ct} and φ_{It} so that, in general, there will be one degree of freedom. To obtain a unique solution we will have to fix one of the three control variables φ_o , φ_r , and φ_I .

Preferences. We use a stochastic differential utility or recursive utility specification for the preferences of the individual so that the utility index V_t^ω associated at time t with a given control process $\omega = (c, \varphi_o, \varphi_r, \varphi_R, \pi_S, \pi_B)$ over the remaining lifetime $[t, T]$ is recursively given by

$$V_t^\omega = \mathbb{E}_t \left[\int_t^T f(z_u^\omega, V_u^\omega) du + \bar{V}_T^\omega \right]. \tag{2.14}$$

Here $z_u^\omega = c_u^\beta \varphi_{Cu}^{1-\beta}$ is the weighted composite consumption at time u with $\beta \in (0, 1)$ defining the relative importance of the two consumption goods, where $\varphi_C = \varphi_o + \varphi_r$ as in (2.13). A unit of housing is assumed to contribute identically to the direct utility whether owned or rented. We assume that the so-called normalized aggregator f is defined by

$$f(z, V) = \begin{cases} \frac{\delta}{1-1/\psi} z^{1-1/\psi} ([1-\gamma]V)^{1-1/\theta} - \delta \theta V, & \text{for } \psi \neq 1 \\ (1-\gamma)\delta V \ln z - \delta V \ln ([1-\gamma]V), & \text{for } \psi = 1 \end{cases} \tag{2.15}$$

where $\theta = (1-\gamma)/(1-\frac{1}{\psi})$. The preferences are characterized by the three parameters δ, γ, ψ . It is well-known that δ is a time preference parameter, $\gamma > 1$ reflects the degree of relative risk aversion towards atemporal bets (on the composite consumption level z in our case), and $\psi > 0$ reflects the elasticity of intertemporal substitution (EIS) towards deterministic consumption plans.⁶ The term \bar{V}_T^ω represents terminal utility and we assume that $\bar{V}_T^\omega = \frac{\varepsilon}{1-\gamma} (X_T^\omega)^{1-\gamma}$, where $\varepsilon \geq 0$ and X_T^ω is the terminal wealth induced by the control process ω . The special case where $\psi = 1/\gamma$ (so that

⁶It is also possible to define a normalized aggregator for $\gamma = 1$ and for $0 < \gamma < 1$ but we focus on the empirically more reasonable case of $\gamma > 1$.

$\theta = 1$) corresponds to the classic time-additive utility with the Cobb-Douglas-style instantaneous utility function⁷ $\frac{1}{1-\gamma} \left(c_u^\beta \varphi_{Cu}^{1-\beta} \right)^{1-\gamma}$.

Let \mathcal{A}_t denote the set of admissible control processes ω over the remaining lifetime $[t, T]$. Constraints on the controls are reflected by \mathcal{A}_t . At any point in time $t < T$, the individual maximizes V_t^ω over all admissible control processes given the values of the state variables at time t . The value function associated with the problem is defined as

$$J(t, x, r, h, y) = \sup \left\{ V_t^\omega \mid (\omega_u)_{u \in [t, T]} \in \mathcal{A}_t, X_t = x, r_t = r, H_t = h, Y_t = y \right\} \quad (2.16)$$

(ignoring y in the retirement phase $t \in [\tilde{T}, T]$). Throughout the analysis we solve the relevant utility maximization problems applying the dynamic programming principle; see [Duffie and Epstein \(1992\)](#) on the validity of this solution technique in the case of stochastic differential utility.

The above utility specification is the continuous-time analogue of the Kreps-Porteus-Epstein-Zin recursive utility defined in a discrete-time setting. Both the discrete-time and the continuous-time versions have been applied in a few recent studies of utility maximization problems involving a single consumption good, cf. [Campbell and Viceira \(1999\)](#), [Campbell, Cocco, Gomes, Maenhout, and Viceira \(2001\)](#), and [Chacko and Viceira \(2005\)](#), and was also applied in a two-good setting related to ours by [Yao and Zhang \(2005b\)](#). Other recent papers modeling related two-good utility maximization problems apply the classic time-additive utility with a Cobb-Douglas-style instantaneous utility function, cf. [Cocco \(2005\)](#), [Yao and Zhang \(2005a\)](#), and [Van Hemert \(2007\)](#).

Benchmark parameter values. When we illustrate our findings in the following sections, we will use the parameter values listed in Table 1 unless otherwise noted. Our benchmark parameter values are roughly in line with those used in similar studies referred to in the introduction. In our illustrations we assume constant $\bar{\mu}_Y$ and σ_Y . This allows us to focus on understanding the impact of the state variables and their interactions on the life-cycle behavior and disregards the more mechanical time-dependence, which is of secondary importance. Whenever we need to specify the bond that the individual invests in, we take it to be a 20-year zero-coupon bond. Unless mentioned otherwise, the results reported presume that the current value of the short-term interest rate is identical to the long-term average, $r = \bar{r}$.

Whenever we need to use levels of current or future house prices, wealth, labor income etc., we use a unit of USD 1 scaled by one plus the inflation rate in the perishable consumption good. For concreteness we think of houses as being fully represented by the number of square feet (of “average quality and location”) and will later use an initial value of $h = 200$ corresponding to USD 200,000 for a house of 1,000 square feet. When the short rate is at its long-term average, the expected growth rate

⁷With $\psi = 1/\gamma$, the recursion (2.14) is satisfied by

$$V_t^\omega = \delta \left(\mathbb{E}_t \left[\int_t^T e^{-\delta(u-t)} \frac{1}{1-\gamma} z_u^{1-\gamma} du + \frac{1}{\delta} e^{-\delta(T-t)} \frac{\varepsilon}{1-\gamma} (X_T^\omega)^{1-\gamma} \right] \right),$$

which is a positive multiple of the traditional time-additive power utility specification. Note that $\varepsilon = \delta$ would correspond to the case where utility of a terminal wealth of X will count roughly as much as the utility of consuming X over the final year.

of house prices is a modest 0.7% per year. This value may seem low given the house price inflation in most developed countries over the last decade, but it is in fact very reasonable considering house price movements over a longer period, cf. the discussions in [Cocco \(2005\)](#) who assumes an expected growth rate of 1% and [Yao and Zhang \(2005a\)](#) who use 0%.

[Table 1 about here.]

3 Solution with fully flexible housing decisions

Assume for now that the individual can continuously and costlessly adjust both the number of housing units consumed and the number of housing units invested in. We shall refer to this situation as “fully flexible housing decisions.” Due to (2.13), we can assume that the individual never has any direct ownership of housing units but continuously adjusts the investment in REITS to obtain the desired housing investment level and continuously adjust the number of housing units rented to achieve the desired housing consumption level. Alternatively, we can disregard REITs and assume a continuously adjusted direct ownership of housing units (admittedly, that may involve substantial transactions costs excluded from the theoretical framework of this paper), as well as a continuously adjusted renting position.

In Appendix B we demonstrate that the value function under fully flexible housing decisions can be separated as

$$J(t, x, r, h, y) = \frac{1}{1-\gamma} g(t, r, h)^\gamma (x + yF(t, r))^{1-\gamma}, \quad (3.1)$$

where g solves a partial differential equation (PDE). This form of the value function has also been found in many simpler cases. The total initial wealth of the individual is the sum of the tangible wealth x and the human capital which, according to Theorem 2.1, equals $yF(t, r)$ with F given by (2.10). As in the existing solutions to similar, but simpler, problems studied in the literature, the g function is determined by the assumed asset price dynamics and will generally depend on variables sufficient to describe relevant variations in the investment opportunity set; see, e.g., [Liu \(2007\)](#). Long-term investors will generally want to hedge variations in investment opportunities as captured by the short-term interest rate and the maximum Sharpe ratio, which together define the location of the instantaneous mean-variance efficient frontier, cf. [Nielsen and Vassalou \(2006\)](#). Since λ_B, λ_S , and λ_H are assumed constant, there are no variations in the maximum Sharpe ratio, so the short-term interest rate alone drives investment opportunities. In addition, a long-term investor who can control her consumption of multiple goods affecting her utility will want to hedge variations in the relative prices of those consumption goods. In our model, the relative price of the two consumption goods is given by H_t . This explains why g is a function of r and h in our setting.

In terms of the functions g and F , the optimal fractions of tangible wealth invested in the stock

and the bond are

$$\pi_S = \frac{1}{\gamma} \frac{\xi_S}{\sigma_S} \frac{x + yF}{x} - \frac{\sigma_Y \zeta_S}{\sigma_S} \frac{yF}{x}, \quad (3.2)$$

$$\pi_B = \frac{1}{\gamma} \frac{\xi_B}{\sigma_B} \frac{x + yF}{x} - \left(\frac{\sigma_Y \zeta_B}{\sigma_B} \frac{yF}{x} - \frac{\sigma_r}{\sigma_B} \frac{yF}{x} \frac{F_r}{F} \right) - \frac{\sigma_r}{\sigma_B} \frac{g_r}{g} \frac{x + yF}{x}, \quad (3.3)$$

respectively, while the optimal units of housing invested in (physically or through REITs) are

$$\varphi_I = \frac{1}{\gamma} \frac{\xi_I}{\sigma_H} \frac{x + yF}{h} - \frac{\sigma_Y \zeta_I}{\sigma_H} \frac{yF}{h} + (x + yF) \frac{g_h}{g}. \quad (3.4)$$

The constants ξ_B, ξ_S, ξ_I are defined in (B.12)-(B.14) in Appendix B in terms of the market prices of risk $\lambda_B, \lambda_S, \lambda_H$ and the pairwise correlations between prices on the bond, the stock, and the house. The constants $\zeta_B, \zeta_S, \zeta_I$ are defined in (A.4)-(A.6) in Appendix A in terms of the pairwise correlations between the bond, the stock, the house, and the labor income.

The first terms in (3.2), (3.3), and (3.4) reflect the speculative demand well-known from the static mean-variance analysis and are determined by wealth, relative risk aversion, variances and covariances, and the market prices of risk.

The second terms in the equations reflect an adjustment of the investments to the risk profile of human wealth. We can think of the individual first determining the desired exposure to all the exogenous shocks—i.e. the standard Brownian motions $W_r, W_S,$ and W_H —and then adjusting for the exposure implicit in the human wealth in order to obtain the desired exposure of the explicit investments towards the shocks. The appropriate adjustment is determined by the instantaneous correlations between the assets and the labor income through the constants $\zeta_B, \zeta_S, \zeta_I$. In addition, human wealth is discounted future labor income and therefore interest rate dependent. From (2.10), it follows that

$$F_r(t, r) = -1_{\{t \leq \bar{T}\}} (1 - b) \int_t^{\bar{T}} \mathcal{B}_\kappa(s - t) e^{-\bar{A}(t, s) - (1 - b) \mathcal{B}_\kappa(s - t)r} ds.$$

Hence, as long as the interest rate sensitivity of the expected income growth rate b is below 1, human wealth is decreasing in the interest rate level and is thus similar to an investment in the bond. If the expected income growth rate is strongly pro-cyclical, i.e. $b > 1$, human wealth is increasing in the interest rate corresponding to an implicit short position in the bond, which is corrected for by a positive explicit demand for the bond. For further discussion of this point, see [Munk and Sørensen \(2008\)](#). The time-dependence of human wealth, as reflected by the function $F(t, r)$, induces a non-constant optimal stock portfolio weight. To be consistent with the popular advice of having “more stocks when you have a long investment horizon”, we need $\xi_S > \gamma \sigma_Y \zeta_S$, which obviously depends on the level of risk aversion and the income volatility, but also on the market prices of risk and numerous correlations embedded in ξ_S and ζ_S .

The last term in (3.3) hedges against variations in future investment opportunities which are summarized by the short-term interest rate and thus hedgeable through a bond investment. At least in the two cases below with a closed-form solution for $g(t, r)$, we find $g_r/g < 0$ so that the intertemporal hedge demand for the bond is positive consistent with intuition and the existing literature. Finally, the

last term in (3.4) represents a hedge against variations in the house price. When house prices increase, the costs of future housing increase. To compensate for that, the individual can invest more in houses so that an increase in house prices will also increase her wealth. Consistent with that interpretation, g_h/g is positive in the closed-form solutions below. An investment in a house is a hedge against future costs of housing consumption.

The optimal consumption rate and the optimal units of housing consumed are given by

$$c = \eta \frac{\beta\nu}{1-\beta} h^k (x + yF) g^{-\frac{\gamma(\psi-1)}{1-\gamma}}, \quad (3.5)$$

$$\varphi_C = \eta h^{k-1} (x + yF) g^{-\frac{\gamma(\psi-1)}{1-\gamma}}, \quad (3.6)$$

where $k = (1 - \psi)(1 - \beta)$ and $\eta = (\delta\beta)^\psi \left(\frac{\beta\nu}{1-\beta}\right)^{k-1}$. This implies that the optimal total expenditure on the two consumption goods is

$$c + \nu h \varphi_C = \delta^\psi \beta^{\beta(\psi-1)} \left(\frac{\nu}{1-\beta}\right)^k h^k (x + yF) g^{-\frac{\gamma(\psi-1)}{1-\gamma}}.$$

The individual distributes the total consumption expenditure to perishable consumption and housing consumption according to the relative weights β and $1 - \beta$ of the goods in the preference specification. The optimal spending on each good is a time- and state-dependent fraction of the total wealth $x + yF$.

It can be shown that (substitute the above expression for total consumption into (B.15)), using the optimal strategies, the dynamics of total wealth $\mathcal{W}_t = X_t + Y_t F(t, r_t)$ will be

$$\begin{aligned} \frac{d\mathcal{W}_t}{\mathcal{W}_t} = & \left(r_t + \frac{1}{\gamma} \tilde{\lambda}^\top \tilde{\lambda} - \sigma_r \lambda_B \frac{g_r}{g} + \sigma_H \lambda'_H H_t \frac{g_h}{g} - \frac{\eta\nu}{1-\beta} H_t^k g^{-\frac{\gamma(\psi-1)}{1-\gamma}} \right) dt \\ & + \frac{1}{\gamma} \tilde{\lambda}^\top dW_t - \frac{g_r}{g} \sigma_r dW_{rt} + H_t \frac{g_h}{g} \sigma_H \tilde{\rho}_H^\top dW_t, \end{aligned} \quad (3.7)$$

where

$$\tilde{\lambda}^\top = \left(\lambda_B, \frac{\lambda_S - \rho_{SB} \lambda_B}{\sqrt{1 - \rho_{SB}^2}}, \frac{1}{\hat{\rho}_H} \left[\lambda'_H - \frac{\rho_{SH} - \rho_{SB} \rho_{HB}}{1 - \rho_{SB}^2} \lambda_S - \frac{\rho_{HB} - \rho_{SH} \rho_{SB}}{1 - \rho_{SB}^2} \lambda_B \right] \right)$$

is the vector of market prices of risk associated with the standard Brownian motion $W = (W_r, W_S, W_H)^\top$, and $\tilde{\rho}_H = (\rho_{HB}, \hat{\rho}_{HS}, \hat{\rho}_H)^\top$. The term $\frac{1}{\gamma} \tilde{\lambda}^\top dW_t$ reflects the optimal risk taking in a setting with constant investment opportunities and the term $\frac{1}{\gamma} \tilde{\lambda}^\top \tilde{\lambda}$ in the drift gives the compensation in terms of excess expected returns for that risk. The shock terms $-\frac{g_r}{g} \sigma_r dW_{rt}$ and $H_t \frac{g_h}{g} \sigma_H \tilde{\rho}_H^\top dW_t$ are the optimal adjustments of the exposure to interest rate risk and house price risk, respectively, due to intertemporal hedging of shifts in investment opportunities, again with appropriate compensation in the drift of wealth. The ratios g_r/g and g_h/g involve the risk aversion and the elasticity of intertemporal substitution (EIS) of the individual.

The specification of the function $g(t, r, h)$ depends on the EIS parameter ψ . When ψ is different from 1, g has to satisfy the non-linear PDE (B.22). However, it is apparently only possible to solve that PDE in closed form in the special case of power utility where $\psi = 1/\gamma$ since the PDE is then linear. We present and discuss that solution next. When $\psi = 1$ and $\varepsilon > 0$, so that the individual

has some utility from terminal wealth, g has to satisfy the PDE (B.31). A closed-form solution and the resulting optimal strategies in that case are discussed in Section 3.2. For the case where the EIS parameter ψ is different from 1 and $1/\gamma$, we present in Section 3.3 a closed-form approximate solution following the approach of [Chacko and Viceira \(2005\)](#).

3.1 Power utility

With time-additive power utility so that $\psi = 1/\gamma$, the optimal consumption strategies simplify to

$$c = \eta \frac{\beta\nu}{1-\beta} h^k \frac{x + yF}{g}, \quad (3.8)$$

$$\varphi_C = \eta h^{k-1} \frac{x + yF}{g}. \quad (3.9)$$

The next theorem states the g function and summarizes the full solution to the problem for power utility.

Theorem 3.1 (Solution, power utility) *For the case where $\psi = 1/\gamma$, the value function is given by (3.1), where F is defined in (2.10) and*

$$g(t, r, h) = \varepsilon^{\frac{1}{\gamma}} e^{-\mathcal{D}_\gamma(T-t) - \frac{\gamma-1}{\gamma} \mathcal{B}_\kappa(T-t)r} + \frac{\eta\nu}{1-\beta} h^k \int_t^T e^{-d_1(u-t) - \beta \frac{\gamma-1}{\gamma} \mathcal{B}_\kappa(u-t)r} du, \quad (3.10)$$

where

$$\begin{aligned} \mathcal{D}_\gamma(\tau) = & \left(\frac{\delta}{\gamma} + \frac{\gamma-1}{2\gamma^2} \tilde{\lambda}^\top \tilde{\lambda} \right) \tau + \left(\bar{r} + \frac{\gamma-1}{\gamma} \frac{\sigma_r \lambda_B}{\kappa} \right) \frac{\gamma-1}{\gamma} (\tau - \mathcal{B}_\kappa(\tau)) \\ & - \frac{1}{2} \frac{\sigma_r^2}{\kappa^2} \left(\frac{\gamma-1}{\gamma} \right)^2 \left(\tau - \mathcal{B}_\kappa(\tau) - \frac{\kappa}{2} \mathcal{B}_\kappa(\tau)^2 \right), \end{aligned} \quad (3.11)$$

$$\begin{aligned} d_1(\tau) = & \left(\frac{\delta}{\gamma} + \frac{\gamma-1}{2\gamma^2} \tilde{\lambda}^\top \tilde{\lambda} - k \left(\frac{1}{\gamma} \sigma_H \lambda'_H - \nu + \frac{1}{2} (k-1) \sigma_H^2 \right) \right) \tau \\ & + \beta \left(\bar{r} + \frac{\gamma-1}{\gamma} \frac{\sigma_r \lambda_B}{\kappa} - \frac{k \sigma_r \sigma_H \rho_{HB}}{\kappa} \right) \frac{\gamma-1}{\gamma} (\tau - \mathcal{B}_\kappa(\tau)) \\ & - \frac{1}{2} \frac{\beta^2 \sigma_r^2}{\kappa^2} \left(\frac{\gamma-1}{\gamma} \right)^2 \left(\tau - \mathcal{B}_\kappa(\tau) - \frac{\kappa}{2} \mathcal{B}_\kappa(\tau)^2 \right), \end{aligned} \quad (3.12)$$

with $\tilde{\lambda}^\top \tilde{\lambda} = \lambda_B \xi_B + \lambda_S \xi_S + \lambda'_H \xi_I$. The optimal controls are given by (3.2)-(3.4) and (3.8)-(3.9).

In the following, we will discuss and illustrate the optimal strategies. Figure 1 shows how the ratio of optimal perishable consumption to total wealth, $c/(x + yF(t, r)) = \frac{\eta\beta\nu}{1-\beta} h^k / g(t, r, h)$, varies with the length of the remaining life-time. The benchmark parameters in Table 1 are applied together with an initial unit house price of $h = 200$ (USD per square foot) and an initial short-term interest rate of $r = \bar{r} = 0.02$. The four curves differ with respect to the value of ε , which indicates the relative preference weighting of terminal wealth and intermediate consumption: a terminal wealth of X will roughly contribute to life-time utility ε/δ times as much as a consumption of X in the final year, cf. footnote 7. For $\varepsilon = 0$, the consumption to wealth ratio goes to infinity as the time horizon goes

to 0 since in that case the individual will want to spend everything before the end. The individual will annually spend around 4-5% of total wealth on perishable consumption goods when young, almost independently of the value of ε . This propensity to consume out of total wealth will then gradually increase as the individual grows older. The consumption-wealth ratio is only little sensitive to the interest rate level and the house price level. The optimal spending on housing consumption equals the perishable consumption multiplied by the factor $(1 - \beta)/\beta$, which is 0.25 with our benchmark parameters.

[Figure 1 about here.]

Next, we derive the expected consumption over the life-cycle. Assume for simplicity that the individual has no utility from terminal wealth, i.e. $\varepsilon = 0$. In this case $g(t, r, h) = \frac{\eta\nu}{1-\beta} h^k G(t, r)$, where $G(t, r) = \int_t^T e^{-d_1(u-t) - \beta(1-\frac{1}{\gamma})\mathcal{B}_\kappa(u-t)r} du$, and the optimal spending on consumption goods will be

$$c_t = \beta \frac{\mathcal{W}_t}{G(t, r_t)}, \quad \varphi_{C_t} \nu H_t = (1 - \beta) \frac{\mathcal{W}_t}{G(t, r_t)}, \quad (3.13)$$

and, in particular, independent of the current house price. The first-order derivatives of g that enter the optimal portfolio weights are then

$$\frac{g_h}{g} = \frac{k}{h} = h^{-1}(1 - \beta) \frac{\gamma - 1}{\gamma}, \quad \frac{g_r}{g} = -\hat{\beta} D(t, r),$$

where $\hat{\beta} = \beta(\gamma - 1)/\gamma$ and $D(t, r) = \left(\int_t^T \mathcal{B}_\kappa(u-t) e^{-d_1(u-t) - \hat{\beta}\mathcal{B}_\kappa(u-t)r} du \right) / G(t, r)$. The dynamics of total wealth in (3.7) simplifies to

$$\begin{aligned} \frac{d\mathcal{W}_t}{\mathcal{W}_t} = & \left(r_t + \frac{1}{\gamma} \tilde{\lambda}^\top \tilde{\lambda} + \sigma_r \lambda_B \hat{\beta} D(t, r_t) + k \sigma_H \lambda'_H - G(t, r_t)^{-1} \right) dt \\ & + \frac{1}{\gamma} \tilde{\lambda}^\top dW_t + \hat{\beta} \sigma_r D(t, r_t) dW_{rt} + k \sigma_H \bar{\rho}_H^\top dW_t. \end{aligned} \quad (3.14)$$

In Appendix B.4 we compute the time 0 expectation of $\mathcal{W}_t/G(t, r_t)$, which leads to the expected spending on the two goods over the life-cycle. Appendix B.4 also contains similar results for $\varepsilon > 0$.

Figure 2 illustrates the expected consumption pattern over the life-cycle. In addition to the benchmark parameters, we have assumed an initial tangible wealth of $X_0 = 20,000$ and an initial income rate of $Y_0 = 20,000$. The figure shows the expected expenditure on each of the two consumption goods on the left scale. The expected perishable consumption grows from around 13,000 to 36,000 (USD per year) over the assumed 40 year horizon. The expected expenditure on housing consumption is again just a $(1 - \beta)/\beta$ multiple of the expected perishable consumption. The expectation of the house price on the left-hand side is⁸

$$E_0[H_t] = H_0 \exp \left\{ (r_0 + \lambda_H \sigma_H - r^{\text{imp}})t + \left(\bar{r} - r_0 + \frac{\sigma_r^2}{2\kappa^2} - \frac{\sigma_r \sigma_H \rho_{HB}}{\kappa} \right) (t - \mathcal{B}_\kappa(t)) - \frac{\sigma_r^2}{4\kappa} \mathcal{B}_\kappa(t)^2 \right\}. \quad (3.15)$$

⁸The house price dynamics (2.6) implies that $H_t = H_0 \exp\{\int_0^t r_u du + (\lambda_H \sigma_H - r^{\text{imp}} - \frac{1}{2}\sigma_H^2)t + \int_0^t \sigma_H \bar{\rho}_H^\top dW_u\}$. Substituting (A.9) and taking expectations, we find the expected house price stated in the text.

Now we can estimate the expected number of housing units consumed as $(1-\beta)E_0[\mathcal{W}_t/G(t, r_t)]/\nu E_0[H_t]$, cf. (3.13). This is illustrated by the blue curve using the right scale in Figure 2. The expected number of housing units consumed grows from about 300 to 685 over the 40 year life-time. Recall that a housing unit can be thought to represent one square foot of housing (of “average quality”) so the above numbers (square feet per person) are of a reasonable magnitude. Finally note that for $\varepsilon > 0$ a little consumption over the life-cycle is given up to generate positive terminal wealth.

[Figure 2 about here.]

Concerning the optimal investments, note that $g_h/g > 0$ so that the risk of higher future housing costs is hedged by an increased investment in houses, and $g_r/g < 0$ so that the intertemporal hedging of shifts in investment opportunities leads to a positive bond demand. Figure 3 shows how the optimal investments as fractions of total wealth vary with the human wealth to total wealth ratio. The fraction of total wealth invested in the *stock* consists of a constant speculative position of 23.0% with an adjustment for labor income which increases linearly from 0 to 4.4% with the relative importance of human wealth; since the auxiliary parameter ζ_S is negative, the income-motivated adjustment of the stock position is positive.

The fraction of total wealth invested in the *bond* consists of four terms: (i) a constant speculative position of -42.3%, (ii) an adjustment due to the instantaneous correlation of income with financial assets, which increases linearly from 0 to 65.8% with the human wealth to total wealth ratio, (iii) an adjustment due to the dependence of human wealth on the interest rate varying from 0 to -35.8% as the human/total wealth ratio goes from 0 to 1, and (iv) an intertemporal hedge against interest rate risk which amounts to 47.6% no matter how the total wealth is decomposed.⁹ The total bond demand varies from 5.3% to 35.4% as the human/total wealth is varied from 0 to 1. Here, the component (iii) depends on F_r/F , the relative sensitivity of human wealth with respect to the interest rate. The numbers just reported and used to generate the figure assume 20 years to retirement in which case $F_r/F \approx -1.8$, but the ratio goes to 0 as retirement is approaching which will slightly increase the fraction of total wealth invested in the bond. The component (iv) depends on the ratio g_r/g , which is approximately -2.3 for a remaining life-time of 40 years. The ratio approaches zero relatively slowly as time passes, which leads to a lower hedge-motivated bond position.

The fraction of total wealth invested in houses (physically or financially) consists of a constant speculative demand of 86%, an income-adjustment term varying from 0 to approximately -100% as the human/total wealth ratio goes from 0 to 1, and an intertemporal hedge against house price risk equal to 15% independent of wealth composition. The large negative income-adjustment is due to the large positive correlation between labor income and house prices. The total investment in houses varies from roughly 100% with no human wealth to roughly 0% with only human wealth.

[Figure 3 about here.]

⁹The hedge demands reported for the bond here and for the house investment below are computed assuming no utility of terminal wealth, but they are only little sensitive to the value of ε .

For a fixed ratio of human wealth to total wealth and $\varepsilon = 0$, the fractions of total wealth invested in the stock and houses are independent of the remaining lifetime, whereas the fraction invested in the bond varies (quite slowly) due to the time-dependence of the ratios F_r/F and g_r/g . The main source of variations in portfolio weights over the life-cycle is that the human/total wealth ratio will decrease to zero as retirement is approaching. According to Figure 3, the individual should therefore through his life increase the fraction of total wealth invested in the house and decrease the fraction of total wealth invested in the stock and in the bond.

Figure 4 shows how the expected total wealth, human wealth, and financial wealth vary over the life-cycle again assuming an initial financial wealth of $X_0 = 20,000$ and an initial labor income rate of $Y_0 = 20,000$. The graph is produced using an approximation of the expected total wealth $E_0[\mathcal{W}_t]$ as given in (B.27) and the approximation $F(t, \bar{r})E_0[Y_t]$ of expected human capital, where the expected income is given by (2.11) in Theorem 2.1. The expected financial wealth is computed residually. When ε is assumed to be zero, all wealth is optimally consumed before the end. Human wealth dominates initially but drops to zero at retirement of course. Financial wealth is hump-shaped since saving is necessary when working in order to finance consumption during retirement.

[Figure 4 about here.]

Figure 5 illustrates how the investments in the stock, the bond, and housing units (physical or through REITs) are expected to evolve over the life of the investor. Early in life human wealth is the major part of total wealth and, in accordance with Figure 3, it is optimal to invest close to nothing in houses and substantial amounts in stocks and long-term bonds, financed in part by short-term borrowing. Due to the large positive correlation between house prices and labor income, the human wealth crowds out housing investments. As human wealth decreases, the housing investment will increase. This trend continues until retirement. At and after retirement, the housing investment is dominated by the large speculative demand which will fall towards zero as the investor consumes out of wealth. The expected stock investment falls steadily with age in line with the standard “more stocks when you are young” advice. The bond demand is more sensitive to the composition of wealth than the stock demand as seen from Figure 3, and this is reflected by variation of the expected bond investment over the life-cycle. Note that we assume that at any date the individual trades in a zero-coupon bond maturing 20 years later. If we had chosen a different bond (or another interest rate dependent asset, e.g. a bond future), the optimal investment in that asset would have been a multiple of the optimal investment in the 20-year bond in order to obtain the same overall exposure to the shocks to the short-term interest rate.

[Figure 5 about here.]

3.2 Unit EIS

For the case where the EIS parameter ψ equals 1, the optimal consumption strategies in (3.5)-(3.6) simplify to

$$c = \delta\beta(x + yF), \quad (3.16)$$

$$\varphi_C = \frac{\delta(1 - \beta)}{\nu} \frac{x + yF}{h}. \quad (3.17)$$

The consumption share of total wealth is now constant over time in contrast to the case of power utility. The age of the individual does not affect the consumption strategy so in a sense such an individual does not take a life-cycle perspective. With our benchmark parameters, the individual will optimally spend 2.4% of total wealth on perishable consumption and 0.6% on housing consumption. The next theorem states the g function and summarizes the full solution to the problem for $\psi = 1$.

Theorem 3.2 (Solution, unit EIS) *For the case where $\psi = 1$ and $\varepsilon > 0$ the value function is given by (3.1), where F is defined in (2.10) and*

$$g(t, r, h) = \varepsilon^{1/\gamma} h^{D_2(T-t)} e^{-D_0(T-t) - D_1(T-t)r}, \quad (3.18)$$

with

$$D_2(\tau) = \hat{k}\delta\mathcal{B}_\delta(\tau), \quad (3.19)$$

$$D_1(\tau) = \left[\frac{\delta\hat{k}}{\kappa} + \frac{\hat{k}}{1 - \beta} \right] \mathcal{B}_{\delta+\kappa}(\tau) - \frac{\delta\hat{k}}{\kappa} \mathcal{B}_\delta(\tau), \quad (3.20)$$

$$D_0(\tau) = K_\delta\mathcal{B}_\delta(\tau) + K_{\delta+\kappa}\mathcal{B}_{\delta+\kappa}(\tau) + K_{2\delta}\mathcal{B}_{2\delta}(\tau) + K_{2\delta+\kappa}\mathcal{B}_{2\delta+\kappa}(\tau) \\ + K_{2(\delta+\kappa)}\mathcal{B}_{2(\delta+\kappa)}(\tau) + K_z(\delta\mathcal{B}_\delta(\tau) - 1)\tau, \quad (3.21)$$

where $\hat{k} = (1 - \frac{1}{\gamma})(1 - \beta)$ and the coefficients in the expression for D_0 are given by (B.32)-(B.37) in the appendix. The optimal controls are given by (3.2), (3.3) with $\frac{g_r}{g} = -D_1(T - t)$, (3.4) with $\frac{g_h}{g} = \frac{D_2(T-t)}{h}$, and (3.16)-(3.17).

From (3.7), we find that the dynamics of optimally managed total wealth is now

$$\frac{d\mathcal{W}_t}{\mathcal{W}_t} = \left(r_t + \frac{1}{\gamma} \tilde{\lambda}^\top \tilde{\lambda} + \sigma_r \lambda_B D_1(T - t) + \sigma_H \lambda_H D_2(T - t) - \delta \right) dt \\ + \frac{1}{\gamma} \tilde{\lambda}^\top dW_t + \sigma_r D_1(T - t) dW_{rt} + \sigma_H D_2(T - t) \tilde{\rho}_H^\top dW_t. \quad (3.22)$$

Optimally managed total wealth is lognormally distributed with an expectation given by (B.38) in Appendix B.5.

It can be shown that (see Appendix B) $D_1(\tau) \geq 0$ and, consequently, $g_r/g < 0$ so that the intertemporal hedge demand for bonds is positive. $D_2(\tau) \geq 0$ and thus $g_h/g > 0$ so that the risk of higher future housing costs is hedged by an increased investment in houses. In contrast to the case of power utility, the term hedging house price risk is time-dependent. Otherwise, the optimal investment strategies are qualitatively and quantitatively as for power utility.

3.3 An approximate solution for other values of the EIS parameter

When the EIS parameter ψ is different from 1 and $1/\gamma$, the PDE (B.22) for $g(t, r, h)$ contains the term $h^k g^{\frac{\gamma\psi-1}{\gamma-1}}$, where the last exponent is different from 0 and 1, so that the PDE has no known closed-form solution. However, following an idea originally put forward by Campbell (1993) in a discrete-time setting and adapted to a continuous-time setting by Chacko and Viceira (2005), it is possible to obtain a closed-form approximate solution. A Taylor approximation of $z \mapsto e^z$ around \hat{z} gives $e^z \approx e^{\hat{z}}(1+z-\hat{z})$. Applying that to $z = k \ln H_t + \frac{\gamma(\psi-1)}{\gamma-1} \ln g_t$, where $g_t = g(t, r_t, H_t)$, implies that

$$\begin{aligned} H_t^k g_t^{\frac{\gamma\psi-1}{\gamma-1}} &= g_t H_t^k g_t^{\frac{\gamma(\psi-1)}{\gamma-1}} = g_t e^{k \ln H_t + \frac{\gamma(\psi-1)}{\gamma-1} \ln g_t} \\ &\approx g_t e^{k \ln \hat{h}(t) + \frac{\gamma(\psi-1)}{\gamma-1} \ln \hat{g}(t)} \left(1 + k[\ln H_t - \ln \hat{h}(t)] + \frac{\gamma(\psi-1)}{\gamma-1} [\ln g_t - \ln \hat{g}(t)] \right) \\ &= g_t \hat{h}(t)^k \hat{g}(t)^{\frac{\gamma(\psi-1)}{\gamma-1}} \left(1 + k[\ln H_t - \ln \hat{h}(t)] + \frac{\gamma(\psi-1)}{\gamma-1} [\ln g_t - \ln \hat{g}(t)] \right). \end{aligned} \quad (3.23)$$

Using that approximation in the PDE (B.22), it will closely resemble the PDE for the case $\psi = 1$, and therefore have a solution of a similar form; see Appendix B.6 for details.

Theorem 3.3 (Approximate solution, EIS different from 1 and $1/\gamma$) *For the case where $\psi \notin \{1, 1/\gamma\}$ and $\varepsilon > 0$ the value function is given by (3.1), where F is defined in (2.10) and g is approximated by*

$$g(t, r, h) = \varepsilon^{1/\gamma} h^{\hat{D}_2(t, T)} e^{-\hat{D}_0(t, T) - \hat{D}_1(t, T)r}, \quad (3.24)$$

with

$$\hat{D}_2(t, T) = \eta\nu \frac{\gamma-1}{\gamma} \int_t^T e^{-\frac{\eta\nu}{1-\beta} \int_t^u \Theta(s) ds} \Theta(u) du, \quad (3.25)$$

$$\hat{D}_1(t, T) = \frac{\gamma-1}{\gamma} \int_t^T e^{-\kappa(u-t) - \frac{\eta\nu}{1-\beta} \int_t^u \Theta(s) ds} du - \eta\nu \frac{\gamma-1}{\gamma} \int_t^T \Theta(u) \mathcal{B}_\kappa(u-t) e^{-\frac{\eta\nu}{1-\beta} \int_t^u \Theta(s) ds} du, \quad (3.26)$$

where $\Theta(t) = \hat{h}(t)^k \hat{g}(t)^{\frac{\gamma(\psi-1)}{\gamma-1}}$. The function $\hat{D}_0(t, T)$ is given in (B.39).

The approximation is most precise when $k \ln H_t + \frac{\gamma(\psi-1)}{\gamma-1} \ln g_t$ is close to $k \ln \hat{h}(t) + \frac{\gamma(\psi-1)}{\gamma-1} \ln \hat{g}(t)$. A promising choice is to let $\ln \hat{h}(t) = \mathbb{E}[\ln H_t] = \ln H_0 + (r_0 + \lambda_H \sigma_H - r^{\text{imp}} - \frac{1}{2} \sigma_H^2) t + (\bar{r} - r_0)(t - \mathcal{B}_\kappa(t))$, and to determine $\hat{g}(t)$ so that

$$\ln \hat{g}(t) = \mathbb{E}[\ln g(t, r_t, H_t)] = \ln \varepsilon^{1/\gamma} - \hat{D}_0(t, T) - \hat{D}_1(t, T) \mathbb{E}[r_t] + \hat{D}_2(t, T) \mathbb{E}[\ln H_t].$$

Since $\hat{D}_0(t, T)$, $\hat{D}_1(t, T)$, and $\hat{D}_2(t, T)$ depend on all $\hat{g}(u)$ for $u \in [t, T]$, this involves a recursive procedure moving backwards from T .

Substituting the approximation (3.24) into the Equations (3.2)–(3.4) and (3.5)–(3.6) gives an approximation to the optimal investment and consumption strategy for general Epstein-Zin preferences. Note that the consumption-wealth ratios will depend on the age of the individual via the g -function so it is only in the special case of $\psi = 1$ considered above that the individual does not take a life-cycle perspective.

4 Robustness to inflexible housing and borrowing constraints

The model of Section 2 was designed with the purpose of allowing for closed-form solutions for the optimal housing, consumption, and investment strategy as found in Section 3. Admittedly, the realism of some of the modeling assumptions are questionable. First, until now we have assumed that the individual was able to adjust the housing consumption and investment positions continuously over time. Clearly, it is practically inefficient to adjust the physical ownership of housing units continuously due to explicit and implicit transaction costs. Continuous adjustment of the investment in REITs may be a reasonable approximation to real life, but if REITs linked to the house prices of interest to the investor are not traded, the individual cannot continuously adjust the housing investment position. If changes in the renting position are costly, continuous adjustment of the housing consumption position will also be inefficient. Second, we have assumed that the investor was able to borrow against future labor income. The optimal unconstrained consumption and investment strategy will lead to situations where the tangible wealth (financial wealth and the value of the housing stock owned) will be negative but the human wealth more than outweighs that so that total wealth is positive. Such a strategy may not be feasible in real life due to moral hazard and asymmetric information issues in the valuation of human wealth.

In this section we consider variations of the basic problem that relaxes these two assumptions. We find strategies that either (a) involves only infrequent adjustments of the housing consumption and/or the investment position in housing, (b) ensures non-negative tangible wealth at all points in time, or (c) do both. Clearly, the expected utility generated by any such strategy, denoted by $\hat{J}(t, x, r, h, y)$, is smaller than the expected utility following the optimal unconstrained strategy, i.e. $J(t, x, r, h, y)$ from Theorem 3.1. We measure the economic importance of the imposed constraint by the percentage decrease in initial financial wealth and labor income (and thus in initial total wealth) necessary to bring the optimal expected utility down to the expected utility obtained with limited flexibility:

$$J(t, x[1 - \ell], r, h, y[1 - \ell]) = \hat{J}(t, x, r, h, y). \quad (4.1)$$

We can interpret ℓ as the percentage wealth loss the individual would incur if he were restricted to the constrained strategy or the percentage of wealth the individual is willing to sacrifice to avoid the constraints. Due to the form of the value function in (3.1), we get

$$\ell = 1 - \left(\frac{\hat{J}(t, x, r, h, y)}{J(t, x, r, h, y)} \right)^{\frac{1}{1-\gamma}}. \quad (4.2)$$

As we will show below, the wealth-equivalent losses associated with the strategies featuring inflexible housing and/or no borrowing are small, which confirms the relevance of our closed-form solutions in realistic settings.

4.1 Constant housing consumption: an explicit solution

First, we discuss a case where we can find an explicit solution to the problem with limited flexibility. We consider a power utility investor ($\psi = 1/\gamma$) who gets no utility from terminal wealth ($\varepsilon = 0$) and

is restricted to a fixed consumption of housing services given by the constant $\bar{\varphi}_C$. Due to (2.13) we can assume that the individual satisfies her housing consumption through renting and does not have direct ownership of housing, i.e. $\varphi_{ot} \equiv 0$. The individual can then obtain the desired exposure to house price risk by investing in REITs as captured by φ_{Rt} . Alternatively, we can ignore REITs and let the individual continuously adjust the direct ownership of houses, but then the housing units rented would have to be adjusted continuously so that the sum $\varphi_{rt} + \varphi_{ot}$ equals the deterministic total housing consumption $\bar{\varphi}_C$. In Appendix C we demonstrate the following result (the superscript “dc” indicates deterministic consumption):

Theorem 4.1 *Assume $\psi = 1/\gamma$, $\varepsilon = 0$, and a fixed housing consumption $\bar{\varphi}_C$. The value function is given by*

$$J^{\text{dc}}(t, x, r, h, y) = \frac{1}{1 - \tilde{\gamma}} g^{\text{dc}}(t, r)^{\tilde{\gamma}} (x + yF(t, r) - \nu h \hat{F}(t))^{1 - \tilde{\gamma}}, \quad (4.3)$$

where $\tilde{\gamma} = 1 - \beta(1 - \gamma) \in (1, \gamma)$, F is defined in (2.10),

$$\hat{F}(t) = \bar{\varphi}_C \mathcal{B}_\nu(T - t), \quad (4.4)$$

$$g^{\text{dc}}(t, r) = (\delta\beta)^{\frac{1}{\tilde{\gamma}}} \bar{\varphi}_C^{1 - \frac{1}{\tilde{\gamma}}} \int_t^T e^{-\mathcal{D}_{\tilde{\gamma}}(u-t) - \frac{\tilde{\gamma}-1}{\tilde{\gamma}} \mathcal{B}_\kappa(u-t)r} du, \quad (4.5)$$

and $\mathcal{D}_{\tilde{\gamma}}$ is given by (3.11) with $\tilde{\gamma}$ replacing γ . The optimal fractions of tangible wealth invested in the stock and the bond are

$$\pi_S = \frac{1}{\tilde{\gamma}} \frac{\xi_S}{\sigma_S} \frac{x + yF - \nu h \hat{F}}{x} - \frac{\sigma_Y \zeta_S}{\sigma_S} \frac{yF}{x}, \quad (4.6)$$

$$\pi_B = \frac{1}{\tilde{\gamma}} \frac{\xi_B}{\sigma_B} \frac{x + yF - \nu h \hat{F}}{x} - \left(\frac{\sigma_Y \zeta_B}{\sigma_B} \frac{yF}{x} - \frac{\sigma_r}{\sigma_B} \frac{yF_r}{x} \right) - \frac{\sigma_r}{\sigma_B} \frac{g_r^{\text{dc}}}{g^{\text{dc}}} \frac{x + yF - \nu h \hat{F}}{x}, \quad (4.7)$$

respectively, while the optimal number of housing units invested in is

$$\varphi_I = \frac{1}{\tilde{\gamma}} \frac{\xi_I}{\sigma_H} \frac{x + yF - \nu h \hat{F}}{h} - \frac{\sigma_Y \zeta_I}{\sigma_H} \frac{yF}{h} + \nu \hat{F} \quad (4.8)$$

and the optimal consumption of the perishable good is

$$c = \left(\int_t^T e^{-\mathcal{D}_{\tilde{\gamma}}(u-t) - \frac{\tilde{\gamma}-1}{\tilde{\gamma}} \mathcal{B}_\kappa(u-t)r} du \right)^{-1} (x + yF - \nu h \hat{F}). \quad (4.9)$$

Relative to the form of the value function in Section 3 there are three differences. *First*, the relevant g -function depends on r , which captures financial investment opportunities, but not on the relative price of consumption goods h since the individual cannot control the consumption of housing. *Second*, note that the present value of all future renting expenses is given by the risk-neutral expectation¹⁰

$$\begin{aligned} \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^T e^{-\int_t^s r_u du} \bar{\varphi}_C \nu H_s ds \right] &= \nu H_t \bar{\varphi}_C \int_t^T \mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^s r_u du} \frac{H_s}{H_t} \right] ds \\ &= \nu H_t \bar{\varphi}_C \int_t^T e^{-\nu(s-t)} ds = \nu H_t \hat{F}(t). \end{aligned} \quad (4.10)$$

¹⁰The risk-neutral drift of house prices is the real-world drift $r_t + \lambda_H \sigma_H - r^{\text{imp}}$ minus the risk premium $\lambda'_H \sigma_H$, which equals $r_t - \nu$.

The total initial wealth at the disposal of the individual is thus the initial tangible wealth plus the human capital minus the present value of future rents, $x + yF(t, r) - \nu h\hat{F}(t)$. *Third*, compared to the separation (3.1) in the previous section, $\tilde{\gamma}$ has replaced γ in the exponents of g and the total present wealth.¹¹

Compared to the optimal investment strategies derived in Section 3, the smaller effective relative risk aversion and the smaller “free wealth” work in opposite directions, and the net effect depends on precise parameter values. The adjustment for the risk exposure of human wealth is the same as before. Concerning the demand for housing investment, the last term in (4.8) ensures that when the present value of future renting costs increases, the tangible wealth increases by at least the same amount. Hence, the individual will always have sufficient wealth to pay the rent. The optimal perishable consumption is a time- and state-dependent fraction of the “free wealth”, a fraction that depends on the constant level of housing consumption. The perishable consumption is decreasing in the housing consumption level as expected.

Theorem 4.1 gives the value function for any constant level of housing consumption, $\bar{\varphi}_C$. We can perform an initial (time $t = 0$) optimization over $\bar{\varphi}_C$ to find the optimal constant consumption of housing services. The first-order condition for the maximization of $J(0, x, r, h, y)$ with respect to φ_C implies

$$\frac{\tilde{\gamma}}{1 - \tilde{\gamma}} \frac{\partial g^{\text{dc}}(0, r)}{\partial \bar{\varphi}_C} (x + yF(0, r)) = \nu h \mathcal{B}_\nu(T) \left(g^{\text{dc}}(0, r) + \frac{\tilde{\gamma}}{1 - \tilde{\gamma}} \frac{\partial g^{\text{dc}}(0, r)}{\partial \bar{\varphi}_C} \bar{\varphi}_C \right), \quad (4.11)$$

and since $\frac{\partial g^{\text{dc}}(0, r)}{\partial \bar{\varphi}_C} \bar{\varphi}_C = (1 - \gamma/\tilde{\gamma})g^{\text{dc}}(0, r)$, we get an optimal housing consumption of

$$\bar{\varphi}_C = (1 - \beta) \frac{X_0 + Y_0 F(0, r_0)}{\nu H_0 \mathcal{B}_\nu(T)}, \quad (4.12)$$

i.e. so that initially the expenditure on house consumption is a fraction $(1 - \beta)/\mathcal{B}_\nu(T)$ of total wealth. With our benchmark parameters and $X_0 = Y_0 = 20,000$, this gives $\bar{\varphi}_C \approx 339$ corresponding initially to 1.24% of total wealth.

The wealth-equivalent of the utility loss an individual suffers by having to stick to a constant consumption of housing is given by (4.2) with \hat{J} replaced by J^{dc} . With the benchmark parameters the loss from applying the optimal constant level of housing consumption instead of the optimal fully flexible housing consumption strategy is 0.66% of total initial wealth, i.e. an individual with full flexibility in housing consumption decisions is willing to give up only 0.66% of his total wealth in order to avoid being restricted to consuming the same level of housing services across time and states. Figure 6 illustrates how the utility loss varies with the chosen fixed housing consumption. Note that the curve is quite flat around the optimum and since the expected units of housing consumed

¹¹The intuition is as follows: In the aggregator capturing the preferences of the individual, the consumption of the two goods enter via the term $z^{1 - \frac{1}{\psi}} = (c^\beta \varphi_C^{1-\beta})^{1 - \frac{1}{\psi}} = (c^\beta \varphi_C^{1-\beta})^{\frac{1-\gamma}{\theta}}$ (focusing on the case $\psi \neq 1$). When we can freely choose c and φ_C , γ will be the effective relative risk aversion, and is therefore the relevant parameter in the exponents of the separation (3.1). Now the assumption is that the individual can only choose how much of the perishable good to consume, and since $z^{1 - \frac{1}{\psi}} = (c^\beta \varphi_C^{1-\beta})^{\frac{1-\gamma}{\theta}} = \bar{\varphi}_C^{\frac{(1-\beta)(1-\gamma)}{\theta}} c^{\frac{\beta(1-\gamma)}{\theta}} = \bar{\varphi}_C^{\frac{(1-\beta)(1-\gamma)}{\theta}} c^{\frac{1-\tilde{\gamma}}{\theta}}$, we can see that $\tilde{\gamma}$ now plays the same role as γ in the preceding section, i.e. $\tilde{\gamma}$ is the effective relative risk aversion.

according to Figure 2 does not fluctuate wildly over life, it should be expected that little can be gained by replacing the optimal constant $\bar{\varphi}_C$ by any deterministic consumption schedule. More surprisingly, the very small wealth loss associated with a constant housing consumption shows that little can be gained by letting the consumption of housing services be state-dependent. The individual can almost completely compensate for the inflexibility in housing consumption via appropriate adjustments of the perishable consumption and the investments in stocks, bonds, and REITs.

[Figure 6 about here.]

4.2 Infrequent housing adjustments: Monte-Carlo results

Next, we implement a Monte Carlo simulation procedure to investigate the welfare loss suffered by an individual restricted to infrequent adjustments of (a) housing consumption φ_{Ct} , (b) housing investment φ_{It} , or (c) both φ_{Ct} and φ_{It} . We experiment with adjustment frequencies of 2 and 5 years. All other controls are adjusted at every time step used in the Monte Carlo simulation of the state variables, i.e. 250 times a year (roughly once per trading day).¹² Whenever a control variable is adjusted, it is reset to the optimal value stated in Theorem 3.1 using the simulated values of wealth, the interest rate, the house price, and the labor income rate. For each simulation path we compute the life-time utility of consumption and terminal wealth and approximate expected utility by averaging over 10,000 paths. The associated welfare loss is then computed from (4.2).¹³

Table 2 reports the wealth-equivalent loss for both adjustment frequencies assuming the benchmark parameter values and $\varepsilon = 0.3$.¹⁴ We see that the loss is very small (below 1.2%) even when both housing consumption and housing investments are reset only every 2 or 5 years. Naturally the loss increases with the period between adjustments. In particular, we see that it has very little value to be able to adjust the housing consumption position frequently, in line with the results of Section 4.1. Frequent adjustments of the housing investment position are more valuable, which indicates that a well-functioning market for REITs or other financial contracts facilitating the hedging of house price risks can have a non-negligible, although relatively moderate, welfare effect. The small losses due to infrequent housing transactions also suggest that transaction costs will be of minor importance to the welfare of the individual in the sense that an investor resetting his housing positions (and paying the associated transaction costs) infrequently to the positions derived in Theorem 3.1 will obtain almost the same utility as if he could continuously adjust the housing positions at zero transaction costs.

[Table 2 about here.]

¹²The results are similar for 12 time steps per year corresponding to monthly rebalancing.

¹³For the purpose of computing the loss, we also compute the expected utility generated by the optimal unconstrained strategy by Monte Carlo simulation using the same set of random numbers as was used to compute the utility of the constrained strategy. This will reduce any bias in the loss due to the simulation procedure.

¹⁴The losses are slightly higher for $\varepsilon = 0.03$.

4.3 Borrowing constraints

The optimal unconstrained consumption and investment strategy derived in Section 3 ensures that total wealth stays non-negative, but there will be situations in which tangible wealth is negative so that the individual is assumed to borrow against future income. We will transform the housing, consumption, and investment strategy shown to be optimal without a borrowing constraint into a strategy that will keep tangible wealth positive. For concreteness, assume time-additive power utility. As long as the tangible wealth X_t is above some (fairly small) number \underline{X} , we follow the optimal unconstrained strategy stated in the Equations (3.2)–(3.4) and (3.8)–(3.9). Whenever $X_t < \underline{X}$, we ignore all the income terms in the investment strategy (i.e. put $F = 0$ in (3.2)–(3.4)). Following this transformed strategy, the tangible wealth will never drop below zero.¹⁵

[Table 3 about here.]

We evaluate the transformed strategy by Monte Carlo simulation as explained in the preceding subsection. We assume again that initial wealth and initial annual income both equal 20,000. We set the critical wealth level \underline{X} to 6,000. The first row of Table 3 shows that following our transformed strategy ensuring positive tangible wealth (updating the strategy “almost-continuously”, i.e. at the simulation frequency) instead of the optimal *unconstrained* strategy leads to a wealth-equivalent loss of only 0.8%. Clearly this is an upper bound on the loss incurred by following the transformed strategy compared to the optimal *constrained* strategy. Hence, our transformation of the closed-form solution must be near-optimal in the constrained setting, emphasizing the relevance of our closed-form solution in more realistic settings. The remaining part of the table shows that limiting the investor to adjustments of the housing consumption and/or investment positions only every 2 or 5 years has a small impact on his expected utility, also in the presence of borrowing constraints.

5 Conclusion

We have provided explicit solutions to quite complicated life-cycle utility maximization problems having many important and realistic features. The explicit consumption and investment strategies are very simple and intuitive and have been discussed and illustrated in the paper. For a calibrated version of the model we find, among other things, that the fairly high correlation between labor income and house prices imply much larger life-cycle variations in the desired exposure to house price risks than in the exposure to the stock and bond markets. While our model involves continuous adjustments of the

¹⁵With truly continuous rebalancing, the critical wealth level \underline{X} could be put equal to zero, but due to the simulation discretization a positive \underline{X} is needed to make sure that wealth does not turn negative over the next time interval in the simulation. Strictly speaking, even when the investment strategy is modified as explained, tangible wealth could become negative if the total consumption expenditures would exceed current income. If that should happen, the consumption would have to be cut back to ensure that tangible wealth stays non-negative. However that never happened in our simulations. This can be explained as follows. Violations of the imposed tangible wealth constraint will typically occur early in life where human wealth is large and the individual still has not built up tangible wealth for consumption after retirement. But early in life the optimal consumption expenditures are below current income.

consumption of housing services and the exposure of wealth to house price risk, we have demonstrated that the derived strategies are still very useful if the housing positions are only reset infrequently and borrowing constraints are imposed. Our results suggest that markets for REITs or other financial contracts facilitating the hedging of house price risks will lead to moderate improvements of welfare.

A Human capital: Proof of Theorem 2.1

In a complete market the human capital is equivalent to a financial asset paying a continuous dividend equal to the income rate. In absence of arbitrage the human capital function $L(t, r, y)$ will therefore satisfy the partial differential equation (PDE)

$$\frac{\partial L}{\partial t} + \mu_r^{\mathbb{Q}}(r)L_r + \frac{1}{2}\sigma_r(r)^2L_{rr} + y\mu_Y^{\mathbb{Q}}(r, t)L_y + \frac{1}{2}y^2\sigma_Y(r, t)^2L_{yy} - \rho_{yB}\sigma_r(r)\sigma_Y(r, t)yL_{ry} + y = rL \quad (\text{A.1})$$

for all (r, y) and $t < \tilde{T}$ with $L(\tilde{T}, r, y) = 0$.

The process $W = (W_r, W_S, W_H)^\top$ is a 3-dimensional standard Brownian motion under the real-world probability measure. Let $\lambda = (\lambda_B, \lambda_S, \lambda'_H)^\top$ and

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ \rho_{SB} & \sqrt{1 - \rho_{SB}^2} & 0 \\ \rho_{HB} & \hat{\rho}_{HS} & \hat{\rho}_H \end{pmatrix}. \quad (\text{A.2})$$

Under the risk-neutral measure, the process $W^{\mathbb{Q}} = (W_r^{\mathbb{Q}}, W_S^{\mathbb{Q}}, W_H^{\mathbb{Q}})^\top$ defined by initial value zero and

$$dW_t^{\mathbb{Q}} = dW_t + \Sigma^{-1}\lambda dt$$

is then a standard Brownian motion. Letting $\vec{\rho}_Y = (\rho_{YB}, \hat{\rho}_{YS}, \hat{\rho}_Y)^\top$, the risk-neutral income dynamics is thus

$$\begin{aligned} dY_t &= Y_t [\mu_Y(r_t, t) dt + \sigma_Y(r_t, t)\vec{\rho}_Y^\top dW_t] \\ &= Y_t \left[(\mu_Y(r_t, t) - \sigma_Y(r_t, t)\vec{\rho}_Y^\top \Sigma^{-1}\lambda) dt + \sigma_Y(r_t, t)\vec{\rho}_Y^\top dW_t^{\mathbb{Q}} \right] \\ &= Y_t \left[(\mu_Y(r_t, t) - \sigma_Y(r_t, t)\lambda_Y) dt + \sigma_Y(r_t, t)\vec{\rho}_Y^\top dW_t^{\mathbb{Q}} \right], \end{aligned}$$

where

$$\lambda_Y = \zeta_B\lambda_B + \zeta_S\lambda_S + \zeta_I\lambda'_H, \quad (\text{A.3})$$

$$\zeta_B = \rho_{YB} - \rho_{SB} \frac{\hat{\rho}_H \hat{\rho}_{YS} - \hat{\rho}_Y \hat{\rho}_{HS}}{\hat{\rho}_H \sqrt{1 - \rho_{SB}^2}} - \rho_{HB} \frac{\hat{\rho}_Y}{\hat{\rho}_H}, \quad (\text{A.4})$$

$$\zeta_S = \frac{\hat{\rho}_H \hat{\rho}_{YS} - \hat{\rho}_Y \hat{\rho}_{HS}}{\hat{\rho}_H \sqrt{1 - \rho_{SB}^2}}, \quad (\text{A.5})$$

$$\zeta_I = \frac{\hat{\rho}_Y}{\hat{\rho}_H}. \quad (\text{A.6})$$

The assumed homogeneity of the income process implies that the human capital will have the form $L(t, r, y) = yF(t, r)$ for some function F . Substituting this into the above PDE, we conclude that F must satisfy

$$\frac{\partial F}{\partial t} + F_r (\kappa[\bar{r} - r] + \sigma_r\lambda_B - \rho_{YB}\sigma_r\sigma_Y(t)) + \frac{1}{2}F_{rr}\sigma_r^2 - F(r - \mu_Y(r, t) + \lambda_Y\sigma_Y(t)) + 1 = 0 \quad (\text{A.7})$$

with $F(\tilde{T}, r) = 0$ for all r .

For the case where

$$\mu_Y(r, t) = \bar{\mu}_Y(t) + br,$$

where $\bar{\mu}_Y$ is a deterministic function and b is a constant, and the income volatility $\sigma_Y(t)$ is a deterministic function,

$$F(t, r) = \int_t^{\bar{T}} \underbrace{\mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^s (r_u - \bar{\mu}_Y(u) - br_u + \lambda_Y \sigma_Y) du} \right]}_{=G(t, s, r)} ds,$$

where

$$dr_t = \kappa(\tilde{r}(t) - r_t)dt - \sigma_r dW_{rt}^{\mathbb{Q}}$$

and $\tilde{r}(t) = \bar{r} + (\sigma_r \lambda_B - \rho_{YB} \sigma_r \sigma_Y(t))/\kappa$. Therefore, for any given s , $G(t, s, r)$ satisfies the PDE (A.7) without the 1 on the left-hand side. Upon substitution into that PDE we find that a solution of the form

$$G(t, s, r) = e^{-\tilde{A}(t, s) - \tilde{B}(t, s)r}$$

works if the functions \tilde{A} and \tilde{B} must satisfy

$$\begin{aligned} \frac{\partial \tilde{B}}{\partial t} &= \kappa \tilde{B} - 1 + b, \\ \frac{\partial \tilde{A}}{\partial t} &= -\kappa \tilde{r}(t) \tilde{B} + 0.5 \sigma_r^2 \tilde{B}^2 + \bar{\mu}_Y(t) - \lambda_Y \sigma_Y. \end{aligned}$$

Therefore,

$$\begin{aligned} \tilde{B}(t, s) &= (1 - b) \mathcal{B}_\kappa(s - t), \\ \tilde{A}(t, s) &= (\kappa \bar{r} + \sigma_r \lambda_B) \int_t^s \tilde{B}(u, s) du - \rho_{YB} \sigma_r \int_t^s \sigma_Y(u) \tilde{B}(u, s) du \\ &\quad - \frac{1}{2} \sigma_r^2 \int_t^s \tilde{B}^2(u, s) du - \int_t^s \bar{\mu}_Y(u) du + \lambda_Y \int_t^s \sigma_Y(u) du \\ &= (\kappa \bar{r} + \sigma_r \lambda_B)(1 - b) \int_t^s \mathcal{B}_\kappa(s - u) du - \rho_{YB} \sigma_r (1 - b) \int_t^s \sigma_Y(u) \mathcal{B}_\kappa(s - u) du \\ &\quad - \frac{1}{2} \sigma_r^2 (1 - b)^2 \int_t^s \mathcal{B}_\kappa^2(s - u) du - \int_t^s \bar{\mu}_Y(u) du + \lambda_Y \int_t^s \sigma_Y(u) du \\ &= (\kappa \bar{r} + \sigma_r \lambda_B)(1 - b) \frac{\tau - \mathcal{B}_\kappa(\tau)}{\kappa} - \rho_{YB} \sigma_r (1 - b) \int_t^s \sigma_Y(u) \mathcal{B}_\kappa(s - u) du \\ &\quad - \frac{1}{2} \sigma_r^2 (1 - b)^2 \frac{1}{\kappa^2} [\tau - 2\mathcal{B}_\kappa(\tau) + \mathcal{B}_{2\kappa}(\tau)] - \int_t^s \bar{\mu}_Y(u) du + \lambda_Y \int_t^s \sigma_Y(u) du, \end{aligned}$$

where $\tau = s - t$ and we have applied some of the results in Appendix D.

The assumed income dynamics implies that

$$Y_t = Y_0 \exp \left\{ \int_0^t \bar{\mu}_Y(u) du + b \int_0^t r_u du - \frac{1}{2} \int_0^t \sigma_Y(u)^2 du + \int_0^t \sigma_Y(u) \bar{\rho}_Y^T dW_u \right\}. \quad (\text{A.8})$$

The interest rate dynamics (2.1) implies that

$$\int_0^t r_u du = r_0 t + (\bar{r} - r_0)(t - \mathcal{B}_\kappa(t)) - \int_0^t \sigma_r \mathcal{B}_\kappa(t - u) dW_{ru}. \quad (\text{A.9})$$

Substituting that into the preceding equation and taking expectations, we get (2.11).

The time 0 expectation of the human capital at time t is

$$\mathbb{E}_0[Y_t F(t, r_t)] = \mathbb{E}_0 \left[Y_t \mathbf{1}_{\{t \leq \bar{T}\}} \int_t^{\bar{T}} e^{-\tilde{A}(t,s) - (1-b)\mathcal{B}_\kappa(s-t)r_t} ds \right].$$

Since F is relatively insensitive to the interest rate, $F(t, r_t) \approx F(t, \bar{r})$ and the expected human capital is approximated by $F(t, \bar{r})\mathbb{E}_0[Y_t]$. However, we can also obtain an exact, but more complicated, expression for the expected human capital by substituting (A.8) and

$$r_t = e^{-\kappa t} r_0 + \bar{r}(1 - e^{-\kappa t}) - \int_0^t \sigma_r e^{-\kappa(t-u)} dW_{r_u}$$

into the relation above. The future human capital is lognormally distributed and tedious computations lead to the expected value $\mathbb{E}_0[Y_t F(t, r_t)] = \bar{F}(t)\mathbb{E}_0[Y_t]$, where

$$\begin{aligned} \bar{F}(t) = & \mathbf{1}_{\{t \leq \bar{T}\}} \exp \left\{ -\frac{b^2 \sigma_r^2}{2\kappa^2} (t - \mathcal{B}_\kappa(t)) + \frac{b^2 \sigma_r^2}{4\kappa} \mathcal{B}_\kappa(t)^2 + b\sigma_r \rho_{YB} \int_0^t \sigma_Y(u) \mathcal{B}_\kappa(t-u) du \right\} \\ & \times \int_t^{\bar{T}} \exp \left\{ \int_t^s \bar{\mu}_Y(u) du - \lambda_Y \int_t^s \sigma_Y(u) du - (1-b)\bar{r} \mathcal{B}_\kappa(s-t) - (1-b)e^{-\kappa t} (r_0 - \bar{r}) \mathcal{B}_\kappa(s-t) \right. \\ & - \left(\bar{r} + \frac{\sigma_r \lambda_B}{\kappa} \right) (1-b)(s-t - \mathcal{B}_\kappa(s-t)) + \frac{\sigma_r^2}{2\kappa^2} (1-b)^2 (s-t - 2\mathcal{B}_\kappa(s-t) + \mathcal{B}_{2\kappa}(s-t)) \\ & + \frac{\sigma_r^2}{2} \mathcal{B}_\kappa(s-t)^2 \mathcal{B}_{2\kappa}(t) - \frac{\sigma_r}{\kappa} b \mathcal{B}_\kappa(s-t) \left(\mathcal{B}_\kappa(t) - e^{-\kappa(s-t)} \mathcal{B}_{2\kappa}(t) \right) \\ & + \frac{\sigma_r^2}{2\kappa^2} b^2 \left(t - \mathcal{B}_\kappa(s) + \mathcal{B}_\kappa(t) - \frac{\kappa}{2} (\mathcal{B}_\kappa(s)^2 - \mathcal{B}_\kappa(s-t)^2) \right) \\ & \left. + \sigma_r \rho_{YB} \left(\int_t^s \sigma_Y(u) \mathcal{B}_\kappa(s-u) du - b \int_0^s \sigma_Y(u) \mathcal{B}_\kappa(s-u) du + \mathcal{B}_\kappa(s-t) \int_0^t \sigma_Y(u) e^{-\kappa(t-u)} du \right) \right\} ds. \end{aligned} \tag{A.10}$$

B Proofs for fully flexible housing decisions

B.1 The HJB equation

Define the scaled controls $\alpha_S = \pi_S \sigma_S x$, $\alpha_B = \pi_B \sigma_B x$, and $\alpha_I = \varphi_I \sigma_H h$. Let $Z = (r, Y, H)^\top$ be the vector of state variables with drift $\mu_Z = (\kappa[\bar{r} - r], y\mu_Y(r, t), h(r + \lambda_H \sigma_H - r^{\text{imp}}))^\top$. Define the vectors $\lambda = (\lambda_B, \lambda_S, \lambda'_H)^\top$ and $\alpha = (\alpha_B, \alpha_S, \alpha_I)^\top$, the matrix Σ as in (A.2), and

$$\Sigma_Z = \begin{pmatrix} -\sigma_r & 0 & 0 \\ 0 & y\sigma_Y & 0 \\ 0 & 0 & h\sigma_H \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \rho_{YB} & \hat{\rho}_{YS} & \hat{\rho}_Y \\ \rho_{HB} & \hat{\rho}_{HS} & \hat{\rho}_H \end{pmatrix} = \begin{pmatrix} -\sigma_r & 0 & 0 \\ y\sigma_Y \rho_{YB} & y\sigma_Y \hat{\rho}_{YS} & y\sigma_Y \hat{\rho}_Y \\ h\sigma_H \rho_{HB} & h\sigma_H \hat{\rho}_{HS} & h\sigma_H \hat{\rho}_H \end{pmatrix}.$$

Σ contains the correlations between the assets $P = (B, S, H)^\top$, and Σ_Z contains the volatilities and correlations of the state variables $Z = (r, Y, H)$. Now the dynamics of the state variables and the

wealth dynamics from (2.12) can be written compactly as

$$dZ_t = \mu_Z(Z_t) dt + \Sigma_Z(Z_t) dW_t, \quad (\text{B.1})$$

$$dX_t = \left(r_t X_t + \alpha_t^\top \lambda_t - \varphi_{Ct} \nu H_t - c_t + 1_{\{t < \tilde{T}\}} Y_t \right) dt + \alpha_t^\top \Sigma dW_t, \quad (\text{B.2})$$

where $W = (W_r, W_s, W_H)^\top$.

The Hamilton-Jacobi-Bellman equation (HJB) associated with the problem can be written as

$$0 = \mathcal{L}_1 J + \mathcal{L}_2 J + \mathcal{L}_3 J, \quad (\text{B.3})$$

where

$$\begin{aligned} \mathcal{L}_1 J &= \max_{c, \varphi_C} \left\{ f \left(c^\beta \varphi_C^{1-\beta}, J \right) - J_x (c + h\nu\varphi_C) \right\}, \\ \mathcal{L}_2 J &= \max_{\alpha} \left\{ J_x \alpha^\top \lambda + \frac{1}{2} J_{xx} \alpha^\top \Sigma \Sigma^\top \alpha + \alpha^\top \Sigma \Sigma_Z^\top J_{xz} \right\}, \\ \mathcal{L}_3 J &= \frac{\partial J}{\partial t} + J_x \left(r x + 1_{\{t < \tilde{T}\}} y \right) + J_z^\top \mu_z + \frac{1}{2} \text{tr} \left(J_{zz} \Sigma_Z \Sigma_Z^\top \right). \end{aligned}$$

Note that the first-order conditions for c and φ_C imply that

$$\frac{\partial f}{\partial z} \left(c^\beta \varphi_C^{1-\beta}, J \right) \beta c^{\beta\gamma-1} \varphi_C^{(1-\beta)\gamma} = J_x, \quad \frac{\partial f}{\partial z} \left(c^\beta \varphi_C^{1-\beta}, J \right) (1-\beta) c^{\beta\gamma} \varphi_C^{(1-\beta)\gamma-1} = \nu h J_x. \quad (\text{B.4})$$

In particular, $c = \frac{\beta}{1-\beta} \nu h \varphi_C$ so that the relation between optimal perishable consumption and optimal housing consumption is proportional to the relative price of the two goods with a proportionality factor determined by the utility weights of the two goods. To proceed with the computation of $\mathcal{L}_1 J$, below we consider the different specifications of the aggregator separately. However, we first derive $\mathcal{L}_2 J$ and the associated optimal α .

B.2 Computation of $\mathcal{L}_2 J$.

The first-order condition reads

$$J_x \lambda + J_{xx} \Sigma \Sigma^\top \alpha + \Sigma \Sigma_Z^\top J_{xz} = 0$$

or

$$\alpha = -\frac{J_x}{J_{xx}} (\Sigma \Sigma^\top)^{-1} \lambda - \frac{1}{J_{xx}} (\Sigma \Sigma^\top)^{-1} \Sigma \Sigma_Z^\top J_{xz} = -\frac{J_x}{J_{xx}} (\Sigma \Sigma^\top)^{-1} \lambda - \frac{1}{J_{xx}} (\Sigma_Z \Sigma^{-1})^\top J_{xz}. \quad (\text{B.5})$$

The dynamics of tangible wealth is then

$$\begin{aligned} dX_t &= \left(r_t X_t - \frac{J_x}{J_{xx}} \tilde{\lambda}^\top \tilde{\lambda} - \tilde{\lambda}^\top \Sigma_Z^\top \frac{J_{xz}}{J_{xx}} - [c_t + \nu H_t \varphi_{Ct}] + 1_{\{t < \tilde{T}\}} Y_t \right) dt \\ &\quad - \frac{J_x}{J_{xx}} \tilde{\lambda}^\top dW_t - \frac{J_{xz}^\top}{J_{xx}} \Sigma_Z dW_t, \end{aligned} \quad (\text{B.6})$$

where $\tilde{\lambda} = \Sigma^{-1} \lambda$. By Itô's Lemma and (A.7),

$$dF(t, r_t) = (-1 + F(t, r_t) [r_t - \mu_Y(r_t, t) + \lambda_Y \sigma_Y(t)] - F_r(r_t, t) [\sigma_r \lambda_B - \rho_{YB} \sigma_r \sigma_Y]) dt - F_r(r_t, t) \sigma_r dW_{rt}.$$

The dynamics of total wealth, $\mathcal{W}_t = X_t + Y_t F(t, r_t)$, now becomes

$$d\mathcal{W}_t = \left(r_t \mathcal{W}_t - \frac{J_x}{J_{xx}} \tilde{\lambda}^\top \tilde{\lambda} - \tilde{\lambda}^\top \Sigma_Z^\top \frac{J_{xz}}{J_{xx}} - [c_t + \nu H_t \varphi_{Ct}] + Y_t F(t, r_t) \lambda_Y \sigma_Y(t) - Y_t F_r(t, r_t) \sigma_r \lambda_B \right) dt - \frac{J_x}{J_{xx}} \tilde{\lambda}^\top dW_t - \frac{J_{xz}^\top}{J_{xx}} \Sigma_Z dW_t - Y_t F_r(t, r_t) \sigma_r dW_{rt} + Y_t F_t \sigma_Y \tilde{\rho}_Y^\top dW_t. \quad (\text{B.7})$$

Substituting the optimal α back into $\mathcal{L}_2 J$ leads to

$$\mathcal{L}_2 J = -\frac{1}{2} \frac{J_x^2}{J_{xx}} \tilde{\lambda}^\top \tilde{\lambda} - \frac{J_x}{J_{xx}} J_{xz}^\top \Sigma_Z \Sigma^{-1} \lambda - \frac{1}{2} \frac{1}{J_{xx}} J_{xz}^\top \Sigma_Z \Sigma_Z^\top J_{xz}, \quad (\text{B.8})$$

where $\tilde{\lambda}^\top \tilde{\lambda} = \lambda^\top (\Sigma \Sigma^\top)^{-1} \lambda$. Next, we compute the relevant matrix products. Firstly, since

$$\begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix}^{-1} = \frac{1}{\det} \begin{pmatrix} 1 - c^2 & bc - a & ac - b \\ bc - a & 1 - b^2 & ab - c \\ ac - b & ab - c & 1 - a^2 \end{pmatrix},$$

where $\det = 1 + 2abc - a^2 - b^2 - c^2$, and

$$\Sigma \Sigma^T = \begin{pmatrix} 1 & \rho_{SB} & \rho_{HB} \\ \rho_{SB} & 1 & \rho_{SH} \\ \rho_{HB} & \rho_{SH} & 1 \end{pmatrix},$$

we have

$$(\Sigma \Sigma^T)^{-1} = \frac{1}{\det} \begin{pmatrix} 1 - \rho_{SH}^2 & -\rho_{SB,H} & -\rho_{BH,S} \\ -\rho_{SB,H} & 1 - \rho_{HB}^2 & -\rho_{SH,B} \\ -\rho_{BH,S} & -\rho_{SH,B} & 1 - \rho_{SB}^2 \end{pmatrix}$$

where $\det = 1 + 2\rho_{SB}\rho_{HB}\rho_{SH} - \rho_{SB}^2 - \rho_{HB}^2 - \rho_{SH}^2$ and for any three variables x, y, z we use the notation $\rho_{xy,z} = \rho_{xy} - \rho_{xz}\rho_{yz}$.

Secondly, disregarding the volatility matrix, Σ and Σ_Z are equal except for the second row. We are thus interested in

$$\begin{pmatrix} 1 & 0 & 0 \\ f & g & h \\ c & d & e \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ a & b & 0 \\ c & d & e \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ f - a \frac{eg-dh}{be} - c \frac{h}{e} & \frac{eg-dh}{be} & \frac{h}{e} \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore,

$$\Sigma_Z \Sigma^{-1} = \begin{pmatrix} -\sigma_r & 0 & 0 \\ 0 & y\sigma_Y & 0 \\ 0 & 0 & h\sigma_H \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \zeta_B & \zeta_S & \zeta_I \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\sigma_r & 0 & 0 \\ y\sigma_Y \zeta_B & y\sigma_Y \zeta_S & y\sigma_Y \zeta_I \\ 0 & 0 & h\sigma_H \end{pmatrix},$$

where $\zeta_B, \zeta_S, \zeta_I$ were defined in (A.4)-(A.6).

Thirdly, simple multiplications lead to

$$\begin{aligned}\Sigma_Z \Sigma_Z^\top &= \begin{pmatrix} -\sigma_r & 0 & 0 \\ 0 & y\sigma_Y & 0 \\ 0 & 0 & h\sigma_H \end{pmatrix} \begin{pmatrix} 1 & \rho_{YB} & \rho_{HB} \\ \rho_{YB} & 1 & \rho_{YH} \\ \rho_{HB} & \rho_{YH} & 1 \end{pmatrix} \begin{pmatrix} -\sigma_r & 0 & 0 \\ 0 & y\sigma_Y & 0 \\ 0 & 0 & h\sigma_H \end{pmatrix} \\ &= \begin{pmatrix} \sigma_r^2 & -\rho_{YB}\sigma_r y\sigma_Y & -\rho_{HB}\sigma_r h\sigma_H \\ -\rho_{YB}\sigma_r y\sigma_Y & y^2\sigma_Y^2 & \rho_{YH}y\sigma_Y h\sigma_H \\ -\rho_{HB}\sigma_r h\sigma_H & \rho_{YH}y\sigma_Y h\sigma_H & h^2\sigma_H^2 \end{pmatrix}.\end{aligned}$$

Substitution of these matrix products into (B.5) gives

$$\alpha_B = -\frac{J_x}{J_{xx}}\xi_B - \frac{yJ_{xy}}{J_{xx}}\sigma_Y\zeta_B + \frac{J_{xr}}{J_{xx}}\sigma_r, \quad (\text{B.9})$$

$$\alpha_S = -\frac{J_x}{J_{xx}}\xi_S - \frac{yJ_{xy}}{J_{xx}}\sigma_Y\zeta_S, \quad (\text{B.10})$$

$$\alpha_I = -\frac{J_x}{J_{xx}}\xi_I - \frac{yJ_{xy}}{J_{xx}}\sigma_Y\zeta_I - \frac{J_{xh}}{J_{xx}}h\sigma_H. \quad (\text{B.11})$$

where

$$\xi_B = \frac{1}{\det} (\lambda_B(1 - \rho_{SH}^2) - \rho_{SB,H}\lambda_S - \rho_{BH,S}\lambda'_H), \quad (\text{B.12})$$

$$\xi_S = \frac{1}{\det} (\lambda_S(1 - \rho_{BH}^2) - \rho_{SB,H}\lambda_B - \rho_{SH,B}\lambda'_H), \quad (\text{B.13})$$

$$\xi_I = \frac{1}{\det} (\lambda'_H(1 - \rho_{SB}^2) - \rho_{SH,B}\lambda_S - \rho_{BH,S}\lambda_B). \quad (\text{B.14})$$

Note that

$$\tilde{\lambda}^\top \tilde{\lambda} = \lambda^\top (\Sigma \Sigma^\top)^{-1} \lambda = \lambda_B \xi_B + \lambda_S \xi_S + \lambda'_H \xi_I.$$

B.3 Simplifications when J has the form in (3.1)

It turns out to be useful to express the derivatives of J in terms of J itself:

$$\begin{aligned}J_x &= \frac{(1-\gamma)J}{x+yF}, & J_{xx} &= -\frac{\gamma(1-\gamma)J}{(x+yF)^2}, \\ J_y &= (1-\gamma)J \frac{F}{x+yF}, & J_{yy} &= -\gamma(1-\gamma)J \frac{F^2}{(x+yF)^2}, \\ J_h &= \gamma J \frac{gh}{g}, & J_{hh} &= \gamma(1-\gamma)J \left[\frac{1}{1-\gamma} \frac{ghh}{g} - \left(\frac{gh}{g} \right)^2 \right], \\ J_{xy} &= -\gamma(1-\gamma)J \frac{F}{(x+yF)^2}, & J_{xh} &= \gamma(1-\gamma)J \frac{1}{x+yF} \frac{gh}{g}, \\ J_{hy} &= \gamma(1-\gamma)J \frac{gh}{g} \frac{F}{x+yF}, & J_r &= (1-\gamma)J \left[\frac{\gamma}{1-\gamma} \frac{g_r}{g} + \frac{yF_r}{x+yF} \right],\end{aligned}$$

$$\begin{aligned}
J_{rr} &= (1-\gamma)J \left[\frac{\gamma}{1-\gamma} \frac{g_{rr}}{g} - \gamma \left(\frac{g_r}{g} \right)^2 + 2\gamma \frac{g_r}{g} \frac{yF_r}{x+yF} - \gamma \left(\frac{yF_r}{x+yF} \right)^2 + \frac{yF_{rr}}{x+yF} \right], \\
J_{xr} &= \gamma(1-\gamma)J \left[\frac{g_r}{g} \frac{1}{x+yF} - \frac{yF_r}{(x+yF)^2} \right], \\
J_{ry} &= (1-\gamma)J \left[\gamma \frac{g_r}{g} \frac{F}{x+yF} + \frac{F_r}{x+yF} - \gamma \frac{yF_r}{(x+yF)^2} \right], \\
J_{rh} &= \gamma(1-\gamma)J \left[\frac{1}{1-\gamma} \frac{g_{rh}}{g} - \frac{g_r g_h}{g^2} + \frac{yF_r}{x+yF} \frac{g_h}{g} \right], \\
\frac{\partial J}{\partial t} &= (1-\gamma)J \left[\frac{\gamma}{1-\gamma} \frac{\partial g}{\partial t} \frac{1}{g} + \frac{y}{x+yF} \frac{\partial F}{\partial t} \right].
\end{aligned}$$

Note that

$$\frac{J_x}{J_{xx}} = -\frac{1}{\gamma}(x+yF), \quad \frac{J_{xr}}{J_{xx}} = yF_r - \frac{g_r}{g}(x+yF), \quad \frac{J_{xy}}{J_{xx}} = F, \quad \frac{J_{xh}}{J_{xx}} = -\frac{g_h}{g}(x+yF).$$

Substituting into (B.9)-(B.11), we get the optimal portfolio weights in (3.2) and (3.3) and the optimal housing investment reflected by (3.4). The dynamics of total wealth in (B.7) simplifies to

$$\begin{aligned}
\frac{d\mathcal{W}_t}{\mathcal{W}_t} &= \left(r_t + \frac{1}{\gamma} \tilde{\lambda}^\top \tilde{\lambda} - \sigma_r \lambda_B \frac{g_r}{g} + \sigma_H \lambda'_H H_t \frac{g_h}{g} - \frac{c_t + \varphi_{Ct} \nu H_t}{\mathcal{W}_t} \right) dt \\
&\quad + \frac{1}{\gamma} \tilde{\lambda}^\top dW_t - \frac{g_r}{g} \sigma_r dW_{rt} + H_t \frac{g_h}{g} \sigma_H \bar{\rho}_H^\top dW_t,
\end{aligned} \tag{B.15}$$

Substituting the relevant derivatives into (B.8) and simplifying, we obtain

$$\begin{aligned}
\mathcal{L}_2 J &= (1-\gamma)J \left\{ \frac{\tilde{\lambda}^\top \tilde{\lambda}}{2\gamma} + \sigma_r \lambda_B \left(\frac{yF_r}{x+yF} - \frac{g_r}{g} \right) - \sigma_Y \lambda_Y \frac{yF}{x+yF} + \sigma_H \lambda'_H h \frac{g_h}{g} \right. \\
&\quad + \frac{\gamma}{2} \sigma_r^2 \left(\frac{g_r}{g} - \frac{yF_r}{x+yF} \right)^2 + \frac{\gamma}{2} \sigma_Y^2 \frac{y^2 F^2}{(x+yF)^2} + \frac{\gamma}{2} \sigma_H^2 h^2 \left(\frac{g_h}{g} \right)^2 \\
&\quad - \gamma \rho_{YB} \sigma_r \sigma_Y \frac{yF}{x+yF} \left(\frac{yF_r}{x+yF} - \frac{g_r}{g} \right) + \gamma \rho_{HB} \sigma_r \sigma_H h \frac{g_h}{g} \left(\frac{yF_r}{x+yF} - \frac{g_r}{g} \right) \\
&\quad \left. - \gamma \rho_{HY} \sigma_H \sigma_Y \frac{yF}{x+yF} h \frac{g_h}{g} \right\}.
\end{aligned}$$

Substituting the relevant derivatives into $\mathcal{L}_3 J$ yields a long expression where a lot of terms are of the form $(1-\gamma)Jy/(x+yF)$ multiplied by one of the terms in the PDE (A.7) for F . Due to the PDE,

all these terms can be reduced to $(1 - \gamma)J[\sigma_Y \lambda_Y F - \sigma_r \lambda_B F_r]y/(x + yF)$. In total, we get

$$\begin{aligned} \mathcal{L}_3 J = (1 - \gamma)J & \left\{ \sigma_Y \lambda_Y \frac{yF}{x + yF} - \sigma_r \lambda_B \frac{yF_r}{x + yF} + \frac{\gamma}{1 - \gamma} \frac{\partial g}{\partial t} \frac{1}{g} + r + \frac{\gamma}{1 - \gamma} \kappa[\bar{r} - r] \frac{g_r}{g} \right. \\ & + \frac{\gamma}{1 - \gamma} (r + \lambda'_H \sigma_H - \nu) h \frac{g_h}{g} + \frac{\gamma}{2} \sigma_r^2 \left[\frac{1}{1 - \gamma} \frac{g_{rr}}{g} - \left(\frac{g_r}{g} - \frac{yF_r}{x + yF} \right)^2 \right] \\ & - \frac{\gamma}{2} \sigma_Y^2 \frac{y^2 F^2}{(x + yF)^2} + \frac{\gamma}{2} \sigma_H^2 h^2 \left[\frac{1}{1 - \gamma} \frac{g_{hh}}{g} - \left(\frac{g_h}{g} \right)^2 \right] \\ & - \gamma \rho_{YB} \sigma_r \sigma_Y \frac{yF}{x + yF} \left(\frac{g_r}{g} - \frac{yF_r}{x + yF} \right) + \gamma \rho_{YH} \sigma_Y \sigma_H h \frac{g_h}{g} \frac{yF}{x + yF} \\ & \left. - \gamma \rho_{HB} \sigma_r \sigma_H h \left[\frac{1}{1 - \gamma} \frac{g_{rh}}{g} + \frac{g_h}{g} \left(\frac{yF_r}{x + yF} - \frac{g_r}{g} \right) \right] \right\}. \end{aligned}$$

Summing up, we get

$$\begin{aligned} \mathcal{L}_2 J + \mathcal{L}_3 J = \gamma J \frac{1}{g} & \left\{ \frac{1}{2} \sigma_r^2 g_{rr} + \frac{1}{2} \sigma_H^2 h^2 g_{hh} - \rho_{HB} \sigma_r \sigma_H h g_{rh} + \left(\kappa[\bar{r} - r] + \frac{\gamma - 1}{\gamma} \sigma_r \lambda_B \right) g_r \right. \\ & \left. + \left(r + \frac{1}{\gamma} \lambda'_H \sigma_H - \nu \right) h g_h + \frac{\partial g}{\partial t} - \frac{\gamma - 1}{\gamma} \left(r + \frac{\tilde{\lambda}^\top \tilde{\lambda}}{2\gamma} \right) g \right\}. \end{aligned}$$

B.4 The case $\psi \neq 1$: Proof of Theorem 3.1

In this case, solving (B.4) for c and φ_C yields

$$c = \eta \frac{\beta \nu}{1 - \beta} h^k J_x^{-\psi} [(1 - \gamma)J]^{\psi(1-1/\theta)}, \quad (\text{B.16})$$

$$\varphi_C = \eta h^{k-1} J_x^{-\psi} [(1 - \gamma)J]^{\psi(1-1/\theta)}, \quad (\text{B.17})$$

where $k = (1 - \psi)(1 - \beta)$ and $\eta = \delta^\psi \beta^\psi \left(\frac{\beta \nu}{1 - \beta} \right)^{\beta(\psi-1)-\psi}$, so that

$$\mathcal{L}_1 J = \frac{\eta \nu}{1 - \beta} \frac{1}{\psi - 1} h^k J_x^{1-\psi} [(1 - \gamma)J]^{\psi(1-1/\theta)} - \delta \theta J. \quad (\text{B.18})$$

Substituting in the conjectured form for J , we get

$$c = \eta \frac{\beta \nu}{1 - \beta} h^k (x + yF) g^{-\frac{\gamma(\psi-1)}{1-\gamma}}, \quad (\text{B.19})$$

$$\varphi_C = \eta h^{k-1} (x + yF) g^{-\frac{\gamma(\psi-1)}{1-\gamma}} \quad (\text{B.20})$$

and

$$\mathcal{L}_1 J = (1 - \gamma)J \frac{1}{g} \left(\frac{\eta \nu}{(\psi - 1)(1 - \beta)} h^k g^{\frac{\gamma\psi-1}{\gamma-1}} - \frac{\delta}{1 - 1/\psi} g \right) \quad (\text{B.21})$$

so that the full HJB-equation reduces to

$$\begin{aligned} 0 = \frac{1}{2} \sigma_r^2 g_{rr} + \frac{1}{2} \sigma_H^2 h^2 g_{hh} - \rho_{HB} \sigma_r \sigma_H h g_{rh} + \left(\kappa[\bar{r} - r] + \frac{\gamma - 1}{\gamma} \sigma_r \lambda_B \right) g_r + \frac{\partial g}{\partial t} \\ + \left(r + \frac{1}{\gamma} \lambda'_H \sigma_H - \nu \right) h g_h + \frac{\eta \nu (1 - \gamma)}{\gamma(\psi - 1)(1 - \beta)} h^k g^{\frac{\gamma\psi-1}{\gamma-1}} + \frac{\gamma - 1}{\gamma} \left(\frac{\delta}{1 - 1/\psi} - r - \frac{\tilde{\lambda}^\top \tilde{\lambda}}{2\gamma} \right) g \end{aligned} \quad (\text{B.22})$$

For general γ, ψ , this PDE is non-linear and it seems impossible to find a closed-form solution. However when $\psi = 1/\gamma$, corresponding to power utility, the non-linearity disappears since $g^{\frac{\gamma\psi-1}{\gamma-1}} = g^0 = 1$.

With power utility we get

$$c = \eta \frac{\beta\nu}{1-\beta} h^k \frac{x+yF}{g}, \quad (\text{B.23})$$

$$\varphi_C = \eta h^{k-1} \frac{x+yF}{g} \quad (\text{B.24})$$

and the PDE (B.22) for g reduces to

$$\begin{aligned} 0 = & \frac{1}{2}\sigma_r^2 g_{rr} + \frac{1}{2}\sigma_H^2 h^2 g_{hh} - \rho_{HB}\sigma_r\sigma_H h g_{rh} + \left(\kappa[\bar{r}-r] + \frac{\gamma-1}{\gamma}\sigma_r\lambda_B\right) g_r \\ & + \left(r + \frac{1}{\gamma}\lambda'_H\sigma_H - \nu\right) h g_h + \frac{\partial g}{\partial t} + \frac{\eta\nu}{1-\beta} h^k + \frac{\gamma-1}{\gamma} \left(\frac{\delta}{1-\gamma} - r - \frac{\tilde{\lambda}^\top \tilde{\lambda}}{2\gamma}\right) g \end{aligned} \quad (\text{B.25})$$

with the terminal condition $g(T, r, h) = \varepsilon^{1/\gamma}$. Conjecturing a solution of the form

$$g(t, r, h) = \varepsilon^{\frac{1}{\gamma}} e^{-d_0(T-t) - \tilde{d}_0(T-t)r} + \frac{\eta\nu}{1-\beta} h^k \int_t^T e^{-d_1(u-t) - \tilde{d}_1(u-t)r} du,$$

we find that d_0 and \tilde{d}_0 must satisfy the ODEs

$$\begin{aligned} \tilde{d}'_0(\tau) + \kappa\tilde{d}_0(\tau) &= \frac{\gamma-1}{\gamma}, \\ d'_0(\tau) &= -\frac{1}{2}\sigma_r^2 \tilde{d}_0(\tau)^2 + \left(\kappa\bar{r} + \frac{\gamma-1}{\gamma}\sigma_r\lambda_B\right) \tilde{d}_0(\tau) + \frac{\delta}{\gamma} + \frac{\gamma-1}{2\gamma^2} \tilde{\lambda}^\top \tilde{\lambda} \end{aligned}$$

with $d_0(0) = \tilde{d}_0(0) = 0$, and d_1 and \tilde{d}_1 must satisfy the ODEs

$$\begin{aligned} \tilde{d}'_1(\tau) + \kappa\tilde{d}_1(\tau) &= \frac{\gamma-1}{\gamma} - k = \beta \frac{\gamma-1}{\gamma}, \\ d'_1(\tau) &= -\frac{1}{2}\sigma_r^2 \tilde{d}_1(\tau)^2 + \left(\kappa\bar{r} + \frac{\gamma-1}{\gamma}\sigma_r\lambda_B - k\sigma_r\sigma_H\rho_{HB}\right) \tilde{d}_1(\tau) \\ &+ \frac{\delta}{\gamma} + \frac{\gamma-1}{2\gamma^2} \tilde{\lambda}^\top \tilde{\lambda} - \frac{1}{2}k(k-1)\sigma_H^2 - k \left[\frac{1}{\gamma}\sigma_H\lambda'_H - \nu\right] \end{aligned}$$

with $d_1(0) = \tilde{d}_1(0) = 0$. The solutions for \tilde{d}_0 and \tilde{d}_1 are

$$\begin{aligned} \tilde{d}_0(\tau) &= \frac{\gamma-1}{\gamma} \frac{1}{\kappa} (1 - e^{-\kappa\tau}) = \frac{\gamma-1}{\gamma} \mathcal{B}_\kappa(\tau), \\ \tilde{d}_1(\tau) &= \beta \frac{\gamma-1}{\gamma} \frac{1}{\kappa} (1 - e^{-\kappa\tau}) = \beta \frac{\gamma-1}{\gamma} \mathcal{B}_\kappa(\tau). \end{aligned}$$

Straightforward integration yields

$$\begin{aligned} d_0(\tau) &= \left(\frac{\delta}{\gamma} + \frac{\gamma-1}{2\gamma^2} \tilde{\lambda}^\top \tilde{\lambda}\right) \tau - \frac{1}{2}\sigma_r^2 \left(\frac{\gamma-1}{\gamma}\right)^2 \int_0^\tau \mathcal{B}_\kappa(u)^2 du \\ &+ \left(\kappa\bar{r} + \frac{\gamma-1}{\gamma}\sigma_r\lambda_B\right) \frac{\gamma-1}{\gamma} \int_0^\tau \mathcal{B}_\kappa(u) du, \end{aligned}$$

which—exploiting the integration results in Appendix D—gives $d_0(\tau) = \mathcal{D}_\gamma(\tau)$ defined in (3.11). The expression (3.12) for $d_1(\tau)$ follows analogously.

Expected consumption and wealth. With $\psi = 1/\gamma$ and $\varepsilon = 0$, the total wealth dynamics is given in (3.14), optimal perishable consumption is $c_t = \beta \mathcal{W}_t / G(t, r_t)$, and optimal expenditure on housing consumption is $\varphi_{C_t} \nu H_t = (1 - \beta) \mathcal{W}_t / G(t, r_t)$. We will use Itô's Lemma to find the dynamics of $\mathcal{W}_t / G(t, r_t)$ and then compute expectations.

First we establish the dynamics of $G(t, r_t) = \int_t^T M_t^u du$, where $M_t^u = \exp\{-d_1(u-t) - \hat{\beta} \mathcal{B}_\kappa(u-t)r_t\}$. Since

$$dM_t^u = M_t^u \left\{ \left(\hat{\beta} r_t + \left(\frac{\gamma-1}{\gamma} \sigma_r \lambda_B - k \sigma_r \sigma_H \rho_{HB} \right) \hat{\beta} \mathcal{B}_\kappa(u-t) + \frac{\delta}{\gamma} + \frac{\gamma-1}{2\gamma^2} \tilde{\lambda}^\top \tilde{\lambda} - \frac{1}{2} k(k-1) \sigma_H^2 - k \left[\frac{1}{\gamma} \sigma_H \lambda'_H - \nu \right] \right) dt + \hat{\beta} \mathcal{B}_\kappa(u-t) \sigma_r dW_{rt} \right\},$$

we get¹⁶

$$\frac{dG(t, r_t)}{G(t, r_t)} = \left[-\frac{1}{G(t, r_t)} + \left(\hat{\beta} r_t + \frac{\delta}{\gamma} + \frac{\gamma-1}{2\gamma^2} \tilde{\lambda}^\top \tilde{\lambda} - \frac{1}{2} k(k-1) \sigma_H^2 - k \left[\frac{1}{\gamma} \sigma_H \lambda'_H - \nu \right] \right) + \hat{\beta} \left(\frac{\gamma-1}{\gamma} \sigma_r \lambda_B - k \sigma_r \sigma_H \rho_{HB} \right) D(t, r_t) \right] dt + \hat{\beta} \sigma_r D(t, r_t) dW_{rt}.$$

Now Itô's Lemma implies

$$\begin{aligned} \frac{d(\mathcal{W}_t / G(t, r_t))}{\mathcal{W}_t / G(t, r_t)} &= \frac{d\mathcal{W}_t}{\mathcal{W}_t} - \frac{dG(t, r_t)}{G(t, r_t)} + \left(\frac{dG(t, r_t)}{G(t, r_t)} \right)^2 - \frac{d\mathcal{W}_t}{\mathcal{W}_t} \frac{dG(t, r_t)}{G(t, r_t)} \\ &= \left[\left(1 - \hat{\beta} \right) r_t + \frac{1+\gamma}{2\gamma^2} \tilde{\lambda}^\top \tilde{\lambda} + k \sigma_H \lambda'_H - \frac{\delta}{\gamma} + \frac{1}{2} k(k-1) \sigma_H^2 + k \left(\frac{1}{\gamma} \sigma_H \lambda'_H - \nu \right) \right] dt \\ &\quad + \frac{1}{\gamma} \tilde{\lambda}^\top dW_t + k \sigma_H \bar{\rho}_H^\top dW_t. \end{aligned} \tag{B.26}$$

We can write $\frac{\mathcal{W}_t}{G(t, r_t)} = \frac{\mathcal{W}_0}{G(0, r_0)} e^{\tilde{z}}$ and use (A.9) to conclude that the random variable \tilde{z} is normally distributed. Taking expectations, we get

$$\mathbb{E}_0 \left[\frac{\mathcal{W}_t}{G(t, r_t)} \right] = \frac{\mathcal{W}_0}{G(0, r_0)} e^{A(t)},$$

where

$$\begin{aligned} A(t) &= \exp \left\{ \left[\left(1 - \hat{\beta} \right) r_0 + \frac{1+\gamma}{2\gamma^2} \tilde{\lambda}^\top \tilde{\lambda} + k \sigma_H \lambda'_H - \frac{\delta}{\gamma} + \frac{1}{2} k(k-1) \sigma_H^2 + k \left(\frac{1}{\gamma} \sigma_H \lambda'_H - \nu \right) \right] t \right. \\ &\quad + \left(1 - \hat{\beta} \right) \left[\left(\bar{r} - r_0 \right) + \frac{\sigma_r^2 (1 - \hat{\beta})}{2\kappa^2} - \frac{\sigma_r}{\kappa} \left(\frac{1}{\gamma} \lambda_B + k \sigma_H \rho_{HB} \right) \right] \left(t - \mathcal{B}_\kappa(t) \right) \\ &\quad \left. - \left(1 - \hat{\beta} \right)^2 \frac{\sigma_r^2}{4\kappa} \mathcal{B}_\kappa(t)^2 \right\}. \end{aligned}$$

¹⁶If $dM_t^u = \mu_t^u du + \sigma_t^u dW_t$ and $G_t = \int_t^T M_t^u du$, then $dG_t = \left(-M_t^t + \int_t^T \mu_t^u du \right) dt + \left(\int_t^T \sigma_t^u du \right) dW_t$.

The expected consumption rate at time t is then $E_0[c_t] = \beta \frac{\mathcal{W}_0}{G(0, r_0)} e^{A(t)}$ and the expected spending on housing consumption is $E_0[\varphi_{Ct} \nu H_t] = (1 - \beta) \frac{\mathcal{W}_0}{G(0, r_0)} e^{A(t)}$.

It is apparently not possible to find a precise explicit expression for the expected total wealth, $E_0[\mathcal{W}_t]$, when total wealth dynamics is given in (3.14), but we can derive an approximate expression as follows. Experiments show that $D(t, r_t)$ and $G(t, r_t)$ are very little sensitive to r_t so we replace them by $D(t, \bar{r})$ and $G(t, \bar{r})$, respectively. Again applying (A.9), we find that future (approximate) total wealth is lognormally distributed with

$$E_0[\mathcal{W}_t] \approx \mathcal{W}_0 \exp \left\{ \left(r_0 + \frac{1}{\gamma} \tilde{\lambda}^\top \tilde{\lambda} + k \sigma_H \lambda'_H \right) t + \left(\bar{r} - r_0 - \frac{\sigma_r}{\kappa} \left[\frac{\lambda_B}{\gamma} + k \sigma_H \rho_{HB} \right] \right) (t - \mathcal{B}_\kappa(t)) \right. \\ \left. + \sigma_r \lambda_B \hat{\beta} \int_0^t D(u, \bar{r}) du - \int_0^t G(u, \bar{r})^{-1} du \right. \\ \left. + \frac{1}{2} \frac{\sigma_r^2}{\kappa^2} \left(t - \mathcal{B}_\kappa(t) - \frac{1}{2} \kappa \mathcal{B}_\kappa(t)^2 \right) - \hat{\beta} \sigma_r^2 \int_0^t D(u, \bar{r}) \mathcal{B}_\kappa(t - u) du \right\}, \quad (\text{B.27})$$

where the integrals have to be evaluated numerically.

With $\varepsilon > 0$, the optimal perishable consumption at time t is

$$c_t = \eta \frac{\beta \nu}{1 - \beta} \frac{\mathcal{W}_t H_t^k}{g(t, r_t, H_t)} = \eta \frac{\beta \nu}{1 - \beta} \frac{\mathcal{W}_t}{\varepsilon^{\frac{1}{\gamma}} H_t^{-k} e^{-\mathcal{D}_\gamma(T-t) - \frac{\gamma-1}{\gamma} \mathcal{B}_\kappa(T-t) r_t} + \frac{\eta \nu}{1-\beta} G(t, r_t)},$$

where g is defined in (3.10) and $G(t, r_t)$ is the same as above. Tedious computations along the same lines as above yield that

$$\frac{d \left(\frac{\mathcal{W}_t H_t^k}{g(t, r_t, H_t)} \right)}{\frac{\mathcal{W}_t H_t^k}{g(t, r_t, H_t)}} = \left[\left(1 - \hat{\beta} \right) r_t + \frac{1 + \gamma}{2\gamma^2} \tilde{\lambda}^\top \tilde{\lambda} + k \sigma_H \lambda'_H - \frac{\delta}{\gamma} + \frac{1}{2} k(k-1) \sigma_H^2 + k \left(\frac{1}{\gamma} \sigma_H \lambda'_H - \nu \right) \right] dt \\ + \frac{1}{\gamma} \tilde{\lambda}^\top dW_t + k \sigma_H \hat{\rho}_H^\top dW_t,$$

analogously to (B.26). Therefore, we can conclude that

$$E_0 \left[\frac{\mathcal{W}_t H_t^k}{g(t, r_t, H_t)} \right] = \frac{\mathcal{W}_0 H_0^k}{g(0, r_0, H_0)} e^{A(t)},$$

where $A(t)$ is as defined above, from which the time 0 expectations of time t consumption follows easily. Expected wealth itself can be approximated similarly as above.

B.5 The case $\psi = 1$: Proof of Theorem 3.2

In the case $\gamma \neq 1, \psi = 1$ the first-order conditions imply

$$c = \delta \beta \frac{(1-\gamma)J}{J_x}, \quad \varphi_C = \frac{\delta(1-\beta)}{\nu} h^{-1} \frac{(1-\gamma)J}{J_x} \quad (\text{B.28})$$

and

$$\mathcal{L}_1 J = \gamma \delta J \left(\frac{1-\gamma}{\gamma} [\ln \tilde{\eta} - 1] + \frac{1-\gamma}{\gamma} (\beta - 1) \ln h - \ln(1-\gamma) - \ln J - \frac{1-\gamma}{\gamma} \ln J_x \right), \quad (\text{B.29})$$

where $\tilde{\eta} = \delta\beta(\beta\nu/[1-\beta])^{\beta-1}$. With the conjectured form of J , we get

$$-\ln J - \frac{1-\gamma}{\gamma} \ln J_x = -\ln g + \ln(1-\gamma)$$

and thus

$$\mathcal{L}_1 J = \gamma\delta J \left(\frac{1-\gamma}{\gamma} [\ln \tilde{\eta} - 1] + \hat{k} \ln h - \ln g \right), \quad (\text{B.30})$$

where $\hat{k} = (1 - \frac{1}{\gamma})(1 - \beta)$. The full HJB equation now reduces to the following PDE for g

$$\begin{aligned} 0 = & \frac{1}{2}\sigma_r^2 g_{rr} + \frac{1}{2}\sigma_H^2 h^2 g_{hh} - \rho_{HB}\sigma_r\sigma_H h g_{rh} + \left(\kappa[\bar{r} - r] + \frac{\gamma-1}{\gamma}\sigma_r\lambda_B \right) g_r + \frac{\partial g}{\partial t} \\ & + \left(r + \frac{1}{\gamma}\lambda'_H\sigma_H - \nu \right) h g_h + \left(\delta\hat{k} \ln h - \delta \ln g - \frac{\gamma-1}{\gamma} \left[r + \frac{\tilde{\lambda}^\tau \tilde{\lambda}}{2\gamma} + \delta(\ln \tilde{\eta} - 1) \right] \right) g \end{aligned} \quad (\text{B.31})$$

with boundary condition $g(T, r, h) = \varepsilon^{1/\gamma}$. Substituting in a candidate solution of the form (3.18), we obtain the following system of ODEs:

$$\begin{aligned} D'_2(\tau) + \delta D_2(\tau) &= \delta\hat{k}, \\ D'_1(\tau) + (\delta + \kappa)D_1(\tau) &= -D_2(\tau) + \left[1 - \frac{1}{\gamma} \right], \\ D'_0(\tau) + \delta D_0(\tau) &= A - \frac{1}{2}\sigma_r^2 D_1(\tau)^2 - \frac{1}{2}\sigma_H^2 D_2(\tau) (D_2(\tau) - 1) - \rho_{HB}\sigma_r\sigma_H D_1(\tau)D_2(\tau) \\ &+ \left(\kappa\bar{r} + \left[1 - \frac{1}{\gamma} \right] \sigma_r\lambda_B \right) D_1(\tau) - \left(\frac{1}{\gamma}\lambda'_H\sigma_H - \nu \right) D_2(\tau), \end{aligned}$$

where $A = \frac{\delta}{\gamma} \ln \varepsilon + \left[1 - \frac{1}{\gamma} \right] \left[\frac{\tilde{\lambda}^\tau \tilde{\lambda}}{2\gamma} + \delta(\ln \tilde{\eta} - 1) \right]$. To fulfill the boundary condition on g , we need $D_2(0) = D_1(0) = D_0(0)$. It can be verified that the solutions to the first two of these ODEs are given by (3.19)-(3.20). Integrating the ODE for D_0 leads to

$$D_0(\tau) = (1) + (2) + (3) + (4) + (5) + (6),$$

where

$$\begin{aligned} (1) &= \int_0^\tau A e^{-\delta(\tau-u)} du = A \mathcal{B}_\delta(\tau), \\ (2) &= -\frac{1}{2}\sigma_r^2 \int_0^\tau e^{-\delta(\tau-u)} D_1(u)^2 du \\ &= -\frac{1}{2}\sigma_r^2 \left\{ \left[\frac{\delta\hat{k}}{\kappa} + 1 - \frac{1}{\gamma} \right]^2 \frac{2}{\delta + \kappa} \left[\frac{\delta + \kappa}{\kappa(\delta + 2\kappa)} \mathcal{B}_\delta(\tau) - \frac{1}{\kappa} \mathcal{B}_{\delta+\kappa}(\tau) + \frac{1}{\delta + 2\kappa} \mathcal{B}_{2(\delta+\kappa)}(\tau) \right] \right. \\ &\quad - 2 \frac{\hat{k}}{\kappa(\delta + \kappa)} \left[\frac{\delta\hat{k}}{\kappa} + 1 - \frac{1}{\gamma} \right] \left[\left(\frac{\kappa}{\delta + \kappa} + \frac{\delta}{\kappa} \right) \mathcal{B}_\delta(\tau) - \frac{\delta + \kappa}{\kappa} \mathcal{B}_{\delta+\kappa}(\tau) + \frac{2\delta + \kappa}{\delta + \kappa} \mathcal{B}_{2\delta+\kappa}(\tau) + (\delta \mathcal{B}_\delta(\tau) - 1)\tau \right] \\ &\quad \left. + 2 \left(\frac{\hat{k}}{\kappa} \right)^2 [\mathcal{B}_{2\delta}(\tau) + (\delta \mathcal{B}_\delta(\tau) - 1)\tau] \right\}, \end{aligned}$$

$$(3) = -\frac{1}{2}\sigma_H^2 \int_0^\tau e^{-\delta(\tau-u)} D_2(u)^2 du = -\sigma_H^2 \hat{k}^2 [\mathcal{B}_{2\delta}(\tau) + (\delta\mathcal{B}_\delta(\tau) - 1)\tau],$$

$$(4) = -\rho_{HB}\sigma_r\sigma_H \int_0^\tau e^{-\delta(\tau-u)} D_1(u)D_2(u) du \\ = -\rho_{HB}\sigma_r\sigma_H \left\{ -\frac{2\hat{k}^2}{\kappa} [\mathcal{B}_{2\delta}(\tau) + (\delta\mathcal{B}_\delta(\tau) - 1)\tau] \right. \\ \left. + \left[\frac{\delta\hat{k}}{\kappa} + 1 - \frac{1}{\gamma} \right] \frac{\hat{k}}{\delta + \kappa} \left[\left(\frac{\kappa}{\delta + \kappa} + \frac{\delta}{\kappa} \right) \mathcal{B}_\delta(\tau) - \frac{\delta + \kappa}{\kappa} \mathcal{B}_{\delta+\kappa}(\tau) + \frac{2\delta + \kappa}{\delta + \kappa} \mathcal{B}_{2\delta+\kappa}(\tau) + (\delta\mathcal{B}_\delta(\tau) - 1)\tau \right] \right\},$$

$$(5) = [\kappa\bar{r} + \frac{\gamma-1}{\gamma}\sigma_r\lambda_B] \int_0^\tau e^{-\delta(\tau-u)} D_1(u) du \\ = \frac{\kappa\bar{r} + \frac{\gamma-1}{\gamma}\sigma_r\lambda_B}{\kappa} \left\{ \left[\frac{\delta\hat{k}}{\kappa} - \hat{k} + \frac{\gamma-1}{\gamma} \right] \mathcal{B}_\delta(\tau) - \left[\frac{\delta\hat{k}}{\kappa} + \frac{\gamma-1}{\gamma} \right] \mathcal{B}_{\delta+\kappa}(\tau) - \hat{k}(\delta\mathcal{B}_\delta(\tau) - 1)\tau \right\},$$

and

$$(6) = \left[\frac{1}{2}\sigma_H^2 - \lambda'_H\sigma_H/\gamma + \nu \right] \int_0^\tau e^{-\delta(\tau-u)} D_2(u) du = \hat{k} \left[\frac{1}{2}\sigma_H^2 - \lambda'_H\sigma_H/\gamma + \nu \right] [\mathcal{B}_\delta(\tau) + (\delta\mathcal{B}_\delta(\tau) - 1)\tau].$$

Here we have used properties of the \mathcal{B} -function summarized in Appendix D. We can thus write D_0 as

$$D_0(\tau) = K_\delta\mathcal{B}_\delta(\tau) + K_{\delta+\kappa}\mathcal{B}_{\delta+\kappa}(\tau) + K_{2\delta}\mathcal{B}_{2\delta}(\tau) + K_{2\delta+\kappa}\mathcal{B}_{2\delta+\kappa}(\tau) + K_{2(\delta+\kappa)}\mathcal{B}_{2(\delta+\kappa)}(\tau) + K_z(\delta\mathcal{B}_\delta(\tau) - 1)\tau,$$

where the constants are given by

$$K_\delta = A - \frac{1}{\kappa(\delta + 2\kappa)} K^2 \sigma_r^2 + \frac{\hat{k}}{\delta + \kappa} \left(\frac{\kappa}{\delta + \kappa} + \frac{\delta}{\kappa} \right) K \sigma_r \left(\frac{\sigma_r}{\kappa} - \rho_{HB}\sigma_H \right) \\ + \frac{\kappa\bar{r} + \frac{\gamma-1}{\gamma}\sigma_r\lambda_B}{\kappa} (K - \hat{k}) + \hat{k} \left[\frac{1}{2}\sigma_H^2 - \lambda'_H\sigma_H/\gamma + \nu \right] \quad (\text{B.32})$$

$$K_{\delta+\kappa} = \frac{1}{\kappa(\delta + \kappa)} K^2 \sigma_r^2 - \hat{k} K \frac{\sigma_r}{\kappa} \left(\frac{\sigma_r}{\kappa} - \rho_{HB}\sigma_H \right) - \frac{\kappa\bar{r} + \frac{\gamma-1}{\gamma}\sigma_r\lambda_B}{\kappa} K \quad (\text{B.33})$$

$$K_{2\delta} = 2 \frac{\hat{k}^2}{\kappa} \rho_{HB}\sigma_r\sigma_H - \left(\frac{\hat{k}}{\kappa} \right)^2 \sigma_r^2 - \hat{k}^2 \sigma_H^2 \quad (\text{B.34})$$

$$K_{2\delta+\kappa} = \frac{\hat{k}(2\delta + \kappa)}{(\delta + \kappa)^2} K \sigma_r \left(\frac{\sigma_r}{\kappa} - \rho_{HB}\sigma_H \right) \quad (\text{B.35})$$

$$K_{2(\delta+\kappa)} = -\frac{1}{(\delta + \kappa)(\delta + 2\kappa)} K^2 \sigma_r^2 \quad (\text{B.36})$$

$$K_z = \sigma_r^2 \frac{\hat{k}}{\kappa} \left(\frac{K}{\delta + \kappa} - \frac{1}{\delta} \right) + K \frac{\hat{k}}{\delta + \kappa} - \hat{k}^2 \sigma_H^2 + 2 \frac{\hat{k}^2}{\kappa} \rho_{HB}\sigma_r\sigma_H - \hat{k} \frac{\kappa\bar{r} + \frac{\gamma-1}{\gamma}\sigma_r\lambda_B}{\kappa} \\ + \hat{k} \left[\frac{1}{2}\sigma_H^2 - \lambda'_H\sigma_H/\gamma + \nu \right] \quad (\text{B.37})$$

with

$$K = \frac{\delta\hat{k}}{\kappa} + \frac{\gamma-1}{\gamma}.$$

In order to see that $D_1(\tau) \geq 0$, note that this will be the case if and only if

$$\left(1 + \frac{\kappa}{\delta(1-\beta)}\right) \frac{1}{\delta + \kappa} \left(1 - e^{-(\delta+\kappa)\tau}\right) \geq \frac{1}{\delta} (1 - e^{-\delta\tau})$$

or, equivalently,

$$\kappa \frac{\beta}{1-\beta} \geq e^{-\delta\tau} \left[\left(\delta + \frac{\kappa}{1-\beta} \right) e^{-\kappa\tau} - (\delta + \kappa) \right].$$

The left-hand side is clearly positive. The maximum of the right-hand side is obtained for $\tau = \tau^*$ satisfying $\left(\delta + \frac{\kappa}{1-\beta} \right) e^{-\kappa\tau^*} = \delta$, so that the maximum equals $-\kappa e^{-\delta\tau^*}$, which is negative.

Expected wealth, expected consumption. With $\psi = 1$, the total wealth dynamics in (3.7) simplifies to

$$\begin{aligned} \frac{d\mathcal{W}_t}{\mathcal{W}_t} &= \left(r_t + \frac{1}{\gamma} \tilde{\lambda}^\top \tilde{\lambda} + \sigma_r \lambda_B D_1(T-t) + \sigma_H \lambda_H D_2(T-t) - \delta \right) dt \\ &\quad + \frac{1}{\gamma} \tilde{\lambda}^\top dW_t + \sigma_r D_1(T-t) dW_{rt} + \sigma_H D_2(T-t) \tilde{\rho}_H^\top dW_t. \end{aligned}$$

Write time t wealth on the form $\mathcal{W}_t = \mathcal{W}_0 e^{\tilde{z}}$ and use (A.9) to conclude that the random variable \tilde{z} is normally distributed. Taking expectations, we get

$$\begin{aligned} E_0[\mathcal{W}_t] &= \mathcal{W}_0 \exp \left\{ \left(r_0 + \frac{1}{\gamma} \tilde{\lambda}^\top \tilde{\lambda} - \delta \right) t + (\bar{r} - r_0)(t - \mathcal{B}_\kappa(t)) + \sigma_r \lambda_B \int_0^t \left[D_1(T-u) - \frac{1}{\gamma} \mathcal{B}_\kappa(t-u) \right] du \right. \\ &\quad + \sigma_H \int_0^t D_2(T-u) [\lambda_H - \rho_{HB} \sigma_r \mathcal{B}_\kappa(t-u)] du \\ &\quad \left. + \frac{1}{2} \sigma_r^2 \int_0^t \mathcal{B}_\kappa(t-u)^2 du - \sigma_r^2 \int_0^t D_1(T-u) \mathcal{B}_\kappa(t-u) du \right\}. \end{aligned} \tag{B.38}$$

With

$$\begin{aligned} \int_0^t \mathcal{B}_b(T-u) du &= \frac{t - \mathcal{B}_b(t) + b \mathcal{B}_b(t) \mathcal{B}_b(T-t)}{b} \\ \int_0^t \mathcal{B}_b(T-u) \mathcal{B}_c(t-u) du &= \frac{t - \mathcal{B}_c(t) - (1 - b \mathcal{B}_b(T-t))(\mathcal{B}_b(t) - \mathcal{B}_{b+c}(t))}{bc} \end{aligned}$$

this representation becomes explicit

$$\begin{aligned}
\mathbb{E}_0[\mathcal{W}_t] &= \mathcal{W}_0 \exp \left((r_0 + \tilde{\lambda}^\top \tilde{\lambda} / \gamma - \delta)t + (\bar{r} - r_0)(t - \mathcal{B}_\kappa(t)) \right. \\
&\quad + \sigma_r \lambda_B \left\{ \left[\frac{\delta \hat{k}}{\kappa} + \frac{\hat{k}}{1 - \beta} \right] \frac{1}{\delta + \kappa} [t - \mathcal{B}_{\delta + \kappa}(t) + (\delta + \kappa) \mathcal{B}_{\delta + \kappa}(t) \mathcal{B}_{\delta + \kappa}(T - t)] \right. \\
&\quad \quad \left. - \frac{\hat{k}}{\kappa} [t - \mathcal{B}_\delta(t) + \delta \mathcal{B}_\delta(t) \mathcal{B}_\delta(T - t)] - \frac{t - \mathcal{B}_\kappa(t)}{\gamma \kappa} \right\} \\
&\quad + \sigma_H \left\{ \lambda_H \hat{k} [t - \mathcal{B}_\delta(t) + \delta \mathcal{B}_\delta(t) \mathcal{B}_\delta(T - t)] \right. \\
&\quad \quad \left. - \frac{\rho_{HB} \sigma_r \hat{k}}{\kappa} [t - \mathcal{B}_\kappa(t) - (1 - \delta \mathcal{B}_\delta(T - t))(\mathcal{B}_\delta(t) - \mathcal{B}_{\delta + \kappa}(t))] \right\} \\
&\quad + \frac{\sigma_r^2}{2\kappa^2} \left\{ t - 2\mathcal{B}_\kappa(t) + \mathcal{B}_{2\kappa}(t) \right\} \\
&\quad - \sigma_r^2 \left\{ \left[\frac{\delta \hat{k}}{\kappa} + \frac{\hat{k}}{1 - \beta} \right] \frac{1}{(\delta + \kappa)\kappa} [t - \mathcal{B}_\kappa(t) - (1 - (\delta + \kappa) \mathcal{B}_{\delta + \kappa}(T - t))(\mathcal{B}_{\delta + \kappa}(t) - \mathcal{B}_{\delta + 2\kappa}(t))] \right. \\
&\quad \quad \left. - \frac{\hat{k}}{\kappa^2} [t - \mathcal{B}_\kappa(t) - (1 - \delta \mathcal{B}_\delta(T - t))(\mathcal{B}_\delta(t) - \mathcal{B}_{\delta + \kappa}(t))] \right\} \Big).
\end{aligned}$$

B.6 The case $\psi \notin \{1, 1/\gamma\}$: Proof of Theorem 3.3

Substitute the approximation (3.23) into the PDE (B.22), apply the conjecture (3.24), and collect terms involving $\ln h$, terms involving r , and the remaining terms. This shows that the functions \hat{D}_0 , \hat{D}_1 , and \hat{D}_2 have to satisfy the following differential equations, where $\Theta(t) = \hat{h}(t)^k \hat{g}(t)^{\frac{\gamma(\psi-1)}{\gamma-1}}$:

$$\begin{aligned}
\frac{\partial \hat{D}_2(t, T)}{\partial t} - \frac{\eta\nu}{1 - \beta} \Theta(t) \hat{D}_2(t, T) &= -\eta\nu \frac{\gamma - 1}{\gamma} \Theta(t), \\
\frac{\partial \hat{D}_1(t, T)}{\partial t} - \left(\kappa + \frac{\eta\nu}{1 - \beta} \Theta(t) \right) \hat{D}_1(t, T) &= \hat{D}_2(t, T) - \frac{\gamma - 1}{\gamma}, \\
\frac{\partial \hat{D}_0(t, T)}{\partial t} - \frac{\eta\nu}{1 - \beta} \Theta(t) \hat{D}_0(t, T) &= \hat{A}(t) + \frac{1}{2} \sigma_r^2 \hat{D}_1(t, T)^2 + \frac{1}{2} \sigma_H^2 \hat{D}_2(t, T) [\hat{D}_2(t, T) - 1] \\
+ \rho_{HB} \sigma_r \sigma_H \hat{D}_1(t, T) \hat{D}_2(t, T) - \left(\kappa \bar{r} + \frac{\gamma - 1}{\gamma} \sigma_r \lambda_B \right) \hat{D}_1(t, T) &+ \left(\frac{1}{\gamma} \lambda'_H \sigma_H - \nu \right) \hat{D}_2(t, T).
\end{aligned}$$

The terminal conditions are $\hat{D}_0(T, T) = \hat{D}_1(T, T) = \hat{D}_2(T, T) = 0$. Here, we have introduced

$$\hat{A}(t) = \frac{\eta\nu(1 - \gamma)\Theta(t)}{\gamma(\psi - 1)(1 - \beta)} \left(1 - k \ln \hat{h}(t) - \frac{\gamma(\psi - 1)}{\gamma - 1} \ln \hat{g}(t) + \frac{\psi - 1}{\gamma - 1} \ln \varepsilon \right) + \frac{\gamma - 1}{\gamma} \left(\frac{\delta}{1 - \frac{1}{\psi}} - \frac{\tilde{\lambda}^\top \tilde{\lambda}}{2\gamma} \right).$$

The solutions to the first differential equations are stated in (3.25) and (3.26), while the solution to the last is

$$\begin{aligned}
\hat{D}_0(t, T) = & - \int_t^T e^{-\frac{\eta\nu}{1-\beta} \int_t^u \Theta(s) ds} \hat{A}(u) du - \frac{\sigma_r^2}{2} \int_t^T e^{-\frac{\eta\nu}{1-\beta} \int_t^u \Theta(s) ds} \hat{D}_1(u, T)^2 du \\
& - \frac{\sigma_H^2}{2} \int_t^T e^{-\frac{\eta\nu}{1-\beta} \int_t^u \Theta(s) ds} \hat{D}_2(u, T) [\hat{D}_2(u, T) - 1] du \\
& - \rho_{HB} \sigma_r \sigma_H \int_t^T e^{-\frac{\eta\nu}{1-\beta} \int_t^u \Theta(s) ds} \hat{D}_1(u, T) \hat{D}_2(u, T) du \\
& + \left(\kappa \bar{r} + \frac{\gamma - 1}{\gamma} \sigma_r \lambda_B \right) \int_t^T e^{-\frac{\eta\nu}{1-\beta} \int_t^u \Theta(s) ds} \hat{D}_1(u, T) du \\
& - \left(\frac{1}{\gamma} \lambda_H' \sigma_H - \nu \right) \int_t^T e^{-\frac{\eta\nu}{1-\beta} \int_t^u \Theta(s) ds} \hat{D}_2(u, T) du.
\end{aligned} \tag{B.39}$$

C Proof of Theorem 4.1 (constant housing consumption)

The HJB equation is again of the form (B.3) with $\mathcal{L}_2 J$ still given by (B.8), while

$$\begin{aligned}
\mathcal{L}_1 J = & \max_c \left\{ f \left(c^\beta \bar{\varphi}_C^{1-\beta}, J \right) - c J_x \right\}, \\
\mathcal{L}_3 J = & \frac{\partial J}{\partial t} + J_x \left(r x + 1_{\{t < \bar{T}\}} y \right) + J_z^\top \mu_z + \frac{1}{2} \text{tr} \left(J_{zz} \Sigma_Z \Sigma_Z^\top \right) - J_x h \nu \bar{\varphi}_C.
\end{aligned}$$

With a conjecture of the form (4.3) for the value function, the derivatives of J can be written in terms of J as follows:

$$\begin{aligned}
J_x = & \frac{(1 - \tilde{\gamma})J}{x + yF - \nu h \hat{F}}, & J_{xx} = & - \frac{\tilde{\gamma}(1 - \tilde{\gamma})J}{(x + yF - \nu h \hat{F})^2}, \\
J_y = & (1 - \tilde{\gamma})J \frac{F}{x + yF - \nu h \hat{F}}, & J_{yy} = & -\tilde{\gamma}(1 - \tilde{\gamma})J \frac{F^2}{(x + yF - \nu h \hat{F})^2}, \\
J_h = & -(1 - \tilde{\gamma})J \frac{\nu \hat{F}}{x + yF - \nu h \hat{F}}, & J_{hh} = & -\tilde{\gamma}(1 - \tilde{\gamma})J \left(\frac{\nu \hat{F}}{x + yF - \nu h \hat{F}} \right)^2, \\
J_{xy} = & -\tilde{\gamma}(1 - \tilde{\gamma})J \frac{F}{(x + yF - \nu h \hat{F})^2}, & J_{xh} = & \tilde{\gamma}(1 - \tilde{\gamma})J \frac{\nu \hat{F}}{(x + yF - \nu h \hat{F})^2}, \\
J_{hy} = & \tilde{\gamma}(1 - \tilde{\gamma})J \frac{\nu F \hat{F}}{(x + yF - \nu h \hat{F})^2}, & J_r = & (1 - \tilde{\gamma})J \left[\frac{\tilde{\gamma}}{1 - \tilde{\gamma}} \frac{g_r}{g} + \frac{y F_r}{x + yF - \nu h \hat{F}} \right],
\end{aligned}$$

$$\begin{aligned}
\frac{\partial J}{\partial t} = & (1 - \tilde{\gamma})J \left[\frac{\tilde{\gamma}}{1 - \tilde{\gamma}} \frac{\partial g}{\partial t} \frac{1}{g} + \frac{1}{x + yF - \nu h \hat{F}} \left(y \frac{\partial F}{\partial t} - \nu h \hat{F}' \right) \right], \\
J_{rr} = & (1 - \tilde{\gamma})J \left[\frac{\tilde{\gamma}}{1 - \tilde{\gamma}} \frac{g_{rr}}{g} - \tilde{\gamma} \left(\frac{g_r}{g} \right)^2 + 2\tilde{\gamma} \frac{g_r}{g} \frac{y F_r}{x + yF - \nu h \hat{F}} \right. \\
& \left. - \tilde{\gamma} \left(\frac{y F_r}{x + yF - \nu h \hat{F}} \right)^2 + \frac{y F_{rr}}{x + yF - \nu h \hat{F}} \right],
\end{aligned}$$

$$\begin{aligned}
J_{xr} &= \tilde{\gamma}(1 - \tilde{\gamma})J \left[\frac{g_r}{g} \frac{1}{x + yF - \nu h\hat{F}} - \frac{yF_r}{(x + yF - \nu h\hat{F})^2} \right], \\
J_{ry} &= (1 - \tilde{\gamma})J \left[\tilde{\gamma} \frac{g_r}{g} \frac{F}{x + yF - \nu h\hat{F}} - \tilde{\gamma} \frac{FyF_r}{(x + yF - \nu h\hat{F})^2} + \frac{F_r}{x + yF - \nu h\hat{F}} \right], \\
J_{rh} &= (1 - \tilde{\gamma})J \left[-\tilde{\gamma} \frac{g_r}{g} \frac{\nu\hat{F}}{x + yF - \nu h\hat{F}} + \tilde{\gamma} \frac{\nu\hat{F}yF_r}{(x + yF - \nu h\hat{F})^2} \right].
\end{aligned}$$

Note that

$$\frac{J_x}{J_{xx}} = -\frac{1}{\tilde{\gamma}}(x + yF - \nu h\hat{F}), \quad \frac{J_{xr}}{J_{xx}} = yF_r - \frac{g_r}{g}(x + yF - \nu h\hat{F}), \quad \frac{J_{xy}}{J_{xx}} = F, \quad \frac{J_{xh}}{J_{xx}} = -\nu\hat{F}.$$

The first-order conditions for α_B , α_S , and α_I are still given by (B.9)-(B.11). Substituting in the relevant derivatives from above, we obtain (4.6)-(4.8).

Substituting the relevant derivatives into (B.8) and simplifying, we obtain

$$\begin{aligned}
\mathcal{L}_2 J &= (1 - \tilde{\gamma})J \left\{ \frac{\tilde{\lambda}^\top \tilde{\lambda}}{2\tilde{\gamma}} + \sigma_r \lambda_B \left(\frac{yF_r}{x + yF} - \frac{g_r}{g} \right) - \sigma_Y \lambda_Y \frac{yF}{x + yF - \nu h\hat{F}} + \sigma_H \lambda'_H \frac{\nu h\hat{F}}{x + yF - \nu h\hat{F}} \right. \\
&\quad + \frac{\tilde{\gamma}}{2} \sigma_r^2 \left(\frac{g_r}{g} - \frac{yF_r}{x + yF - \nu h\hat{F}} \right)^2 + \frac{\tilde{\gamma}}{2} \sigma_Y^2 \frac{y^2 F^2}{(x + yF - \nu h\hat{F})^2} + \frac{\tilde{\gamma}}{2} \sigma_H^2 \left(\frac{\nu h\hat{F}}{x + yF - \nu h\hat{F}} \right)^2 \\
&\quad - \tilde{\gamma} \rho_{YB} \sigma_r \sigma_Y \frac{yF}{x + yF - \nu h\hat{F}} \left(\frac{yF_r}{x + yF - \nu h\hat{F}} - \frac{g_r}{g} \right) \\
&\quad \left. - \tilde{\gamma} \rho_{HB} \sigma_r \sigma_H \frac{\nu h\hat{F}}{x + yF - \nu h\hat{F}} \left(\frac{g_r}{g} - \frac{yF_r}{x + yF - \nu h\hat{F}} \right) - \tilde{\gamma} \rho_{HY} \sigma_H \sigma_Y \frac{yF \nu h\hat{F}}{(x + yF - \nu h\hat{F})^2} \right\}.
\end{aligned}$$

Substituting the relevant derivatives into $\mathcal{L}_3 J$ and applying the PDE (A.7) for F and the fact that $\hat{F}'(t) = -\bar{\varphi}_C + \nu\hat{F}(t)$, we obtain

$$\begin{aligned}
\mathcal{L}_3 J &= (1 - \tilde{\gamma})J \left\{ \sigma_Y \lambda_Y \frac{yF}{x + yF - \nu h\hat{F}} - \sigma_r \lambda_B \frac{yF_r - \nu h\hat{F}}{x + yF - \nu h\hat{F}} - \sigma_H \lambda'_H \frac{\nu h\hat{F}}{x + yF - \nu h\hat{F}} \right. \\
&\quad + \frac{\tilde{\gamma}}{1 - \tilde{\gamma}} \frac{\partial g}{\partial t} \frac{1}{g} + r + \frac{\tilde{\gamma}}{1 - \tilde{\gamma}} \kappa[\bar{r} - r] \frac{g_r}{g} + \frac{\tilde{\gamma}}{2} \sigma_r^2 \left[\frac{1}{1 - \tilde{\gamma}} \frac{g_{rr}}{g} - \left(\frac{g_r}{g} - \frac{yF_r}{x + yF - \nu h\hat{F}} \right)^2 \right] \\
&\quad - \frac{\tilde{\gamma}}{2} \sigma_Y^2 \frac{y^2 F^2}{(x + yF - \nu h\hat{F})^2} - \frac{\tilde{\gamma}}{2} \sigma_H^2 \left(\frac{\nu h\hat{F}}{x + yF - \nu h\hat{F}} \right)^2 \\
&\quad + \tilde{\gamma} \rho_{YB} \sigma_r \sigma_Y \frac{yF}{x + yF - \nu h\hat{F}} \left(\frac{yF_r}{x + yF - \nu h\hat{F}} - \frac{g_r}{g} \right) \\
&\quad \left. + \tilde{\gamma} \rho_{HB} \sigma_r \sigma_H \frac{\nu h\hat{F}}{x + yF - \nu h\hat{F}} \left(\frac{g_r}{g} - \frac{yF_r}{x + yF - \nu h\hat{F}} \right) + \tilde{\gamma} \rho_{YH} \sigma_Y \sigma_H \frac{yF \nu h\hat{F}}{(x + yF - \nu h\hat{F})^2} \right\}.
\end{aligned}$$

Summing up, we get

$$\mathcal{L}_2 J + \mathcal{L}_3 J = \tilde{\gamma} J \frac{1}{g} \left\{ \frac{1}{2} \sigma_r^2 g_{rr} + \left(\kappa[\bar{r} - r] + \frac{\tilde{\gamma} - 1}{\tilde{\gamma}} \sigma_r \lambda_B \right) g_r + \frac{\partial g}{\partial t} - \frac{\tilde{\gamma} - 1}{\tilde{\gamma}} \left(r + \frac{\tilde{\lambda}^\top \tilde{\lambda}}{2\tilde{\gamma}} \right) g \right\}.$$

Since $\psi = 1/\gamma$ we have

$$\mathcal{L}_1 J = \max_c \left\{ \frac{\delta}{1-\gamma} c^{\beta(1-\gamma)} \bar{\varphi}_C^{(1-\beta)(1-\gamma)} - \delta J - c J_x \right\}$$

giving a first-order condition implying that

$$c = (\delta\beta)^{1/\tilde{\gamma}} \bar{\varphi}_C^{1-\gamma/\tilde{\gamma}} J_x^{-1/\tilde{\gamma}} \quad (\text{C.1})$$

from which we obtain

$$\mathcal{L}_1 J = \frac{\tilde{\gamma}}{1-\tilde{\gamma}} (\delta\beta)^{1/\tilde{\gamma}} \bar{\varphi}_C^{1-\gamma/\tilde{\gamma}} J_x^{1-1/\tilde{\gamma}} - \delta J.$$

With the conjecture for J , we get the expression (4.9) for c and

$$\mathcal{L}_1 J = \tilde{\gamma} J \frac{1}{g} \left\{ -\frac{\delta}{\tilde{\gamma}} g + (\delta\beta)^{1/\tilde{\gamma}} \bar{\varphi}_C^{1-\frac{\gamma}{\tilde{\gamma}}} \right\}.$$

Substituting into $0 = \mathcal{L}_1 J + \mathcal{L}_2 J + \mathcal{L}_3 J$, we see that g must satisfy the PDE

$$0 = \frac{1}{2} \sigma_r^2 g_{rr} + \left(\kappa[\bar{r} - r] + \frac{\tilde{\gamma} - 1}{\tilde{\gamma}} \sigma_r \lambda_B \right) g_r + \frac{\partial g}{\partial t} - \left(\frac{\delta}{\tilde{\gamma}} + \frac{\tilde{\gamma} - 1}{\tilde{\gamma}} \left[r + \frac{\tilde{\lambda}^\top \tilde{\lambda}}{2\tilde{\gamma}} \right] \right) g + (\delta\beta)^{1/\tilde{\gamma}} \bar{\varphi}_C^{1-\frac{\gamma}{\tilde{\gamma}}} \quad (\text{C.2})$$

with the terminal condition $g(T, r) = 0$ (since $\varepsilon = 0$). Since this resembles the pricing PDE for an asset with a zero terminal payment and a continuous dividend of $(\delta\beta)^{1/\tilde{\gamma}} \bar{\varphi}_C^{1-\frac{\gamma}{\tilde{\gamma}}}$, we try a solution of the form

$$g(t, r) = (\delta\beta)^{1/\tilde{\gamma}} \bar{\varphi}_C(s)^{1-\frac{\gamma}{\tilde{\gamma}}} \int_t^T e^{-d(s-t) - \tilde{d}(s-t)r} ds.$$

We need $d(0) = \tilde{d}(0) = 0$ and

$$\begin{aligned} \tilde{d}'(\tau) + \kappa \tilde{d}(\tau) &= 1 - \frac{1}{\tilde{\gamma}}, \\ d'(\tau) &= -\frac{1}{2} \sigma_r^2 \tilde{d}(\tau)^2 + \left(\kappa \bar{r} + \frac{\tilde{\gamma} - 1}{\tilde{\gamma}} \sigma_r \lambda_B \right) \tilde{d}(\tau) + \frac{\delta}{\tilde{\gamma}} + \frac{\tilde{\gamma} - 1}{\tilde{\gamma}} \frac{\tilde{\lambda}^\top \tilde{\lambda}}{2\tilde{\gamma}}. \end{aligned}$$

The solution to the first ODE is $\tilde{d}(\tau) = \frac{\tilde{\gamma}-1}{\tilde{\gamma}} \mathcal{B}_\kappa(\tau)$ and then $d(\tau)$ follows from the second ODE by integration:

$$d(\tau) = -\frac{1}{2} \sigma_r^2 \left(\frac{\tilde{\gamma} - 1}{\tilde{\gamma}} \right)^2 \int_0^\tau \mathcal{B}_\kappa(u)^2 du + \frac{\tilde{\gamma} - 1}{\tilde{\gamma}} \left(\kappa \bar{r} + \frac{\tilde{\gamma} - 1}{\tilde{\gamma}} \sigma_r \lambda_B \right) \int_0^\tau \mathcal{B}_\kappa(u) du + \left(\frac{\delta}{\tilde{\gamma}} + \frac{\tilde{\gamma} - 1}{\tilde{\gamma}} \frac{\tilde{\lambda}^\top \tilde{\lambda}}{2\tilde{\gamma}} \right) \tau.$$

Using the integration results of Appendix D we obtain $d(\tau) = \mathcal{D}_{\tilde{\gamma}}(\tau)$ defined through (3.11).

D Some properties of the \mathcal{B} -function

Recall the definition

$$\mathcal{B}_m(\tau) = \frac{1 - e^{-m\tau}}{m}$$

Lemma D.1 (Multiplying \mathcal{B} s) *The following equations hold*

$$\begin{aligned}\mathcal{B}_b\mathcal{B}_c &= \frac{b\mathcal{B}_b + c\mathcal{B}_c - (b+c)\mathcal{B}_{b+c}}{bc} \\ \mathcal{B}_{b+c} &= \frac{b\mathcal{B}_b + c\mathcal{B}_c - bc\mathcal{B}_b\mathcal{B}_c}{b+c}\end{aligned}\tag{D.1}$$

Proof. Follows from the definition of \mathcal{B} . \square

Remark. Equation (D.1) shows that second-order terms of the form $\mathcal{B}_b\mathcal{B}_c$ can be transformed into a sum of first-order terms. Note that, in particular, $\mathcal{B}_{2b} = \mathcal{B}_b - \frac{b}{2}\mathcal{B}_b^2$.

We set $\tau = T - t$.

Lemma D.2 (Integrating \mathcal{B} s) *(i) Assuming $a \neq 0$, we obtain¹⁷*

$$\int_t^T e^{-a(s-t)}\mathcal{B}_b(T-s) ds = \begin{cases} \frac{\mathcal{B}_a(\tau) - \mathcal{B}_b(\tau)}{b-a} & \text{if } b \neq a, \\ \frac{\mathcal{B}_a(\tau) + (a\mathcal{B}_a(\tau) - 1)\tau}{a} & \text{if } b = a, \end{cases}\tag{D.2}$$

$$\begin{aligned}& \int_t^T e^{-a(s-t)}\mathcal{B}_b(T-s)\mathcal{B}_c(T-s) ds \\ &= \begin{cases} \frac{1}{bc} \left[\frac{bc(b+c-2a)}{(b-a)(c-a)(b+c-a)}\mathcal{B}_a(\tau) - \frac{b}{b-a}\mathcal{B}_b(\tau) - \frac{c}{c-a}\mathcal{B}_c(\tau) + \frac{b+c}{b+c-a}\mathcal{B}_{b+c}(\tau) \right] & \text{if } b \neq a \neq c, \\ \frac{1}{ab} \left[\left(\frac{b-a}{b} + \frac{a}{b-a} \right) \mathcal{B}_a(\tau) - \frac{b}{b-a}\mathcal{B}_b(\tau) + \frac{a+b}{b}\mathcal{B}_{a+b}(\tau) + (a\mathcal{B}_a(\tau) - 1)\tau \right] & \text{if } b \neq a = c, \\ \frac{2}{a^2} [\mathcal{B}_{2a}(\tau) + (a\mathcal{B}_a(\tau) - 1)\tau] & \text{if } b = a = c. \end{cases}\end{aligned}\tag{D.3}$$

(ii) Furthermore,

$$\begin{aligned}\int_t^T \mathcal{B}_b(T-s) ds &= \frac{\tau - \mathcal{B}_b(\tau)}{b} \\ \int_t^T \mathcal{B}_b(T-s)\mathcal{B}_c(T-s) ds &= \frac{\tau - \mathcal{B}_b(\tau) - \mathcal{B}_c(\tau) + \mathcal{B}_{b+c}(\tau)}{bc}\end{aligned}\tag{D.4}$$

Proof. Equation (D.2) follows by simple integration. To show (D.3), one can use (D.1) and then apply (D.2). The proof of (ii) works similarly. \square

Remarks. a) In the special case $a \neq b = c$, equation (D.3) simplifies into

$$\frac{2}{b} \left[\frac{b}{(b-a)(2b-a)}\mathcal{B}_a(\tau) - \frac{1}{b-a}\mathcal{B}_b(\tau) + \frac{1}{2b-a}\mathcal{B}_{2b}(\tau) \right].$$

b) In the special case $b = c$, equation (D.4) simplifies into

$$\frac{1}{b^2} [\tau - 2\mathcal{B}_b(\tau) + \mathcal{B}_{2b}(\tau)] = \frac{1}{b^2} \left[\tau - \mathcal{B}_b(\tau) - \frac{b}{2}\mathcal{B}_b(\tau)^2 \right].$$

¹⁷Note that $b \neq a \neq c$ does not exclude cases with $b = c$.

c) We can use (D.1) to rewrite (D.3) as follows:

$$\begin{aligned}
& \int_t^T e^{-a(s-t)} \mathcal{B}_b(T-s) \mathcal{B}_c(T-s) ds \\
&= \begin{cases} \frac{1}{b+c-a} \left[\frac{b+c-2a}{(b-a)(c-a)} \mathcal{B}_a(\tau) - \frac{1}{b-a} \mathcal{B}_b(\tau) - \frac{1}{c-a} \mathcal{B}_c(\tau) - \mathcal{B}_b(\tau) \mathcal{B}_c(\tau) \right] & \text{if } b \neq a \neq c, \\ \frac{1}{ab} \left[\frac{b}{b-a} \mathcal{B}_a(\tau) - \frac{a}{b-a} \mathcal{B}_b(\tau) - a \mathcal{B}_a(\tau) \mathcal{B}_b(\tau) + (a \mathcal{B}_a(\tau) - 1) \tau \right] & \text{if } b \neq a = c, \\ \frac{1}{a^2} [(2 - a \mathcal{B}_a(\tau)) \mathcal{B}_a(\tau) + 2(a \mathcal{B}_a(\tau) - 1) \tau] & \text{if } b = a = c. \end{cases}
\end{aligned}$$

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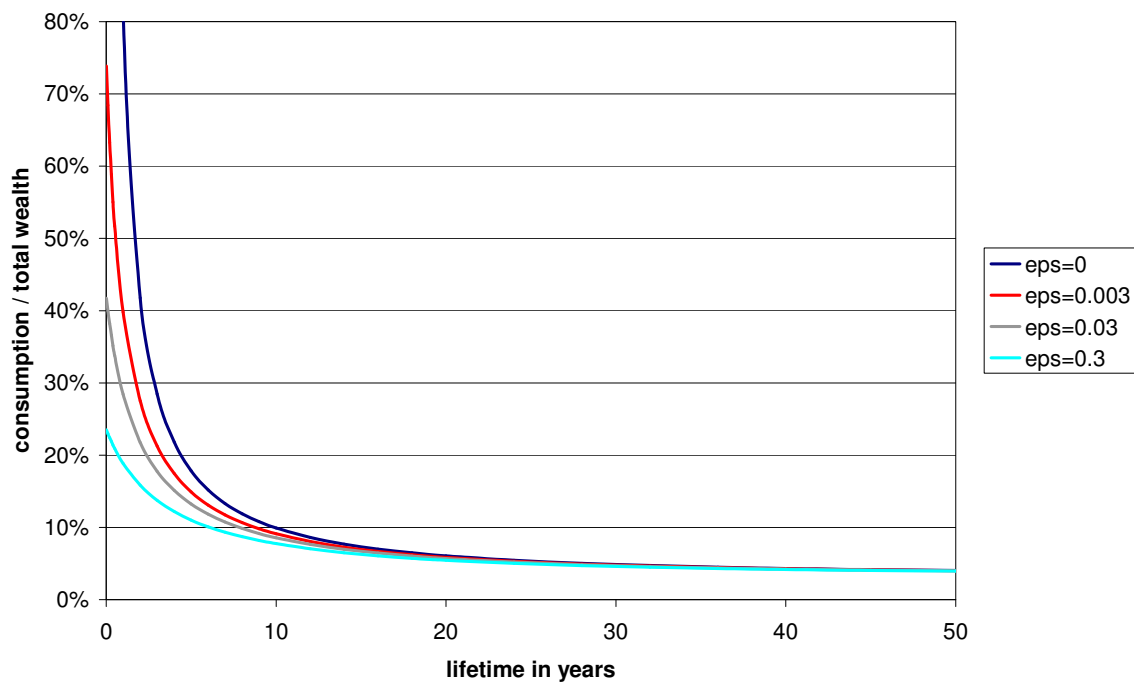


Figure 1: **Optimal consumption/wealth ratio and the time horizon.** The figure shows how the optimal ratio between perishable consumption and total wealth varies over the life-cycle. Time-additive power utility is assumed with different weights associated with utility of terminal wealth as indicated by ϵ . The benchmark parameters in Table 1 are used. The current short rate is set to the long-term level, $r_0 = \bar{r}$, and the current unit house price is set to $h = 200$.

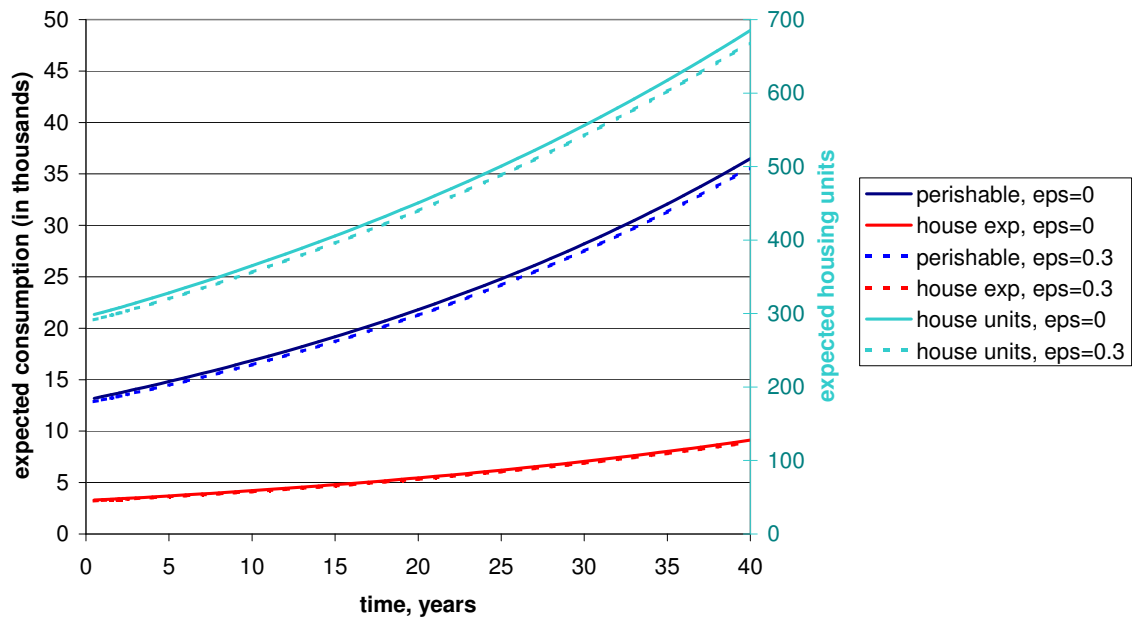


Figure 2: **Expected consumption over a life-time.** The figure shows the expectation of optimal consumption over the life-time of the individual. Time-additive power utility is assumed with no utility from terminal wealth. The benchmark parameters in Table 1 are used. The current short rate is set to the long-term level, $r_0 = \bar{r}$, the current unit house price is set to $H_0 = 200$, the current tangible wealth is set to $X_0 = 20,000$, and the current income rate is set to $Y_0 = 20,000$. The solid [dashed] lines are for the case without [with] utility of terminal wealth.

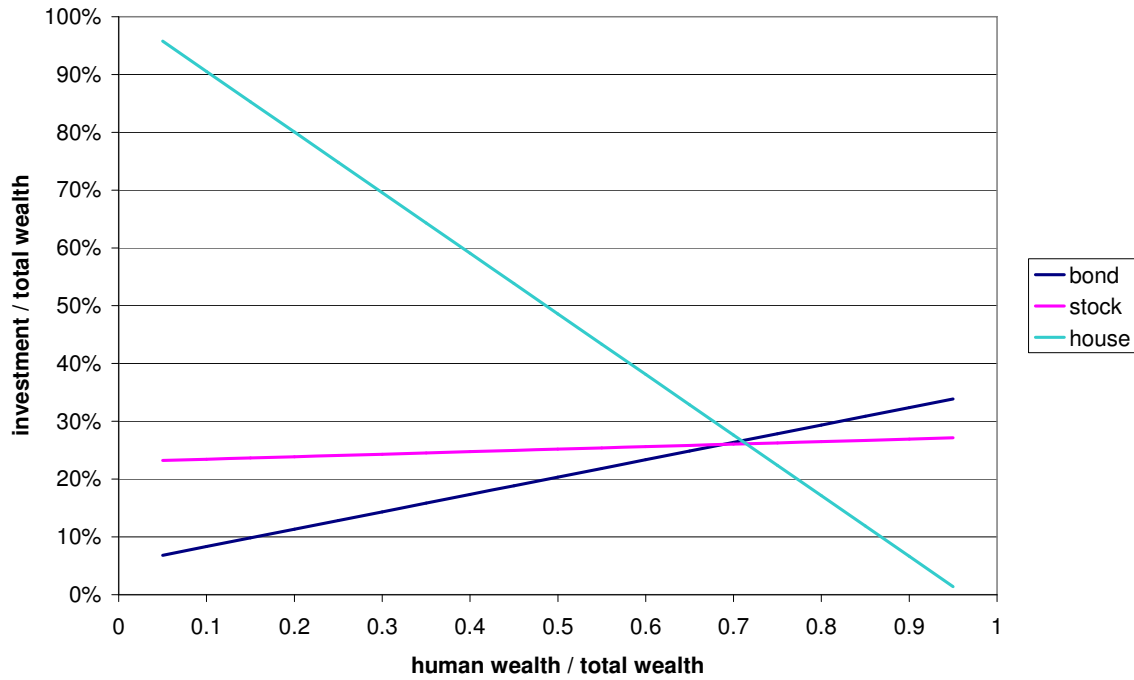


Figure 3: **Optimal investments and the composition of wealth.** The figure shows how the optimal fractions of total wealth invested in bonds, stocks, and houses (physically or financially) vary with the ratio of human wealth $yF(t, r)$ to total wealth $\mathcal{W} = x + yF(t, r)$. Time-additive power utility is assumed with no utility of terminal wealth. The benchmark parameters in Table 1 are used. The ratios F_r/F and g_r/g are computed assuming 20 years to retirement and a retirement period of 20 years. The current short rate is set to the long-term level, $r = \bar{r}$.

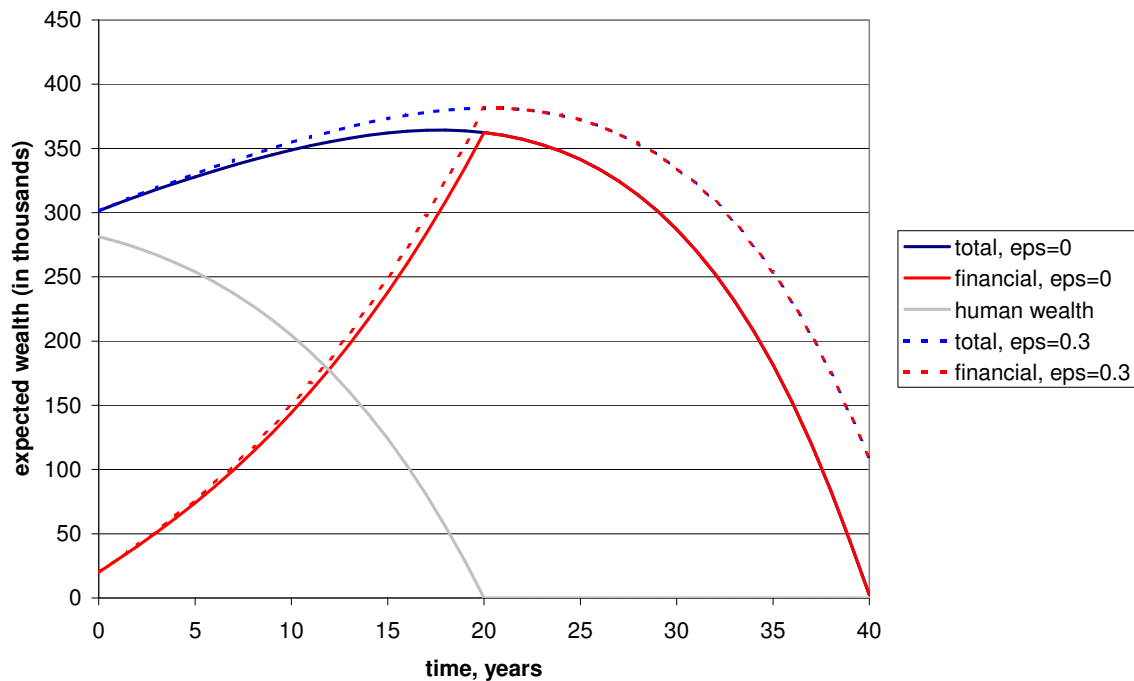


Figure 4: **Expected wealth over the life-cycle.** The figure shows the initial expectations of total wealth, financial wealth, and human wealth over the life-cycle. Time-additive power utility is assumed with no utility of terminal wealth. The benchmark parameters in Table 1 are used. The current short rate is set to the long-term level, $r_0 = \bar{r}$, the current unit house price is set to $H_0 = 200$, the current tangible wealth is set to $X_0 = 20,000$, and the current income rate is set to $Y_0 = 20,000$. The solid [dashed] lines are for the case without [with] utility of terminal wealth.

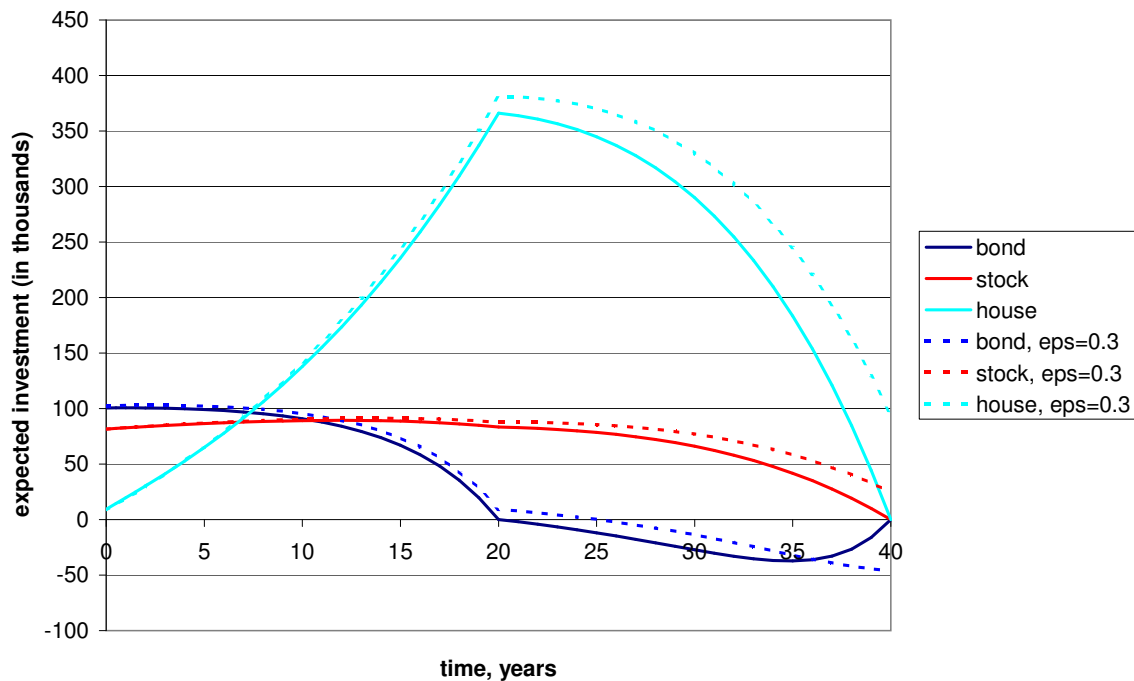


Figure 5: **Expected investments over the life-cycle.** The figure shows the initial expectations of the investments in bonds, stocks, and housing units over the life-cycle. Time-additive power utility is assumed with no utility of terminal wealth. The benchmark parameters in Table 1 are used. The current short rate is set to the long-term level, $r_0 = \bar{r}$, the current unit house price is set to $H_0 = 200$, the current tangible wealth is set to $X_0 = 20,000$, and the current income rate is set to $Y_0 = 20,000$. The solid [dashed] lines are for the case without [with] utility of terminal wealth.

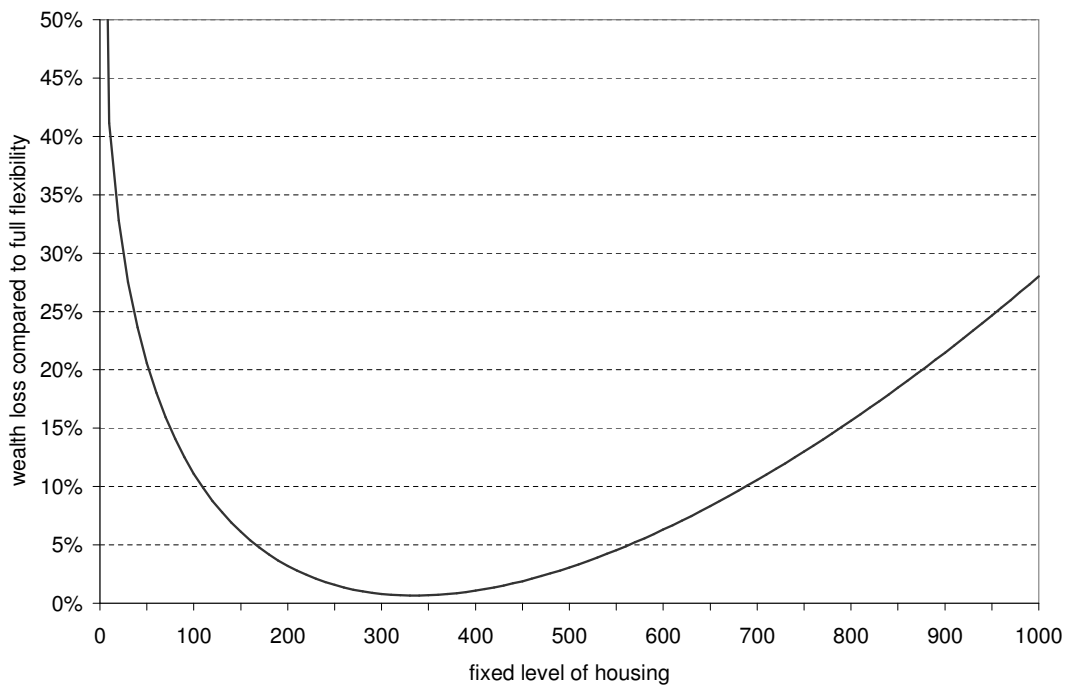


Figure 6: **The utility loss due to a fixed level of housing consumption.** The figure shows how the utility loss measured by the ℓ defined in (4.2) with $\hat{J} = J^{\text{dc}}$ varies with an assumed fixed level of housing consumption throughout life. Benchmark parameters and initial state variables $X_0 = Y_0 = 20,000$, $r_0 = 0.02$, and $H_0 = 200$ are used.

The preferences

δ	γ	ψ	β	\tilde{T}	T	$\tilde{\gamma}$
0.03	4	0.25 or 1	0.8	20	40	3.4

The interest rate and the bond

κ	\bar{r}	σ_r	λ_B	T_{bond}	σ_B	ξ_B	ζ_B
0.2	0.02	0.02	0.1	20	0.0982	-0.1661	-0.6462

The stock

σ_S	λ_S	ρ_{SB}	ξ_S	ζ_S
0.2	0.25	0.1	0.1840	-0.0870

Housing

σ_H	λ_H	r^{imp}	ν	ρ_{HB}	ρ_{HS}	λ'_H	ξ_I	ζ_I
0.12	0.35	0.055	0.055	0.6	0.2	0.35	0.4129	1.2582

The labor income

b	$\bar{\mu}_Y$	σ_Y	ρ_{YB}	ρ_{YS}	ρ_{HY}	λ_Y
0.5	0.01	0.1	0.1	0.1	0.8531	0.3540

Table 1: **Benchmark parameter values.** Exogenous parameters are stated to the left of the double vertical line, while the parameters to the right are derived from the exogenous parameters.

	Adjustment frequency	
	2 years	5 years
infrequent φ_C , frequent φ_I	0.05%	0.12%
infrequent φ_I , frequent φ_C	0.29%	1.08%
infrequent φ_C and φ_I	0.35%	1.19%

Table 2: **Loss due to infrequent adjustments of housing consumption and investment.** The benchmark parameters in Table 1 are assumed together with $\psi = 1/\gamma$ and $\varepsilon = 0.3$. The current short rate is set to the long-term level, $r_0 = \bar{r}$, the current unit house price is set to $H_0 = 200$, the current tangible wealth is set to $X_0 = 20,000$, and the current income rate is set to $Y_0 = 20,000$. The loss is defined in (4.2), where \hat{J} is computed using Monte Carlo simulations. “Frequent” means that the control is updated at the simulation frequency, i.e. 250 times per year. The results are based on 10,000 simulated paths.

	Adjustment frequency	
	2 years	5 years
frequent φ_C and φ_I	0.80%	0.80%
infrequent φ_C , frequent φ_I	0.91%	0.96%
infrequent φ_I , frequent φ_C	1.11%	1.73%
infrequent φ_C and φ_I	1.30%	1.93%

Table 3: **Loss due to borrowing constraints and infrequent adjustments of housing consumption and investment.** The benchmark parameters in Table 1 are assumed together with $\psi = 1/\gamma$ and $\varepsilon = 0.3$. The current short rate is set to the long-term level, $r_0 = \bar{r}$, the current unit house price is set to $H_0 = 200$, the current tangible wealth is set to $X_0 = 20,000$, and the current income rate is set to $Y_0 = 20,000$. The investment strategy is transformed to make sure that tangible wealth stays positive. In the transformation we put $\underline{X} = 6,000$. The loss is defined in (4.2), where \hat{J} is computed using Monte Carlo simulations. “Frequent” means that the control is updated at the simulation frequency, i.e. 250 times per year. The results are based on 10,000 simulated paths.

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