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Vernon L. Smith
Chapman University, vsmith@chapman.edu

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Optimal Insurance Coverage

Vernon L. Smith*

Brown University

The literature of expected utility theory has treated extensively the problem of optimal portfolio investment, but there is limited treatment of the parallel problem of the optimal protection of assets against casualty or liability loss (Arrow, 1963, 1965). The problem of optimal insurance coverage is formally similar to the problem of optimal inventory stockage under uncertainty. To inventory a product is to "insure" against sales loss—the larger the inventory, given the distribution of demand, the greater the "insurance coverage." If casualty or liability loss (demand) is less than the insurance coverage (inventory level), excessive insurance cost (inventory holding cost) is incurred. If casualty or liability loss (demand) is greater than the insurance coverage (inventory level), one must absorb the cost of the unrecoverable loss (sales loss). These two components of loss must be balanced in determining optimal insurance (inventory) levels.

In the analysis to follow, we will use V to denote the given value of an individual's property or assets which are insurable against loss. In deciding how much insurance to buy, an individual must choose $\lambda \geq 0$, the fraction (or multiple) of V which is to be protected against loss. That is, he chooses an amount of insurance or coverage level, λV. We assume, throughout, that the individual can buy as much insurance as he pleases at a fixed price, $m \geq 0$ in dollars per dollar of protection, for a given time interval of exposure to risk of loss. His premium for that time interval is then $P = m\lambda V$ if he buys λV dollars of insurance.

The analysis will be divided into two sections, the first dealing with insurance against casualty losses of physical property due to fire, wind, storm, vandalism, and so on, in which it is assumed that the loss cannot exceed the value of the property, V. The second section will deal with insurance against liability claims on an individual's tangible and intangible assets, in which it is assumed that the liability claim can exceed the value of assets, but cannot exceed the value of assets plus insurance coverage $(1 + \lambda)V$.

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1. Optimal Casualty Loss Insurance

We assume that an individual's property is subject to the risk of casualty loss, the value of said loss, X, being a random variable over which is defined a probability mass or density function. In general, it will not do to describe this risk in terms only of a continuous density function on $0 \leq X \leq V$. This is because it may and perhaps often will be the case that there is a non-zero probability mass at $X = 0$ and at $X = V$. That is, it may be rather likely that no casualty loss at all will be suffered. In general, it may also be the case that there is a non-zero probability associated with total loss $X = V$. However, on the open interval $0 < X < V$, there may be no positive probability mass associated with any point, so that it is reasonable to assign probabilities to subintervals using a continuous density function, $f(X)$.

Consequently, we assume a mixed mass and density function $D(X)$, defined:

$$D(X) = \begin{cases} \underline{\pi}, & \text{if } X = 0 \\ f(X), & \text{if } 0 < X < V \\ \bar{\pi}, & \text{if } X = V. \end{cases} \tag{1}$$

Where $\underline{\pi} \geq 0, f(X) > 0, \bar{\pi} \geq 0$, and

$$\underline{\pi} + \int_0^V f(X)dX + \bar{\pi} = 1.$$

Casualty loss insurance has the property that one cannot normally claim more than the value of the loss or the value of the coverage purchased, whichever is smaller. If we let R be the total recovery from insurance claims, then:

$$R = \begin{cases} X, & \text{if } 0 \leq X < \lambda V \\ \lambda V, & \text{if } \lambda V \leq X \leq V. \end{cases} \tag{2}$$

If we let W be an individual's insured wealth state net of non-insurable assets, then:

$$W = V - P - X + R = V(1 - \lambda m) + (R - X). \tag{3}$$

Since X is a random variable, so also are R and W.[1] We assume a utility function of insured wealth, $U(W)$ with $U'(W) > 0$, $U''(W) \leq 0$ for all W. Hence, expected utility, z, under these assumptions can be written:

$$z(\lambda) = E[U(W)] = \underline{\pi}U[V(1 - m\lambda)] + \int_0^{\lambda V} U[V(1 - m\lambda)]f(X)dX$$
$$+ \int_{\lambda V}^{V} U[V(1 - m\lambda) + (\lambda V - X)]f(X)dX + \bar{\pi}U[\lambda V(1 - m)]. \tag{4}$$

[1] Alternatively, we could write (2) as $R = \min(X, \lambda V)$, $0 \leq X \leq V$, $0 \leq \lambda \leq 1$; and (3) as $W = V(1 - \lambda m) + \min(0, \lambda V - X)$.

We assume that an individual desires to maximize $z(\lambda)$ subject to $0 \leq \lambda \leq 1$.[2]

The theorems below are very easily proved by inspection of the first and second derivatives of $z(\lambda)$, which are:

$$\begin{aligned} z'(\lambda) = & -mVU'[V(1-m\lambda)]\left[\underline{\pi} + \int_0^{\lambda V} f(X)dX\right] \\ & + V(1-m)\int_{\lambda V}^{V} U'[V(1-m\lambda+\lambda)-X]f(X)dX \\ & + \bar{\pi}V(1-m)U'[\lambda V(1-m)]. \end{aligned} \tag{5}$$

$$\begin{aligned} z''(\lambda) = & -V^2U'[V(1-m\lambda)]f(\lambda V) \\ & + m^2V^2U''[V(1-m\lambda)]\left[\underline{\pi} + \int_0^{\lambda V} f(X)dX\right] \\ & + V^2(1-m)^2\int_{\lambda V}^{V} U''[V(1-m\lambda+\lambda)-X]f(X)dX \\ & + \bar{\pi}V^2(1-m)^2U''[\lambda V(1-m)]. \end{aligned} \tag{6}$$

Theorem 1

(*a*) If $m < 1$, $\underline{\pi} = 0$, that is, some casualty loss is certain, then it is always optimal to buy some positive amount of insurance. (*b*) If $m < 1$, $\underline{\pi} > 0$, then optimality may require the purchase of no insurance (one "self-insures"). This follows if we evaluate $z'(0)$, that is,

$$\begin{aligned} z'(0) = & -mVU'(V)\underline{\pi} + V(1-m)\int_0^V U'(V-X)f(X)dX \\ & + \bar{\pi}V(1-m)U'(0)\begin{cases} > 0, & \text{if } \underline{\pi} = 0, \\ \lesseqgtr 0, & \text{if } \underline{\pi} > 0. \end{cases} \end{aligned}$$

Theorem 2

(*a*) If the probability of total loss is not smaller than the insurance price, $\bar{\pi} \geq m$, full insurance coverage is optimal. (*b*) If $\bar{\pi} < m$, less than full

[2] For policy contracts calling for damage deductibility, D, in place of (4) we have

$$\begin{aligned} z(\lambda) = & U[V(1-m\lambda)]\underline{\pi} + \int_0^D U[V(1-m\lambda)-X]f(X)dX \\ & + \int_D^{\lambda V} U[V(1-m\lambda)-D]f(X)dX + \int_{\lambda V}^{V} U[V(1-m\lambda)-D \\ & + (\lambda V - X)]f(X)dX + \bar{\pi}U[\lambda V(1-m)-D]. \end{aligned}$$

coverage is optimal. This follows if we evaluate $z'(1)$, that is, $z'(1) = (\bar{\pi} - m)VU'[V(1 - m)] \gtreqless 0$ according as $\bar{\pi} \gtreqless m$.

Discussion.—On the supply side of the market, it is reasonable to assume that insurance companies will use actuarial principles in determining m. Thus, anyone buying a coverage λV presents an expected claim loss to the insurance company equal to:

$$\bar{\pi}\lambda V + \int_0^{\lambda V} Xf(X)dX + \int_{\lambda V}^{V} \lambda V f(X)dX,$$

which must be less than the premium $m\lambda V$. Average (that is, per dollar of coverage) expected claim cost to the insurance company is

$$a(\lambda) = \bar{\pi} + \frac{1}{\lambda V}\int_0^{\lambda V} Xf(X)dX + \int_{\lambda V}^{V} f(X)dX, \tag{7}$$

and profitable insurance operations require $m > a(\lambda) > \bar{\pi}$, $0 \leq \lambda \leq 1$.

Imposing $\bar{\pi} < m$, we conclude that $z'(1) < 0$, provided only that $VU'[V(1 - m)] > 0$, and *it is always optimal to buy less than full insurance coverage.*

Theorem 3

If $U'' \leq 0$, than z is strictly concave, that is, $z''(\lambda) < 0$. The theorem follows from (6), for $U'' \leq 0$.

From these three theorems, with $U'' < 0$, the following three cases exhaust the possibilities for optimal insurance coverage $\lambda = \lambda^0$:

i) If $\underline{\pi} \geq 0$, $\bar{\pi} \geq m$, then $\lambda^0 = 1$. This case is excluded if we impose $\bar{\pi} < m$ (Fig. 1).

ii) If $\underline{\pi} > 0$, $\bar{\pi} < m$, then $0 \leq \lambda^0 < 1$ (Fig. 2*a* and 2*b*).

iii) If $\underline{\pi} = 0$, $\bar{\pi} < m$, then $0 < \lambda^0 < 1$ (Fig. 3).

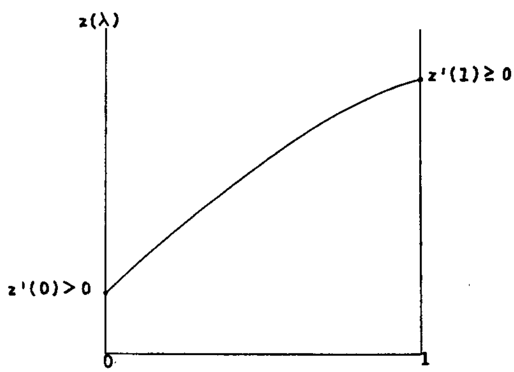

FIG. 1

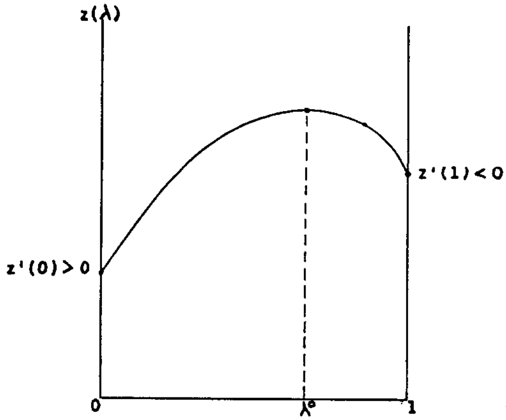

FIG. 2*a*

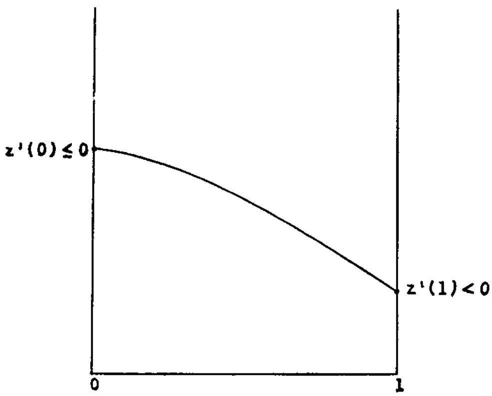

FIG. 2*b*

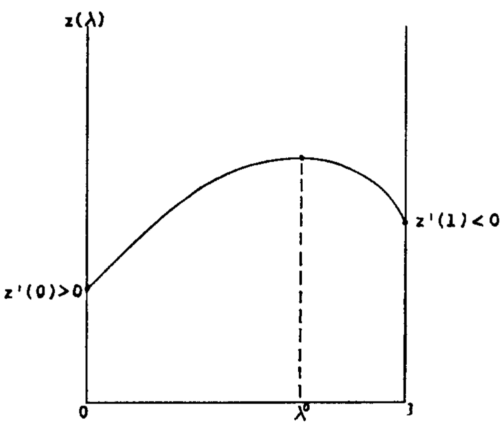

FIG. 3

Theorem 4

(*a*) If $\underline{\pi} + m < 1$, and $\bar{\pi} < m$, an expected utility maximizer (utility linear in wealth) will buy a coverage $\lambda^0 V$, satisfying: $1 - \underline{\pi} - m = F(\lambda^0 V)$, $0 < \lambda^0 < 1$, where

$$F(\lambda V) = \int_0^{\lambda V} f(X)dX,$$

and $0 \leq F(\lambda V) \leq 1 - \underline{\pi} - \bar{\pi}, 0 \leq \lambda \leq 1$. (*b*) If $\underline{\pi} + m \geq 1$, then $\lambda^0 = 0$.

This follows if we evaluate $z'(\lambda)$, when $U'(W) \equiv 1$;

$$z'(\lambda)\,|_{U' \equiv 1} = V[1 - m - \underline{\pi} - F(\lambda V)].$$

Hence,

$$z'(0)\,|_{U' \equiv 1} = V(1 - m - \underline{\pi}) > 0, \quad \text{if} \quad \underline{\pi} + m < 1,$$
$$z'(1)\,|_{U' \equiv 1} = V(\bar{\pi} - m) < 0, \quad \text{if} \quad \bar{\pi} < m.$$

Hence,

$$z'(\lambda^0)\,|_{U' \equiv 1} = 0, \; 0 < \lambda^0 < 1, \quad \text{or} \quad 1 - m - \underline{\pi} = F(\lambda^0 V).$$
$$\text{If} \quad \underline{\pi} + m \geq 1, \quad \text{then} \quad z'(0)\,|_{U' \equiv 1} \leq 0, \quad \text{and} \quad \lambda^0 = 0.$$

Returning now to the general case, $U''(W) \leq 0$, suppose we have a solution $0 < \lambda^0 < 1$, given by $z'(\lambda^0) = 0$. This condition defines an individual's demand function for insurance, $\lambda^0 = D(m)$. By differentiating $z'(\lambda^0)$ with respect to m, we determine:

$$\frac{d\lambda^0}{dm} = -\left[\frac{\partial^2 z/\partial m \partial \lambda^0}{z''(\lambda^0)}\right] < 0, \quad \text{if} \quad \frac{\partial^2 z}{\partial m \partial \lambda^0} < 0, \quad \text{and} \quad U'' \leq 0.$$

By computing $(\partial^2 z/\partial m \partial \lambda^0)$, one can readily verify that $U'' < 0$ is not sufficient to guarantee $(\partial^2 z/\partial m \partial \lambda^0) < 0$. However, the conditions as stated give $(d\lambda^0/dm) < 0$ and we have a downward-sloping demand for insurance.

Finally we state:

Theorem 5

(*a*) If $m = 1$, $\underline{\pi} \geq 0$, $\bar{\pi} \geq 0$, it is optimal to self-insure. (*b*) If $m = 0$, $\underline{\pi} \geq 0, \bar{\pi} \geq 0$, it is optimal to buy full coverage.

This result follows if we evaluate $z'(\lambda)$ at $m = 1$ and $m = 0$, that is:

$$z'(\lambda)\,|_{m=1} = -VU'[V(1 - \lambda)]\left[\underline{\pi} + \int_0^{\lambda V} f(X)dX\right] < 0,$$
$$\underline{\pi} \geq 0, \quad \text{for all } \lambda.$$

Hence, $\lambda^0 = 0$.

$$z'(\lambda)\,|_{m=0} = V\int_{\lambda V}^{V} U'[V(1 + \lambda) - X]f(X)dX$$
$$+ \bar{\pi} VU'(\lambda V) > 0, \; \bar{\pi} \geq 0, \quad \text{for all } \lambda.$$

Hence, $\lambda^0 = 1$.

If we let $d(\lambda)$ be the inverse of the demand function for insurance when utility is linear in wealth, then from Theorem 4 we have

$$m = d(\lambda) = 1 - \underline{\pi} - F(\lambda V). \tag{8}$$

From (8) we have,

$$d(0) = 1 - \underline{\pi}, \quad d(1) = \bar{\pi};$$
$$d'(\lambda) = -Vf(\lambda V) < 0, \quad d''(\lambda) = -V^2 f'(\lambda V) \gtreqless 0.$$

From (7), and these calculations from (8), we have

$$\lim_{\lambda \to 0} a(\lambda) = \bar{\pi} + F(V) = 1 - \underline{\pi} = d(0),$$

$$a(1) = \bar{\pi} + \frac{1}{V}\int_0^V Xf(X)dX = \frac{E(X)}{V} > \bar{\pi} = d(1),$$

$$a'(\lambda) = -\frac{1}{\lambda^2 V}\int_0^{\lambda V} Xf(X)dX < 0, \quad 0 \leq \lambda \leq 1,$$

$$\lim_{\lambda \to 0} a'(\lambda) = -\frac{Vf(0)}{2} = \frac{1}{2}d'(0).$$

Finally, from (7) and (8), we have $a(\lambda) \geq \bar{\pi} + F(V) - F(\lambda V) = d(\lambda)$, on $0 \leq \lambda \leq 1$, and the graphs of $d(\lambda)$ and $a(\lambda)$ may be represented as in Figure 4.

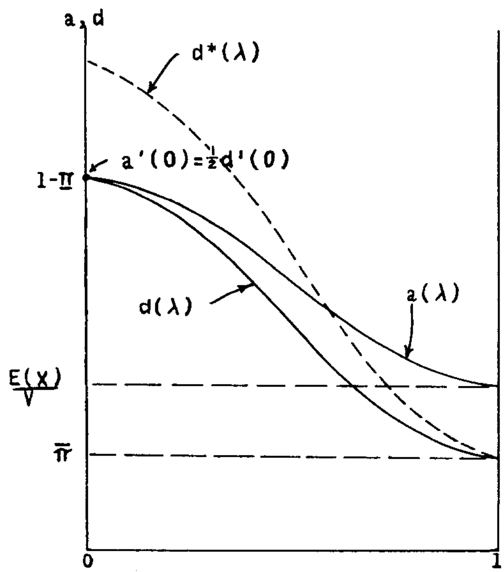

FIG. 4

From these "average revenue" and "average cost" curves for the insurance firm, it is evident that the profit-maximizing sale to an expected wealth maximizer is zero. The demand for insurance is nowhere above its average cost.

From equation (5), we can show that the average revenue function, call it $d^*(\lambda)$, when $U'' < 0$, $U' > 0$ is not below $d(\lambda)$—the average revenue when $U'' = 0$, $U' > 0$. Using the inequality,

$$U'[V(1 - m\lambda)] \leq U'[V(1 - m\lambda + \lambda) - X] \leq U'[\lambda V(1 - m)],$$

$\lambda V \leq X \leq V$, from (5), for optimal λ corresponding to prices d^*, we can write:

$$\begin{aligned} 0 \geq z'(\lambda) = & -d^* VU'[V(1 - d^*\lambda)][\underline{\pi} + F(\lambda V)] \\ & + V(1 - d^*) \int_{\lambda V}^{V} U'[V(1 - d^*\lambda + \lambda) - X]f(X)dX \\ & + \bar{\pi} V(1 - d^*)U'[\lambda V(1 - d^*)] \geq VU'[V(1 - d^*\lambda)] \\ & \times [1 - \underline{\pi} - F(\lambda V) - d^*(1 - 2\underline{\pi})]. \end{aligned}$$

Therefore, $d^*(\lambda) \geq [1 - \underline{\pi} - F(\lambda V)]/(1 - 2\underline{\pi}) \geq d(\lambda)$. A possible $d^*(\lambda)$ graph is shown dotted in Figure 4.[3,4]

2. Optimal Liability Insurance[5]

Consider now that an individual's total tangible and intangible assets, V, are subject to the risk of liability claims. Since liability claims can also be levied against future income, V for this case must include the present worth of remaining lifetime labor income. Also, X in this application must be interpreted as the random value of total liability claims, with $X \geq 0$, and not necessarily bounded above. Conceivably, there may be no upper limit to the value of liability claims, but the amount that can be collected cannot exceed assets plus insurance coverage $(1 + \lambda)V$. This case includes the

[3] Note that if a uniform price, m, is to be charged all customers, the optimal price for the insurance company may intersect the set of points $d(\lambda)$. In that case, the insurance company must set a coverage minimum that prevents policies from being written in the region below $a(\lambda)$.

[4] This is a good place to record the result if $U'' > 0$, $U' > 0$. In this case we have the inequalities, $U'(0) \leq U'(V - X) \leq U'(V)$ for $0 \leq X \leq V$. Consequently from (5), $z'(\lambda) \leq -mVU'[V(1 - m\lambda)][\underline{\pi} + F(\lambda V)] + V(1 - m)U'[V(1 - m\lambda] \times [F(V) - F(\lambda V)] + \bar{\pi} V(1 - m)U'[V(1 - m\lambda)] = VU'[V(1 - m\lambda)][1 - \underline{\pi} - F(\lambda V) - m]$. Now, for any m, if the insurance company is to make a positive expected profit it must write a policy whose coverage satisfies $m > a(\lambda)$. But we have already shown that $a(\lambda) \geq d(\lambda) = 1 - \underline{\pi} - F(\lambda V)$, $0 \leq \lambda \leq 1$. Hence, $1 - \underline{\pi} - F(\lambda V) - m < 0$, and we conclude that $z'(\lambda) < 0$, $0 \leq \lambda \leq 1$, that is, a "riskophile" will never buy any positive insurance coverage.

[5] I am indebted to Cliff Lloyd for pointing out the possibility that with liability insurance one might lose both his assets and the insurance, thus providing the motivation for this section.

problem of optimal medical and hospital insurance, since the contingencies of bodily injury and ill health are means by which (medical and hospital fee) liabilities are incurred.

The mixed density and mass function on X is now:

$$D(X) = \begin{cases} \underline{\pi}, & \text{if } X = 0 \\ f(X), & \text{if } X > 0, \end{cases} \tag{9}$$

where $\underline{\pi} \geq 0, f(X) > 0$, and

$$\underline{\pi} + \int_0^V f(X)dX = 1.$$

With this definition of X, let L be total liability losses (payments against claims), and R be the insurance recovery, as before. Then:

$$L = \begin{cases} X, & \text{if } 0 \leq X < (1 + \lambda)V \\ (1 + \lambda)V, & \text{if } X \geq (1 + \lambda); \end{cases} \tag{10}$$

$$R = \begin{cases} L, & \text{if } 0 \leq X < \lambda V \\ \lambda V, & \text{if } X \geq \lambda V; \end{cases} \tag{11}$$

and the insured wealth state is:

$$W = V - P - L + R = V(1 - \lambda m) + (R - L). \tag{12}$$

Expected utility $Z(\lambda)$ can be written:

$$\begin{aligned} Z(\lambda) = E[U(W)] = U[V(1 - m\lambda)]\left[\underline{\pi} + \int_0^{\lambda V} f(X)dX\right] \\ + \int_{\lambda V}^{(1+\lambda)V} U[V(1 - m\lambda) + (\lambda V - X)]f(X)dX \\ + \int_{(1+\lambda)V}^{\infty} U[-m\lambda V]f(X)dX, \end{aligned} \tag{13}$$

with:

$$\begin{aligned} Z'(\lambda) = -mVU'[V(1 - m\lambda)]\left[\underline{\pi} + \int_0^{\lambda V} f(X)dX\right] \\ + \int_{\lambda V}^{(1+\lambda)V} V(1 - m)U'[V(1 - m\lambda) + (\lambda V - X)]f(X)dX \\ - mV\int_{(1+\lambda)V}^{\infty} U'(-m\lambda V)f(X)dX; \end{aligned} \tag{14}$$

$$\begin{aligned} Z''(\lambda) = m^2V^2U''[V(1 - m\lambda)][\underline{\pi} + F(\lambda V)] \\ + V^2U'(-m\lambda V)f[(1 + \lambda)V] - V^2U'[V(1 - m\lambda)]f(\lambda V) \\ + V^2(1 - m)^2\int_{\lambda V}^{(1+\lambda)V} U''[V(1 - m\lambda) + (\lambda V - X)]f(X)dX \\ + m^2V^2\int_{(1+\lambda)V}^{\infty} U''(-m\lambda V)f(X)dX. \end{aligned} \tag{15}$$

In (15), note that $U'' \leq 0$ is not sufficient to insure $Z''(\lambda) < 0$. Thus, for a relative interior maximum λ^0, it is necessary to postulate $Z''(\lambda^0) < 0$.

Theorem 6

If $m < 1$, $\underline{\pi} \geq 0$, optimality may require the purchase of any amount of insurance from zero to a coverage in excess of the asset value V. This follows from (14) by noting that

$$Z'(0) \gtreqless 0, \quad Z'(1) \gtreqless 0, \quad \underline{\pi} \geq 0.$$

Theorem 7

If $Z''(\lambda) < 0$, an expected wealth maximizer ($U'' = 0$, $U' = 1$) will buy a coverage $\lambda^0 V$ such that: $F[(1 + \lambda^0)V] \leq m + F(\lambda^0 V)$. This result follows from the Kuhn-Tucker conditions, $Z'(\lambda^0) = VF[(1 + \lambda^0)V] - VF(\lambda^0 V) - Vm \leqq 0$, if $<$, $\lambda^0 = 0$. In this case, we have, from (15), $Z''(\lambda) = V^2\{f[(1 + \lambda)V] - f(\lambda V)\}$, and a relative interior maxima requires $f[(1 + \lambda)V] < f(\lambda V)$. As in Theorem 4, profitable insurance operations require

$$m > A(\lambda) = \frac{1}{\lambda V}\int_0^{\lambda V} Xf(X)dX + \int_{\lambda V}^{\infty} f(X)dX,$$

and

$$\lim_{\lambda \to 0} A(\lambda) = F(\infty) = 1 - \underline{\pi}.$$

Therefore, $F[(1 + \lambda^0)V] - F(\lambda^0 V) < F(\infty) \leq m$, and $\lambda^0 = 0$.

If for every m in (14), there is a λ^0 satisfying $Z'(\lambda^0) \leq 0$, then this condition defines implicitly, the demand for liability insurance. Furthermore,

$$Z'(\lambda)\,|_{m=1} < 0 \quad \text{and} \quad Z'(\lambda)\,|_{m=0} > 0, \quad \text{for all } \lambda.$$

Hence, the domain of the demand function for liability insurance is the price interval $0 \leq m \leq 1$.

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