Optimal Inattention to the Stock Market with Information Costs and Transactions Costs*

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Abstract

If it is costly to obtain and apply information about the value of wealth, consumers will optimally be inattentive to their wealth for finite intervals of time. We develop a model with separate costs of observing wealth and of conducting asset transactions. In general, transactions need not occur even when observations do, and the inattention span between observations is state dependent. Surprisingly, if the fixed component of transactions costs is sufficiently small, then eventually a purely time-dependent rule emerges. Under this time-dependent rule, transactions take place whenever observations occur, and the inattention span is constant. If the fixed component of transactions costs is large, the optimal rule remains state-dependent indefinitely.

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A pervasive finding in studies of microeconomic choice is that adjustment to economic news tends to be sluggish and infrequent. Investors rebalance their portfolios and revisit their spending behavior at discrete and potentially infrequent points of time. Between these times, inaction is the rule. If individuals take several months or even years to adjust their portfolios and their spending plans, the standard predictions of the consumption smoothing and portfolio choice theories might fail, and the standard intertemporal Euler equation relating asset returns and consumption growth may not hold.[1] Similar sorts of inaction also characterize the financing, investment, and pricing behavior of firms. These observations have led economists to formulate models that are consistent with infrequent adjustment.[2]

Formal models of infrequent adjustment are often described as either time dependent or state dependent. In time-dependent models, adjustment is triggered simply by calendar time. In state-dependent models, adjustment takes place only when a particular state variable reaches some trigger value, so the timing of adjustments depends on factors other than, or in addition to, calendar time alone. A classic example of state-dependent adjustment is the (S,s) model. The distinction between time-dependent and state-dependent models can have crucial implications for important economic questions. For instance, monetary policy has substantial real effects that persist for several quarters if firms change their prices according to a time-dependent rule. However, if firms adjust their prices according to a state-dependent rule, then monetary policy may have little or no effect on the real economy. (See e.g. Caplin and Spulber (1987) and Golosov and Lucas (2007).)

In this paper we develop and analyze an optimizing model that can generate both time-dependent adjustment and state-dependent adjustment. The economic context is an infinite-horizon continuous-time model of consumption and portfolio choice that builds on the framework of Merton (1971). We augment Merton's model by requiring consumption to be purchased with the liquid asset and by introducing two sorts of costs – a utility cost of observing the value of the consumer's wealth;[3] and a resource cost of transferring assets between a transactions account consisting of liquid assets and an investment portfolio consisting of risky equity and riskless bonds. We model the resource cost of transferring assets as the sum of two components: (1) a component that is proportional to the amount of assets transferred; and (2) a component that is a homogeneous linear function of the balances in

[1] See, for example, Lynch (1996) and Gabaix and Laibson (2002).

[2] Stokey (2009) presents a comprehensive analysis of issues related to inaction and infrequent adjustment.

[3] We call this cost the "observation cost," though it summarizes all costs associated with obtaining and applying the information necessary to choose consumption and the allocation of assets.

the transactions account and in the investment portfolio. Since the second component is independent of the amount of assets transferred, we refer to it as a fixed resource cost of transferring assets.

Because it is costly to observe the value of wealth, the consumer chooses to observe this value only at discretely-spaced points in time. At these observation times, the consumer chooses when next to observe the value of wealth, executes any transfers between the investment portfolio and the transactions account, chooses the risky share of the investment portfolio, and chooses the path of consumption until the next observation date. During intervals of time between consecutive observations, the consumer remains inattentive to the value of equities in her portfolio and thus follows a consumption path that is unresponsive to any news about the value of equities.

In the absence of any transactions costs, optimal behavior would be time-dependent as described in Abel, Eberly, and Panageas (2007). Specifically, the consumer would run down the transactions balance to zero on each observation date and then would transfer a constant fraction of the investment portfolio to the transactions account immediately after observing the value of equities. In addition, the time between consecutive observations would be constant, so that the optimal policy is a purely time-dependent rule.

In our current framework with transactions costs in addition to observation costs, optimal behavior is, in general, state dependent. The relevant state of the consumer's balance sheet at time t is x_t, which is defined as the ratio of the balance in the transactions account to the contemporaneous value of the investment portfolio. When the transactions account is large relative to the investment portfolio on observation date t_j, so that x_{t_j} is high, the consumer will transfer some of these assets in the transactions account to the investment portfolio. Alternatively, when the transactions account is small relative to the investment portfolio on observation date t_j, so that x_{t_j} is low, the consumer will sell some assets from the investment portfolio to replenish the transactions account in order to finance consumption until the next observation date. However, when x_{t_j} has an intermediate value on an observation date, the consumer will not find it worthwhile to pay the costs associated with transferring assets between the investment portfolio and the transactions account.

Because the timing (and direction) of asset transfers depends on the value of x_{t_j}, these transfers are state dependent. A surprising result of our analysis, however, is that, if the fixed resource cost of transferring assets is not large, eventually an optimally inattentive consumer's asset transfers are purely time dependent, with a constant length of time between consecutive observations, and a transfer from the investment portfolio to the transactions account on every observation date. We demonstrate this finding by showing that eventually optimal behavior by a consumer facing observation costs leads to a low value of x_{t_j} on an observation date. Once a low value of x_{t_j} is realized on an observation date, the consumer transfers only enough assets to the transactions account to finance consumption until the next observation date, provided that the fixed resource cost of transferring assets is not too large. This behavior is optimal because it is costly to transfer assets, and the liquid asset in the transactions account earns a lower rate of return than the riskless bond in the investment portfolio. In this case, the consumer plans to hold a zero balance in the transactions account on the next observation date, so that x_{t_j} will equal zero on the next observation date, and on all subsequent observation dates.

This paper is related to two strands of literature. The first strand is the large literature on transactions costs. In Baumol (1952) and Tobin (1956), which are the forerunners of the cash-in-advance model used in macroeconomics, consumers can hold two riskless assets that pay different rates of return: money, which pays zero interest, and a riskless bond that pays a positive rate of interest. As in our paper, consumers are willing to hold money, despite the fact that its rate of return is dominated by the rate of return on riskless bonds, because goods have to be purchased with money. That is, money offers liquidity services.

A more recent literature on portfolio transactions costs, including Constantinides (1986) and Davis and Norman (1990), models the cost of transferring assets between stocks and bonds in the investment portfolio as proportional to the size of the transfers. Here we also include proportional transactions costs, but these costs apply only to transfers between the liquid asset in the transactions account on the one hand and the investment portfolio of stocks and bonds on the other. We do not model the costs of reallocating stocks and bonds within the investment portfolio. For a retired consumer who finances consumption by withdrawing assets from a tax-deferred retirement account, the cost of withdrawing assets from the investment portfolio includes taxes paid at the time of withdrawal. For most consumers in this situation, the marginal tax rate, which is part of the cost of transferring assets from the investment portfolio to the transactions account, is likely to be far greater than any costs associated with reallocating stocks and bonds within the investment portfolio.[4]

A second strand of the literature analyzes optimally inattentive behavior by consumers or

[4] Bilias, Georgarakos, and Haliassos (2010) find panel data evidence of substantial inertia in household asset adjustments, particularly among retirement accounts. Brunnermeier and Nagel (2008) also use panel data to show that risky asset holdings exhibit substantial inertia, which they determine to be "the dominant factor in determining changes in asset allocation" (page 715).

firms. Two distinct approaches to modeling inattention appear in this strand of literature. One approach, introduced by Sims (2003), and used by Moscarini (2004), Woodford (2009), and Mackowiak and Wiederholt (2009), uses the information-theoretic concept of entropy to model rational inattention as the outcome of the limited ability of people to infer the true values of decision-relevant variables. In those papers, the decisionmaker generally receives noisy information and can choose the timing and information content of signals about these variables. The other approach specifies the cost of observing decision-relevant variables. In this approach, the decisionmaker optimally conserves on observation costs by observing these variables only at discretely-spaced points of time. Two considerations led us to pursue the observation-cost approach rather than the entropy-based approach. The first consideration is tractability. Existing applications of the entropy-based approach have not incorporated adjustment costs of any sort, and the non-convex transactions cost we analyze would be particularly problematic in the entropy-based approach. However, by pursuing the observation-cost based approach, we develop a tractable framework that easily accommodates non-convex transactions costs. More importantly, whether the optimal state-dependent rule evolves to a purely time-dependent rule depends on a comparison of the sizes of transactions costs and observation costs. This comparison is readily apparent in the observation-cost based approach, and would appear to be strained, at best, in the entropy-based approach.

The two closest antecedents to our current paper[5] are Duffie and Sun (1990) and Abel, Eberly, and Panageas (2007).[6,7] These papers, as well as the current paper require consumption to be purchased with a liquid asset, such as cash. In addition, because these papers

[5]Reis (2006) develops and analyzes a model of optimal inattention for a consumer with constant absolute risk aversion who faces a cost of observing additive income, such as labor income. In that model, the consumer can hold only a single riskless asset so there is no asset allocation problem.

[6]Gabaix and Laibson (2002) is very similar to Abel, Eberly, and Panageas (2007). An important difference, however, is that (unlike our formulation in Abel, Eberly, and Panageas (2007) and in the current paper) the formulation of the observation cost in Gabaix and Laibson does not preserve homogeneity of the value function. Therefore, Gabaix and Laibson compute an approximate solution.

[7]Huang and Liu (2007) apply the concept of rational inattention to study the optimal portfolio decision of an investor who can obtain costly noisy signals about a state variable governing the expected growth rate of stock prices. Huang and Liu do not include any costs of trading assets and they allow continuous observation of stock prices so that the investor continuously trades assets within the investment portfolio. However, our modeling of transfer costs and infrequent observation of stock prices leads to infrequent transfers of assets. Finally, and more importantly, Huang and Liu impose a time-dependent rule for what they call "periodic news" because they assume a constant interval of time between the acquisition of periodic news. Thus they cannot address the distinction between state-dependent and time-dependent behavior.

include an observation cost, the consumer will not continuously observe the value of the stock market. In Abel, Eberly, and Panageas (2007),[8] which includes explicit observation costs, the consumer transfers assets from the investment portfolio to the transactions account on every observation date, because, in contrast to the current paper, there are no transactions costs incurred after the consumer incurs the observation cost. In Duffie and Sun (1990), the transactions dates and observation dates are perfectly synchronized by the assumption that "the agent observes his or her current wealth only when making a transaction" (p. 35). In both of these papers the synchronization of observations and transactions follows directly from the assumptions underlying the respective framework, but it is endogenous in our model. Existing models of infrequent adjustment — including both transactions cost models and inattention models — are not capable of addressing the larger question of whether optimal behavior is time dependent or state dependent. Specifically, models that include transactions costs (such as Constantinides (1986), Davis and Norman (1990)[9]), but no inattention, will generate infrequent adjustment that is state dependent. On the other hand, models of inattention based on observation frictions (such as Moscarini (2004), Reis (2006), Huang and Liu (2007), and Abel, Eberly, and Panageas (2007)) generate optimal behavior that is time dependent. By including separate[10] costs for transactions and observations in our model, we can determine endogenously whether the optimal timing of adjustment is time dependent or state dependent, as well as whether observations and transactions are synchronized.

[8]In Abel, Eberly, and Panageas (2007), the observation cost reduces the value of wealth and thus, indirectly, reduces utility. In the current paper, the observation cost directly reduces utility without reducing wealth. The major results of the paper do not depend on whether observation costs are utility costs or resource costs, and we have adopted a utility cost because it seems to capture the effort and hassle of gathering and interpreting relevant information.

[9]Vayanos (1998) and Lo, Mamaysky, and Wang (2004) present generalizations to general equilibrium setups featuring constant absolute risk aversion and normally distributed dividends.

[10]We emphasize that the observation costs and transactions costs are separate, so that in principle, costly observations can occur at times without transactions, and costly transactions can occur at times without observation. In contrast, as we have mentioned, Duffie and Sun (1990) *assume* that transactions and observations are synchronized. Similarly, in the context of a pricing problem, Woodford (2009) assumes that "the menu cost is also the fixed cost of obtaining new (complete) information about the state of the economy." (p.104) Furthermore, the setup in Woodford (2009) precludes a study of the distinction between time- and state-dependent adjustment since "The assumption that memory is (at least) as costly as information about current conditions external to the firm implies that under an optimal policy, the timing of price reviews is (stochastically) state-dependent, but not time-dependent, just as in full-information menu-cost models.... If, instead, memory were costless, the optimal hazard under a stationary optimal plan would also depend on the number of periods n since the last price review" (p.106)

Perhaps surprisingly, we show that if transactions costs are relatively small, the optimal rule evolves endogenously to a purely time-dependent rule, characterized by a constant interval of time between observations and a transaction on every observation date. Although this pure time-dependent rule is reached with probability one, the optimal behavior starts out as state dependent. While the ultimate emergence of a time-dependent rule occurs for sure if the transactions costs are sufficiently small, we also show that optimal behavior will remain state dependent, and transactions and observations will not be synchronized, if transactions costs are large.

A recent paper by Alvarez, Guiso, and Lippi (2010) analyzes data on Italian investors and finds that the median investor observes the value of her portfolio about once per month, which is consistent with the calculations we report in Section 4 using small observation costs. Alvarez, Guiso, and Lippi also find that investors typically observe the values of their portfolios about three times as frequently as they transact. This finding is consistent with our model, if, as pointed out above, transactions costs are large relative to observation costs, as assumed in Alvarez, Guiso, and Lippi.

Section 1 sets up the consumer's decision problem. Section 2 characterizes the optimal trigger and return values for the state variable x_t. In addition, this section contains a detailed discussion of a typical indifference curve of the value function to illustrate various aspects of optimal adjustment behavior. The dynamic evolution of x_t is analyzed in Section 3, which also characterizes the long-run situation that is eventually attained if the fixed component of transactions costs is sufficiently small. In addition, Section 4 presents a numerical illustration of the constant length of time between consecutive observations in the long run, followed by a discussion of the Euler equation in Section 4. Section 5 concludes. The online Appendix contains proofs of all lemmas and propositions.

1 Consumer's Decision Problem

Consider an infinitely-lived consumer who does not earn any labor income but has wealth that consists of risky equity, riskless bonds, and a riskless liquid asset. Risky equity and riskless bonds are held in an investment portfolio, and the consumer is not permitted to take either a leveraged or a negative position in equity. Consumption must be purchased with the liquid asset, which the consumer holds in a transactions account.

1.1 Asset Returns

Equity is a non-dividend-paying stock with a price P_t that evolves according to a geometric Brownian motion

$$\frac{dP_t}{P_t} = \mu dt + \sigma dz, \tag{1}$$

where $\mu > 0$ is the mean rate of return and σ is the instantaneous standard deviation. The riskless bond in the investment portfolio has a constant rate of return $r_f < \mu$. The total value of the investment portfolio, consisting of equity and riskless bonds, is S_t at time t. At time t, the consumer holds X_t in the liquid asset in the transactions account, which pays a riskless rate of return r_L, where $r_L < r_f$ because the liquid asset provides transactions services not provided by the bond in the investment portfolio.

Suppose the consumer observes the value of the investment portfolio at time t_j and next observes its value at time $t_{j+1} = t_j + \tau_j$. Upon observing the values of S_{t_j} and X_{t_j},[11] the consumer may transfer assets between the investment portfolio and the transactions account (at a cost described below) so that at time t_j^+ the value of the investment portfolio is $S_{t_j^+}$. The consumer chooses to hold a fraction ϕ_j of $S_{t_j^+}$ in risky equity and a fraction $1 - \phi_j$ in riskless bonds and does not rebalance the investment portfolio before the next observation.[12] Since the consumer cannot take a negative position or a leveraged position in equity, $0 \leq \phi_j \leq 1$. When the consumer next observes the value of the investment portfolio, at time $t_{j+1} = t_j + \tau_j$, its value is

$$S_{t_{j+1}} = R(t_j, \tau_j) S_{t_j^+} \tag{2}$$

where

$$R(t_j, \tau_j) \equiv \phi_j \frac{P_{t_{j+1}}}{P_{t_j}} + (1 - \phi_j) e^{r_f \tau_j}. \tag{3}$$

1.2 Costs of Transferring Assets

The consumer can transfer assets between the investment portfolio and the transactions account by incurring a resource cost that is proportional to the size of the transfer and

[11] Because the transactions account does not include any risky assets, the consumer continuously knows the value of X_t.

[12] The consumer does not observe any new information between time t_j^+ and time t_{j+1} and hence cannot adjust consumption or the holdings of assets at any time before t_{j+1}in response to news that arrives during this interval of inattention. Proposition 4 addresses the case in which the consumer can nonetheless decide at time t_j^+ to transfer funds between between the investment portfolio and the transactions account at some time(s) before t_{j+1}.

a "fixed" resource cost that is independent of the size of the transfer. Specifically, if the consumer sells $-y^s \geq 0$ dollars of assets from the investment portfolio, there is a proportional transfer cost of $-\psi_s y^s$ dollars, where $0 \leq \psi_s < 1$, so that a sale of $-y^s$ dollars from the investment portfolio is accompanied by an increase in X of $-(1-\psi_s)\,y^s$ dollars. For transfers in the other direction, an increase of $y^b \geq 0$ dollars in the investment portfolio is accompanied by a decrease in X of $(1+\psi_b)\,y^b$ dollars, where $\psi_b \geq 0$. Assume that $\psi_s + \psi_b > 0$ so that at least one of the proportional transfer cost parameters is positive. One interpretation of ψ_s and ψ_b is that they represent brokerage fees. Another interpretation arises if the investment portfolio is a tax-deferred account such as a 401k account. In this case, the consumer must pay a tax on withdrawals from the investment portfolio, and ψ_s would include the consumer's income tax rate, which would be substantially higher than a brokerage fee.[13]

The fixed resource cost is independent of the size of the asset transfer but is a homogeneous linear function of X_t and S_t. Specifically, the fixed resource cost is $\theta_X X_t + \theta_S S_t$, where $0 \leq \theta_X < \overline{\theta_X} < 1$, with $\overline{\theta_X}$ as defined in equation (27), and $0 \leq \theta_S < 1 - \psi_s$.[14] This formulation of the fixed resource cost scales the cost to the components of wealth; technically, it preserves the homogeneity of the value function in X and S. Assume that $\theta_X X$ is paid from the transactions account and $\theta_S S$ is paid from the investment portfolio.[15] If the consumer buys $y^b(t_j) \geq 0$ dollars of assets in the investment portfolio or sells $-y^s(t_j) \geq 0$ dollars of assets from the investment portfolio, then

$$X_{t_j^+} = \left[1 - \left(\mathbf{1}_{\{y^b(t_j)>0\}} + \mathbf{1}_{\{y^s(t_j)<0\}}\right)\theta_X\right] X_{t_j} - (1+\psi_b)\,y^b(t_j) - (1-\psi_s)\,y^s(t_j) \qquad (4)$$

[13]This interpretation of ψ_s as a tax rate is most plausible if the consumer only withdraws money from the investment portfolio and never transfers assets into the investment portfolio. As we will see in Section 3, the long run is characterized by precisely this situation, if the fixed component of the transfer cost is sufficiently small.

[14]We assume that θ_X is small enough so that if $X > 0$ and $S = 0$, the consumer will not be deterred from transferring at least some assets from the transactions account to the investment portfolio. We assume that $\psi_s + \theta_S < 1$ to prevent assets from becoming "trapped" in the investment portfolio if the consumer were to try to sell assets from the investment portfolio at a time when $X = 0$. If, instead, $\psi_s + \theta_S$ were greater than or equal to one, then an attempt to sell a dollar of assets from the investment portfolio would cost at least one dollar and the consumer would not receive any liquid assets as a result of this transaction.

[15]Duffie and Sun (1990) assume that on each observation date the consumer pays a portfolio management fee that is proportional to total wealth. In their model, optimal behavior implies that $X = 0$ on each observation date, so the fixed transaction cost $\theta_X X + \theta_S S$ is simply $\theta_S S$; hence, they do not need to explicitly specify the value of θ_X.

and

$$S_{t_j^+} = \left[1 - \left(\mathbf{1}_{\left\{y^b(t_j)>0\right\}} + \mathbf{1}_{\{y^s(t_j)<0\}}\right)\theta_S\right] S_{t_j} + y^b\left(t_j\right) + y^s\left(t_j\right), \tag{5}$$

where $\mathbf{1}_{\left\{y^b(t_j)>0\right\}}$ is an indicator function that equals 1 if $y^b\left(t_j\right) > 0$ and equals 0 otherwise, and $\mathbf{1}_{\{y^s(t_j)<0\}}$ is an indicator function that equals 1 if $y^s\left(t_j\right) < 0$ and equals 0 otherwise.

1.3 The Utility Function

Suppose that the consumer observes the value of the investment portfolio only at discretely-spaced points in time $t_0, t_1, t_2, \ldots$. At observation date t_j, after observing the value of the investment portfolio, lifetime utility is

$$E_{t_j}\left\{\int_{t_j}^{\infty} \frac{1}{1-\alpha} c_t^{1-\alpha} e^{-\rho(t-t_j)} dt - \sum_{i=j}^{\infty} A\left(t_i, \tau_i\right) e^{-\rho(t_i+\tau_i-t_j)}\right\}, \tag{6}$$

where c_t is consumption at time t, $0 < \alpha \neq 1$ measures risk aversion, the rate of time preference, $\rho > 0$, is large enough so that

$$e^{-\rho\tau_j} E_{t_j}\left\{\left[R\left(t_{j,}\tau_j\right)\right]^{1-\alpha}\right\} < 1, \tag{7}$$

and $A\left(t_i, \tau_i\right)$ is the utility cost of observing the investment portfolio at time $t_i + \tau_i$, given that the preceding observation was at date t_i.

We scale the utility cost of an observation to be a stationary fraction of the consumer's utility from consumption over the interval of time between observations. This property prevents the observation cost from asymptotically becoming prohibitively large or vanishingly small when measured in consumption-equivalent units.[16] In particular,

$$A\left(t_i, \tau_i\right) = \kappa \widetilde{b}\left(\tau_i\right) \int_{t_i}^{t_i+\tau_i} c_t^{1-\alpha} e^{-\rho(t-t_i)} dt, \tag{8}$$

where $\widetilde{b}\left(\tau_i\right) > 0$ for $\tau_i > 0$, and $\kappa > 0$. We want $A\left(t_i, \tau_i\right)$ to capture the notion that it is costly to increase the frequency of observation. We also want this function to be well-behaved for arbitrarily short or long inattention intervals. Therefore, we also require, for any path $c_t > 0$, $t_i < t \leq t_i + \tau_i$, and $\int_{t_i}^{t_i+\tau_i} c_t^{1-\alpha} e^{-\rho(t-t_i)} dt < \infty$, that $A\left(t_i, \tau_i\right)$ has the following three properties

[16] This property is reminiscent of the specification in King, Plosser, and Rebelo (1988) in which the disutility of labor is a stationary fraction of the utility from consumption, with the implication that hours of labor can be stationary even though consumption is nonstationary.

$$0 < \lim_{\tau_i \to 0} A\left(t_i, \tau_i\right) < \infty \tag{9a}$$

$$\lim_{\tau_i \to \infty} e^{-\rho\tau_i} A\left(t_i, \tau_i\right) = 0 \tag{9b}$$

$$e^{-\rho\tau_i} A\left(t_i, \tau_i\right) + e^{-\rho(\tau_i+\tau_{i+1})} A\left(t_{i+1}, \tau_{i+1}\right) > e^{-\rho(\tau_i+\tau_{i+1})} A\left(t_i, \tau_i + \tau_{i+1}\right). \tag{9c}$$

Equation (9a) states that as the interval of time between consecutive observations vanishes, the utility cost per observation approaches a finite positive value. Therefore, the cost of continuous observation is infinite, and hence it is not optimal to observe the value of the investment portfolio continuously. Equation (9b) states that as the length of time until the next observation grows without bound, the discounted value of the utility cost of that observation goes to zero; equivalently, the observation cost does not grow faster than the rate of time preference. Finally, the left hand side of equation (9c) is the discounted (to time t_i) utility cost of observing the investment portfolio twice during the interval $(t_i, t_i + \tau_i + \tau_{i+1}]$: once at time $t_i + \tau_i$ and once at time $t_i + \tau_i + \tau_{i+1}$. The right hand side of equation (9c) is the discounted (to time t_i) utility cost of observing the investment portfolio only once during this interval, at the end of the interval. The inequality in (9c) states that for a given interval of time, two observations are more costly than one observation. The properties of the observation cost in equations (9a), (9b), and (9c) imply restrictions on the function $\widetilde{b}\left(\tau_i\right)$. Rather than work directly with the function $\widetilde{b}\left(\tau_i\right)$, it will be more convenient to work with the function $b\left(\tau_i\right)$ defined as

$$b\left(\tau\right) \equiv e^{-\rho\tau}\widetilde{b}\left(\tau\right). \tag{10}$$

Multiplying both sides of equation (8) by $e^{-\rho\tau_i}$ and using the definition of $b\left(\tau\right)$ from equation (10) yields

$$e^{-\rho\tau_i} A\left(t_i, \tau_i\right) = \kappa b\left(\tau_i\right) \int_{t_i}^{t_i+\tau_i} c_t^{1-\alpha} e^{-\rho(t-t_i)} dt. \tag{11}$$

The following Lemma presents some necessary properties of $b(\tau)$.

Lemma 1 *Suppose that $A\left(t_i, \tau_i\right)$ satisfies equation (11) and has the properties in equations (9a), (9b), and (9c). Then*

1. *$b\left(\tau\right)$ is non-increasing.*

2. *$0 < \lim_{\tau \to 0} \tau b\left(\tau\right) < \infty$, which implies $\lim_{\tau \to 0} b\left(\tau\right) = \infty$ and $\lim_{\tau \to 0} \frac{\tau b'(\tau)}{b(\tau)} = -1$.*

3. *$\lim_{\tau \to \infty} b\left(\tau\right) = 0$, if $\lim_{\tau \to \infty} \int_{t_i}^{t_i+\tau} c_t^{1-\alpha} e^{-\rho(t-t_i)} dt > 0$ is finite.*

Finally, we adopt the normalization $b(1) = 1$. As an illustration of the function $b(\tau)$, suppose that $A(t_i, \tau_i)$ is proportional to the average rate at which (discounted) utility from consumption is accrued over the interval $(t_i, t_i + \tau_i]$. Thus, $\widetilde{b}(\tau_i)$ in equation (8) is proportional to $\frac{1}{\tau_i}$, and normalizing $b(\tau) \equiv e^{-\rho\tau}\widetilde{b}(\tau)$ so that $b(1) = 1$, we have

$$b(\tau) = e^{-\rho(\tau-1)}\frac{1}{\tau}. \tag{12}$$

It is straightforward to verify that $b(\tau)$ in equation (12) satisfies conditions (1) to (3) in Lemma 1. In the numerical example in Section 4, we use the specification of $b(\tau)$ in equation (12), but everywhere else in the paper we allow any $b(\tau) > 0$ that satisfies the properties in statements (1) to (3) in Lemma 1.

Substitute the discounted observation cost from equation (11) into the lifetime utility function in equation (6) to obtain

$$\frac{1}{1-\alpha}E_{t_j}\left\{\sum_{i=j}^{\infty} e^{-\rho(t_i-t_j)}\left[1-(1-\alpha)\kappa b(\tau_i)\right]\int_{t_i}^{t_i+\tau_i} c_t^{1-\alpha}e^{-\rho(t-t_i)}dt\right\}. \tag{13}$$

Since the consumer will not observe any new information between times t_j and t_{j+1}, she can, at time t_j, plan the entire path of consumption from time t_j^+ to time t_{j+1}. Let $C(t_j, \tau_j)$ be the present value, discounted at rate r_L, of the (deterministic) flow of consumption over the interval of time from t_j^+ until the next observation date, $t_{j+1} \equiv t_j + \tau_j$. Specifically,

$$C(t_j, \tau_j) = \int_{t_j^+}^{t_{j+1}} c_s e^{-r_L(s-t_j)}ds, \tag{14}$$

where the path of consumption c_s, $t_j^+ \leq s \leq t_{j+1}$, is chosen to maximize the discounted value of utility over the interval from t_j^+ to t_{j+1}. Let

$$U(C(t_j, \tau_j)) = \max_{\{c_s\}_{s=t_j^+}^{t_{j+1}}} \int_{t_j^+}^{t_{j+1}} \frac{1}{1-\alpha}c_s^{1-\alpha}e^{-\rho(s-t_j)}ds, \tag{15}$$

subject to a given value of $C(t_j, \tau_j)$ in equation (14). It is straightforward to show that[17]

$$U(C(t_j, \tau_j)) = \frac{1}{1-\alpha}\left[h(\tau_j)\right]^{\alpha}\left[C(t_j, \tau_j)\right]^{1-\alpha}, \tag{16}$$

[17] During the interval of time from t_j^+ to t_{j+1} the (deterministic) Euler equation implies that optimal values of consumption satisfy

$$c_s = e^{-\frac{\rho-r_L}{\alpha}(s-t_j^+)}c_{t_j^+}, \text{ for } t_j^+ \leq s \leq t_{j+1}. \tag{*}$$

Substituting c_s from equation (*) into equation (14) in the text yields

$$C(t_j, \tau_j) = h(\tau_j)\, c_{t_j^+}, \tag{**}$$

where

$$h(\tau_j) \equiv \int_0^{\tau_j} e^{-\chi s} ds = \frac{1 - e^{-\chi \tau_j}}{\chi} \tag{17}$$

and we assume that

$$\chi \equiv \frac{\rho - (1-\alpha) r_L}{\alpha} > 0. \tag{18}$$

Since consumption during the interval of time from t_j^+ to t_{j+1} is financed from the transactions account, which earns an instantaneous riskless rate of return r_L, we have

$$X_{t_{j+1}} = e^{r_L \tau_j} \left(X_{t_j^+} - C(t_j, \tau_j) \right). \tag{19}$$

Use equation (16) and the expression for lifetime utility in (13) to obtain the value function[18] at observation date t_j, immediately after observing the value of the investment portfolio at date t_j,

$$\begin{aligned} V\left(X_{t_j}, S_{t_j}\right) = & \max_{C(t_j,\tau_j), y^b(t_j), y^s(t_j), \phi_j, \tau_j} \left[1 - (1-\alpha)\kappa b(\tau_j)\right] U\left(C(t_j, \tau_j)\right) \\ & + e^{-\rho \tau_j} E_{t_j} \left\{ V\left(e^{r_L \tau_j} \left(X_{t_j^+} - C(t_j, \tau_j) \right), R(t_j, \tau_j) S_{t_j^+} \right) \right\}, \end{aligned} \tag{20}$$

where the maximization in equation (20) is subject to equations (4) and (5) and the inequality constraints $C(t_j, \tau_j) \leq X_{t_j^+}$, $0 \leq \phi_j \leq 1$, $y^b(t_j) \geq 0$, and $y^s(t_j) \leq 0$.

The value function in equation (20) is homogeneous of degree $1-\alpha$ in X_{t_j} and S_{t_j}, and consequently it can be written as

$$V\left(X_{t_j}, S_{t_j}\right) = \frac{1}{1-\alpha} S_{t_j}^{1-\alpha} v\left(x_{t_j}\right), \tag{21}$$

where $\frac{1}{1-\alpha} v\left(x_{t_j}\right)$ is strictly increasing in x_t and

$$x_t \equiv \frac{X_t}{S_t} \tag{22}$$

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where $h(\tau_j)$ is defined in equation (17) in the text. Equations (*) and (**) imply that

$$c_s = \left[h(\tau_j)\right]^{-1} e^{-\frac{\rho - r_L}{\alpha}\left(s - t_j^+\right)} C(t_j, \tau_j), \qquad \text{for } t_j^+ \leq s \leq t_{j+1}. \tag{***}$$

Substituting equation (***) into equation (15), and using the definition of $h(\tau_j)$ in equation (17) yields $U(C(t_j, \tau_j)) = \frac{1}{1-\alpha}\left[h(\tau_j)\right]^{\alpha}\left[C(t_j, \tau_j)\right]^{1-\alpha}$, which, along with equation (**), implies that $U'(C(t_j, \tau_j)) = c_{t_j^+}^{-\alpha}$.

[18] If $\alpha > 1$, then $\left[1 - (1-\alpha)\kappa b(\tau_i)\right] > 0$ for all $\tau > 0$; as we show in the online Appendix, optimality implies that τ will be large enough so that $\left[1 - (1-\alpha)\kappa b(\tau_i)\right]$ is positive even when $\alpha < 1$. Equation (16) gives the maximized value of $\frac{1}{1-\alpha}\int_{t_i}^{t_{i+1}} c_t^{1-\alpha} e^{-\rho(t - t_i)} dt$ in equation (13) subject to equation (14). Since $\left[1 - (1-\alpha)\kappa b(\tau_i)\right] > 0$, we can substitute equation (16) into the continuous-time optimization problem in equation (13) to obtain the discrete-time problem in equation (20).

is the ratio of the transactions account to the investment portfolio. The optimal length of time between consecutive observation dates t_j and t_{j+1}, τ_j, is a function of x_{t_j}.

2 Trigger and Return Values of x

The value of $x_{t_j} \equiv \frac{X_{t_j}}{S_{t_j}}$ on an observation date t_j determines whether, in which direction, and what amounts of assets the consumer transfers between the investment portfolio and the transactions account. There are two trigger values of x, ω_1 and ω_2, that determine whether the consumer transfers assets, and there are two return values of x, π_1 and π_2, that characterize the optimal value of $x_{t_j^+}$ immediately after a transfer.

To define and characterize the trigger values, ω_1 and ω_2, first define the restricted value function $\widetilde{V}\left(X_{t_j}, S_{t_j}\right)$ at observation date t_j as the maximized expected value of utility over the infinite future, subject to the restriction that *the consumer does not transfer any assets between the transactions account and the investment portfolio at time* t_j (but optimally transfers assets between the transactions account and the investment portfolio at all future observation dates). Formally,

$$\widetilde{V}\left(X_{t_j}, S_{t_j}\right) = \max_{C(t_j,\tau_j),\phi_j,\tau_j} \left[1-(1-\alpha)\kappa b\left(\tau_j\right)\right] U\left(C\left(t_j,\tau_j\right)\right) + e^{-\rho\tau_j} E_{t_j}\left\{V\left(e^{r_L\tau_j}\left(X_{t_j}-C\left(t_j,\tau_j\right)\right), R\left(t_j,\tau_j\right) S_{t_j}\right)\right\}, \tag{23}$$

subject to $C\left(t_j,\tau_j\right) \leq X_{t_j}$ and $0 \leq \phi_j \leq 1$. For the remainder of this section, we will suppress the time subscripts, with the understanding that the results apply at any observation date. Like the value function, the restricted value function is homogeneous of degree $1-\alpha$ and can be written as

$$\widetilde{V}\left(X,S\right) = \frac{1}{1-\alpha} S^{1-\alpha}\widetilde{v}\left(x\right), \tag{24}$$

where $\frac{1}{1-\alpha}\widetilde{v}\left(x\right)$ is strictly increasing in x. On any observation date, $\widetilde{V}\left(X,S\right) \leq V\left(X,S\right)$, with equality only if the optimal values of y^b and y^s are both zero.

Define

$$\omega_1 \equiv \inf x > 0 : \widetilde{v}\left(x\right) = v\left(x\right) \tag{25}$$

and

$$\omega_2 \equiv \sup x > 0 : \widetilde{v}\left(x\right) = v\left(x\right). \tag{26}$$

The proposition below shows that ω_1 and ω_2 are trigger values for x in the sense that if x is less than ω_1 on an observation date, the consumer will transfer assets to the transactions account, and if x exceeds ω_2 on an observation date, the consumer will transfer assets to the investment portfolio. To ensure that ω_2 is finite, we assume that κ and θ_X are small enough that a consumer who holds all of her wealth in the transactions account on an observation date will not be deterred from transferring some assets from the transactions account to the investment portfolio. Specifically, we assume

$$\theta_X < \overline{\theta_X} \equiv \left[(1-\theta_S)\,\frac{1-\psi_s}{1+\psi_b}\frac{\chi}{r_f - r_L + \chi}\right]^{\frac{\chi}{r_f - r_L}} \frac{r_f - r_L}{r_f - r_L + \chi} < 1 \tag{27}$$

and

$$\kappa < \overline{\kappa} \equiv \frac{\left(\frac{\theta_X}{\overline{\theta_X}}\right)^{-\frac{r_f - r_L}{\chi}(1-\alpha)} - 1}{(1-\alpha)\, b\left(\widehat{T}\right)\left(\exp\left(\chi \widehat{T}\right) - 1\right)}, \tag{28}$$

where $\widehat{T} \equiv -\frac{1}{\chi} \ln \left[\left(1 + \frac{\chi}{r_f - r_L}\right)\theta_X\right] > 0$. We also define

$$\pi_1 \equiv \sup \left\{ \begin{array}{r} x \geq 0 : \forall z \in \left(0, \frac{xS}{1-\psi_s}\right], \;\; (1)\; V(xS, S) \geq V(xS - (1-\psi_s)\, z, S + z) \\ \text{and } (2)\; V(xS, S) > \widetilde{V}(xS - (1-\psi_s)\, z, S + z) \end{array} \right\} \tag{29}$$

and

$$\pi_2 \equiv \inf \left\{ \begin{array}{r} x \geq 0 : \forall z \in (0, S], \;\; (1)\; V(xS, S) \geq V(xS + (1+\psi_b)\, z, S - z) \\ \text{and } (2)\; V(xS, S) > \widetilde{V}(xS + (1+\psi_b)\, z, S - z) \end{array} \right\}. \tag{30}$$

The proposition below shows that π_1 and π_2 are return values for x. Specifically, if $x \leq \omega_1$, the consumer will transfer enough assets from the investment portfolio to the transactions account to increase x to π_1. Alternatively, if $x \geq \omega_2$, the consumer will use the transactions account to buy enough assets in the investment portfolio to decrease x to π_2.

Proposition 1 *Assume that $\kappa < \overline{\kappa}$ and $\theta_X < \overline{\theta_X}$. Then*

1. $0 < \omega_1 \leq \pi_1 \leq \pi_2 \leq \omega_2 < \infty$.

2. *If $x_{t_j} < \omega_1$, then (a) $y^s(t_j) < 0$, (b) $x_{t_j^+} = \pi_1$, (c) $m\left(x_{t_j}\right) \equiv \frac{V_S\left(X_{t_j}, S_{t_j}\right)}{V_X\left(X_{t_j}, S_{t_j}\right)} = (1-\psi_s)\,\frac{1-\theta_S}{1-\theta_X}$, (d) $v\left(x_{t_j}\right) = \left[\frac{(1-\theta_X)x_{t_j} + (1-\theta_S)(1-\psi_s)}{(1-\theta_X)\omega_1 + (1-\theta_S)(1-\psi_s)}\right]^{1-\alpha} v(\omega_1)$*

3. $v(\omega_1) = \left[\frac{(1-\theta_X)\omega_1 + (1-\theta_S)(1-\psi_s)}{\pi_1 + 1 - \psi_s}\right]^{1-\alpha} v(\pi_1)$

4. If $x_{t_j} > \omega_2$*, then (a)* $y^b(t_j) > 0$*, (b)* $x_{t_j^+} = \pi_2$*, (c)* $m\left(x_{t_j}\right) \equiv \frac{V_S\left(X_{t_j},S_{t_j}\right)}{V_X\left(X_{t_j},S_{t_j}\right)} = (1+\psi_b)\,\frac{1-\theta_S}{1-\theta_X}$*,*
(d) $v\left(x_{t_j}\right) = \left[\frac{(1-\theta_X)x_{t_j}+(1-\theta_S)(1+\psi_b)}{(1-\theta_X)\omega_2+(1-\theta_S)(1+\psi_b)}\right]^{1-\alpha} v\left(\omega_2\right)$

5. $v\left(\omega_2\right) = \left[\frac{(1-\theta_X)\omega_2+(1-\theta_S)(1+\psi_b)}{\pi_2+1+\psi_b}\right]^{1-\alpha} v\left(\pi_2\right)$

Proposition 1 is proved in the online Appendix. Here we use the indifference curves in Figure 1 to illustrate this proposition and the definitions of the trigger and return points. For simplicity, Figure 1 is drawn for the case in which $\theta_X = \theta_S$. The indifference curve of the value function $V\left(X,S\right)$ passes through points A, B, C, D, E, and F, and the indifference curve of the restricted value function $\widetilde{V}\left(X,S\right)$ passes through points K, B, C, D, E, and J. In Regions II, III, and IV, the two indifference curves are identical, reflecting the fact that $V\left(X,S\right) = \widetilde{V}\left(X,S\right)$. Therefore, Regions II, III, and IV represent the "inaction region" in which the consumer can attain $V\left(X,S\right)$ without transferring any assets between the investment portfolio and the transactions account.

The consumer will transfer assets if $V\left(X,S\right) > \widetilde{V}\left(X,S\right)$, which is the case in Regions I and V. For instance, in Region I, the indifference curve of the restricted value function passes through point B and lies above the indifference curve of the value function that also passes through point B, thereby implying that $V\left(X,S\right) > \widetilde{V}\left(X,S\right)$ in this region.[19] In order to attain the maximized value of expected lifetime utility, the consumer must transfer assets between the investment portfolio and the transactions account. As shown in statement 2a of Proposition 1, $y^s < 0$ so the consumer sells assets from the investment portfolio to increase the amount of liquid assets in the transactions account. Similarly, according to statement 4a, if the consumer is in Region V on an observation date, the optimal policy is to use some of the liquid assets in the transactions account to purchase additional assets in the investment portfolio.

Now consider the return value π_1. We proceed in two steps. First, assume that the consumer has already paid the fixed transfer cost $\theta_2\left(X+S\right)$, where θ_2 is the common value of $\theta_X = \theta_S$, and that the consumer is choosing the size of the asset transfer from the investment portfolio to the transactions account. In the second step, we consider the impact of the fixed transfer cost, $\theta_2\left(X+S\right)$, on the optimal transfer.

[19] To see that $V\left(X,S\right) > \widetilde{V}\left(X,S\right)$ in Region I, use the fact that $V\left(X,S\right)$ is strictly increasing in X and S to obtain $V^K > V^A = V^B = \widetilde{V}^B = \widetilde{V}^K$, where V^i is the value of $V\left(X,S\right)$ at point i and $\widetilde{V}^j$ is the value of $\widetilde{V}\left(X,S\right)$ at point j in the figure.

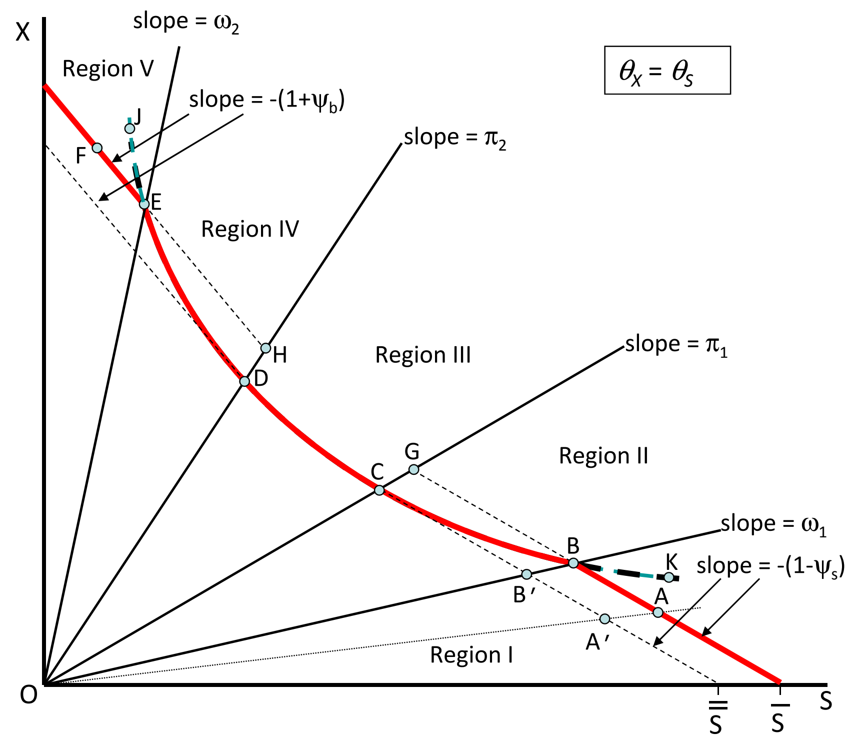

FIGURE 1: Indifference Curve of the Value Function When $\theta_X = \theta_S$.

Suppose that, after paying the fixed cost $\theta_2 (X + S)$, the consumer is located somewhere to the right of point C along the dashed line through point C with slope $-(1 - \psi_s)$. For instance, suppose that the consumer is at point A'. Having already paid the fixed cost, the consumer can move instantaneously to any point up and to the left of point A' along the dashed line with slope $-(1 - \psi_s)$ by reducing S by $-y^s > 0$ dollars and increasing X by $(1 - \psi_s)(-y^s)$ dollars. The consumer will sell assets from the investment portfolio, until (X, S) reaches point C, where the dashed line with slope $-(1 - \psi_s)$ is tangent to the indifference curve, which is essentially a smooth-pasting condition. At point C, the ratio of X to S, i.e., x, is equal to π_1, as indicated by the line through points O, C, and G, which has slope equal to π_1.

Now consider the impact of the fixed cost $\theta_2 (X + S)$ on the optimal transfer of assets. If $\theta_2 > 0$, the consumer cannot move from point A' to point C. To see the impact of $\theta_2 > 0$, consider the line through points G, B, and A, which is parallel to the line through points C, B', and A', and hence has slope $-(1 - \psi_s)$. Point G lies on the half-line through the origin with slope π_1 and is located so that the length of $\overline{OC}$ is $1-\theta_2$ times the length of $\overline{OG}$. The properties of similar triangles imply that the length of $\overline{OB'}$ is $1-\theta_2$ times the length of $\overline{OB}$ and that the length of $\overline{OA'}$ is $1-\theta_2$ times the length of $\overline{OA}$.

Now suppose that the consumer starts at point A and transfers $-y^s > 0$ dollars from the investment portfolio, thereby incurring a cost of $\theta_2(X+S) - \psi_s y^s$ dollars. The fixed cost of $\theta_2(X+S)$ dollars reduces both X and S by the fraction θ_2 and can be represented by the movement from point A to point A'; the transfer of $-y^s > 0$ dollars from the investment portfolio can be represented by a movement from point A' upward and leftward along the dashed line through points C, B', and A'. The consumer will be willing to move from A to point C only if doing so increases (or at least does not lower) the value of the value function. That is, the gain in value from moving to an improved allocation between X and S, with $x = \pi_1$, must outweigh the fixed cost $\theta_2(X+S)$ represented by the movement downward and leftward from the line through points G, B, and A to the line through points C, B', and A'. For a large change in the ratio x, such as the change in moving from point A to point C, the net gain in value is positive. For a small change in x, the change is not worthwhile. At point B, the gain from the improved allocation between X and S is exactly offset by the cost of moving from the line through points G, B, and A to the line through points C, B', and A'. Formally, this equality of gain and benefit is represented by statement 3 in Proposition 1, which is essentially a value-matching condition.

For points along the segment $\overline{GB}$, the change in the value of x is small enough that the improved allocation between X and S is outweighed by the fixed cost $\theta_2(X+S)$. Therefore, the consumer will not transfer assets from any points along this segment. The fact that the consumer will not move from points along segment $\overline{GB}$ to point C is illustrated by the fact that these points lie above the indifference curve of the value function that passes through point C. Alternatively, for points below and to the right of point B along the line through points A and B, the improved asset allocation made possible by moving to point C, and the associated increase in value, are large enough to compensate for the fixed transfer cost, and the consumer will move from any of these points to C (statements 2a and 2b). Since the consumer ends up at the same point, namely point C, from any point below and to the right of point B, all of these points have the same value. Thus, all of these points lie on the same indifference curve (statement 2d), so that indifference curve has slope equal to $-(1-\psi_s)$ below and to the right of point B, which is statement 2c in Proposition 1.[20]

[20] If we relax the assumption that $\theta_X = \theta_S$, then statement 2c of Proposition 1 implies that the slope of the linear portion of the indifference curve through points B and A is $-(1-\psi_s)\frac{1-\theta_S}{1-\theta_X}$ while the slope of the

We have used Figure 1 to illustrate the trigger point ω_1 and the return point π_1 when the consumer chooses to transfer assets from investment portfolio to the transactions account. A similar set of arguments can explain the trigger point ω_2 and the return point π_2 when the consumer chooses to transfer assets from the transactions account to the investment portfolio.

We conclude this section with the following corollary to Proposition 1.

Corollary 1 $\omega_1 \leq x_{t_j^+} \leq \omega_2$.

The value of x_t immediately following any observation date t_j (and following any optimal asset transfers at that date) is confined to the closed interval $[\omega_1, \omega_2]$. This result will be useful when we analyze the dynamic behavior of asset holdings in the next section.

3 Dynamic Behavior

We have shown that the direction of the optimal transfer on an observation date depends on the value of x_{t_j}. In this section, we examine the dynamic behavior of the stochastic process for x_{t_j}. If the value of X_{t_j} is positive on an observation date, then, depending on the outcome of the stochastic process for S, the value of x_{t_j} could be in any of the five regions in Figure 1. However, the stochastic process for x_{t_j} will eventually be absorbed at $x_{t_j} = 0$ provided that θ_S is sufficiently small.

Proposition 2 *There exists $\underline{\theta_S} > 0$, such that for any non-negative $\theta_S < \underline{\theta_S}$, if $x_{t_j} < \omega_1$ on observation date t_j, then $x_{t_k} = 0$ on all subsequent observation dates $t_k > t_j$.*

The proof of Proposition 2 is in the online Appendix. Here we provide an intuitive argument. First, consider the case in which $\theta_X = \theta_S = 0$. If $x_{t_j} < \omega_1$ on observation date t_j, the optimal transfer is from the investment portfolio to the transactions account so that $x_{t_j^+}$ increases to π_1. Since each additional dollar that is transferred from the investment

dashed line through points C, B', and A' remains $-(1-\psi_s)$. The horizontal intercept of the indifference curve, $\overline{S}$, is $\frac{1}{1-\theta_S} \geq 1$ times as large as $\overline{\overline{S}}$, the horizontal intercept of the dashed line through points C, B', and A' because starting from $(X, S) = \left(0, \overline{S}\right)$ the fixed transaction cost moves the allocation (X, S) to $\left(0, (1-\theta_S)\,\overline{S}\right) = \left(0, \overline{\overline{S}}\right)$. Therefore, even if $\theta_X > \theta_S$, so that the linear portion of the indifference curve slopes downward more steeply than the dashed line, the linear portion of the indifference curve will not cross the dashed line for any non-negative values of X. Also, statement 4c of Proposition 1 implies that the slope of the indifference curve through points E and F is $-(1+\psi_b)\,\frac{1-\theta_S}{1-\theta_X}$. The vertical intercept of the indifference curve is $\frac{1}{1-\theta_X} \geq 1$ times as large as the vertical intercept of the dashed line through point D and thus the indifference curve does not cross this dashed line for non-negative values of S.

portfolio to the transactions account incurs a transactions cost ψ_s, and since the transactions account earns a lower riskless rate of return than the riskless rate of return on bonds in the investment portfolio, the consumer would never transfer more assets from the investment portfolio than are needed to finance consumption until the next observation date. Thus, the consumer will arrive at the next observation date with zero liquid assets, so that $x_{t_{j+1}}$ will be zero. Since $x_{t_{j+1}} = 0 < \omega_1$, the process will repeat itself *ad infinitum* with $x_{t_k} = 0$ on every observation date $t_k > t_j$.

If at least one of θ_X and θ_S is positive, then we need to consider the possibility that the consumer would want to arrive at the next observation date with enough liquid assets in the transactions account to avoid transferring assets from the investment portfolio and thus avoid paying the fixed transactions cost at that date. As the proof of Proposition 2 shows, if θ_S is small enough, the consumer will still optimally choose to arrive at the next observation date with a zero balance in the transactions account, even though this action necessitates payment of the fixed transaction cost at the next observation date.[21] Alternatively, if θ_S is large, the consumer may choose to arrive at observation dates with a positive balance in the transactions account; holding a positive transactions balance gives the consumer the option to avoid paying a transaction cost if $\omega_1 < x_{t_{j+1}} < \omega_2$ on observation date t_{j+1} and this option becomes valuable when the fixed cost of transactions is large.

The following lemma together with Proposition 2 allows us to prove that the stochastic process for x_{t_j} is eventually absorbed at zero, if θ_S is sufficiently small.

Lemma 2 *Eventually, $x_{t_j} < \omega_1$ on an observation date.*

The proof of Lemma 2 is in the online Appendix. Here we provide an intuitive argument. Because the expected rate of return on equity, μ, exceeds the riskless rate of return, r_f, on bonds in the investment portfolio, the optimal share of equity, ϕ_j, is positive. Therefore, during any given inattention interval, there is a chance that $R(t_j, \tau_j)$ will be sufficiently high that $x_{t_{j+1}} = \frac{e^{r_L \tau_j}\left(X_{t_j^+} - C(t_j, \tau_j)\right)}{R(t_j, \tau_j) S_{t_j^+}}$ will be less than ω_1. After sufficiently many spells of inattention, eventually this event will occur.

Proposition 3 *There exists $\underline{\theta_S} > 0$, such that for any non-negative $\theta_S < \underline{\theta_S}$, eventually the stochastic process for x_{t_j} is absorbed at zero and the time between consecutive observations is constant.*

[21] Of course, a positive value of θ_X will not induce the consumer to try to avoid the fixed transactions cost on the next observation date because $\theta_X X$ will be zero under the policy described here.

Proposition 3 implies that, in the long run, optimal asset holdings have a Baumol-Tobin flavor, if $\theta_S \geq 0$ is sufficiently small. Specifically, the consumer will arrive at each observation date having just exhausted the liquid assets in the transactions account and will liquidate just enough assets from the investment portfolio to finance consumption until the next observation date. Observations and transfers are perfectly synchronized and a constant amount of time elapses between asset transfers.[22] We will refer to this situation as the *long run*. Proposition 3 is robust to a change in an important assumption that we have maintained to this point. Specifically, we have assumed that transfers between the investment portfolio and the transactions account can occur only on observation dates. For the remainder of this section only, we consider the impact of allowing "automatic" transactions *between* observation dates.[23] The essence of inattention is that between observation dates, the consumer does not observe the realization of random returns and does not change consumption in response to events since the most recent observation. Because the consumer would not know in advance the proceeds of any automatic transfer that depends on the stock price at the time of the transaction, she could not make any adjustments to her plans at that time. Accordingly, there would be no reason for the consumer to transfer assets from stocks to the transactions account, which is dominated in rate of return by the investment portfolio. In general, any optimal transfer from the investment portfolio to the transactions account between observation dates must be known as of the most recent observation date. Specifically, the consumer may consider asset transfers at times between observation dates t_j and t_{j+1} as long as (1) the amounts are known as of time t_j, and (2) $X_t \geq 0$ and $S_t \geq 0$ for all t. As an example, an automatic transfer could specify that the consumer transfer a given amount from the bond holdings in the investment portfolio to the transactions account at time $t_j + \frac{1}{2}\tau_j$.

Proposition 4 *Define a plan of automatic transfers as*

$$\left\{ y^b(t) \geq 0 \text{ and } y^s(t) \leq 0 \text{ for } t \in (t_j, t_{j+1}) : \begin{array}{l} (1)\ y^b(t) \text{ and } y^s(t) \text{ are } F_{t_j}\text{-measurable, and} \\ (2)\ X_t \geq 0 \text{ and } S_t \geq 0 \text{ for any path of } P_t. \end{array} \right\}$$

Assume that $\theta_X > 0$. Suppose that we allow automatic transfers between observation dates. Then there exists $\underline{\theta_S} > 0$, such that for any non-negative $\theta_S < \underline{\theta_S}$, eventually the stochastic process for x_{t_j} is absorbed at zero and the time between consecutive observations is constant.

[22] The model in Duffie and Sun (1990) shares this property because it assumes that the consumer starts with $x_t = 0$.

[23] In a price-setting framework, Bonomo, Carvalho, and Garcia (2010) analyze "uninformed adjustments," which are price adjustments that occur between observation dates. These uninformed adjustments are analogous to our "automatic" transactions in the consumer's allocation of assets.

4 Long-Run Behavior

Table 1 presents the optimal time between consecutive observation dates in the long run for the case in which $\theta_X = \theta_S = \theta_2$, there are no automatic transfers, and the parameter values are given in the table's caption. For these numerical exercises, we specify $b(\tau)$ as in equation (12), so that the utility cost $A(t_i, \tau_i)$ is proportional to the average discounted utility of consumption accrued over the inattention interval. This formulation allows us to present both the observation cost and the fixed component of the transactions cost in terms of dollars.[24] For all the numerical calculations we assume that the consumer has \$1 million in the investment portfolio on an observation date. The observation cost in Column (1) is the dollar equivalent of the reduction in utility associated with the observation cost. In the baseline case, the observation cost is \$2.30 per observation. Column (2) reports the optimal time between consecutive observations when $\theta_2 = 0$ so that fixed cost parameters θ_X and θ_S are both zero. The time between observations is measured in years, so in the baseline case, the optimal time between observations is slightly longer than one month. Column (3) reports θ_2^*, which is the largest value of $\theta_X = \theta_S = \theta_2$ such that the time between consecutive observations eventually becomes constant. For values of $\theta_X = \theta_S = \theta_2$ larger than θ_2^*, the optimal rule remains state dependent indefinitely and the frequency of observations will exceed the frequency of transactions indefinitely. The values reported in column (3) are actually $\theta_2^* \times 10^6$ so that, for instance, in the baseline case, the fixed transactions cost is \$6.60 for a millionaire. Finally, column (4) reports the time between consecutive observations when $\theta_2 = \theta_2^*$.

Table 1 allows us to draw two broad conclusions. First, even tiny observation costs can

[24] In order to obtain the equivalent dollar cost, we use the fact that the utility cost of an observation is $A(t_j, \tau_j) = \kappa e^{\rho} \times \frac{1}{\tau_j} \int_{t_j}^{t_j+\tau_j} c_t^{1-\alpha} e^{-\rho(t-t_j)} dt = (1-\alpha) \kappa e^{\rho} \frac{1}{\tau_j} U(C(t_j, \tau_j))$. In the long run, $C(t_j, \tau_j) = X_{t_j^+}$, so the utility cost of an observation is $(1-\alpha) \kappa e^{\rho} \frac{1}{\tau_j} U\left(X_{t_j^+}\right)$. We want to compute the reduction in the transactions balance at time t_j^+ that would cause the same loss in utility over the interval $(t_j, t_j + \tau_j]$ as would the observation cost. Writing the reduction in the transactions balance as $\lambda X_{t_j^+}$, we find the value of λ such that $U\left(X_{t_j^+}\right) - U\left((1-\lambda) X_{t_j^+}\right) = (1-\alpha) \kappa e^{\rho} \frac{1}{\tau_j} U\left(X_{t_i^+}\right)$. Since $U()$ is homogeneous of degree $1-\alpha$, we have $1 - (1-\lambda)^{1-\alpha} = (1-\alpha) \kappa e^{\rho} \frac{1}{\tau_j}$, which implies $\lambda = 1 - \left[1 - (1-\alpha) \frac{\kappa}{\tau_j} e^{\rho}\right]^{\frac{1}{1-\alpha}}$. On any observation date in the long run, $X_{t_j} = 0$, so that $X_{t_j^+} = \pi_1 S_{t_j^+}$. Equations (4) and (5), using the fact that $X_{t_j} = 0$ and $y^b(t_j) = 0$, imply $S_{t_j^+} = \frac{1-\psi_s}{1-\psi_s+\pi_1}(1-\theta_S) S_{t_j}$ so that we have $X_{t_j^+} = \pi_1 \frac{1-\psi_s}{1-\psi_s+\pi_1}(1-\theta_S) S_{t_j}$. Therefore, for a consumer who has wealth of 10^6 dollars on an observation date, the observation cost is $\lambda \pi_1 \frac{1-\psi_s}{1-\psi_s+\pi_1}(1-\theta_S) 10^6$ dollars. (Although the length of the optimal inattention interval is invariant to ψ_s, the dollar-equivalent observation cost depends on ψ_s. For this calculation, we have set $\psi_s = 0.01$.)

	(1) Observation cost (dollar equivalent)	(2) $\tau^*, \theta_2 = 0$ (years)	(3) $\theta_2^* \times 10^6$ (dollar equivalent)	(4) $\tau^*, \theta_2 = \theta_2^*$ (years)
Baseline	2.3	0.097	6.5	0.190
$\kappa = 0.001$	23.1	0.309	63.6	0.593
$\rho = 0.02$	2.6	0.098	7.8	0.198
$\alpha = 3$	2.4	0.092	5.9	0.174
$r_L = 0$	2.3	0.080	11.3	0.194
$r_f = 0.03$	2.8	0.084	27.3	0.281
$\mu = 0.07$	2.7	0.089	6.1	0.161
$\sigma = 0.2$	2.1	0.097	8.1	0.218

TABLE 1: θ_2^* is the largest value of $\theta_2 = \theta_X = \theta_S$ that leads to constant optimal inattention spans. Baseline Parameters: $\alpha = 4, \rho = 0.01, r_L = 0.01, r_f = 0.02, \mu = 0.06, \sigma = 0.16,\ \kappa = 0.0001$.

lead to substantial inattention intervals. Column (2) shows that even when the fixed costs of transacting are zero ($\theta_X = \theta_S = 0$), a consumer who owns one million dollars, and incurs an observation cost equivalent to about two dollars, will observe her portfolio at approximately a monthly frequency, which is the empirical frequency reported by Alvarez, Guiso, and Lippi (2010). Second, fixed transaction costs can significantly magnify the effect of observation costs to produce even larger inattention spans. The inattention spans in column (4) are about twice as large as the inattention spans in column (2). Intuitively, when fixed transaction costs are not too large compared to the observation costs, the consumer will find it optimal to transact on every observation date, in order to avoid "wasting" observation costs without using the obtained information to undertake a transaction. Because of this synchronization, the optimal inattention interval is determined as if fixed transaction costs and observation costs are bundled together, effectively magnifying the impact of the observation cost. For instance, with an observation cost of \$2.30, the optimal time between observations can be more than two months, if $\theta_2 = \theta_2^*$.

The calculations reported in Table 1 are invariant to the proportional transaction cost parameters ψ_b and ψ_s. The irrelevance of ψ_b results from the fact that in the long run the consumer does not ever transfer any assets from the transactions account to the investment portfolio and thus never incurs any cost $\psi_b y^b$. On any observation date in the long run, all of the consumer's wealth is in the investment portfolio. In order to consume any of this wealth the consumer effectively must pay a tax at rate ψ_s to transfer the wealth to the transactions account. Thus ψ_s is a pure consumption tax and hence reduces the path of consumption by a fraction ψ_s while leaving the timing of transfers unchanged. This result is formalized in Proposition 6 in the online Appendix.

Proposition 3 implies that in the long run the consumer will transfer assets in the same direction (from the investment portfolio to the transactions account) on every observation date. Therefore, if the consumer is sufficiently risk averse[25] so that optimal ϕ_j is interior to $[0,1]$, then an Euler equation, described in the following proposition, holds in the long run.[26]

Proposition 5 *There exists* $\underline{\theta_S} > 0$*, such that if* $\theta_S < \underline{\theta_S}$*, and* $\alpha > \frac{\mu - r_f}{\sigma^2}$*, then in the long run* $E_{t_j}\left\{c_{t_{j+1}^+}^{-\alpha}\left(\frac{P_{t_{j+1}}}{P_{t_j}} - e^{r_f \tau_j}\right)\right\} = 0$.

The Euler equation in Proposition 5, which is proved in the online Appendix, resembles a standard Euler equation, but it is important to note that here the Euler equation applies *only to intervals of time that begin and end on observation dates.* This implication of the model is consistent with the evidence reported in Jagannathan and Wang (2007), where they find that the consumption Euler equation is empirically more successful on dates and at frequencies where decisions are likely to be made.

5 Concluding Remarks

Rules governing infrequent adjustment are typically categorized as time dependent or state dependent. Time-dependent rules depend only on calendar time and can optimally result from costs of gathering and processing information. State-dependent rules depend on the value of some state variable, typically reaching some trigger threshold, and can be the optimal response to a transactions cost. Our model combines costly information and costly transactions. In general, on any observation date, the consumer chooses the length of time until the next date at which to gather information and re-optimize, but that length of time may be state dependent. Moreover, conditional on the information observed at that future date, the agent's action (or lack thereof) may also be state dependent. Thus, in general, the model has elements of both state- and time-dependent rules.

[25] It is worth noting that "sufficiently risk-averse" need not require a very high value of α. For instance, if the expected equity premium is $\mu - r_f = 0.04$ and and the standard deviation of the rate of return on equity is $\sigma = 0.16$, then any value of α greater than 1.5625 will be sufficiently risk averse.

[26] Eberly (1994) shows that a version of the consumption Euler equation also holds in a model with a fixed cost of adjusting the stock of durables, by considering consumption at consecutive adjustment dates.

If the fixed component of the transactions cost is sufficiently small, the optimal behavior converges to a rule that is purely time dependent. Once the consumer arrives at an observation date with a sufficiently small balance in the transactions account, she will optimally choose to arrive at all subsequent observation dates with zero liquid assets in the transactions account. In our model, this behavior results from the facts that (1) the consumer can save on costs by synchronizing observation and transactions dates and (2) the consumer would prefer to hold as little as possible of her wealth in the liquid asset because the return on the transactions account is dominated by the return on the investment account.

The endogenous emergence of a purely time-dependent rule is a novel feature of our model. However, there are forces that could prevent this situation from arising, even within the model. As we have pointed out, if the fixed component of the transactions cost is large, the consumer may choose to arrive at observation dates with a positive balance in the transactions account. And if the consumer arrives at an observation date with a positive amount of liquid assets, then the state variable x_t could potentially take on any positive value, so that a purely time-dependent rule would not be optimal, even in the long run. Outside the model, one might consider allowing for the arrival of labor income in the transactions account or the occurrence of attention-grabbing events that occur when the consumer is not at a planned observation date.[27]

We offer a more general view of time dependence by thinking of the distribution of the length of inattention intervals. With sufficiently small transactions costs, the long run is characterized by a constant length of inattention intervals and thus the distribution is degenerate. More generally, even if the model is configured or amended so that pure time dependence does not eventually emerge, the value of x_{t_j} will frequently be below the lower trigger value. Whenever x_{t_j} is lower than the lower trigger value, the length of time until the next observation date will be the same regardless of the value of x_{t_j}. Therefore, the distribution of inattention intervals will have a mass at that length of time.[28] This mass point in the distribution of inattention intervals can be viewed as a generalization of the eventual emergence of a purely time-dependent rule that we have analyzed in this paper.

[27] Recent work by Yu (2008) has documented that investors appear to react to news that the stock market has reached a new peak.

[28] A similar argument applies to the inattention interval associated with optimal behavior for x_{t_j} above the upper trigger value.

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A Online Appendix

Proof of Lemma 1. Since $e^{-\rho\tau_i}A(t_i,\tau_i)=\kappa b(\tau_i)\int_{t_i}^{t_i+\tau_i}c_t^{1-\alpha}e^{-\rho(t-t_i)}dt$, we have

$$\lim_{\tau_i\to 0}\tau_i b(\tau_i)=\lim_{\tau_i\to 0}\frac{e^{-\rho\tau_i}A(t_i,\tau_i)}{\frac{\kappa}{\tau_i}\int_{t_i}^{t_i+\tau_i}c_t^{1-\alpha}e^{-\rho(t-t_i)}dt}. \tag{A.1}$$

Equation (9a) implies that the numerator on the right hand side of equation (A.1) has a positive finite limit as $\tau_i\to 0$. The limit of the denominator is $\lim_{\tau_i\to 0}\frac{\kappa}{\tau_i}\int_{t_i}^{t_i+\tau_i}c_t^{1-\alpha}e^{-\rho(t-t_i)}dt=c_{t_i^+}^{1-\alpha}$, which is positive and finite since we are confining attention to cases with positive (and finite) consumption. Therefore, statement 2 holds.[29] Statement 3 follows from the fact that $e^{-\rho\tau_i}A(t_i,\tau_i)=\kappa b(\tau_i)\int_{t_i}^{t_i+\tau_i}c_t^{1-\alpha}e^{-\rho(t-t_i)}dt$ and equation (9b) along with the assumptions that $\kappa>0$ and $c_t>0$.

Equation (11) and $\kappa>0$ can be used to rewrite equation (9c) as

$$\begin{aligned}&b(\tau_i)\int_{t_i}^{t_i+\tau_i}c_t^{1-\alpha}e^{-\rho(t-t_i)}dt+e^{-\rho\tau_i}b(\tau_{i+1})\int_{t_{i+1}}^{t_{i+1}+\tau_{i+1}}c_t^{1-\alpha}e^{-\rho(t-t_{i+1})}dt\\&>b(\tau_i+\tau_{i+1})\int_{t_i}^{t_i+\tau_i+\tau_{i+1}}c_t^{1-\alpha}e^{-\rho(t-t_i)}dt.\end{aligned} \tag{A.2}$$

To see the implications of equation (A.2) for $b(\tau_i)$, we first state the following lemma.

Lemma 3 *Suppose $q_1b(z_1)+q_2b(z_2)>(q_1+q_2)b(z_1+z_2)$ for all positive q_i and z_i, $i=1,2$, and that $b(z)>0$ for all $z>0$. Then $b(z)$ is non-increasing.*

Proof of Lemma 3. The assumption that $q_1b(z_1)+q_2b(z_2)>(q_1+q_2)b(z_1+z_2)$ for all positive q_i and z_i, $i=1,2$, implies that $q_1[b(z_1)-b(z_1+z_2)]+q_2[b(z_2)-b(z_1+z_2)]>0$ for all positive q_i and z_i, $i=1,2$. Suppose that, contrary to what is to be proved, for some positive z_1 and z_2, $b(z_1)<b(z_1+z_2)$. Then for any $q_1>-q_2\frac{b(z_2)-b(z_1+z_2)}{b(z_1)-b(z_1+z_2)}$, $q_1[b(z_1)-b(z_1+z_2)]+q_2[b(z_2)-b(z_1+z_2)]<0$, which is a contradiction. Therefore, $b(z_1)\geq b(z_1+z_2)$ for any positive z_1 and z_2. ■

Applying Lemma 3 to equation (A.2) while setting $z_1=\tau_i$, $z_2=\tau_{i+1}$, $q_1=\int_{t_i}^{t_i+\tau_i}c_t^{1-\alpha}e^{-\rho(t-t_i)}dt$, and $q_2{=}e^{-\rho\tau_i}\int_{t_{i+1}}^{t_{i+1}+\tau_{i+1}}c_t^{1-\alpha}e^{-\rho(t-t_i)}dt$, , implies that $b(\tau)$ is non-increasing, which is statement 1 in Lemma 1. ■

Proof of Proposition 1. We start by proving the following Lemma.

[29] Let $\gamma=\lim_{\tau\to 0}\tau b(\tau)=\lim_{\tau\to 0}\frac{\tau}{\frac{1}{b(\tau)}}$, which, by L'Hopital's Rule, implies $\gamma=\frac{1}{\lim_{\tau\to 0}-\frac{b'(\tau)}{b[(\tau)]^2}}$, or $\lim_{\tau\to 0}\frac{b'(\tau)}{b[(\tau)]^2}=-\gamma^{-1}$. Then $\lim_{\tau\to 0}\frac{\tau b'(\tau)}{b(\tau)}=\lim_{\tau\to 0}\frac{\tau b(\tau)b'(\tau)}{[b(\tau)]^2}=[\lim_{\tau\to 0}\tau b(\tau)]\left[\lim_{\tau\to 0}\frac{b'(\tau)}{[b(\tau)]^2}\right]=\gamma\left(-\gamma^{-1}\right)=-1$.

Lemma 4 *Optimal behavior requires $y^s y^b = 0$. If the optimal asset transfer increases x, then $y^s < 0$. If the optimal transfer decreases x, then $y^b > 0$.*

Proof of Lemma 4. To prove that $y^s y^b = 0$, suppose $y^s y^b \neq 0$, which implies that $y^s < 0$ and $y^b > 0$. Now consider reducing y^b by ε and increasing y^s by ε, which will have no effect on the value of S relative to the original transfer but will increase X by $(\psi_s + \psi_b)\,\varepsilon$ relative to the original transfer by reducing the amount of proportional transactions cost incurred. Therefore, it could not have been optimal for $y^s y^b \neq 0$. Hence, $y^s y^b = 0$.

The value function $V(X, S)$ is strictly increasing in X and S, so an optimal transfer will never decrease both X and S. Therefore, if the optimal transfer increases $x \equiv \frac{X}{S}$, then the optimal transfer cannot decrease X and must decrease S, which implies that $y^b = 0$ and $y^s < 0$. Similarly, if the optimal transfer decreases $x \equiv \frac{X}{S}$, then the optimal transfer cannot decrease S and must decrease X, which implies that $y^s = 0$ and $y^b > 0$. ■

Proof of statement 2a. Suppose that $x < \omega_1$. The definition of ω_1 in equation (25) implies that $v(x) \neq \widetilde{v}(x)$. The optimal asset transfer will change the value of x to some value z for which $v(z) = \widetilde{v}(z)$. The definition of ω_1 implies that such a z cannot be less than ω_1, so the optimal transfer increases x. Lemma 4 implies that $y^s < 0$. ■

Proof of statement 2b. Suppose that on an observation date, normalized to be $t = 0$, $X_0 < \omega_1 S_0$. Statement 2a implies that optimal $y^s < 0$. Let $\overline{y^s}$ be the optimal value of y^s. Therefore,

$$V\left((1-\theta_X) X_0 - (1-\psi_s)\,\overline{y^s}, (1-\theta_S) S_0 + \overline{y^s}\right) = \widetilde{V}\left((1-\theta_X) X_0 - (1-\psi_s)\,\overline{y^s}, (1-\theta_S) S_0 + \overline{y^s}\right) \tag{A.3}$$

and

$$\begin{aligned} &V\left((1-\theta_X) X_0 - (1-\psi_s)\,\overline{y^s}, (1-\theta_S) S_0 + \overline{y^s}\right) \\ &\geq V\left((1-\theta_X) X_0 - (1-\psi_s)\left(\overline{y^s} - \zeta\right), (1-\theta_S) S_0 + \overline{y^s} - \zeta\right), \end{aligned} \tag{A.4}$$

for $\zeta \in [0, (1-\theta_S) S_0 + \overline{y^s}]$.

Define y^{s*} as the value of y^s that will lead to $x_{0+} = \pi_1$. Use equations (4) and (5) with $y^b = 0$ and $y^s = y^{s*} < 0$ to obtain $X_{0+} = (1-\theta_X) X_0 - (1-\psi_s)\, y^{s*}$ and $S_{0+} = (1-\theta_S) S_0 + y^{s*}$, which can be rearranged to obtain

$$[x_{0+} + 1 - \psi_s]\,\frac{S_{0+}}{S_0} = (1-\theta_X)\, x_0 + (1-\psi_s)(1-\theta_S). \tag{A.5}$$

Set $x_{0+} = \pi_1$ in equation (A.5) and rearrange to obtain

$$y^{s*} = S_{0+} - (1-\theta_S) S_0 = \left[\frac{(1-\theta_X)\, x_0 - (1-\theta_S)\,\pi_1}{\pi_1 + 1 - \psi_s}\right] S_0.$$

From this point onward, the proof proceeds by contradiction. Assume $\overline{y^s} > y^{s*}$ so that the magnitude of the transfer $\overline{y^s}$ is smaller than the transfer needed to increase x_{0+} to π_1. Since $\overline{y^s}$ is optimal, equations (A.3) and (A.4) imply that

$$\begin{aligned}\widetilde{V}\left(\left(1-\theta_X\right)X_0-\left(1-\psi_s\right)\overline{y^s},\left(1-\theta_S\right)S_0+\overline{y^s}\right) &= V\left(\left(1-\theta_X\right)X_0-\left(1-\psi_s\right)\overline{y^s},\left(1-\theta_S\right)S_0+\overline{y^s}\right) \\ &\geq V\left(\left(1-\theta_X\right)X_0-\left(1-\psi_s\right)y^{s*},\left(1-\theta_S\right)S_0+y^{s*}\right)\end{aligned} \tag{A.6}$$

But the definition of π_1 implies that

$$\widetilde{V}\left(\left(1-\theta_X\right)X_0-\left(1-\psi_s\right)\overline{y^s},\left(1-\theta_S\right)S_0+\overline{y^s}\right) < V\left(\left(1-\theta_X\right)X_0-\left(1-\psi_s\right)y^{s*},\left(1-\theta_S\right)S_0+y^{s*}\right),$$

which contradicts equation (A.6). ■

Proof of statement 2c. Consider the point (X_0, S_0) with $x_0 \equiv \frac{X_0}{S_0} = \omega_1$ and define D as the set of (X, S) for which $x < \omega_1$ and from which the consumer can instantaneously move to (X_0, S_0) by transferring assets from the investment portfolio to the transactions account. Specifically,

$$D \equiv \left\{ \begin{array}{c} (X,S) \text{ with } X < \omega_1 S : \\ \exists y^s < 0 \text{ for which } \left(1-\theta_X\right)X - \left(1-\psi_s\right)y^s = X_0 \text{ and } \left(1-\theta_S\right)S + y^s = S_0 \end{array} \right\}. \tag{A.7}$$

Define F as the set of (X, S) for which $x \geq \omega_1$ and to which the consumer can instantaneously move from any point in D by transferring assets from the investment portfolio to the transactions account. Specifically,

$$F \equiv \left\{ \begin{array}{c} (X,S) \text{ with } X \geq \omega_1 S : \\ \exists y^s < 0 \text{ for which } X = X_0 - \left(1-\psi_s\right)y^s \text{ and } S = S_0 + y^s \geq 0 \end{array} \right\}. \tag{A.8}$$

Consider two arbitrary points (X_1, S_1) and (X_2, S_2) in set D. Since $x_1 < \omega_1$ and $x_2 < \omega_1$, the optimal value of y^s will be strictly negative starting from either point. Moreover, y^s must be large enough in absolute value so that the post-transfer value of (X, S) satisfies $x \equiv \frac{X}{S} \geq \omega_1$ because it is always optimal to transfer assets from the investment portfolio to the transactions account from any point in set D. Therefore, the post-transfer value of (X, S) will be an element of set F. Thus, regardless of whether the consumer starts from point (X_1, S_1) or (X_2, S_2), the consumer's choice of asset transfer can be described as choosing $(X^+, S^+) \in F$ to maximize the value function. Therefore, $V(X_1, S_1) = V(X_2, S_2)$, so all of the points in set D lie on the same indifference curve of $V(X, S)$. The slope of this indifference curve is $\frac{dX}{dS} = \frac{dX}{dy^s}\frac{dy^s}{dS} = -\left(1-\psi_s\right)\frac{1-\theta_S}{1-\theta_X}$, which proves statement 2c. ■

Proof of statement 2d. We have shown that if $x < \omega_1$, then $m(x) = (1-\psi_s)\frac{1-\theta_S}{1-\theta_X}$. The expression for $V\left(X_{t_j}, S_{t_j}\right)$ in equation (21) can be used to rewrite the marginal rate of substitution, $m\left(x_{t_j}\right) \equiv \frac{V_S\left(X_{t_j}, S_{t_j}\right)}{V_X\left(X_{t_j}, S_{t_j}\right)}$, as $m\left(x_{t_j}\right) = \frac{(1-\alpha)v\left(x_{t_j}\right)}{v'\left(x_{t_j}\right)} - x_{t_j}$, so that

$$\frac{(1-\alpha)\,v(x)}{v'(x)} - x = (1-\psi_s)\,\frac{1-\theta_S}{1-\theta_X}, \text{ for } 0 \leq x < \omega_1, \tag{A.9}$$

which implies

$$v(x) = \left[\frac{(1-\theta_X)\,x + (1-\theta_S)(1-\psi_s)}{(1-\theta_X)\,\omega_1 + (1-\theta_S)(1-\psi_s)}\right]^{1-\alpha} v(\omega_1)\,, \text{ for } 0 \leq x \leq \omega_1. \tag{A.10}$$

■

Proof of statement 1. We start by proving the following Lemma.

Lemma 5 *For sufficiently small* $\overline{x} > 0$, $\frac{1}{1-\alpha}\widetilde{v}(x) < \frac{1}{1-\alpha}v(x)$ *for all* $x \in (0, \overline{x})$.

Proof of Lemma 5. Substitute the expression for $U(C(t_j, \tau_j))$ from equation (16) into the restricted value function in equation (23) to obtain

$$\begin{aligned}\widetilde{V}\left(X_{t_j}, S_{t_j}\right) = &\max_{C(t_j,\tau_j),\phi_j,\tau_j} \left[1-(1-\alpha)\,\kappa b(\tau_j)\right]\frac{1}{1-\alpha}\left[h(\tau_j)\right]^{\alpha}\left[C(t_j,\tau_j)\right]^{1-\alpha} \\ &+ e^{-\rho\tau_j}E_{t_j}\left\{V\left(e^{r_L\tau_j}\left(X_{t_j} - C(t_j,\tau_j)\right), R(t_j,\tau_j)\,S_{t_j}\right)\right\}.\end{aligned} \tag{A.11}$$

Equation (**) in footnote 17 states that $C(t_j,\tau_j) = h(\tau_j)\,c_{t_j^+}$, so that

$$\left[1-(1-\alpha)\,\kappa b(\tau_j)\right]\frac{1}{1-\alpha}\left[h(\tau_j)\right]^{\alpha}\left[C(t_j,\tau_j)\right]^{1-\alpha} = \frac{1}{1-\alpha}\left[1-(1-\alpha)\,\kappa b(\tau_j)\right]h(\tau_j)\,c_{t_j^+}^{1-\alpha}. \tag{A.12}$$

Substitute equation (A.12) into equation (A.11) to obtain

$$\widetilde{V}\left(X_{t_j}, S_{t_j}\right) = \max_{C(t_j,\tau_j),\phi_j,\tau_j} \frac{1}{1-\alpha}\left[1-(1-\alpha)\,\kappa b(\tau_j)\right]h(\tau_j)\,c_{t_j^+}^{1-\alpha} + e^{-\rho\tau_j}E_{t_j}\left\{V\left(e^{r_L\tau_j}\left(X_{t_j} - C(t_j,\tau_j)\right), R(t_j,\tau_j)\,S_{t_j}\right)\right\}. \tag{A.13}$$

Because the choice of $C(t_j,\tau_j)$ must satisfy the constraint $X_{t_j} - C(t_j,\tau_j) \geq 0$, the partial derivative with respect to $C(t_j,\tau_j)$ of the maximand on the right hand side of (A.11) must be non-negative. Therefore, differentiation of this maximand with respect to $C(t_j,\tau_j)$ yields

$$\left[1-(1-\alpha)\,\kappa b(\tau_j)\right]\left[h(\tau_j)\right]^{\alpha}\left[C(t_j,\tau_j)\right]^{-\alpha} - e^{-(\rho-r_L)\tau_j}E_{t_j}\left\{V_X\left(e^{r_L\tau_j}\left(X_{t_j} - C(t_j,\tau_j)\right), R(t_j,\tau_j)\,S_{t_j}\right)\right\} \geq 0. \tag{A.14}$$

Since $V_X() > 0$, $\left[h(\tau_j)\right]^{\alpha}\left[C(t_j,\tau_j)\right]^{-\alpha} > 0$, and $e^{-(\rho-r_L)\tau_j} > 0$, equation (A.14) implies that

$$1-(1-\alpha)\,\kappa b(\tau_j) > 0, \tag{A.15}$$

where τ_j^* is the value of τ_j that maximizes the restricted value function. Equation (A.15) implies that we can confine attention to value of τ_j that are greater than $\overline{\tau} \equiv \inf\{\tau > 0 : \kappa(1-\alpha)\,b(\tau) < 1\}$.

If $\alpha > 1$, then $1 - \kappa(1-\alpha) b(\tau_j) > 0$ for any positive value of τ_j so $\overline{\tau} = 0$. However, if $\alpha < 1$, Lemma 3 implies $\overline{\tau} > 0$.

Now we consider the cases in which $\alpha < 1$ and $\alpha > 1$ separately.

Case I: $\alpha < 1$. When $\alpha < 1$, $\tau^* > \overline{\tau} > 0$. Since $C(t_j, \tau_j) = h(\tau_j) c_{t_j^+}$,

$$c_{t_j^+} = \frac{C\left(t_j, \tau_j^*\right)}{h\left(\tau_j^*\right)} < \frac{X_{t_j}}{h(\overline{\tau})}, \tag{A.16}$$

where the inequality follows from the constraint $C\left(t_j, \tau_j^*\right) \leq X_{t_j}$ and the facts that $h(\tau_j)$ is strictly increasing in τ_j and $\tau_j^* > \overline{\tau}$. Equation (A.16) implies $\lim_{X_{t_j} \to 0} c_{t_j^+} = 0$. Therefore, taking the limits of both sides of equation (A.13) as $X_{t_j} \to 0$, and using the facts that $0 \leq C\left(t_j, \tau_j^*\right) \leq X_{t_j}$ and $\tau_j^* > \overline{\tau} > 0$ implies

$$\lim_{X_{t_j} \to 0} \widetilde{V}\left(X_{t_j}, S_{t_j}\right) = \lim_{X_{t_j} \to 0} e^{-\rho \tau_j} E_{t_j}\left\{V\left(0, R(t_j,\tau_j) S_{t_j}\right)\right\} = \lim_{X_{t_j} \to 0} e^{-\rho \tau_j^*} E_{t_j}\left\{\left[R\left(t_j,\tau_j^*\right)\right]^{1-\alpha}\right\} \frac{1}{1-\alpha} S_{t_j}^{1-\alpha} v(0) \tag{A.17}$$

Use equation (7) and the fact that $\tau^* > \overline{\tau}$ to obtain

$$\lim_{X_{t_j} \to 0} \widetilde{V}\left(X_{t_j}, S_{t_j}\right) < \frac{1}{1-\alpha} S_{t_j}^{1-\alpha} v(0) = V\left(0, S_{t_j}\right). \tag{A.18}$$

Case II: $\alpha > 1$. In the case with $\alpha > 1$, $c_{t_j^+}$ does not go to zero as X_{t_j} approaches 0, because the instantaneous flow of utility would be unboundedly negative. Thus, $\overline{c} \equiv \lim_{X_{t_j} \to 0} c_{t_j^+} > 0$. Since $X_{t_j} \geq C(t_j, \tau_j) = h(\tau_j) c_{t_j^+}$ and $\lim_{\tau_j \to 0} h(\tau_j) = 0$, $\lim_{X_{t_j} \to 0} \tau_j = 0$ (which is consistent with equation (A.15) because $\overline{\tau} = 0$ when $\alpha > 1$). We now that show that $\overline{c} < \infty$, i.e., the consumption flow $c_{t_j^+}$ approaches a finite limit. To see this, define $\Psi(\tau_j)$ as

$$\Psi(\tau_j) \equiv \frac{h'(\tau_j)}{h(\tau_j)} \left(-\frac{(1-\alpha)\kappa b(\tau_j)}{1-(1-\alpha)\kappa b(\tau_j)} \frac{b'(\tau_j) h(\tau_j)}{b(\tau_j) h'(\tau_j)} + \alpha\right), \tag{A.19}$$

and note that the first-order condition of (A.11) with respect to τ_j can be expressed as[30]

$$\Psi(\tau_j) \frac{1}{1-\alpha} \left[1-(1-\alpha)\kappa b(\tau_j)\right] \left[h(\tau_j)\right]^{\alpha} \left[C(t_j, \tau_j)\right]^{1-\alpha} = -\frac{d}{d\tau_j}\left[e^{-\rho\tau_j} E_{t_j}\left\{V\left(e^{r_L \tau_j}\left(X_{t_j} - C(t_j,\tau_j)\right), R(t_j,\tau_j) S_{t_j}\right)\right\}\right]. \tag{A.20}$$

[30]Let Δ be the partial derivative of the first term on the right hand side of equation (A.11) with respect to τ_j, holding $C(t_j, \tau_j)$ fixed. Therefore,

$$\begin{aligned} \Delta = & \frac{h'(\tau_j)}{h(\tau_j)} \left(-\frac{(1-\alpha)\kappa b(\tau_j)}{\left[1-(1-\alpha)\kappa b(\tau_j)\right]} \frac{b'(\tau_j) h(\tau_j)}{b(\tau_j) h'(\tau_j)} + \alpha\right) \\ & \times \frac{1}{1-\alpha} \left[1-(1-\alpha)\kappa b(\tau_j)\right] \left[h(\tau_j)\right]^{\alpha} \left[C(t_j, \tau_j)\right]^{1-\alpha} \end{aligned}$$

As $X_{t_j} \to 0$, equation (21) implies that the right hand side of (A.20) approaches $-\frac{1}{1-\alpha}v(0)S_{t_j}^{1-\alpha}\frac{d}{d\tau_j}\left[e^{-\rho\tau_j}E_{t_j}\left\{\left[R\left(t_j,\tau_j\right)\right]^{1-\alpha}\right\}\right]$, which is finite. Hence, the left hand side of (A.20) must also approach a finite limit. Now suppose (counterfactually) that $\lim_{X_{t_j}\to 0} c_{t_j^+} = \infty$, and re-write the left hand side of (A.20) as $\left\{h\left(\tau_j\right)\Psi\left(\tau_j\right)\frac{1}{1-\alpha}\right\} \times \left\{\left[1-(1-\alpha)\kappa b\left(\tau_j\right)\right]\left[\frac{C(t_j,\tau_j)}{h(\tau_j)}\right]^{-\alpha}\right\} \times \left(\frac{C(t_j,\tau_j)}{h(\tau_j)}\right)$. By equations (A.19) and (17), $\lim_{X_{t_j}\to 0}\left\{h\left(\tau_j\right)\Psi\left(\tau_j\right)\frac{1}{1-\alpha}\right\} = \lim_{\tau_j\to 0}\left\{h\left(\tau_j\right)\Psi\left(\tau_j\right)\frac{1}{1-\alpha}\right\}$ $= \lim_{\tau_j\to 0} h'\left(\tau_j\right)\left(-\frac{(1-\alpha)\kappa b(\tau_j)}{[1-(1-\alpha)\kappa b(\tau_j)]}\frac{b'(\tau_j)h(\tau_j)}{b(\tau_j)h'(\tau_j)}+\alpha\right)\frac{1}{1-\alpha}$. Use the facts that $\lim_{\tau\to 0} b(\tau) = \infty$ and $\lim_{\tau\to 0}\frac{\tau b'(\tau)}{b(\tau)} = -1$ (from statement 2 of Lemma 1) along with $\lim_{\tau\to 0} h'(\tau) = 1$ and $\lim_{\tau\to 0}\frac{h(\tau)}{\tau h'(\tau)} = 1$ (from the definition of $h(\tau)$ in equation (17)) to obtain $\lim_{X_{t_j}\to 0}\left\{h\left(\tau_j\right)\Psi\left(\tau_j\right)\frac{1}{1-\alpha}\right\} = -1$. Equation (A.14) implies that $\lim_{X_{t_j}\to 0}\left\{\left[1-(1-\alpha)\kappa b\left(\tau_j\right)\right]\left[\frac{C(t_j,\tau_j)}{h(\tau_j)}\right]^{-\alpha}\right\} > 0$, and the (counterfactual) assumption that $\lim_{X_{t_j}\to 0} c_{t_j^+} = \infty$ implies that $\lim_{X_{t_j}\to 0}\frac{C(t_j,\tau_j)}{h(\tau_j)} = \infty$ by since $C\left(t_j,\tau_j\right) = h\left(\tau_j\right)c_{t_j^+}$. Hence, the left hand side of (A.20) approaches $-\infty$, so that equation (A.20) cannot hold. Therefore, $\lim_{X_{t_j}\to 0} c_{t_j^+} = \overline{c} < \infty$.

Now, taking the limits of both sides of equation (A.13) as $X_{t_j} \to 0$, and using the facts[31] that $0 \le C\left(t_j,\tau_j\right) \le X_{t_j}$, $\lim_{\tau_j\to 0} R\left(t_j,\tau_j\right) = 1$, and $\lim_{\tau_j\to 0} b\left(\tau_j\right)h\left(\tau_j\right) = \lim_{\tau_j\to 0}\tau_j b\left(\tau_j\right)$ implies

$$\lim_{X_{t_j}\to 0}\widetilde{V}\left(X_{t_j},S_{t_j}\right) = -\left(\lim_{\tau_j\to 0}\tau_j b\left(\tau_j\right)\right)\overline{c}^{1-\alpha} + V\left(0,S_{t_j}\right) < V\left(0,S_{t_j}\right),$$

where the inequality follows from the facts that $\lim_{\tau_j\to 0}\tau_j b\left(\tau_j\right) > 0$ (from statement 2 in Lemma 1) and $\overline{c} > 0$. ■

Proof of $\boldsymbol{\omega}_1 > 0$. Since Lemma 5 implies that $\lim_{x_{t_j}\to 0}\frac{1}{1-\alpha}\widetilde{v}\left(x_{t_j}\right) < \frac{1}{1-\alpha}v(0)$, $\exists\ \overline{x} > 0$ s.t. $\frac{1}{1-\alpha}\widetilde{v}(x) < \frac{1}{1-\alpha}v(0) \le \frac{1}{1-\alpha}v(x)\ \forall x \in [0,\overline{x}]$. Therefore, $\omega_1 \ge \overline{x} > 0$. ■

Proof of $\boldsymbol{\pi}_2 \ge \boldsymbol{\pi}_1$. To prove that $\pi_2 \ge \pi_1$, suppose the contrary, i.e., that $\pi_1 > \pi_2$, and consider three points (X_A,S_A), (X_B,S_B), and (X_C,S_C), where $X_A = \pi_1 S_A$, $(X_B,S_B) = (\pi_1 S_A - (1-\psi_s)z^*, S_A + z^*)$ where $z^* \equiv \frac{\pi_1-\pi_2}{\pi_2+1-\psi_s}S_A$, which implies $X_B = \pi_2 S_B$, $(X_C,S_C) = (\pi_2 S_B + (1+\psi_b)z^{**}, S_B - z^{**})$ where $z^{**} \equiv \frac{\pi_1-\pi_2}{\pi_1+1+\psi_b}S_B$, which implies $X_C = \pi_1 S_C$. The definition of π_1 implies that $V(X_A,S_A) \ge V(X_B,S_B)$ and the definition of π_2 implies that $V(X_B,S_B) \ge$

Now use the definition of $\Psi\left(\tau_j\right)$ in equation (A.19) to obtain

$$\Delta = \Psi\left(\tau_j\right)\frac{1}{1-\alpha}\left[1-(1-\alpha)\kappa b\left(\tau_j\right)\right]\left[h\left(\tau_j\right)\right]^{\alpha}\left[C\left(t_j,\tau_j\right)\right]^{1-\alpha}.$$

[31] $\lim_{\tau_j\to 0} b\left(\tau_j\right)h\left(\tau_j\right) = \left(\lim_{\tau_j\to 0}\tau_j b\left(\tau_j\right)\right)\left(\lim_{\tau_j\to 0}\frac{h(\tau_j)}{\tau_j h'(\tau_j)}\right)\lim_{\tau_j\to 0} h'\left(\tau_j\right)$. Since $h\left(\tau_j\right) = \frac{1-e^{-\chi\tau_j}}{\chi}$, $h'\left(\tau_j\right) = e^{-\chi\tau_j}$. Therefore, $\lim_{\tau_j\to 0}\frac{h(\tau_j)}{\tau_j h'(\tau_j)} = \frac{1-e^{-\chi\tau_j}}{\chi\tau_j e^{-\chi\tau_j}}$ and L'Hopital's Rule implies $\lim_{\tau_j\to 0}\frac{h(\tau_j)}{\tau_j h'(\tau_j)} = 1$. Also $\lim_{\tau_j\to 0} h'\left(\tau_j\right) = 1$. Therefore, $\lim_{\tau_j\to 0} b\left(\tau_j\right)h\left(\tau_j\right) = \lim_{\tau_j\to 0}\tau_j b\left(\tau_j\right)$.]

$V(X_C, S_C)$ so that $V(X_A, S_A) \geq V(X_C, S_C)$. But $S_C = S_B - z^{**} = S_B - \frac{\pi_1 - \pi_2}{\pi_1 + 1 + \psi_b} S_B = \frac{\pi_2 + 1 + \psi_b}{\pi_1 + 1 + \psi_b} S_B$ $= \frac{\pi_2 + 1 + \psi_b}{\pi_1 + 1 + \psi_b} \frac{\pi_1 + 1 - \psi_s}{\pi_2 + 1 - \psi_s} S_A = \left(\frac{(\pi_1 - \pi_2)(\psi_s + \psi_b)}{(\pi_1 + 1 + \psi_b)(\pi_2 + 1 - \psi_s)} + 1 \right) S_A > S_A$, since $\psi_s + \psi_b > 0$. Therefore, since $X_C = \pi_1 S_C$ and $X_A = \pi_1 S_A$, we have $X_C > X_A$. Hence, since $V(X, S)$ is strictly increasing in X and S, we have $V(X_C, S_C) > V(X_A, S_A)$, which contradicts the earlier statement that $V(X_A, S_A) \geq V(X_C, S_C)$. ∎

Proof of $\boldsymbol{\omega_1 \leq \pi_1}$. We will prove this statement using a geometric argument to show that $\omega_1 > \pi_1$ leads to a contradiction. We consider three cases: $\theta_S < \theta_X$, $\theta_S > \theta_X$, and $\theta_S = \theta_X$.

Suppose that $\omega_1 > \pi_1$ and consider the case in which $\theta_S < \theta_X$, so that in Figure 2(a) the line through points B, C, and E, which has slope $-(1 - \psi_s) \frac{1 - \theta_S}{1 - \theta_X}$, is steeper than the line through points C and D, which has slope $-(1 - \psi_s)$. Statement 2c of Proposition 1 implies that for values of $x \equiv \frac{X}{S}$ less than ω_1, indifference curves of the value function are straight lines with slope $-(1 - \psi_s) \frac{1 - \theta_S}{1 - \theta_X}$. Therefore, $V(B) = V(C) = V(E)$, where the notation $V(J)$ indicates the value of the value function evaluated at point J. The definition of π_1 implies that $V(C) \geq V(D)$. Therefore, $V(E) \geq V(D)$, which contradicts strict monotonicity of the value function since both X and S are larger at point D than at point E. Therefore, $\omega_1 \leq \pi_1$ if $\theta_S < \theta_X$.

Suppose that $\omega_1 > \pi_1$ and consider the case in which $\theta_S > \theta_X$, so that in Figure 2(b) the line through points D and E, which has slope $-(1 - \psi_s) \frac{1 - \theta_S}{1 - \theta_X}$, is less steep than the line through points C and E, which has slope $-(1 - \psi_s)$. Statement 2c of Proposition 1 implies that the line from point D through point E is an indifference curve and all points on this indifference curve are preferred to all points below and to the left of the indifference curve for which $x < \omega_1$. In particular, point E is preferred to all points below point E along the line through points E and C. Since the value of x at point E is higher than π_1, the fact that the value function evaluated at point E is greater than the value function, and hence greater than the restricted value function, evaluated at all points below point E with slope $-(1 - \psi_s)$ contradicts the definition of π_1. Therefore, $\omega_1 \leq \pi_1$ if $\theta_S > \theta_X$.

Suppose that $\omega_1 > \pi_1$ and consider the case in which $\theta_S = \theta_X$, so that in Figure 2(c) the slope of the line through points C and E is $-(1 - \psi_s) \frac{1 - \theta_S}{1 - \theta_X} = -(1 - \psi_s)$. Statement 2c of Proposition 1 implies that for values of $x \equiv \frac{X}{S} < \omega_1$, indifference curves of the value function are straight lines with slope $-(1 - \psi_s) \frac{1 - \theta_S}{1 - \theta_X}$ so points E and C are on the same indifference curve. Indeed, point E yields the same value of the value function as all points below point E on the line through points E and C. That is, for any point J below point E along the line through points E and C with $X \geq 0$, $V(E) = V(J)$. Since $x < \omega_1$ at point J, the definition of ω_1 implies that $V(J) > \widetilde{V}(J)$. Therefore, $V(E) = V(J) > \widetilde{V}(J)$. Since $x > \pi_1$ at point E, the facts that for arbitrary point J we have $V(E) = V(J)$ and $V(E) > \widetilde{V}(J)$ contradict the definition of π_1. Therefore, $\omega_1 \leq \pi_1$ if

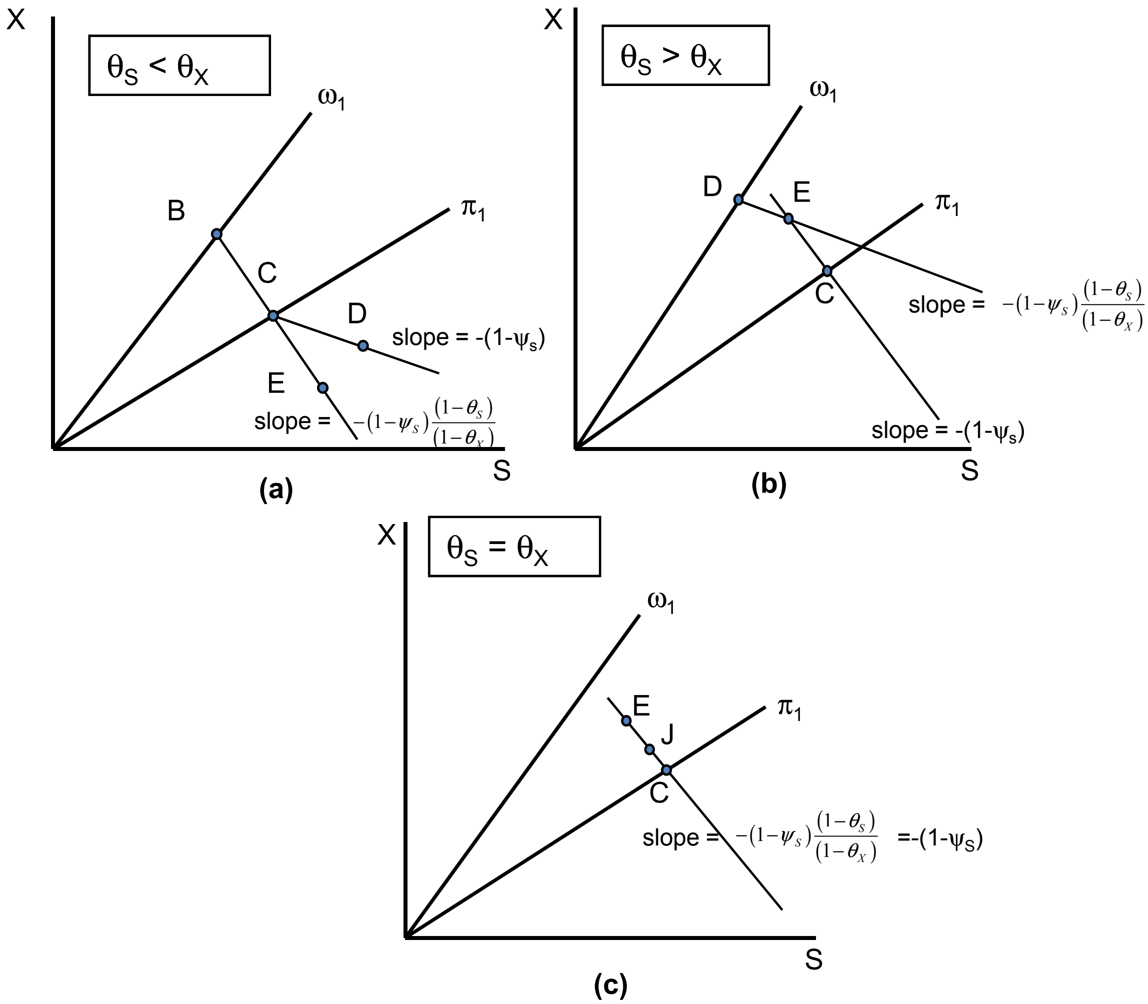

FIGURE 2: Proof of $\omega_1 \leq \pi_1$

$\theta_S = \theta_X$.

Putting together the cases in which $\theta_S < \theta_X$, $\theta_S > \theta_X$, and $\theta_S = \theta_X$, we have proved that $\omega_1 \leq \pi_1$. ■

Proof of $\omega_2 \geq \pi_2$**.** Use a set of arguments similar to the proof that $\omega_1 \leq \pi_1$. ■

Proof of $\omega_2 < \infty$**.** We will prove that ω_2 is finite by showing that if the investment portfolio has zero value on an observation date, the consumer will use some of the liquid assets in the transactions account to buy assets for the investment portfolio. We use proof by contradiction. That is, suppose that time 0 is an observation date, and that at this observation date, the transactions account has a balance $X_0 > 0$ and the investment portfolio has a zero balance so that $S_0 = 0$ and x_0 is infinite. Suppose that whenever the investment portfolio has zero value on an observation date, the consumer does not transfer any assets to the investment portfolio. Then the consumer will simply consume from the transactions account over the infinite future, never incurring any observation costs or transactions costs. In this case, with the values of the variables denoted with asterisks, $c_{0^+}^* = \frac{X_0}{h(\infty)} = \chi X_0$, $c_t^* = \exp\left(-\frac{\rho - r_L}{\alpha} t\right) c_{0^+}^* = \chi X_t^*$, so $X_t^* = \exp\left(-\frac{\rho - r_L}{\alpha} t\right) X_0$. Equation (16) implies that lifetime utility is

$$U^* = \frac{1}{1-\alpha} \left[h(\infty)\right]^\alpha X_0^{1-\alpha} = \frac{1}{1-\alpha} \chi^{-\alpha} X_0^{1-\alpha}. \tag{A.21}$$

Now consider an alternative feasible path that sets $c_t = c_t^*$ for $0 < t \leq T$ and at time 0^+ transfers to the investment portfolio any liquid assets in the transactions account that will not be needed to finance consumption until time T. Under this alternative policy, the present value of consumption up to date T is $h(T) c_{0^+}^* = h(T) \chi X_0$, so

$$X_{0^+} = h(T) \chi X_0. \tag{A.22}$$

The consumer uses $(1 - \theta_X - \chi h(T)) X_0$ liquid assets to purchase assets in the investment portfolio. After paying the transactions cost,

$$S_{0^+} = \frac{1 - \theta_X - \chi h(T)}{1 + \psi_b} X_0. \tag{A.23}$$

Suppose that the consumer invests the investment portfolio entirely in the riskless bond. At time T, the transactions account has a zero balance, and the investment portfolio is worth $S_T = \exp(r_f T) \frac{1 - \theta_X - \chi h(T)}{1 + \psi_b} X_0$. The consumer transfers the entire investment portfolio to the transactions account, so that after paying the transactions costs, the balance in the transactions account is

$$X_{T^+} = (1 - \theta_S) \frac{1 - \psi_s}{1 + \psi_b} \exp(r_f T) \left[1 - \theta_X - \chi h(T)\right] X_0. \tag{A.24}$$

Define $P \equiv \frac{X_{T^+}}{X_T^*}$ as the ratio of the transactions account balance at time T^+ under this alternative policy to the transactions account balance under the initial policy. Use equation (A.24) and $X_T^* = \exp\left(-\frac{\rho - r_L}{\alpha} T\right) X_0$, along with $\chi \equiv \frac{\rho - (1-\alpha) r_L}{\alpha}$, to obtain

$$P \equiv \frac{X_{T^+}}{X_T^*} = (1 - \theta_S) \frac{1 - \psi_s}{1 + \psi_b} F(T), \tag{A.25}$$

where

$$F(T) \equiv \exp\left[(r_f - r_L) T\right] \left[1 - \theta_X \exp(\chi T)\right]. \tag{A.26}$$

Equation (A.25) and $X_T^* = \exp\left(-\frac{\rho - r_L}{\alpha} T\right) X_0$ implies

$$X_{T^+} = (1 - \theta_S) \frac{1 - \psi_s}{1 + \psi_b} F(T) \exp\left(-\frac{\rho - r_L}{\alpha} T\right) X_0 \tag{A.27}$$

Now choose T to maximize $F(T)$. Differentiate $F(T)$ and set the derivative equal to zero to obtain

$$\exp\left(-\chi\widehat{T}\right) = \left(1+\frac{\chi}{r_f - r_L}\right)\theta_X < 1, \tag{A.28}$$

where $\widehat{T}$ is the optimal value of T and the inequality follows from the assumption that $\theta_X < \overline{\theta_X}$ and the fact that $\frac{\chi}{r_f - r_L} > 0$.[32] Use equation (A.28) to evaluate $F\left(\widehat{T}\right)$ to obtain

$$F\left(\widehat{T}\right) = \left(1+\frac{\chi}{r_f - r_L}\right)^{-1-\frac{r_f - r_L}{\chi}} \frac{\chi}{r_f - r_L}\theta_X^{-\frac{r_f - r_L}{\chi}}. \tag{A.29}$$

Use equation (A.28) and the definition of $h(T)$ to obtain

$$\chi h\left(\widehat{T}\right) = 1 - \left(1+\frac{\chi}{r_f - r_L}\right)\theta_X. \tag{A.30}$$

The present value of lifetime utility under the alternative plan is

$$U = \left[1-(1-\alpha)\kappa b\left(\widehat{T}\right)\right]\frac{1}{1-\alpha}\left[h\left(\widehat{T}\right)\right]^{\alpha}\left[X_{0+}\right]^{1-\alpha} + \exp\left(-\rho\widehat{T}\right)\frac{1}{1-\alpha}\left[h(\infty)\right]^{\alpha}\left[X_{\widehat{T}+}\right]^{1-\alpha}. \tag{A.31}$$

Substitute equations (A.22) and (A.27) into equation (A.31) and use the fact that $h(\infty) = \frac{1}{\chi}$ to obtain

$$\begin{aligned} U = {} & \left[1-(1-\alpha)\kappa b\left(\widehat{T}\right)\right]\frac{1}{1-\alpha}h\left(\widehat{T}\right)\left[\chi X_0\right]^{1-\alpha} \\ & + \exp\left(-\rho\widehat{T}\right)\frac{1}{1-\alpha}\chi^{-\alpha}\left[(1-\theta_S)\frac{1-\psi_s}{1+\psi_b}F\left(\widehat{T}\right)\exp\left(-\frac{\rho - r_L}{\alpha}\widehat{T}\right)X_0\right]^{1-\alpha}. \end{aligned} \tag{A.32}$$

Now divide the utility under the alternative plan in equation (A.32) by utility under the initial plan in equation (A.21) and use the definition of χ and the fact that $\chi h(T) = 1 - \exp(-\chi T)$ to obtain

$$\frac{U}{U^*} = \left[1-(1-\alpha)\kappa b\left(\widehat{T}\right)\right]\left[1-\exp\left(-\chi\widehat{T}\right)\right] + \exp\left(-\chi\widehat{T}\right)\left[(1-\theta_S)\frac{1-\psi_s}{1+\psi_b}F\left(\widehat{T}\right)\right]^{1-\alpha}, \tag{A.33}$$

and then rearrange to obtain

$$\frac{U}{U^*} = 1 + \left(\left[(1-\theta_S)\frac{1-\psi_s}{1+\psi_b}F\left(\widehat{T}\right)\right]^{1-\alpha} - \left[1+(1-\alpha)\kappa b\left(\widehat{T}\right)\left(\exp\left(\chi\widehat{T}\right)-1\right)\right]\right)\exp\left(-\chi\widehat{T}\right). \tag{A.34}$$

[32] From equation (27), $\overline{\theta_X} \equiv \left[(1-\theta_S)\frac{1-\psi_s}{1+\psi_b}\frac{\chi}{r_f - r_L + \chi}\right]^{\frac{\chi}{r_f - r_L}}\frac{r_f - r_L}{r_f - r_L + \chi}$, which implies $\left(1+\frac{\chi}{r_f - r_L}\right)\overline{\theta_X} = \left[(1-\theta_S)\frac{1-\psi_s}{1+\psi_b}\frac{\chi}{r_f - r_L + \chi}\right]^{\frac{\chi}{r_f - r_L}} < 1$ because $(1-\theta_S)\frac{1-\psi_s}{1+\psi_b} < 1$, $\frac{\chi}{r_f - r_L} > 0$, and hence $\frac{\chi}{r_f - r_L + \chi} < 1$.

If $\alpha < 1$, utility under the alternative plan, U, will exceed U^* if $\frac{U}{U^*} > 1$; if $\alpha > 1$, utility under the alternative plan, U, will exceed U^* if $\frac{U}{U^*} < 1$. A sufficient condition for U to exceed U^*, regardless of whether α is less than or greater than one, is[33]

$$\left[(1-\theta_S)\frac{1-\psi_s}{1+\psi_b}\right] F\left(\widehat{T}\right) > \left[1+(1-\alpha)\,\kappa b\left(\widehat{T}\right)\left(\exp\left(\chi\widehat{T}\right)-1\right)\right]^{\frac{1}{1-\alpha}}. \tag{A.35}$$

Multiply both sides of equation (A.29) by $(1-\theta_S)\frac{1-\psi_s}{1+\psi_b}$ to obtain

$$\left[(1-\theta_S)\frac{1-\psi_s}{1+\psi_b}\right] F\left(\widehat{T}\right) = \left[(1-\theta_S)\frac{1-\psi_s}{1+\psi_b}\frac{\chi}{r_f-r_L+\chi}\right]\left(\frac{r_f-r_L}{r_f-r_L+\chi}\right)^{\frac{r_f-r_L}{\chi}}\theta_X^{-\frac{r_f-r_L}{\chi}} \tag{A.36}$$

Use the definition of $\overline{\theta_X}$ in equation (27) and the assumption that $\theta_X < \overline{\theta_X}$ to write equation (A.36) as

$$\left[(1-\theta_S)\frac{1-\psi_s}{1+\psi_b}\right] F\left(\widehat{T}\right) = \left(\frac{\theta_X}{\overline{\theta_X}}\right)^{-\frac{r_f-r_L}{\chi}} > 1. \tag{A.37}$$

Substitute equation (A.37) into equation (A.35) to obtain the following sufficient condition for U to exceed U^*

$$\left(\frac{\theta_X}{\overline{\theta_X}}\right)^{-\frac{r_f-r_L}{\chi}} > \left[1+(1-\alpha)\,\kappa b\left(\widehat{T}\right)\left(\exp\left(\chi\widehat{T}\right)-1\right)\right]^{\frac{1}{1-\alpha}} \tag{A.38}$$

Regardless of whether α is larger or smaller than one, the condition in equation (A.38) is satisfied if $\theta_X < \overline{\theta_X}$ and $\kappa < \overline{\kappa}$, where

$$\overline{\kappa} \equiv \frac{\left(\frac{\theta_X}{\overline{\theta_X}}\right)^{-\frac{r_f-r_L}{\chi}(1-\alpha)} - 1}{(1-\alpha)\,b\left(\widehat{T}\right)\left(\exp\left(\chi\widehat{T}\right)-1\right)}. \tag{A.39}$$

Since $\theta_X < \overline{\theta_X}$ and $\kappa < \overline{\kappa}$, the original plan, in which the consumer does not buy any assets in the investment portfolio, is not optimal. ■

The proof of statement 1 is now complete. ■

Proof of statement 3. Suppose that on observation date t_j the consumer has $\left(X_{t_j}, S_{t_j}\right) = \left(\omega_1 S_{t_j}, S_{t_j}\right)$ so that $x_{t_j} = \omega_1$. The proof of statement 2b implies that if the consumer sells assets from the investment portfolio, he will choose $\left(X_{t_j^+}, S_{t_j^+}\right) = \left(\pi_1 S_{t_j^+}, S_{t_j^+}\right)$ so that $x_{t_j^+} = \pi_1 \geq \omega_1$.

[33]If $\alpha > 1$, then κ must be less than $\widehat{\kappa} \equiv \frac{1}{\alpha-1}\frac{1}{b\left(\widehat{T}\right)\left(\exp\left(\chi\widehat{T}\right)-1\right)}$ so that the right hand side of equation (A.35) is defined. Since we assume that $\kappa < \overline{\kappa}$ in equation (28) and $\widehat{\kappa} = \left[1-\left(\frac{\theta_X}{\overline{\theta_X}}\right)^{-\frac{r_f-r_L}{\chi}(1-\alpha)}\right]^{-1}\overline{\kappa} > \overline{\kappa}$, we have $\kappa < \widehat{\kappa}$.

Applying equation (A.5) at observation date t_j rather than time 0, and setting $x_0 = \omega_1$ and $x_{0^+} = \pi_1$ implies that

$$\frac{S_{t_j^+}}{S_{t_j}} = \frac{(1-\theta_X)\,\omega_1 + (1-\theta_S)\,(1-\psi_s)}{\pi_1 + 1 - \psi_s}. \tag{A.40}$$

The value-matching condition states that the consumer is indifferent between the initial allocation with $x_{t_j} = \omega_1$ and the new allocation with $x_{t_j^+} = \pi_1$ so that

$$V\left(\omega_1 S_{t_j}, S_{t_j}\right) = V\left(\pi_1 S_{t_j^+}, S_{t_j^+}\right). \tag{A.41}$$

Use equation (21), which is based on the homogeneity of the value function, to obtain

$$S_{t_j}^{1-\alpha} v\left(\omega_1\right) = S_{t_j^+}^{1-\alpha} v\left(\pi_1\right). \tag{A.42}$$

Divide both sides of equation (A.42) by $S_{t_j}^{1-\alpha}$ and use equation (A.40) to obtain

$$v\left(\omega_1\right) = \left[\frac{(1-\theta_X)\,\omega_1 + (1-\theta_S)\,(1-\psi_s)}{\pi_1 + 1 - \psi_s}\right]^{1-\alpha} v\left(\pi_1\right). \tag{A.43}$$

Observe from equation (A.43) that if $\theta_X = \theta_S = 0$, then $\omega_1 = \pi_1$. In this case, π_1 is both a trigger value and a return value. That is, if $x_{t_j} < \pi_1$ on observation date t_j, the consumer will sell assets from the investment portfolio to make $x_{t_j^+} = \pi_1$. ■

Proof of statements 4 and 5. The proof of statement 4 follows the proof of statement 2, and the proof of statement 5 follows the proof of statement 3. ■

The proof of Proposition 1 is now complete. ■

To prepare for the proof of Proposition 2, we state and prove the following Lemma.

Lemma 6 *If $C\left(t_j, \tau_t\right) \leq X_{t_j}$, then, for sufficiently small $\theta_S \geq 0$, $y^s\left(t_j\right) = 0$.*

Proof of Lemma 6. Consider some path for X_t, S_t, $y^s(t)$, and $y^b(t)$, $t \in [t_j, t_{j+1}]$, and let X_t^0, S_t^0, $y^{s,0}(t)$, and $y^{b,0}(t)$ denote the values of these variables along this path. Suppose that $C\left(t_j, \tau_t\right) \leq X_{t_j}^0$ and (contrary to what is to be proved) that $y^{s,0}\left(t_j\right) < 0$. Consider a deviation from $y^{s,0}\left(t_j\right) < 0$ that reduces $-y^s\left(t_j\right)$ to zero so that $X_{t_j^+}$ falls by $-y^{s,0}\left(t_j\right)\left(1-\psi_s\right) - \theta_X X_{t_j}^0$ and $S_{t_j^+}$ increases by $-y^{s,0}\left(t_j\right) + \theta_S S_{t_j}^0$. Suppose that the consumer invests the additional assets in the investment portfolio in the riskless bond, which pays a rate of return r_f. Thus, at the next observation date t_{j+1}, the transaction account will have fallen by $\left[-y^{s,0}\left(t_j\right)\left(1-\psi_s\right) - \theta_X X_{t_j}^0\right] e^{r_L \tau_j}$ and the investment portfolio will have increased by $\left[-y^{s,0}\left(t_j\right) + \theta_S S_{t_j}^0\right] e^{r_f \tau_j}$, relative to the original path. The deviation at time t_{j+1} depends on the direction of the transfer along the original path at time t_{j+1}. (1) If $y^{s,0}\left(t_{j+1}\right) < 0$, increase $-y^s\left(t_{j+1}\right)$ by $\left(1-\theta_S\right)\left[-y^{s,0}\left(t_j\right) + \theta_S S_{t_j}^0\right] e^{r_f \tau_j}$, which makes the value of the investment portfolio under the deviation equal to the value under the original path. Compared to the original path, the transactions account at time t_{j+1}^{+} changes by $\xi \equiv -(1-\theta_X)\left[-y^{s,0}(t_j)(1-\psi_s)-\theta_X X_{t_j}^0\right]e^{r_L\tau_j}+(1-\psi_s)(1-\theta_S)\left[-y^{s,0}(t_j)+\theta_S S_{t_j}^0\right]e^{r_f\tau_j}$. $\lim_{\theta_S\to 0}\xi=(1-\theta_X)\theta_X X_{t_j}^0 e^{r_L\tau_j}+\left(e^{r_f\tau_j}-(1-\theta_X)e^{r_L\tau_j}\right)(1-\psi_s)\left[-y^{s,0}(t_j)\right]>0$. (2) If the consumer would not have transferred assets in either direction between the investment portfolio and the transactions account at time t_{j+1}, set $y^s(t_{j+1})$ equal to $-(1-\theta_S)\left[-y^{s,0}(t_j)+\theta_S S_{t_j}^0\right]e^{r_f\tau_j}-\theta_S S_{t_{j+1}}^0$ (which is negative for θ_S sufficiently close to zero), so that the value of the investment portfolio under the deviation equals the value under the original path. Compared to the original path, the transactions account at time t_{j+1}^{+} changes by $\xi \equiv -\left[-y^{s,0}(t_j)(1-\psi_s)-\theta_X X_{t_j}^0\right]e^{r_L\tau_j}-\theta_X X_{t_{j+1}}^0+(1-\psi_s)\left((1-\theta_S)\left[-y^{s,0}(t_j)+\theta_S S_{t_j}^0\right]e^{r_f\tau_j}+\theta_S S_{t_{j+1}}^0\right)$. $\lim_{\theta_S\to 0}\xi=\left[e^{r_f\tau_j}-e^{r_L\tau_j}\right](1-\psi_s)\left[-y^{s,0}(t_j)\right]+\theta_X X_{t_j}^0 e^{r_L\tau_j}-\theta_X X_{t_{j+1}}^0$. Since $y^s(t_j)=0$ under the deviation, $X_{t_{j+1}}=\left[X_{t_j}^0-C(t_j,\tau_j)\right]e^{r_L\tau_j}$, which implies that $\lim_{\theta_S\to 0}\xi=\left[e^{r_f\tau_j}-e^{r_L\tau_j}\right](1-\psi_s)\left[-y^{s,0}(t_j)\right]+\theta_X C(t_j,\tau_j)e^{r_L\tau_j}>0$. (3) If $y^{b,0}(t_{j+1})>0$, the deviation depends on whether $(1-\theta_S)\left[-y^{s,0}(t_j)+\theta_S S_{t_j}^0\right]e^{r_f\tau_j}$ is larger or smaller than $y^{b,0}(t_{j+1})$. (3a) If $(1-\theta_S)\left[-y^{s,0}(t_j)+\theta_S S_{t_j}^0\right]e^{r_f\tau_j}$ is greater than $y^{b,0}(t_{j+1})$, set $y^s(t_{j+1})=-(1-\theta_S)\left[-y^{s,0}(t_j)+\theta_S S_{t_j}^0\right]e^{r_f\tau_j}+y^{b,0}(t_{j+1})$ and set $y^b(t_{j+1})=0$, so that the value of the investment portfolio at time t_{j+1}^{+} is the same for the deviation and for the original path. Compared to the original path, the transactions account at time t_{j+1}^{+} changes by $\xi \equiv -(1-\theta_X)\left[-y^{s,0}(t_j)(1-\psi_s)-\theta_X X_{t_j}^0\right]e^{r_L\tau_j}+(1-\psi_s)\left((1-\theta_S)\left[-y^{s,0}(t_j)+\theta_S S_{t_j}^0\right]e^{r_f\tau_j}-y^{b,0}(t_{j+1})\right)+(1+\psi_b)y^{b,0}(t_{j+1})$. $\lim_{\theta_S\to 0}\xi=\left[e^{r_f\tau_j}-(1-\theta_X)e^{r_L\tau_j}\right](1-\psi_s)\left[-y^{s,0}(t_j)\right]+(\psi_s+\psi_b)y^{b,0}(t_{j+1})+(1-\theta_X)\theta_X X_{t_j}^0 e^{r_L\tau_j}>0$. (3b) If $(1-\theta_S)\left[-y^{s,0}(t_j)+\theta_S S_{t_j}^0\right]e^{r_f\tau_j}$ is less than $y^{b,0}(t_{j+1})$, set $y^b(t_{j+1})=y^{b,0}(t_{j+1})-(1-\theta_S)\left[-y^{s,0}(t_j)+\theta_S S_{t_j}^0\right]e^{r_f\tau_j}$ and set $y^s(t_{j+1})=0$ so that the value of the investment portfolio at time t_{j+1}^{+} is the same for the deviation and for the original path. Compared to the original path, the transactions account at time t_{j+1}^{+} changes by

$$\xi \equiv -(1-\theta_X)\left[-y^{s,0}(t_j)(1-\psi_s)-\theta_X X_{t_j}^0\right]e^{r_L\tau_j}+(1+\psi_b)(1-\theta_S)\left[-y^{s,0}(t_j)+\theta_S S_{t_j}^0\right]e^{r_f\tau_j}.$$

$\lim_{\theta_S\to 0}\xi=\left[(1+\psi_b)e^{r_f\tau_j}-(1-\psi_s)e^{r_L\tau_j}\right](1-\theta_X)\left[-y^{s,0}(t_j)\right]+(1-\theta_X)\theta_X X_{t_j}^0 e^{r_L\tau_j}>0$.

(3c) If $(1-\theta_S)\left[-y^{s,0}(t_j)+\theta_S S_{t_j}^0\right]e^{r_f\tau_j}$ equals $y^{b,0}(t_{j+1})$, set $y^b(t_{j+1})=y^s(t_{j+1})=0$ so that the value of the investment portfolio at time t_{j+1}^{+} is $\theta_S S_{t_{j+1}}^0$ higher for the deviation than for the original path. Compared to the original path, the transactions account at time t_{j+1}^{+} changes by $\xi \equiv -\left[-y^{s,0}(t_j)(1-\psi_s)-\theta_X X_{t_j}^0\right]e^{r_L\tau_j}+\theta_X X_{t_{j+1}}^0+(1+\psi_b)y^{b,0}(t_{j+1})$. Since

$$(1-\theta_S)\left[-y^{s,0}(t_j)+\theta_S S_{t_j}^0\right]e^{r_f\tau_j}=y^{b,0}(t_{j+1}),$$

we have $\lim_{\theta_S\to 0}-y^{s,0}(t_j)e^{r_f\tau_j}=y^{b,0}(t_{j+1})$ so that $\lim_{\theta_S\to 0}\xi=(\psi_b+\psi_s)y^{b,0}(t_{j+1})+\theta_X X_{t_j}^0 e^{r_L\tau_j}+\theta_X X_{t_{j+1}}^0>0$. Therefore, the deviation dominates the original path in all cases, so $y^{s,0}(t_j)<0$ cannot be optimal. ■

Proof of Proposition 2. Consider some path for X_t, S_t, $y^s(t)$, and $y^b(t)$, $t \in [t_j, t_{j+1}]$, and let X_t^0, S_t^0, $y^{s,0}(t)$, and $y^{b,0}(t)$ denote the values of these variables along this path. Suppose that $x_{t_j} < \omega_1$ and (contrary to what is to be proved) $X_{t_j+1}^0 > 0$. Since $\theta_1 > 0$, the consumer will not continuously observe the value of the investment portfolio. That is, $\tau_j > 0$. If $x_{t_j} < \omega_1$ on an observation date t_j, then Proposition 1 implies that optimal $y^s(t_j) < 0$. Since $X_{t_j^+}^0 = X_{t_j}^0 - (1-\psi_s)\, y^{s,0}(t_j) - \theta_X X_{t_j}^0$, we have $-y^{s,0}(t_j) = \frac{1}{1-\psi_s}\left[X_{t_j^+}^0 - X_{t_j}^0 + \theta_X X_{t_j}^0\right] = \frac{1}{1-\psi_s}\left[X_{t_j^+}^0 - C(t_j,\tau_j) + C(t_j,\tau_j) - X_{t_j}^0 + \theta_X X_{t_j}^0\right]$. Then use the fact that $e^{-r_L \tau_j} X_{t_j+1}^0 = X_{t_j^+}^0 - C(t_j,\tau_j)$ and Lemma 6 (which implies that since $y^{s,0}(t_j) < 0$, $C(t_j,\tau_t) > X_{t_j}^0$) to deduce that $-y^{s,0}(t_j) = \frac{1}{1-\psi_s}\left[e^{-r_L\tau_j} X_{t_{j+1}}^0 + \left(C(t_j,\tau_j) - X_{t_j}^0\right) + \theta_X X_{t_j}^0\right] > \frac{1}{1-\psi_s} e^{-r_L \tau_j} X_{t_j+1}^0 > 0$. We will show that there exists a deviation from this choice that will increase the consumer's expected lifetime utility, and hence $X_{t_{j+1}}^0 > 0$ cannot be optimal.

Consider a deviation in which the consumer reduces $y^s(t_j)$ by $\frac{X_{t_j^+}^0 - C(t_j,\tau_j)}{1-\psi_s} = \frac{e^{-r_L\tau_j} X_{t_j+1}^0}{1-\psi_s}$ and invests this amount in the riskless bond in the investment portfolio. With this deviation, the value of the investment portfolio at time t_{j+1} will exceed its value under the original policy by $\frac{X_{t_{j+1}}^0}{1-\psi_s} e^{(r_f - r_L)\tau_j}$ and the transactions account will have a zero balance at time t_{j+1}.

The deviation from the original path at time t_{j+1} depends on the whether, and in which direction, the consumer would transfer assets between the transactions account and the investment portfolio under the original path at that time. First, consider the case in which the consumer transfers assets from the investment portfolio to the transactions account at time t_{j+1}. In this case, the consumer can increase $-y^s\left(t_{t_{j+1}}\right)$ by $(1-\theta_S)\frac{X_{t_{j+1}}^0}{1-\psi_s} e^{(r_f-r_L)\tau_j}$, which leaves the value of the investment portfolio at time t_{j+1}^+ equal to its value on the original path. Compared to the original path, this deviation will increase the balance in the transactions account at time t_{j+1}^+ by $-(1-\theta_X) X_{t_{j+1}}^0 + (1-\theta_S) X_{t_{j+1}}^0 e^{(r_f-r_L)\tau_j} = \left[(1-\theta_S) e^{(r_f-r_L)\tau_j} - (1-\theta_X)\right] X_{t_{j+1}}^0$, which is positive for sufficiently small $\theta_S \geq 0$. Therefore, the deviation dominates the original path in this case when θ_S is sufficiently small.

Second, consider the case in which the consumer would not make any transfers between the investment portfolio and the transactions account at time t_{j+1} under the original policy. In this case, the consumer sets $-y^s(t_{j+1}) = (1-\theta_S)\frac{X_{t_{j+1}}^0}{1-\psi_s} e^{(r_f-r_L)\tau_j} - \theta_S S_{t_{j+1}}^0$, which is positive for sufficiently small $\theta_S \geq 0$. With this transfer, the value of assets in the investment portfolio at time t_{j+1}^+ will be the same under the deviation as under the original path. Compared to the original path, this deviation will increase the balance in the transactions account at time t_{j+1}^+ by $-(1-\theta_X) X_{t_{j+1}}^0 + (1-\theta_S) X_{t_{j+1}}^0 e^{(r_f-r_L)\tau_j} - (1-\psi_s)\theta_S S_{t_{j+1}}^0 = \left[(1-\theta_S) e^{(r_f-r_L)\tau_j} - (1-\theta_X)\right] X_{t_{j+1}}^0 - (1-\psi_s)\theta_S S_{t_{j+1}}^0$, which is positive for sufficiently small $\theta_S \geq 0$. Therefore, the deviation dominates the original path in this case when θ_S is sufficiently small.

Third, consider the case in which the consumer transfers assets from the transactions account to the investment portfolio at time t_{j+1}. If $y^{b,0}(t_{j+1}) \geq (1-\theta_S)\frac{X^0_{t_{j+1}}}{1-\psi_s}e^{(r_f-r_L)\tau_j}$, the deviation reduces $y^b(t_{j+1})$ by $(1-\theta_S)\frac{X^0_{t_{j+1}}}{1-\psi_s}e^{(r_f-r_L)\tau_j}$ and sets $y^s(t_{j+1})=0$, which will leave the value of the investment portfolio at time t^+_{j+1} under the deviation equal to its value on the original path. Compared to the original path, this deviation will increase the balance in the transactions account at time t^+_{j+1} by $-(1-\theta_X)X^0_{t_{j+1}} + (1+\psi_b)(1-\theta_S)\frac{X^0_{t_{j+1}}}{1-\psi_s}e^{(r_f-r_L)\tau_j} = \left[(1-\theta_S)\frac{1+\psi_b}{1-\psi_s}e^{(r_f-r_L)\tau_j} - (1-\theta_X)\right]X^0_{t_{j+1}}$, which is positive for sufficiently small $\theta_S \geq 0$. Therefore, the deviation dominates the original path in this case when θ_S is sufficiently small. If $y^{b,0}(t_{j+1}) \leq (1-\theta_S)\frac{X^0_{t_{j+1}}}{1-\psi_s}e^{(r_f-r_L)\tau_j}$, the deviation sets $y^b(t_{j+1})=0$ and sets $-y^s(t_{j+1}) = (1-\theta_S)\frac{X^0_{t_{j+1}}}{1-\psi_s}e^{(r_f-r_L)\tau_j} - y^{b,0}(t_{j+1}) \geq 0$, which will leave the value of the investment portfolio at time t^+_{j+1} under the deviation equal to its value on the original path. Compared to the original path, this deviation will increase the balance in the transactions account at time t^+_{j+1} by $-(1-\theta_X)X^0_{t_{j+1}} + (1+\psi_b)y^{b,0}(t_{j+1}) + (1-\psi_s)\left[(1-\theta_S)\frac{X^0_{t_{j+1}}}{1-\psi_s}e^{(r_f-r_L)\tau_j} - y^{b,0}(t_{j+1})\right] = \left[(1-\theta_S)e^{(r_f-r_L)\tau_j} - (1-\theta_X)\right]X^0_{t_{j+1}} + (\psi_b+\psi_s)y^{b,0}(t_{j+1})$, which is positive for sufficiently small $\theta_S \geq 0$. Therefore, the deviation dominates the original path in this case when θ_S is sufficiently small.

We have shown that if $x_{t_j} < \omega_1$, then optimal $X_{t_{j+1}} = 0$. Therefore, $x_{t_{j+1}} = 0 < \omega_1$, which implies $x_{t_{j+2}} = 0$ and so on, *ad infinitum.* ■

To prepare for the proof of Proposition 4, we state and prove the following three Lemmas.

Lemma 7 *If $y^s(t_j) < 0$ immediately after observation date t_j, then $y^b(t) = 0$ for all $t \in (t_j, t_{j+1})$.*

Proof of Lemma 7. Consider some path for X_t, S_t, $y^s(t)$, and $y^b(t)$, $t \in [t_j, t_{j+1}]$, and let X^0_t, S^0_t, $y^{s,0}(t)$, and $y^{b,0}(t)$ denote the values of these variables along this path. Suppose that $y^s(t_j) < 0$ and (contrary to what is to be proved) that $y^b(t) > 0$ for some $t \in (t_j, t_{j+1})$. Define $\overline{t} \equiv \min\left\{t \in (t_j, t_{j+1}) : y^b(t) > 0\right\}$ as the first time during the inattention interval at which $y^b > 0$, and define $\widehat{t} \equiv \max\left\{t \in \left[t^+_j, \overline{t}\right) : y^s(t) < 0\right\}$ as the last time before $\overline{t}$ at which $y^s(t) < 0$. We will examine different deviations for the three cases in which $-y^{s,0}\left(\widehat{t}\right)$ is respectively, less than, greater than, or equal to $e^{-r_L\left(\overline{t}-\widehat{t}\right)}y^{b,0}\left(\overline{t}\right)$.

If $-y^{s,0}\left(\widehat{t}\right) < e^{-r_f\left(\overline{t}-\widehat{t}\right)}y^{b,0}\left(\overline{t}\right)$, the deviation sets $-y^s\left(\widehat{t}\right) = 0$ which reduces the value of the transactions account at time $\widehat{t}^+$ by $(1-\psi_s)\left[-y^{s,0}\left(\widehat{t}\right)\right] - \theta_X X^0_{\widehat{t}}$ and increases the value of the investment portfolio at time $\widehat{t}^+$ by $\left[-y^{s,0}\left(\widehat{t}\right)\right] + \theta_X S^0_{\widehat{t}}$. Suppose that the consumer invests the additional assets in the investment portfolio in the riskless bond, which pays a rate of return r_f. At time

$\bar{t}$, the deviation reduces $y^b\left(\bar{t}\right)$ by $(1-\theta_x)\left(\left[-y^{s,0}\left(\widehat{t}\right)\right]+\theta_X S^0_{\widehat{t}}\right)e^{r_f\left(\bar{t}-\widehat{t}\right)}$, which makes the value of the investment portfolio at time $\bar{t}^+$ under the deviation equal to its value along the original path. Compared to the original path, the deviation increases the value of the transactions account at time $\bar{t}^+$ by $(1-\theta_X)\left(-(1-\psi_s)\left[-y^{s,0}\left(\widehat{t}\right)\right]+\theta_X X^0_{\widehat{t}}\right)e^{r_L\left(\bar{t}-\widehat{t}\right)}+(1+\psi_b)(1-\theta_x)\left(\left[-y^{s,0}\left(\widehat{t}\right)\right]+\theta_X S^0_{\widehat{t}}\right)e^{r_f\left(\bar{t}-\widehat{t}\right)}=$ $(1-\theta_X)\left[-y^{s,0}\left(\widehat{t}\right)\right]\left[(1+\psi_b)e^{r_f\left(\bar{t}-\widehat{t}\right)}-(1-\psi_s)e^{r_L\left(\bar{t}-\widehat{t}\right)}\right]+(1-\theta_X)\theta_X X^0_{\widehat{t}}e^{r_L\left(\bar{t}-\widehat{t}\right)}+(1+\psi_b)(1-\theta_x)\theta_X S^0_{\widehat{t}}e^{r_f\left(\bar{t}-\widehat{t}\right)}>$ 0. If $-y^{s,0}\left(\widehat{t}\right)>e^{-r_f\left(\bar{t}-\widehat{t}\right)}y^{b,0}\left(\bar{t}\right)$, the deviation reduces $-y^s\left(\widehat{t}\right)$ by $e^{-r_f\left(\bar{t}-\widehat{t}\right)}y^{b,0}\left(\bar{t}\right)$, which reduces the value of the transactions account at time $\widehat{t}^+$ by $(1-\psi_s)\left[e^{-r_f\left(\bar{t}-\widehat{t}\right)}y^{b,0}\left(\bar{t}\right)\right]$ and increases the value of the investment portfolio at time $\widehat{t}^+$ by $e^{-r_f\left(\bar{t}-\widehat{t}\right)}y^{b,0}\left(\bar{t}\right)$. Again suppose that the consumer invests the additional assets in the investment portfolio in the riskless bond, which pays a rate of return r_f. At time $\bar{t}^+$, the deviation sets $y^b\left(\bar{t}\right)=0$, which increases the value of the investment portfolio at time $\bar{t}^+$ by $\theta_S S^0_{\bar{t}}$ under the deviation relative to its value along the original path. Compared to the original path, the deviation increases the value of the transactions account at time $\bar{t}^+$ increases by $-(1-\psi_s)\left[e^{-r_f\left(\bar{t}-\widehat{t}\right)}y^{b,0}\left(\bar{t}\right)\right]e^{r_L\left(\bar{t}-\widehat{t}\right)}+(1+\psi_b)y^{b,0}\left(\bar{t}\right)+\theta_X X^0_{\bar{t}}=$ $\left[(1+\psi_b)-(1-\psi_s)e^{-(r_f-r_L)\left(\bar{t}-\widehat{t}\right)}\right]y^{b,0}\left(\bar{t}\right)+\theta_X X^0_{\bar{t}}>0$.

If $-y^{s,0}\left(\widehat{t}\right)=e^{-r_f\left(\bar{t}-\widehat{t}\right)}y^{b,0}\left(\bar{t}\right)$, the deviation sets $y^{s,0}\left(\widehat{t}\right)=y^{b,0}\left(\bar{t}\right)=0$, which reduces the value of the transactions account at time $\widehat{t}^+$ by $-y^{s,0}\left(\widehat{t}\right)(1-\psi_s)-\theta_X X^0_{\widehat{t}}$ and increases the value of the investment portfolio at time $\widehat{t}^+$ by $y^{s,0}\left(\widehat{t}\right)+\theta_s S^0_{\widehat{t}}$. Again, suppose that the consumer invests the additional assets in the investment portfolio in the riskless bond, which pays a rate of return r_f. Compared to the original path, the deviation increases the value of the investment portfolio at time $\bar{t}^+$ by $\left[y^{s,0}\left(\widehat{t}\right)+\theta_s S^0_{\widehat{t}}\right]e^{r_f\left(\bar{t}-\widehat{t}\right)}-y^{b,0}\left(\bar{t}\right)+\theta_S S^0_{\bar{t}^+}=\theta_s S^0_{\widehat{t}}e^{r_f\left(\bar{t}-\widehat{t}\right)}+\theta_S S^0_{\bar{t}^+}>0$ and increases the value of the transactions account at time $\bar{t}^+$ by $\left[y^{s,0}\left(\widehat{t}\right)(1-\psi_s)+\theta_X X^0_{\widehat{t}}\right]e^{r_L\left(\bar{t}-\widehat{t}\right)}+(1+\psi_b)y^{b,0}\left(\bar{t}\right)+\theta_X X^0_{\bar{t}^+}$ $=\left[(1+\psi_b)-e^{-(r_f-r_L)\left(\bar{t}-\widehat{t}\right)}(1-\psi_s)\right]y^{b,0}\left(\bar{t}\right)+\theta_X X^0_{\widehat{t}}e^{r_L\left(\bar{t}-\widehat{t}\right)}+\theta_X X^0_{\bar{t}^+}>0$. Therefore, the original path could not be optimal and we have proved that optimal $y^b(t)=0$ for all $t\in(t_j,t_{j+1})$. ■

Lemma 8 *Suppose that $\theta_X>0$. If $y^s(t_j)<0$ immediately after observation date t_j, then, for any non-negative $\theta_S<\underline{\theta_S}$, where $\underline{\theta_S}>0$ is sufficiently small, $y^s(t)X_t=0$ for all $t\in(t_j,t_{j+1})$.*

Proof of Lemma 8. Consider some path for X_t, S_t, $y^s(t)$, and $y^b(t)$, $t\in[t_j,t_{j+1}]$, and let X^0_t, S^0_t, $y^{s,0}(t)$, and $y^{b,0}(t)$ denote the values of these variables along this path. Suppose that $y^s(t_j)<0$ and (contrary to what is to be proved) that $y^s(t)X_t<0$ for some $t\in(t_j,t_{j+1})$. Define $\bar{t}$ to be the largest such t in that interval, i.e., $\bar{t}\equiv\max\{t\in(t_j,t_{j+1}):y^s(t)X_t<0\}$. Consider the following deviation: Eliminate the transfer at time $\bar{t}$ and instead transfer assets from the investment portfolio to the transactions account at time $\widehat{t}>\bar{t}$ where $\widehat{t}\equiv\min\left[t_{j+1},t:\int_{\bar{t}}^{t}c_u e^{-r_L\left(u-\bar{t}\right)}du=X^0_{\bar{t}}\right]$.[34] This

[34]Lemma 7 and the definition of $\widehat{t}$ imply that there will no transfers between the investment portfolio and

deviation increases $S_{\bar{t}^+}$ by $-y^{s,0}\left(\bar{t}\right)+\theta_S S_{\bar{t}}^0$ and reduces $X_{\bar{t}^+}$ by $-\left(1-\psi_s\right)y^{s,0}\left(\bar{t}\right)-\theta_X X_{\bar{t}}^0$. Suppose that the consumer invests the additional assets in the investment portfolio in the riskless bond, which pays a rate of return r_f. Therefore, the deviation increases $S_{\widehat{t}}$ by $\left[-y^{s,0}\left(\bar{t}\right)+\theta_S S_{\bar{t}}^0\right]e^{r_f\left(\widehat{t}-\bar{t}\right)}$ and reduces $X_{\widehat{t}}$ by $\left[-\left(1-\psi_s\right)y^{s,0}\left(\bar{t}\right)-\theta_X X_{\bar{t}}^0\right]e^{r_L\left(\widehat{t}-\bar{t}\right)}$.

If the consumer were not going to transfer any assets at time $\widehat{t}$ along the original path, then set $-y^s\left(\widehat{t}\right)=\left(1-\theta_S\right)\left[-y^{s,0}\left(\bar{t}\right)+\theta_S S_{\bar{t}}^0\right]e^{r_f\left(\widehat{t}-\bar{t}\right)}-\theta_S S_{\widehat{t}}^0$, which will make the value of $S_{\widehat{t}^+}$ the same under the deviation as on the original path. This deviation will allow the consumer to maintain an unchanged path of consumption through time $\widehat{t}$, and will increase the consumer's balance in the transactions account at time $\widehat{t}^+$ by $\xi_1\equiv\left[\left(1-\psi_s\right)y^{s,0}\left(\bar{t}\right)+\theta_X X_{\bar{t}}^0\right]e^{r_L\left(\widehat{t}-\bar{t}\right)}-\theta_X X_{\widehat{t}}^0+\left(1-\psi_s\right)\left[\left(1-\theta_S\right)\left[-y^{s,0}\left(\bar{t}\right)+\theta_S S_{\bar{t}}^0\right]e^{r_f\left(\widehat{t}-\bar{t}\right)}-\theta_S S_{\widehat{t}}^0\right]$. $\lim_{\theta_S\to 0}\xi_1=\left(1-\psi_s\right)\left[-y^{s,0}\left(\bar{t}\right)\right]\left[e^{r_f\left(\widehat{t}-\bar{t}\right)}-e^{r_L\left(\widehat{t}-\bar{t}\right)}\right]+\theta_X\left[X_{\bar{t}}^0 e^{r_L\left(\widehat{t}-\bar{t}\right)}-X_{\widehat{t}}^0\right]\geq 0$ because $X_{\bar{t}}^0 e^{r_L\left(\widehat{t}-\bar{t}\right)}-X_{\widehat{t}}^0=\int_{\bar{t}}^{\widehat{t}}c_u e^{r_L\left(\widehat{t}-u\right)}du\geq 0$. If the consumer were going to transfer any assets from the investment portfolio to the transactions account at time $\widehat{t}$ along the original path, then set $-y^s\left(\widehat{t}\right)=\left(1-\theta_S\right)\left[-y^{s,0}\left(\bar{t}\right)+\theta_S S_{\bar{t}}^0\right]e^{r_f\left(\widehat{t}-\bar{t}\right)}$, which will make the value of $S_{\widehat{t}^+}$ the same under the deviation as on the original path. This deviation will allow the consumer to maintain an unchanged path of consumption through time $\widehat{t}$, while leaving the balance in the investment portfolio unchanged at that time, and will increase the consumer's balance in the transactions account at time $\widehat{t}^+$ by $\xi_2\equiv\left(1-\theta_X\right)\left[\left(1-\psi_s\right)y^{s,0}\left(\bar{t}\right)+\theta_X X_{\bar{t}}^0\right]e^{r_L\left(\widehat{t}-\bar{t}\right)}+\left(1-\psi_s\right)\left(1-\theta_S\right)\left[-y^{s,0}\left(\bar{t}\right)+\theta_S S_{\bar{t}}^0\right]e^{r_f\left(\widehat{t}-\bar{t}\right)}$. $\lim_{\theta_S\to 0}\xi_2=\left(1-\psi_s\right)\left[-y^{s,0}\left(\bar{t}\right)\right]\left[e^{r_f\left(\widehat{t}-\bar{t}\right)}-\left(1-\theta_X\right)e^{r_L\left(\widehat{t}-\bar{t}\right)}\right]+\left(1-\theta_X\right)\theta_X X_{\bar{t}}^0 e^{r_L\left(\widehat{t}-\bar{t}\right)}>0$. Finally, if the consumer were going to transfer any assets from the transactions account to the investment portfolio at time $\widehat{t}$ along the original path (which is possible if $\widehat{t}=t_{j+1}$ and the stock market falls so sharply in value that $x_{t_{j+1}}>\omega_2$), then consider three cases: (1) if $y^{b,0}\left(\widehat{t}\right)>\left(1-\theta_S\right)\left[-y^{s,0}\left(\bar{t}\right)+\theta_S S_{\bar{t}}^0\right]e^{r_f\left(\widehat{t}-\bar{t}\right)}$, reduce $y^{b,0}\left(\widehat{t}\right)$ by $\left(1-\theta_S\right)\left[-y^{s,0}\left(\bar{t}\right)+\theta_S S_{\bar{t}}^0\right]e^{r_f\left(\widehat{t}-\bar{t}\right)}$ and set $y^s\left(\widehat{t}\right)=0$, which will leave the value of the investment portfolio at time $\widehat{t}^+$ under the deviation equal to its value under the original policy. Compared to the original policy the deviation will increase the value of the transactions account at time $\widehat{t}^+$ by $\xi_3\equiv\left(1-\theta_X\right)\left[\left(1-\psi_s\right)y^{s,0}\left(\bar{t}\right)+\theta_X X_{\bar{t}}^0\right]e^{r_L\left(\widehat{t}-\bar{t}\right)}+\left(1+\psi_b\right)\left(1-\theta_S\right)\left[-y^{s,0}\left(\bar{t}\right)+\theta_S S_{\bar{t}}^0\right]e^{r_f\left(\widehat{t}-\bar{t}\right)}$. $\lim_{\theta_S\to 0}\xi_3=\left[-y^{s,0}\left(\bar{t}\right)\right]\left[\left(1+\psi_b\right)e^{r_f\left(\widehat{t}-\bar{t}\right)}-\left(1-\theta_X\right)\left(1-\psi_s\right)e^{r_L\left(\widehat{t}-\bar{t}\right)}\right]+\theta_X\left(1-\theta_X\right)X_{\bar{t}}^0 e^{r_L\left(\widehat{t}-\bar{t}\right)}>0$. (2) if $y^{b,0}\left(\widehat{t}\right)<\left(1-\theta_S\right)\left[-y^{s,0}\left(\bar{t}\right)+\theta_S S_{\bar{t}}^0\right]e^{r_f\left(\widehat{t}-\bar{t}\right)}$, set $y^b\left(\widehat{t}\right)=0$ and set $-y^s\left(\widehat{t}\right)=\left(1-\theta_S\right)\left[-y^{s,0}\left(\bar{t}\right)+\theta_S S_{\bar{t}}^0\right]e^{r_f\left(\widehat{t}-\bar{t}\right)}-y^{b,0}\left(\widehat{t}\right)$, which will leave the value of the investment portfolio at time $\widehat{t}^+$ under the deviation equal to its value under the original policy. Compared to the original policy the deviation will increase the value of the transactions account

the transactions account between $\bar{t}$ and $\widehat{t}$.

at time $\widehat{t}^{+}$ by

$$\begin{aligned}\xi_4 \equiv & (1-\theta_X)\left[(1-\psi_s)\, y^{s,0}\left(\overline{t}\right)+\theta_X X^0_{\overline{t}}\right] e^{r_L\left(\widehat{t}-\overline{t}\right)} + (1+\psi_b)\, y^{b,0}\left(\widehat{t}\right) \\ & + (1-\psi_s)\left[(1-\theta_S)\left[-y^{s,0}\left(\overline{t}\right)+\theta_S S^0_{\overline{t}}\right] e^{r_f\left(\widehat{t}-\overline{t}\right)} - y^{b,0}\left(\widehat{t}\right)\right] \\ = & (1-\theta_X)\left[(1-\psi_s)\, y^{s,0}\left(\overline{t}\right)+\theta_X X^0_{\overline{t}}\right] e^{r_L\left(\widehat{t}-\overline{t}\right)} \\ & + (1-\psi_s)(1-\theta_S)\left[-y^{s,0}\left(\overline{t}\right)+\theta_S S^0_{\overline{t}}\right] e^{r_f\left(\widehat{t}-\overline{t}\right)} + (\psi_b+\psi_s)\, y^{b,0}\left(\widehat{t}\right).\end{aligned}$$

Accordingly,

$$\begin{aligned}\lim_{\theta_S\to 0}\xi_4 = & (1-\psi_s)\left[-y^{s,0}\left(\overline{t}\right)\right]\left[e^{r_f\left(\widehat{t}-\overline{t}\right)} - (1-\theta_X)\, e^{r_L\left(\widehat{t}-\overline{t}\right)}\right] \\ & + \theta_X (1-\theta_X) X^0_{\overline{t}} e^{r_L\left(\widehat{t}-\overline{t}\right)} + (\psi_b+\psi_s)\, y^{b,0}\left(\widehat{t}\right) > 0.\end{aligned}$$

(3) $y^{b,0}\left(\widehat{t}\right) = (1-\theta_S)\left[-y^{s,0}\left(\overline{t}\right)+\theta_S S^0_{\overline{t}}\right] e^{r_f\left(\widehat{t}-\overline{t}\right)}$, set $y^s\left(\widehat{t}\right) = y^b\left(\widehat{t}\right) = 0$, which will leave the value of investment portfolio at time $\widehat{t}^{+}$ under the deviation equal to its value under the original policy. Compared to the original policy the deviation will increase the value of the transactions account by at time $\widehat{t}^{+}$ $\xi_5 \equiv \left[(1-\psi_s)\, y^{s,0}\left(\overline{t}\right)+\theta_X X^0_{\overline{t}}\right] e^{r_L\left(\widehat{t}-\overline{t}\right)} + \theta_X X^0_{\widehat{t}} + (1+\psi_b)\, y^{b,0}\left(\widehat{t}\right) = \left[(1-\psi_s)\, y^{s,0}\left(\overline{t}\right)+\theta_X X^0_{\overline{t}}\right] e^{r_L\left(\widehat{t}-\overline{t}\right)}+\theta_X X^0_{\widehat{t}}+(1+\psi_b)(1-\theta_S)\left[-y^{s,0}\left(\overline{t}\right)+\theta_S S^0_{\overline{t}}\right] e^{r_f\left(\widehat{t}-\overline{t}\right)}$. $\lim_{\theta_S\to 0}\xi_5 = \left[-y^{s,0}\left(\overline{t}\right)\right]\left[(1+\psi_b)\, e^{r_f\left(\widehat{t}-\overline{t}\right)} - (1-\psi_s)\, e^{r_L\left(\widehat{t}-\overline{t}\right)}\right] + \theta_X X^0_{\overline{t}} e^{r_L\left(\widehat{t}-\overline{t}\right)} + \theta_X X^0_{\widehat{t}} > 0$.

The final step is to show that the deviation dominates the original path both in the case in which $\widehat{t} < t_{j+1}$ and in the case in which $\widehat{t} = t_{j+1}$. First, consider the case in which $\widehat{t} < t_{j+1}$ so that $\widehat{t} = \widetilde{t}$ where $\widetilde{t}$ is such that $\int_{\overline{t}}^{\widetilde{t}} c_u e^{-r_L\left(u-\overline{t}\right)} du = X^0_{\overline{t}}$. For any value of $X^0_{\overline{t}} > 0$, $\widetilde{t}-\overline{t} > 0$, even in the limit as θ_S approaches zero. Therefore, $\lim_{\theta_S\to 0}\xi_1 > 0$ and since $\lim_{\theta_S\to 0}\xi_i > 0$ for $i = 2,3,4,5$, we have shown that the deviation dominates the original path when $\widehat{t} < t_{j+1}$. Finally, consider the case in which $\widehat{t} = t_{j+1}$. In this case, there is a positive probability that the consumer will transfer assets in one direction or the other between the investment portfolio and the transactions account and will benefit because $\lim_{\theta_S\to 0}\xi_i > 0$ for $i = 2,3,4,5$. Therefore, the original path cannot be optimal and we have shown that there is no $\overline{t} \in (t_j, t_{j+1})$ for which $y^s\left(\overline{t}\right) X_{\overline{t}} < 0$. Therefore, $y^s\left(\overline{t}\right) X_{\overline{t}} = 0$ for all $\overline{t} \in (t_j, t_{j+1})$. ■

Lemma 9 *Suppose that $\theta_X > 0$. If $y^s(t_j) < 0$ and $\widehat{t} \equiv \max\left\{t \in \left[t_j^+, t_{j+1}\right) : y^s(t) < 0\right\} > t_j^+$, then for any non-negative $\theta_S < \underline{\theta_S}$, where $\underline{\theta_S} > 0$ is sufficiently small, optimal $X_{t_{j+1}} = 0$.*

Proof of Lemma 9. Since $y^s(t_j) < 0$, Lemma 8 implies $X_{\widehat{t}} = 0$. Lemma 7 and the definition of $\widehat{t}$ imply $y^b(t) = y^s(t) = 0$ for all $t \in \left(\widehat{t}, t_{j+1}\right)$. Therefore, $-(1-\psi_s)\, y^s\left(\widehat{t}\right) = \int_{\widehat{t}}^{t_{j+1}} c_u e^{-r_L\left(u-\widehat{t}\right)} du + e^{-r_L\left(t_{j+1}-\widehat{t}\right)} X_{t_{j+1}}$, which implies $-y^s\left(\widehat{t}\right) \geq \frac{1}{1-\psi_s} e^{-r_L\left(t_{j+1}-\widehat{t}\right)} X_{t_{j+1}}$.

Consider some path for X_t, S_t, $y^s(t)$, and $y^b(t)$, $t \in [t_j, t_{j+1}]$, and let X^0_t, S^0_t, $y^{s,0}(t)$, and $y^{b,0}(t)$ denote the values of these variables along this path. Suppose that (contrary to what is to be proved) $X^0_{t_{j+1}} > 0$. We will show that there exists a deviation from this choice that will increase the consumer's expected lifetime utility, and hence $X_{t_{j+1}} > 0$ cannot be optimal. Specifically, consider a deviation in which the consumer reduces $y^s\left(\widehat{t}\right)$ by $\frac{1}{1-\psi_s} e^{-r_L\left(t_{j+1}-\widehat{t}\right)} X^0_{t_{j+1}}$ and invests this amount in the riskless bond in the investment portfolio. With this deviation, the value of the investment portfolio at time t_{j+1} will exceed its value under the original policy by $\frac{X^0_{t_{j+1}}}{1-\psi_s} e^{(r_f-r_L)\left(t_{j+1}-\widehat{t}\right)}$ and the transactions account will have a zero balance at time t_{j+1}.

The deviation from the original path at time t_{j+1} depends on whether, and in which direction, the consumer would transfer assets between the transactions account and the investment portfolio under the original path at that time. First, consider the case in which the consumer transfers assets from the investment portfolio to the transactions account at time t_{j+1}. In this case, the consumer can increase $-y^s\left(t_{t_{j+1}}\right)$ by $(1-\theta_S)\frac{X^0_{t_{j+1}}}{1-\psi_s} e^{(r_f-r_L)\left(t_{j+1}-\widehat{t}\right)}$, which leaves the value of the investment portfolio at time t^+_{j+1} equal to its value on the original path. Compared to the original path, this deviation will increase the balance in the transactions account at time t^+_{j+1} by $-(1-\theta_X) X^0_{t_{j+1}} + (1-\theta_S) X^0_{t_{j+1}} e^{(r_f-r_L)\left(t_{j+1}-\widehat{t}\right)} = \left[(1-\theta_S) e^{(r_f-r_L)\left(t_{j+1}-\widehat{t}\right)} - (1-\theta_X)\right] X^0_{t_{j+1}}$, which is positive for sufficiently small $\theta_S \geq 0$. Therefore, in this case, the deviation dominates the original path when θ_S is sufficiently small.

Second, consider the case in which the consumer would not make any transfers between the investment portfolio and the transactions account at time t_{j+1} under the original policy. In this case, the consumer sets $-y^s(t_{j+1}) = (1-\theta_S)\frac{X^0_{t_{j+1}}}{1-\psi_s} e^{(r_f-r_L)\left(t_{j+1}-\widehat{t}\right)} - \theta_S S^0_{t_{j+1}}$, which is positive for sufficiently small $\theta_S \geq 0$. With this transfer, the value of assets in the investment portfolio at time t^+_{j+1} will be the same under the deviation as under the original path. Compared to the original path, this deviation will increase the balance in the transactions account at time t^+_{j+1} by $-(1-\theta_X) X^0_{t_{j+1}} + (1-\theta_S) X^0_{t_{j+1}} e^{(r_f-r_L)\left(t_{j+1}-\widehat{t}\right)} - (1-\psi_s)\theta_S S^0_{t_{j+1}} = \left[(1-\theta_S) e^{(r_f-r_L)\left(t_{j+1}-\widehat{t}\right)} - (1-\theta_X)\right] X^0_{t_{j+1}} - (1-\psi_s)\theta_S S^0_{t_{j+1}}$, which is positive for sufficiently small $\theta_S \geq 0$. Therefore, the deviation dominates the original path in this case when θ_S is sufficiently small.

Third, consider the case in which the consumer transfers assets from the transactions account to the investment portfolio at time t_{j+1}. If $y^{b,0}(t_{j+1}) \geq (1-\theta_S)\frac{X^0_{t_{j+1}}}{1-\psi_s} e^{(r_f-r_L)\left(t_{j+1}-\widehat{t}\right)}$, the consumer can reduce $y^b(t_{j+1})$ by $(1-\theta_S)\frac{X^0_{t_{j+1}}}{1-\psi_s} e^{(r_f-r_L)\left(t_{j+1}-\widehat{t}\right)}$ and set $y^s(t_{j+1}) = 0$, which will leave the value of the investment portfolio at time t^+_{j+1} under the deviation equal to its value on the original path. Compared to the original path, this deviation will increase the balance in the transactions account at time t^+_{j+1} by $-(1-\theta_X) X^0_{t_{j+1}} + (1+\psi_b)(1-\theta_S)\frac{X^0_{t_{j+1}}}{1-\psi_s} e^{(r_f-r_L)\left(t_{j+1}-\widehat{t}\right)} = \left[(1-\theta_S)\frac{1+\psi_b}{1-\psi_s} e^{(r_f-r_L)\left(t_{j+1}-\widehat{t}\right)} - (1-\theta_X)\right] X^0_{t_{j+1}}$, which is positive for sufficiently small $\theta_S \geq 0$. Therefore, the deviation dominates the original path in this case when θ_S is sufficiently small. If

$y^{b,0}(t_{j+1}) \leq (1-\theta_S)\frac{X^0_{t_{j+1}}}{1-\psi_s}e^{(r_f-r_L)(t_{j+1}-\widehat{t})}$, the consumer can set $y^b(t_{j+1})=0$ and set $-y^s(t_{j+1}) = (1-\theta_S)\frac{X^0_{t_{j+1}}}{1-\psi_s}e^{(r_f-r_L)(t_{j+1}-\widehat{t})} - y^{b,0}(t_{j+1}) \geq 0$, which will leave the value of the investment portfolio at time t^+_{j+1} under the deviation equal to its value on the original path. Compared to the original path, this deviation will increase the balance in the transactions account at time t^+_{j+1} by $-(1-\theta_X)X^0_{t_{j+1}} + (1+\psi_b)y^{b,0}(t_{j+1}) + (1-\psi_s)\left[(1-\theta_S)\frac{X^0_{t_{j+1}}}{1-\psi_s}e^{(r_f-r_L)(t_{j+1}-\widehat{t})} - y^{b,0}(t_{j+1})\right] = \left[(1-\theta_S)e^{(r_f-r_L)(t_{j+1}-\widehat{t})} - (1-\theta_X)\right]X^0_{t_{j+1}} + (\psi_b+\psi_s)y^{b,0}(t_{j+1})$, which is positive for sufficiently small $\theta_S \geq 0$. Therefore, the deviation dominates the original path in this case when θ_S is sufficiently small. Therefore, we have shown that optimal $X_{t_{j+1}} = 0$. ■

Proof of Lemma 2. Lemma 10 states that the optimal value of ϕ_j is positive. Since $\tau_j > 0$ as a consequence of the observation cost, there exists some $\delta > 0$ such that between any two consecutive observation dates, t_j and $t_{j+1} = t_j + \tau_j$, $\Pr\left\{e^{-r_L\tau_j}R(t_j,\tau_j) > \frac{\omega_2}{\omega_1}\right\} \geq \delta$. Therefore, since $x_{t_{j+1}} \equiv \frac{X_{t_{j+1}}}{S_{t_{j+1}}} = \frac{e^{r_L\tau_j}}{R(t_j,\tau_j)}\frac{X_{t_j^+}-C(t_j,\tau_j)}{S_{t_j^+}} < \frac{e^{r_L\tau_j}}{R(t_j,\tau_j)}\frac{X_{t_j^+}}{S_{t_j^+}} = \frac{x_{t_j^+}}{e^{-r_L\tau_j}R(t_j,\tau_j)} \leq \frac{\omega_2}{e^{-r_L\tau_j}R(t_j,\tau_j)}$ (where the final inequality follows from Corollary 1), $\Pr\left\{x_{t_{j+1}} < \omega_1\right\} \geq \delta$. Let $t_k \geq t_j$ be the first observation date at which $x_{t_k} < \omega_1$. Then by Williams[35] (1991), p. 233, $\Pr\{t_k < \infty\} = 1$ and $E\{t_k\} < \infty$. ■

Proof of Proposition 3. Lemma 2 states that eventually $x_{t_j} < \omega_1$ on an observation date. Proposition 2 implies that when this event occurs, $x_{t_{j+1}} = 0$ on the next observation date and on all subsequent observation dates, provided that $\theta_S \geq 0$ is sufficiently small. Since the optimal value of τ_j is simply a function of x_{t_j}, τ_j will be constant when x_{t_j} becomes constant. ■

Proof of Proposition 4. Lemma 2 implies that eventually $x_{t_j} < \omega_1$ on an observation date. Therefore, Proposition 1 implies that $y^s(t_j) < 0$. Lemma 7 implies that $y^b(t) = 0$ for all t in the inattention interval (t_j, t_{j+1}). Therefore, any automatic transfers during this interval will be from the investment portfolio to the transactions account at some time $\widehat{t}$ after t^+_j. Therefore, Lemma 9 implies if automatic transfers take place between observation dates, then $X_{t+1} = 0$ and hence $x_{t_{j+1}} = 0 < \omega_1$. Therefore, $x_{t_{j+2}} = 0$ and so on, *ad infinitum*. Since the optimal value of τ_j is simply a function of x_{t_j}, τ_j will be constant when x_{t_j} becomes constant. We conclude the proof by noting that if automatic transfers do not take place between observation dates, Proposition 3 implies that eventually x_{t_j} is absorbed at zero and the time between consecutive observations is constant. ■

Proposition 6 *Define* $V\left(0, S_{t_j}; \psi_s\right)$ *as the value function, for a given value of the transactions*

[35] D. Williams (1991): "Probability Theory with Martingales," Cambridge Mathematical Textbooks, Cambridge University Press.

cost parameter ψ_s*, on observation date* t_j *when* $\left(X_{t_j}, S_{t_j}\right) = \left(0, S_{t_j}\right)$*, and define* $\pi_1(\psi_s)$ *as the optimal return value of* $x_{t_j^+}$ *for* $x_{t_j} < \omega_1$*. Suppose that* θ_S *is sufficiently small that for any admissible value of* ψ_s*, if* $x_{t_j} < \omega_1$ *on observation date* t_j*, then on all subsequent observation dates* $x_{t_{j+1}} = 0$.

1. $V\left(0, S_{t_j}; \psi_s\right) = (1-\psi_s)^{1-\alpha} V\left(0, S_{t_j}; 0\right)$
2. *the optimal observation dates* $t_k = t_j + (k-j)\tau^*$*, for* $k \geq j$*, are invariant to* ψ_s
3. $\pi_1(\psi_s) = (1-\psi_s)\pi_1(0)$.

Proof of Proposition 6. Suppose that $\psi_s = 0$ and let $\{S_t^*\}_{t=t_j}^{t=\infty}$ be the path of the S_t under the optimal policy starting from observation date t_j when the consumer observes $X_{t_j} = 0$ and $S_{t_j} = S_{t_j}^*$. Let τ^* be the constant optimal interval of time between consecutive observations so that observation date $t_k = t_j + (k-j)\tau^*$, for $k \geq j$. For any observation date $t_k \geq t_j$, the transactions account balance will be $X_{t_k} = 0$, and immediately after each observation date the transactions account balance will be $X_{t_k^+} = X_{t_k^+}^* \equiv \pi_1(0) S_{t_k^+}^*$. Since $0 = X_{t_{k+1}}^* = e^{r_L\tau^*}\left(X_{t_k^+}^* - C(t_k,\tau^*)\right)$, we have $C(t_k,\tau^*) = X_{t_k^+}^*$.

Now let ψ_s take an arbitrary admissible value and suppose that the consumer continues to observe the value of the investment portfolio on dates $t_k = t_j + (k-j)\tau^*$, for $k \geq j$, and maintains the same path of S_t, i.e., that $S_t = S_t^*$ for $t \geq t_j$. Since the consumer will make the same transfers out of the investment portfolio as in the initial case with $\psi_s = 0$, a feasible path of the transaction account balance immediately after each observation date would be $X_{t_k^+} = (1-\psi_s) X_{t_k^+}^*$, which supports a feasible path of consumption of $C(t_k,\tau^*) = (1-\psi_s) X_{t_k^+}^*$. Therefore, $V\left(0, S_{t_j}; \psi_s\right) \geq (1-\psi_s)^{1-\alpha} V\left(0, S_{t_j}; 0\right)$.

A similar argument starting with an arbitrary admissible value of ψ_s less than one implies $V\left(0, S_{t_j}; 0\right) \geq \left(\frac{1}{1-\psi_s}\right)^{1-\alpha} V\left(0, S_{t_j}; \psi_s\right)$. Therefore, $V\left(0, S_{t_j}; \psi_s\right) \geq (1-\psi_s)^{1-\alpha} V\left(0, S_{t_j}; 0\right) \geq V\left(0, S_{t_j}; \psi_s\right)$, which implies $V\left(0, S_{t_j}; \psi_s\right) = (1-\psi_s)^{1-\alpha} V\left(0, S_{t_j}; 0\right)$ (statement 1). We showed that by maintaining the same observation dates when ψ_s is positive as when $\psi_s = 0$ allows a path of consumption that achieves $V\left(0, S_{t_j}; \psi_s\right) \geq (1-\psi_s)^{1-\alpha} V\left(0, S_{t_j}; 0\right) = V\left(0, S_{t_j}; \psi_s\right)$. Similarly, by maintaining the same observation dates when $\psi_s = 0$ as when ψ_s is positive allows a path of consumption that achieves $V\left(0, S_{t_j}; 0\right) \geq \left(\frac{1}{1-\psi_s}\right)^{1-\alpha} V\left(0, S_{t_j}; \psi_s\right) = V\left(0, S_{t_j}; 0\right)$. Therefore, we have proven statement 2. For any observation date $t_k \geq t_j$, $x_{t_k^+} = \pi_1(\psi_s)$. Therefore, $\pi_1(\psi_s) = \frac{X_{t_k^+}}{S_{t_k^+}} = \frac{(1-\psi_s) X_{t_k^+}^*}{S_{t_k^+}^*} = (1-\psi_s)\pi_1(0)$, which proves statement 3. ■

Proof of Proposition 5. At each observation date t_j the consumer chooses the share ϕ_j of the investment portfolio to allocate to equity to maximize $E_{t_j}\left\{V\left(X_{t_{j+1}}, S_{t_{j+1}}\right)\right\}$ subject to the constraints $0 \leq \phi_j \leq 1$. Using equations (2) and (3), we can write the Lagrangian for this constrained maximization as

$$\mathcal{L}_j = E_{t_j}\left\{V\left(X_{t_{j+1}}, \phi_j \frac{P_{t_{j+1}}}{P_{t_j}} S_{t_j^+} + (1-\phi_j) e^{r_f \tau_j} S_{t_j^+}\right)\right\} + \delta_j S_{t_j^+}\phi_j + \xi_j S_{t_j^+}(1-\phi_j) \tag{A.44}$$

where $\delta_j S_{t_j^+} \geq 0$ is the Lagrange multiplier on the constraint $\phi_j \geq 0$ and $\xi_j S_{t_j^+} \geq 0$ is the Lagrange multiplier on the constraint $\phi_j \leq 1$. Differentiating the Lagrangian in equation (A.44) with respect to ϕ_j, setting the derivative equal to zero, and then dividing both sides by $S_{t_j^+}$ yields

$$E_{t_j}\left\{V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)\left(\frac{P_{t_{j+1}}}{P_{t_j}} - e^{r_f \tau_j}\right)\right\} = \xi_j - \delta_j. \tag{A.45}$$

Next, we prove the following lemma.

Lemma 10 *$\phi_j > 0$ and $\delta_j = 0$.*

Proof of Lemma 10. We will proceed by contradiction. Suppose that $\phi_j = 0$, which implies that $\xi_j = 0$ and that $S_{t_{j+1}}$ is known at time t_j. Therefore, equation (A.45) can be written as $V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right) E_{t_j}\left\{\left(\frac{P_{t_{j+1}}}{P_{t_j}} - e^{r_f \tau_j}\right)\right\} = -\delta_j \leq 0$, which is a contradiction because $V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right) > 0$ and, by assumption, the expected equity premium, $E_{t_j}\left\{\left(\frac{P_{t_{j+1}}}{P_{t_j}} - e^{r_f \tau_j}\right)\right\}$, is positive. Therefore, ϕ_j must be positive, which implies $\delta_j = 0$. ■

To replace the marginal valuation of the investment portfolio, $V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)$, by a function of the marginal utility of consumption, first use the definition of the marginal rate of substitution $m\left(x_{t_{j+1}}\right)$ to obtain

$$V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right) = m\left(x_{t_{j+1}}\right) V_X\left(X_{t_{j+1}}, S_{t_{j+1}}\right). \tag{A.46}$$

Then use the envelope theorem to obtain

$$V_X\left(X_{t_{j+1}}, S_{t_{j+1}}\right) = \left[1 - \left(\mathbf{1}_{\{y^b(t_{j+1})>0\}} + \mathbf{1}_{\{y^s(t_{j+1})<0\}}\right)\theta_X\right]\left(1 - (1-\alpha)\kappa b(\tau_{j+1})\right) U'\left(C\left(t_{j+1}, \tau_{j+1}\right)\right) \tag{A.47}$$

which implies that $V_X\left(X_{t_{j+1}}, S_{t_{j+1}}\right)$, the increase in expected lifetime utility made possible by a one-dollar increase in $X_{t_{j+1}}$, equals the increase in utility that would accompany an increase of $1 - \left(\mathbf{1}_{\{y^b(t_{j+1})>0\}} + \mathbf{1}_{\{y^s(t_{j+1})<0\}}\right)\theta_X$ dollars in $C\left(t_{j+1}, \tau_{j+1}\right)$. That is, if consumer transfers assets between the investment portfolio and the transactions account at time t_{j+1}, a one-dollar increase in $X_{t_{j+1}}$ would allow $C\left(t_{j+1}, \tau_{j+1}\right)$ to increase by $1 - \theta_X$ dollars; otherwise, $C\left(t_{j+1}, \tau_{j+1}\right)$ can increase by one dollar. Differentiate equation (16) with respect to $C\left(t_j, \tau_j\right)$ and use equation (**) in footnote 17 to obtain

$$U'\left(C\left(t_j, \tau_j\right)\right) = c_{t_j^+}^{-\alpha}. \tag{A.48}$$

Substitute equation (A.47) into equation (A.46) and use equation (A.48) to obtain

$$V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right) = m\left(x_{t_{j+1}}\right)\left[1 - \left(\mathbf{1}_{\left\{y^b(t_{j+1})>0\right\}} + \mathbf{1}_{\{y^s(t_{j+1})<0\}}\right)\theta_X\right]\left(1-(1-\alpha)\kappa b\left(\tau_{j+1}\right)\right)c_{t_{j+1}^+}^{-\alpha}. \tag{A.49}$$

Substituting the right hand side of equation (A.49) for $V_S\left(X_{t_{j+1}}, S_{t_{j+1}}\right)$ in equation (A.45) and using Lemma 10 to set $\delta_j = 0$ yields

$$E_{t_j}\left\{m\left(x_{t_{j+1}}\right)\left[1 - \left(\mathbf{1}_{\left\{y^b(t_{j+1})>0\right\}} + \mathbf{1}_{\{y^s(t_{j+1})<0\}}\right)\theta_X\right]\left(1-(1-\alpha)\kappa b\left(\tau_{j+1}\right)\right)c_{t_{j+1}^+}^{-\alpha}\left(\frac{P_{t_{j+1}}}{P_{t_j}} - e^{r_f\tau_j}\right)\right\} = \xi_j. \tag{A.50}$$

In standard models without observation costs and transfer costs, and without the constraints $0 \leq \phi_j \leq 1$, the corresponding Euler equation, which is widely used in financial economics, is

$$E_t\left\{c_s^{-\alpha}\left(\frac{P_s}{P_t} - e^{r_f(s-t)}\right)\right\} = 0 \quad \text{for } s > t. \tag{A.51}$$

In general, the Euler equation in the presence of observation costs and transactions costs in equation (A.50) differs from the standard Euler equation in equation (A.51) in three ways: (1) the Euler equation in equation (A.50) contains the Lagrange multiplier on the constraint $\phi_j \leq 1$ but this Lagrange multiplier does not appear in the standard Euler equation; (2) the Euler equation in equation (A.50) contains the marginal rate of substitution $m\left(x_{t_{j+1}}\right)$, which is a random variable, but this marginal rate of substitution is absent (or implicitly equal to a constant) in the standard Euler equation;[36] (3) the Euler equation in equation (A.50) contains the term $1 - \left(\mathbf{1}_{\left\{y^b(t_{j+1})>0\right\}} + \mathbf{1}_{\{y^s(t_{j+1})<0\}}\right)\theta_X$, which reflects the additional fixed transfer cost associated with having an additional dollar in the transactions account; (4) the Euler equation in equation (A.50) contains the term $1-(1-\alpha)\kappa b\left(\tau_{j+1}\right)$, which reflects the utility cost of the next observation; and (5) in the presence of observation costs, the Euler equation holds only for rates of return between observation dates, whereas the Euler equation in the standard case holds for rates of return between any arbitrary pair of dates because all dates are observation dates in the standard case. We show that in the long run in an interesting special case, the first four of these differences disappear. Before showing this result, we prove the following lemma.

Lemma 11 *Suppose that θ_S is sufficiently small, in the sense described in the proof of Proposition 2. If $x_{t_j} \leq \omega_1$, then (i) $\phi_j < 1$ if $\alpha > \frac{\mu - r_f}{\sigma^2}$ and (ii) $\phi_j = 1$ if $\alpha \leq \frac{\mu - r_f}{\sigma^2}$.*

[36] If assets could be transferred without any resource costs (i.e., if $\theta_X = \theta_S = \psi_s = \psi_b = 0$), then $m\left(x_{t_j}\right) = 1$ at all observation dates, and hence can be eliminated from equation (A.50).

Proof of Lemma 11. Proposition 2 implies that if $x_{t_j} \leq \pi_1$, then $x_{t_{j+1}} = 0$. The optimal value of ϕ_j, $0 \leq \phi_j \leq 1$, maximizes $E_{t_j}\left\{V\left(X_{t_{j+1}}, S_{t_{j+1}}\right)\right\} = \frac{1}{1-\alpha}E_{t_j}\left\{S_{t_{j+1}}^{1-\alpha}v(0)\right\}$, which is equivalent to maximizing $\varphi(\phi_j;\alpha) \equiv \frac{1}{1-\alpha}E_{t_j}\left\{\left[\phi_j \frac{P_{t_j+\tau_j}}{P_{t_j}} + (1-\phi_j)e^{r_f\tau_j}\right]^{1-\alpha}\right\}$. Define α^* such that $\arg\max_{\phi_j}\varphi(\phi_j;\alpha^*) = 1$ and note that $\varphi'(1;\alpha^*) = 0$.

Differentiating the definition of $\varphi(\phi_j;\alpha)$ with respect to ϕ_j and setting $\phi_j = 1$ yields

$$\varphi'(1;\alpha) = E_{t_j}\left\{\left(\frac{P_{t_j+\tau_j}}{P_{t_j}}\right)^{1-\alpha}\right\} - e^{r_f\tau_j}E_{t_j}\left\{\left(\frac{P_{t_j+\tau_j}}{P_{t_j}}\right)^{-\alpha}\right\}.$$

Use the fact that $\frac{P_{t_j+\tau_j}}{P_{t_j}}$ is lognormal to obtain

$$\varphi'(1;\alpha) = \exp\left[(1-\alpha)\left(\mu - \frac{1}{2}\alpha\sigma^2\right)\tau_j\right] - e^{r_f\tau_j}\exp\left[-\alpha\left(\mu + \frac{1}{2}(-\alpha-1)\sigma^2\right)\tau_j\right].$$

Further rearrangement yields

$$\varphi'(1;\alpha) = \exp\left[\left(-\alpha\mu + r_f - \frac{1}{2}\alpha(1-\alpha)\sigma^2\right)\tau_j\right] \times \left[\exp\left((\mu - r_f)\tau_j\right) - \exp\left(\alpha\sigma^2\tau_j\right)\right],$$

which implies that

$$\varphi'(1;\alpha) \lesseqqgtr 0 \text{ as } \alpha \gtreqqless \alpha^* \equiv (\mu - r_f)/\sigma^2.$$

Differentiate $\varphi(\phi_j;\alpha)$ twice with respect to ϕ_j to obtain

$$\varphi''(\phi_j;\alpha) = -\alpha E_{t_j}\left\{\left(\phi_j\frac{P_{t_j+\tau_j}}{P_{t_j}} + (1-\phi_j)e^{r_f\tau_j}\right)^{-\alpha-1}\left(\frac{P_{t_j+\tau_j}}{P_{t_j}} - e^{r_f\tau_j}\right)^2\right\} < 0,$$

which implies that $\varphi(\phi_j;\alpha)$ is concave. If $\alpha > \alpha^*$, then $\varphi'(1;\alpha) < 0$, so the concavity of $\varphi(\phi_j;\alpha)$ implies that the optimal value of ϕ_j is less than one and the Lagrange multiplier on the constraint $\phi_j \leq 1$ is $\xi_j = 0$. If $\alpha \leq \alpha^*$, then $\varphi'(1;\alpha) \geq 0$, so the concavity of $\varphi(\phi_j;\alpha)$ implies that the optimal value of ϕ_j equals one. If $\alpha < \alpha^*$, the Lagrange multiplier on the constraint $\phi_j \leq 1$ is $\xi_j > 0$. ■

Suppose that θ_S is sufficiently small so that in the long run, the stochastic process for x_{t_j} is absorbed at zero. Lemma 11 implies that if the coefficient of relative risk aversion α exceeds $\frac{\mu - r_f}{\sigma^2}$, then in the long run the constraint $\phi_j \leq 1$ does not bind, and hence $\xi_j = 0$. In this case, the first of the five differences between the Euler equation in equation (A.50) and the standard Euler equation disappears. In addition, in the long run $x_{t_j} = 0$ on each observation date t_j so (1) $m\left(x_{t_j}\right) = (1-\psi_s)\frac{1-\theta_S}{1-\theta_X}$ on each observation date, (2) the consumer sells assets from the investment portfolio on each observation date so $1 - \left(\mathbf{1}_{\left\{y^b_{t_{j+1}}>0\right\}} + \mathbf{1}_{\left\{y^s_{t_{j+1}}<0\right\}}\right)\theta_X = 1 - \theta_X$ on each observation date, and (3) the time between consecutive observations is constant so $1 - (1-\alpha)\kappa b(\tau_{j+1})$ is constant. Using the fact that $\xi_j = 0$ and dividing both sides of equation (A.50) by $(1-\psi_s)(1-\theta_S)(1-(1-\alpha)\kappa b(\tau_{j+1}))$, proves proposition 5. ■