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Optimal investment in derivative securities

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Abstract. We consider the problem of optimal investment in a risky asset, and in derivatives written on the price process of this asset, when the underlying asset price process is a pure jump Lévy process. The duality approach of Karatzas and Shreve is used to derive the optimal consumption and investment plans. In our economy, the optimal derivative payoff can be constructed from dynamic trading in the risky asset and in European options of all strikes. Specific closed forms illustrate the optimal derivative contracts when the utility function is in the HARA class and when the statistical and risk-neutral price processes are in the variance gamma (VG) class. In this case, we observe that the optimal derivative contract pays a function of the price relatives continuously through time.

Key words: Lévy process, market completeness, stochastic duality, option pricing, variance gamma model

JEL Classification: G11,C61

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1 Introduction

In a classic paper, Merton [23] derived the optimal consumption and investment rules for investors maximizing the expected utility of their consumption in an economy consisting of a riskless asset and risky assets whose prices follow geometric Brownian motion. He showed that when investors have HARA (hyperbolic absolute risk aversion) utility functions, then one can solve for the optimal consumption and investment rules in closed form. Merton's analysis

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relied on Markov state processes and sought to obtain explicit solutions to the Hamilton-Jacobi-Bellman equation in this context. Subsequently, Pliska [26], Cox and Huang [5], and Karatzas et al. [16] all showed how to solve these problems in a non-Markovian context by applying stochastic duality theory in the context of complete markets. Our interest here is in determining optimal strategies for investing in derivative assets such as options of various strikes. In Merton's economy, the continuity of the underlying stock price process and the resulting completeness of markets renders options redundant. In contrast to Merton, we consider an economy in which dynamic trading in options is necessary and sufficient for market completeness. We accomplish this by assuming that both the statistical and risk-neutral processes for the underlying risky asset price are pure jump processes of finite variation which are infinitely divisible with independent increments. Our solution methods follow the duality approach as described in Karatzas and Shreve [17]. To generate explicit illustrative solutions, we focus on a particularly tractable subclass of processes called the variance gamma (VG) process, which is discussed in detail in [22], [21], and [20].

While the demand for options can arise from many sources, our focus on jumps stems from fundamental considerations regarding the nature of price processes in an arbitrage-free economy. Recently, Madan [19] has argued that all arbitrage-free continuous time price processes must be both semi-martingales and time-changed Brownian motion. Furthermore, it is argued that if the time change is not locally deterministic, then the resulting price process is discontinuous. If the price is modelled as a pure jump Lévy process with infinity activity, then the need for a continuous martingale component is obviated, since there will be an infinite number of small jumps in any time interval. Our focus on processes of finite variation is motivated by the observation in [19] that for such processes, the requirement of equivalence between the statistical and risk-neutral probabilities imposes no integrability or parametric restrictions. Thus, the resulting class of processes is quite rich in its ability to describe both the statistical and risk-neutral dynamics of market data, as is evidenced by the special VG model studied in [20].

For HARA investors in an economy for which the statistical and risk-neutral price processes are both VG processes, we show that the optimal derivative security is a contract written on the underlying asset's instantaneous returns (as measured by log price relatives), rather than on its final price. The optimal contract pays a function of this return when prices change. We also show how such a payoff may be synthesized by dynamic trading in options of all strikes and a single maturity. For statistical volatilities below risk-neutral ones, we show that the optimal derivative contract for intermediate levels of risk aversion is a collar that finances downside protection against drops in the market by selling upside gain. We also note in passing that certain houses have recently started offering derivatives whose payoffs are tied to returns in response to investor demand[1].

[1] For example, three recent innovations which satisfy these criteria are at-the-money forward start options, passport options and variance swaps.

The outline of this paper is as follows. Section 2 presents the financial market model. Section 3 gives a brief description of the martingale representation theorem in our pure jump context, which is used to generate our completeness results later. Self-financing strategies and the budget constraint are described in detail in Sect. 4. The finite horizon problem is formulated and solved in Sect. 5, while Sect. 6 considers the infinite horizon problem. The completeness of markets with respect to trading strategies is discussed in Sect. 7. Explicit solutions for the case of HARA utility are presented in Sect. 8. Closed form solutions for the context of VG dynamics for the statistical and risk-neutral log stock price process are given in Sect. 9. The finite and infinite horizon solutions are both discussed in Sect. 10, while Sect. 11 summarizes the paper and suggests extensions.

2 The financial market model

The time interval for the economy is $[0, \Upsilon]$, where Υ may be infinite. The economy has a money market account which pays interest at a constant continuously compounded interest rate r and has a time t unit account value of $B(t) = e^{rt}$. Also trading is a non-dividend paying stock, and European calls on the stock of all strikes $K > 0$. In both the finite and the infinite horizon case, we suppose that option markets are open at each $t > 0$ with an expiry date $T > t$ such that the time to maturity $T - t$ is bounded below by $\tau_1 > 0$ and bounded above by $\tau_2 > \tau_1$. For example, if τ_1 is one month and τ_2 is two months, then we only take positions in options with time to maturity between one and two months. As the time to maturity drops below a month, we change the expiry date of the options which we invest in. Thus, our interest is in determining the optimal investment in stock and in calls of some finite time to maturity bounded between two constants.

Let $X(t)$ be a pure jump finite variation Lévy process of homogeneous and independent increments with a Lévy density $k_P(x)$ and characteristic function[2]:

$$\phi_{X(t)}(u) = E\left[e^{iuX(t)}\right] = \exp\left[-t\int_{-\infty}^{\infty}(1 - e^{iux})k_P(x)dx\right]. \tag{1}$$

We suppose that the probability space $(\Omega, \Im, P)$ with stochastic basis $\mathbf{F} = \{\Im_{\mathbf{t}} | \mathbf{0} \leq t \leq \Upsilon\}$ is the canonical one generated by the process $X(t)$. The only uncertainty in our economy is that of the paths of the process $X(t)$. We further suppose that $\phi_{X(t)}(-i) < \infty$ for all $t < \Upsilon$, i.e. that the exponential of $X(t)$ has a finite expectation for all $t \in [0, \Upsilon]$.

The process for the stock price in our economy is adapted to the Lévy process $X(t)$ and has a constant mean rate of return μ. Specifically, we suppose that the stock price process is given by:

$$S(t) = S(0)\exp(\mu t)\exp\left(t\int_{-\infty}^{\infty}(1 - e^{x})\,k_P(x)dx\right)\exp(X(t)). \tag{2}$$

[2] The assumption of finite variation justifies the simple form of the characteristic function we employ as the Lévy density $k_P(x)$ integrates $|x| \wedge 1$ in a neighbourhood of 0.

The continuously compounded mean rate of return is μ by construction, while deviations of the continuously compounded rate of return from this mean are given by the compensated jumps of the Lévy process $X(t)$. Thus, the process for the stock price is the product of the drift factor $\exp(\mu t)$ with the stochastic exponential of the compensated process $X(t)$.

We ensure that we have an arbitrage-free economy by assuming the existence of an equivalent martingale measure Q.

Assumption. We suppose that under an equivalent probability Q, $X(t)$ is a finite variation Lévy process with Lévy density $k_Q(x)$ and characteristic function:

$$\phi^Q_{X(t)}(u) = E^Q\left[e^{iuX(t)}\right] = \exp\left[-t\int_{-\infty}^{\infty}(1-e^{iux})k_Q(x)dx\right].$$

We further suppose that $\phi^Q_{X(t)}(-i) < \infty$ for $t < \Upsilon$. Under Q the money market discounted stock price process is a Q martingale and we have that:

$$S(t) = S(0)\exp(rt)\exp\left(t\int_{-\infty}^{\infty}(1-e^x)\,k_Q(x)dx\right)\exp(X(t)). \tag{3}$$

Let T_t denote the maturity of the call at any time $t \in [0,\Upsilon]$ and let $C(t;K,T_t)$ denote the market price for the European call option of strike K and maturity T_t. It is important to note that T_t is a piecewise constant function of t and at the time when it jumps, a portfolio of options expiring at the earlier maturity is traded for a portfolio expiring at the later maturity. In particular, the derivative of T with respect to t is almost everywhere 0. Since the call options are also discounted martingales under the measure Q,we have::

$$\begin{aligned} C\,(t;K,T_t) &= E^Q\{e^{-r(T_t-t)}[S(T_t)-K]^+|\Im_t\} \\ &= \psi(t,S(t);K,T_t), \end{aligned} \tag{4}$$

where the function $\psi(t,S(t);K,T_t)$ is the call pricing function of the economy. The fact that ψ depends on just time and the current stock price follows from our assumption that the stock price is a Markov process under Q. This completes the description of our financial market model.

3 Martingale representation under Q

The process for the underlying uncertainty $X(t)$ under Q is by construction a multivariate point process, and the filtration of the economy is that generated by the process $X(t)$. It follows from Theorem 4.37 of Chapt. 3 and Proposition 1.28 of Chapt. 2, of Jacod and Shiryaev [15] that the martingale representation theorem holds for the economy and all local martingales $M(t)$ have the form:

$$M(t) = M(0) + \int_0^t\int_{-\infty}^{\infty}H(s,x)[m(dx,ds)-k_Q(x)dxds], \tag{5}$$

where $m(dx, ds)$ is the integer valued random measure associated with the process $X(t)$ and $H(s, x)$ is predictable and satisfies the condition that $\int_0^t \int_{-\infty}^{\infty} |H(s, x)| k_Q(x) dxds$ is locally integrable. This result is at the core of the market completeness for our economy as we shall later describe.

4 Self-financing and the budget constraint

Our economy is one of continuous trading in the money market account, the stock, and options of all strikes. A trading strategy is a triple $\pi = (\alpha, \beta, \gamma)$ where α, β are predictable measurable real valued processes and γ is a *predictable* process that takes values in the space $L^1([0, \Upsilon] X \mathbb{R}^+)$ of Lebesgue integrable functions on the positive half line such that:

$$\begin{aligned} \int_0^{\Upsilon} |\alpha(u)| du &< \infty \\ \int_0^{\Upsilon} |\beta(u)| du &< \infty \\ \int_0^{\Upsilon} \int_0^{\infty} |\gamma(u; K, T_u)| dK du &< \infty \end{aligned}$$

holds almost surely under P. The process $\alpha(t)$ specifies the number of shares held in the money market account at time t, valued at e^{rt}. Similarly, $\beta(t)$ is the number of stocks held at time t while $\gamma(t; K, T_t)$ is then number of calls of strike K and expiry T_t held at time t. The gains process $G(t)$ associated with this trading strategy is defined as the process:

$$\begin{aligned} G(t) &= \int_0^t \alpha(u) r \exp(ru) du + \int_0^t \beta(u) dS(u) \\ &\quad + \int_0^t \int_0^{\infty} \gamma(u; K, T_u) dC(u; K, T_u) dK. \end{aligned} \tag{6}$$

The trading strategy is said to be self-financed if:

$$G(t) = \alpha(t) e^{rt} + \beta(t) S(t) + \int_0^{\infty} \gamma(t; K, T_u) C(t; K, T_t) dK \quad \text{for all } t \in [0, \Upsilon], \tag{7}$$

i.e., if the value of the portfolio at any time is just the accumulation of investment gains from positions in the money market, stock, and options markets. By standard arguments, one may show that for self-financed trading strategies, the discounted gains process is the following compensated jump martingale that recognizes that no discounted gains arise from investment in the money market account:

$$e^{-rt}G(t) = \int_0^t \beta(u)e^{-ru}S(u_-)\int_{-\infty}^{\infty}(e^x - 1)\left[m(dx,du) - k_Q(x)dxdu\right] +$$
$$\int_0^t \int_{-\infty}^{\infty}\left\{\int_0^{\infty}\gamma(u;K)e^{-ru}[\psi(u,S(u_-)e^x;K)\right.$$
$$\left. -\psi(u,S(u_-);K]dK\right\}\left[m(dx,du) - k_Q(x)dxdu\right],$$

where the maturity argument T_u has been dropped to economize notation.

For our infinite horizon economy, we shall be interested in optimal derivative investment *and consumption* policies, and thus we define a consumption process as an $\Im_t$ − *progressively measurable* non-negative process satisfying almost surely that $\int_0^t c(u)du < \infty$ for all $t < \infty$. The wealth process $W(t)$ corresponding to a trading strategy $\pi = (\alpha, \beta, \gamma)$, a consumption process c, and initial wealth z satisfies the equation:

$$e^{-rt}W^{z,c,\pi}(t) = z - \int_0^t e^{-ru}c(u)du +$$
$$\int_0^t \beta(u)e^{-ru}S(u_-)\int_{-\infty}^{\infty}(e^x - 1)\left[m(dx,du) - k_Q(x)dxdu\right] +$$
$$\int_0^t \int_{-\infty}^{\infty}\left\{\int_0^{\infty}\gamma(u;K)e^{-ru}[\psi(u,S(u_-)e^x;K)\right.$$
$$\left. -\psi(u,S(u_-);K)]dK\right\}\left[m(dx,du) - k_Q(x)dxdu\right]. \tag{8}$$

Given initial wealth $W(0) = z$, we say that a consumption process c and trading strategy $\pi = (\alpha, \beta, \gamma)$ is admissible at z and write $(c, \pi) \in \mathscr{A}^t(z)$ if the wealth process corresponding to z, c, π satisfies:

$$W^{z,c,\pi}(t) \geq 0, \quad \text{for } t \in [0, \Upsilon],$$

almost surely.

Instead of seeking optimal trading strategies directly, we shall initially formulate our investment policies in terms of optimal wealth response functions, which determine the jump in the logarithm of wealth in response to a jump of size x in the logarithm of the stock price. This formulation ensures that the wealth process is bounded below by 0. Formally, a wealth response function $w(\cdot, t)$ is a predictable process taking values in the space of real valued functions on the real line, such that $\int_0^t \int_{-\infty}^{\infty} W(s_-)|e^{w(x,t)} - 1|k_Q(x)dxds$ is locally integrable.

Definition 1 *(Wealth Response delivered by a Trading Strategy): The wealth response $w(\cdot, t)$ delivered by a trading strategy $\pi = (\alpha, \beta, \gamma)$ is defined implicitly by:*

$$e^{-rt}W(t_-)[e^{w(x,t)} - 1] = \beta(t)e^{-rt}S(t_-)(e^x - 1)$$
$$+ \int_0^{\infty}\gamma(t;K)e^{-rt}[\psi(t,S(t_-)e^x;K) - \psi(t,S(t_-);K)]dK. \tag{9}$$

Once again, note that there is no gain attained in discounted wealth by investment in the money market account, so that all gains in discounted wealth arise purely from stock and option investment. The position in the money market account is always the residual between one's wealth level and the cost of the exposure sought. Hence, on substituting Eq. (9) into (8), one may define the exposure design budget constraint as follows:

Definition 2 *(Exposure Design Budget Constraint): The wealth process $W(t)$ corresponding to a wealth response function w, a consumption process c, and initial wealth z, satisfies:*

$$e^{-rt}W^{z,c,w}(t) = z - \int_0^t e^{-ru}c(u)du + \tag{10}$$
$$\int_0^t e^{-ru}W(u_-)\int_{-\infty}^{\infty}[e^{w(x,u)} - 1][m(dx,du) - k_Q(x)dxdu].$$

It is useful to note the relationship between the wealth response function and the predictable process $H(s,x)$ that represents the discounted wealth martingale and this is given by:

$$e^{-ru}W(u_-)[e^{w(x,u)} - 1] = H(u,x),$$

or equivalently that:

$$w(x,u) = \log\left[1 + \frac{e^{ru}H(u,x)}{W(u-)}\right],$$

where the positivity of the wealth process guarantees that the logarithm is a well-defined real number.

Given initial wealth $W(0) = z$, we say that a consumption process c and wealth response function w is admissible at z and write $(c,w) \in \mathscr{A}^e(z)$, if the wealth process corresponding to z, c, w satisfies:

$$W^{z,c,w}(t) \geq 0, \quad \text{for } t \in [0, \Upsilon],$$

almost surely.

5 The finite horizon investment problem

We first formulate and solve the investment problem in a finite horizon context with no intermediate consumption. For this, we follow the approach of Karatzas and Shreve [17] that relies on the completeness of markets which we now establish. A claim Y is said to be attainable by an exposure design from an initial investment $W(0)$ if there exists a wealth response function $w(x,t)$ such that $e^{-r\Upsilon}W(\Upsilon) = Y$ and $(0,w) \in \mathscr{A}^e(W(0))$. Furthermore, a financial market is called complete with respect to exposure designs if each $Y \in L^1(Q), Y > 0$ is attainable by an exposure design. In a similar way, we define a claim Y to be attainable by a trading strategy from an initial investment $W(0)$ if there exists a trading strategy $\pi = (\alpha, \beta, \gamma)$ such that $e^{-r\Upsilon}W(\Upsilon) = Y$ and $(0, \pi) \in \mathscr{A}^t(W(0))$. A financial market is called complete with respect to trading strategies if each $Y \in L^1(Q), Y > 0$ is attainable by a trading strategy.

Theorem 3 *The financial market is complete with respect to exposure designs.*

Proof Let $Y \in L^1(Q), Y > 0$ and define by $Y(t)$ the positive martingale:

$$Y(t) = E^Q \left[Y | \Im_t \right].$$

By the martingale representation theorem, there exists $H(s, x)$ such that:

$$Y(t) = Y(0) + \int_0^t \int_{-\infty}^{\infty} H(s, x)[m(dx, ds) - k_Q(x)dxds].$$

Define the exposure design $w(x, u)$ by:

$$Y(u_-)[e^{w(x,u)} - 1] = H(u, x),$$

and note that the resulting discounted wealth process $e^{-ru}W(u) = Y(u)$, whereby $e^{-r\Upsilon}W(\Upsilon) = Y(\Upsilon)$. The required condition that $\int_0^t \int_{-\infty}^{\infty} W(s_-)|e^{w(x,t)} - 1|k_Q(x)dxds$ be locally integrable is satisfied by virtue of the representation theorem and $(0, w) \in \mathscr{A}^e(W(0))$. □

In the finite horizon problem, the investor's objective function is:

$$U = E^P \left\{ u[W(\Upsilon)] \right\}, \tag{11}$$

for a horizon of Υ and a strictly increasing, strictly concave utility function $u[\cdot]$, with $\lim_{w \to \infty} u'(w) = 0$, defined over terminal wealth. The investor's dynamic exposure design problem may now be formalized as:

Program A

$$\begin{aligned}
\max_{[w(\cdot)]} U &= E^P \left\{ u[W(\Upsilon)] \right\} \\
\textit{subject to} &: \\
e^{-rt}W(t) &= W_0 + \int_0^t \int_{-\infty}^{\infty} W(s_-) \left[e^{w(x,s)} - 1 \right] [m(\omega; dx, ds) - k_Q(x)dxds], \\
& \quad \textit{and } W(t) \geq 0 \textit{ almost surely}.
\end{aligned}$$

Due to the completeness of the financial market, it is well-known (Cox and Huang [5]; Karatzas and Shreve [17]) that the investor's dynamic investment problem can be converted into the following static variational problem:

Program A$'$

$$\begin{aligned}&Max_{\xi} E^{P}\left[u(\xi)\right]\\&subject\ \ to:\\&E^{Q}\left[e^{-r\Upsilon}\xi\right]=W(0).\end{aligned}$$

In exactly the same manner as that of Chapt. 3 of Karatzas and Shreve [17], we can establish the following result:

Theorem 4 *Suppose that:*

$$\chi(y)=E^{Q}\left\{e^{-r\Upsilon}I\left[ye^{-r\Upsilon}\left(\frac{dQ}{dP}\right)_{\Upsilon}\right]\right\}<\infty$$

for any $y\in(0,\infty)$, *where:*

$$I[y]=(u')^{-1}[y].$$

Then program A' *has an optimal solution* $w(x,t)$ *such that its corresponding terminal wealth:*

$$W(\Upsilon)=I\left[y(W(0))e^{-r\Upsilon}\left(\frac{dQ}{dP}\right)_{\Upsilon}\right],$$

where $y(W(0))$ *satisfies:*

$$\chi(y(W(0)))=W(0).$$

Moreover, the optimal wealth process can be expressed as:

$$W(t)=e^{-r(\Upsilon-t)}\left(\frac{dQ}{dP}\right)_{t}^{-1}E^{P}\left[\left(\frac{dQ}{dP}\right)_{\Upsilon}W(\Upsilon)\,|\Im_{t}\right],$$

for $t\in[0,\Upsilon]$.

Proof See Karatzas and Shreve [17]. □

6 The infinite horizon consumption and investment problems

For infinite time horizons with intermediate consumption, we first need to identify the class of consumption processes that may be financed by a wealth response function. Specifically, a consumption process c is financed by a wealth response function w and an initial wealth z if $(c,w)\in\mathscr{A}^{e}(z)$.

Theorem 5 *Let* $z>0$ *be given and let* $c(\cdot)$ *be a consumption process such that:* $E^{Q}\left[\int_{0}^{\infty}e^{-ru}c(u)du\right]=z$.

Then there exists a wealth response function $w(s,x)$ *such that the pair* $(c,w)\in\mathscr{A}^{e}(z)$.

Proof Let us define $J(t) = \int_0^t e^{-ru}(\frac{dQ}{dP})_u c(u)du$ and consider the nonnegative martingale:

$$M(t) = E\left[J(\infty)\,|\Im_t\right].$$

We observe that:

$$\begin{aligned} M(t) &= \lim_{T\to\infty} E\left[\int_0^T \left(\frac{dQ}{dP}\right)_u e^{-ru}c(u)du\,|\Im_t\right] \\ &= \lim_{T\to\infty} E\left[\int_0^T \left(\frac{dQ}{dP}\right) e^{-ru}c(u)du\,|\Im_t\right] \\ &= E^Q\left[\int_0^\infty e^{-ru}c(u)du|\Im_t\right]. \end{aligned} \quad (12)$$

Hence, $M(t)$ is a Q martingale and so by the martingale representation theorem referred to above, there exists a predictable process $H(u,x)$ such that $\int_0^t \int_{-\infty}^{\infty} |H(u,x)|k_Q(x)dxds$ is locally integrable and:

$$M(t) = z + \int_0^t H(u,x)[m(dx,ds) - k_Q(x)dxds].$$

We also observe from (12) that:

$$M(t) = \int_0^t e^{-ru}c(u)du + E^Q\left[\int_t^\infty e^{-ru}c(u)du|\Im_t\right].$$

It follows that:

$$\begin{aligned} E^Q\left[\int_t^\infty e^{-ru}c(u)du|\Im_t\right] &= z - \int_0^t e^{-ru}c(u)du \\ &+ \int_0^t \int_{-\infty}^{\infty} H(u,x)[m(dx,ds) - k_Q(x)dxds]. \end{aligned}$$

Define:

$$e^{-rt}W(t) = E^Q\left[\int_t^\infty e^{-ru}c(u)du|\Im_t\right]$$

as a nonnegative process, and let:

$$w(x,u) = \log\left[1 + \frac{e^{ru}H(u,x)}{W(u-)}\right].$$

We then have that:

$$\begin{aligned} e^{-rt}W(t) &= z - \int_0^t e^{-ru}c(u)du + \\ &\int_0^t e^{-ru}W(u_-)[e^{w(x,u)} - 1][m(dx,ds) - k_Q(x)dxds], \end{aligned}$$

or that $(c,w) \in \mathscr{A}^e(z)$. □

We now formulate the infinite time horizon consumption and investment problem. Following Merton[23], we suppose that an investor in the economy of Sect. 2.1 has a preference ordering over potential consumption streams $c = \{c(t), t > 0\}$ given by the expected utility of the consumption stream. Let the instantaneous flow rate of utility at time t be $u[c(t)]$ for some concave utility function $u[\cdot]$. The discounted utility of the consumption stream c over the infinite horizon is:

$$U = \int_0^\infty e^{-\beta t} u[c(t)]dt, \tag{13}$$

where β is the pure rate of time preference for the investor.

The investor's infinite horizon problem may now be formalized for the stock price driven by a Lévy process as:

Program B

$$\begin{aligned}
\max_{[c(\cdot),w(\cdot)]} U &= E^P \left\{ \int_0^\infty e^{-\beta s} u[c(s)]ds \right\} \\
\textit{subject} \quad \textit{to} &: \\
e^{-rt} W(t) &= W_0 - \int_0^t e^{-rs} c(s)ds \\
&\quad + \int_0^t \int_{-\infty}^\infty e^{-rs} W(s_-) \left[e^{w(x,s)} - 1 \right] [m(\omega; dx, ds) - k_Q(x)dxds], \\
\textit{and} \ W(t) &\geq 0 \ \textit{almost everywhere}.
\end{aligned}$$

This problem can be solved as in Sect. 3.9 of Karatzas and Shreve [17].

Theorem 6 *Suppose that there exists an initial wealth $W(0)$ such that:*

$$V_\infty(W(0)) = \sup_{(c,w)\in\mathscr{A}^e(W(0))} E\left[\int_0^\infty u[c(t)]dt\right] < \infty.$$

Let:

$$\chi(y) = E\left\{\int_0^\infty e^{-rt} \left(\frac{dQ}{dP}\right)_t I\left[ye^{-rt}\left(\frac{dQ}{dP}\right)_t\right] dt\right\} < \infty \ \textit{for} \ 0 < y < \infty$$

Then:

$$c_*(t) = I\left[y_*(W(0))e^{-rt}\left(\frac{dQ}{dP}\right)_t\right]$$

is the optimal consumption process where:

$$\chi(y_*(W(0)) = W(0).$$

Furthermore, the optimal wealth process is:

$$W(t) = E^Q \left[\int_t^\infty e^{-r(u-t)} c(u) du | \Im_t \right].$$

Proof See Karatzas and Shreve [17]. □

7 Completeness with respect to trading strategies

The ability to dynamically trade in the money market account, the stock, and in European calls of all strikes can under certain conditions allow an investor to generate any desired consumption stream and wealth response function consistent with initial wealth. This section addresses the conditions under which the financial market is complete with respect to trading strategies, given that it is complete with respect to exposure designs. For both the finite and the infinite horizon, what is at issue is the determination of a portfolio $\pi = (\alpha, \beta, \gamma)$ such that the wealth response to market jumps given by a wealth response function $w(x, u)$ matches the response of the portfolio values. Hence we seek $\pi = (\alpha, \beta, \gamma)$, consistent with (9), such that the positions in stocks and options access the desired exposure:

$$\begin{aligned} W(u_-)[e^{w(x,u)} - 1] &= \beta S(u_-)(e^x - 1) \\ &+ \int_0^\infty \gamma(u; K, T_u)[\psi(u, S(u_-)e^x; K, T_u) - \psi(u, S(u_-); K, T_u)] dK, \end{aligned} \tag{14}$$

and $e^{ru}\alpha = W(u_-) - \beta S(u_-) - \int_0^\infty \gamma(u; K, T_u)\psi(u, S(u_-); K, T_u) dK$.

In this section, we identify the portfolio position that delivers the desired wealth response at all times $u \in [0, \Upsilon]$. For this purpose, we employ a result from Carr and Madan [6], where it is shown that for any twice differentiable function $\phi_u(S)$, thought of as the payoff at time T_u from a contingent claim on the stock price, one may construct a position in unit face value bonds, the stock, and calls on this stock that replicate this claim as follows

$$\phi_u(S) = \phi_u(0) + \phi_u'(0) S + \int_0^\infty \phi_u''(K)(S - K)^+ dK. \tag{15}$$

As we are interested in replicating an exposure design and the bond provides no response to jumps in the stock price, we may set the bond position to zero without loss of generality. Hence, our interest is in the stock and option positions that accomplish an exposure design.

Let $V^\phi(u, S)$ be the value of the claim ϕ at time u, written as function of the stock price prevailing at time u. If we can determine the claim ϕ such that

$$W(u_-)[e^{w(x,u)} - 1] = V^\phi(u, S(u_-)e^x) - V^\phi(u, S(u_-)), \tag{16}$$

then it follows that:

$$\begin{aligned}\alpha(u) &= W(u_-) - \phi_u'(0)S(u_-) - \int_0^\infty \phi_u''(K)\psi(u, S(u_-); K, T_u)dK \\ \beta(u) &= \phi_u'(0) \\ \gamma(u; K, T_u) &= \phi_u''(K).\end{aligned}$$

To determine ϕ, we proceed as follows. From (3), the time T_u stock price is given by:

$$S_{T_u} = S_u e^{\eta(T_u - u) + Y_{u,T_u}}, \tag{17}$$

where $\eta = r + \int_{-\infty}^{\infty}(1 - e^x)k_Q(x)dx$, $Y_{u,T_u} \equiv X_{T_u} - X_u$. It follows that $V^\phi(u, S)$ is:

$$V^\phi(u, S) \equiv e^{-r(T-u)} \int_{-\infty}^{\infty} \phi_u(Se^{\eta(T_u - u)+y})q(y)dy,$$

where $q(y)$ is the risk-neutral probability density function of Y_{u,T_u}. Hence the function ϕ is such that:

$$\begin{aligned}W(u_-)[e^{w(x,u)} - 1] = e^{-r(T_u - u)} \int_{-\infty}^{\infty} [\phi_u(S(u_-)e^{x+\eta(T_u-u)+y}) \\ -\phi_u(S(u_-)e^{\eta(T_u-u)+y})]q(y)dy.\end{aligned} \tag{18}$$

To explicitly solve for ϕ_u, we define:

$$g(x) = e^{r(T_u-u)}W(u_-)[e^{w(x,u)} - 1],$$

and write:

$$h(y) = \phi_u(S(u_-)e^{\eta(T_u-u)+y}),$$

whereby we have that:

$$\phi_u(S) = h(\log(S/S(u_-)) - \eta(T_u - u)). \tag{19}$$

The Eq. (18) defining ϕ may then be written in terms of h and g as:

$$g(x) = \int_{-\infty}^{\infty} [h(x + y) - h(y)]q(y)dy,$$

or that:

$$g'(x) = \int_{-\infty}^{\infty} h'(x + y)q(y)dy.$$

Let $v = x + y$ be a change of variable in the integral and let $\tilde{q}(y) = q(-y)$:

$$g'(x) = \int_{-\infty}^{\infty} h'(v)\tilde{q}(x - v)dv.$$

Taking Fourier transforms of both sides:

$$\mathscr{F}(g') = \mathscr{F}(h')\mathscr{F}(\tilde{q})$$

yields the Fourier transform of h' as the ratio of the transforms of g' and $\tilde{q}$. The function h can be identified on Fourier inversion and integration with $h(0) = 0$. The function ϕ_u follows from Eq. (19) and this determines the position in options. We note that this position changes as t evolves, implying that dynamic trading in the stock and options is required in general.

8 Explicit solutions of the consumption/investment problems

We first present explicit solutions for the infinite horizon problem, and then consider the finite horizon problem. We restrict attention to utility functions in the HARA (Hyperbolic Absolute Risk Aversion) class whereby:

$$u[c] = \frac{\gamma}{1-\gamma}\left(\frac{\alpha}{\gamma}c - A\right)^{1-\gamma}.$$

For such utility functions, absolute risk aversion is hyperbolic in consumption:

$$-\frac{u''[c]}{u'[c]} = \frac{1}{\frac{c}{\gamma} - \frac{A}{\alpha}}. \tag{20}$$

Alternatively, risk tolerance is linear in consumption with the slope parameter (cautiousness) being $1/\gamma$. The utility function is only defined for values of $c > \gamma A/\alpha$.

For this utility function, we have that:

$$I[y] = U'^{-1}[y] = \frac{\gamma}{\alpha}\left[\left(\frac{y}{\alpha}\right)^{-1/\gamma} + A\right].$$

From Theorems 4 and 2, we may infer the optimal consumption and final wealth for the infinite and finite horizon problems respectively. One may then compute the form of the optimized utility function and observe that for the infinite horizon case, we have a differentiable function of the current wealth level, while in the finite horizon case, we have a differentiable function of current wealth and time. The exact structure of the optimal local exposure design function $w(x,t)$ is then determined by solving explicitly the associated Hamilton-Jacobi-Bellman equations. We follow this procedure for our explicit solutions.

8.1 Explicit solution of the infinite horizon problem

In the infinite horizon problem, we have from Theorem 4 that the optimized initial expected utility is $V_\infty(W)$, for $W = W(0)$, which we now denote as $J(W)$ in keeping with the notation used in Merton [23]. Theorem 4 provides us with an explicit computation for this optimized expected utility function $J(W)$. In particular, let:

$$\xi_t = e^{-rt}\left(\frac{dQ}{dP}\right)_t.$$

Then $y_*(W)$ satisfies:

$$\begin{aligned} W &= E\{\int_0^\infty \xi_t I[y_*(W)\xi_t]dt\} \\ &= E\{\int_0^\infty \xi_t \frac{\gamma}{\alpha}\left[\left(\frac{y_*(W)\xi_t}{\alpha}\right)^{-1/\gamma} + A\right] dt\} \\ &= y_*(W)^{-1/\gamma}\gamma\alpha^{1/\gamma-1}E\left[\int_0^\infty \xi_t^{1-1/\gamma}dt\right] + \frac{A\gamma}{\alpha r}. \end{aligned}$$

We let:

$$M_0 = E\left[\int_0^\infty \xi_t^{1-1/\gamma} dt\right]$$

and require that M_0 be finite for the change of measure process we work with and then write:

$$y_*(W) = \left(\frac{W - \frac{A\gamma}{\alpha r}}{\gamma\alpha^{1/\gamma-1}M_0}\right)^{-\gamma}.$$

It follows that:

$$c_*(t) = \frac{\xi_t^{-1/\gamma}\left(W - \frac{A\gamma}{\alpha r}\right)}{M_0} + \frac{A\gamma}{\alpha}.$$

We may now evaluate $J(W)$ as:

$$J(W) = \frac{\gamma}{1-\gamma}\left(\frac{\alpha}{\gamma M_0}\right)^{1-\gamma}\left(W - \frac{A\gamma}{\alpha r}\right)^{1-\gamma} E\left[\int_0^\infty e^{-\beta s}\xi_s^{1-1/\gamma} ds\right],$$

and observe that $J(W)$ is differentiable in W and is in the *HARA* class itself.

For an explicit determination of the exposure design function, we employ the framework of the Hamilton-Jacobi-Bellman equation and relate the exposure design explicitly to the derivatives of the optimized wealth function $J(W)$. For this purpose, we let the infinitessimal generator of the Markov wealth process under the measure P and under the controls $[c(\cdot), w(\cdot)]$ be denoted by $A^{c,w}$. Given our assumption that the process is one dimensional Markov, this generator is defined by (see Garroni and Menaldi [10], page 50, or Gihman and Skorohod[12], page 291):

$$\begin{aligned} A^{c,w}[\varphi](W) \equiv & \left\{ rW - c - \int_{-\infty}^{\infty} W\left[e^{w(x)} - 1\right] k_Q(x)dx \right\} \varphi_W \\ & + \int_{-\infty}^{\infty} \left[\varphi(We^{w(x)}) - \varphi(W)\right] k_P(x)dx. \end{aligned} \tag{21}$$

It is shown in Rishel[27], Eqs. 8.20 and 8.21, that under the optimal controls $[c^*(\cdot), w^*(\cdot)]$:

$$A^{c^*,w^*}[J] - \beta J + u[c^*(\cdot)] = 0, \tag{22}$$

and the optimal controls are given by:

$$c^*, w^* = \arg\max_{c,w}\left\{A^{c,w}[J] - \beta J + u[c]\right\}. \tag{23}$$

Substitution of the generator (21) into (23) defines the instantaneous optimization problem determining the optimal controls c^*, w^* as:

$$c^*, w^* \quad = \quad \arg\max_{c,w} \left\{ \left\{ rW - c - \int_{-\infty}^{\infty} W \left[e^{w(x)} - 1 \right] k_Q(x) dx \right\} J_W + \int_{-\infty}^{\infty} \left[J(We^{w(x)}) - J(W) \right] k_P(x) dx - \beta J + u[c] \right\}. \tag{24}$$

Differentiating with respect to c and w yields the first order conditions:

$$J_W = u'[c^*], \tag{25}$$

and:

$$J_W(We^{w^*(x)}) \frac{k_P(x)}{k_Q(x)} = J_W(W), \tag{26}$$

respectively. Solving for the optimal consumption and the optimal wealth response yields:

$$c^* = (u')^{-1}[J_W] \tag{27}$$

and:

$$w^*(x) = \ln \left[(J_W)^{-1} \left(J_W(W) \frac{k_Q(x)}{k_P(x)} \right) \right] - \ln(W). \tag{28}$$

Equation (26) defining the optimal wealth response function has a useful economic interpretation. The random marginal utility per dollar when the jump size is x and the wealth response function is $w^*(x)$ is $J_W(We^{w^*(x)})$. The marginal utility *expected* in this state is $J_W(We^{w(x)})k_P(x)$, since $k_P(x)$ provides the arrival rate of the state. The ex-ante cost of obtaining ex-post payoffs in state x is $k_Q(x)$. Thus the ratio $J_W(We^{w^*(x)})k_P(x)/k_Q(x)$ is the expected marginal utility per ex-ante dollar invested in state x. Hence, (26) expresses the classical Marshallian principle that the optimal policy is determined so that the expected marginal utility earned per ex-ante dollar spent is equal across all states.

8.2 Explicit solution of finite horizon problem

For the finite horizon problem, we have from Theorem 2 that the optimized initial expected utility is of the form $J(W, t)$, where in particular:

$$J(W, t) = E\{u[W(\Upsilon)] \,|\Im_t\},$$

and the terminal wealth position is given by:

$$W(\Upsilon) = I\left[y(W)\xi_\Upsilon\right].$$

We also know that $y(W)$ is implicitly defined by:

$$\begin{aligned} W &= E\{\xi_\Upsilon I[y(W)\xi_\Upsilon]\} \\ &= y(W)^{-1/\gamma}\alpha^{1/\gamma-1}\gamma E\left[\xi_\Upsilon^{1-1/\gamma}\right] + \frac{A\gamma}{\alpha}e^{-r\Upsilon}. \end{aligned}$$

We now require that:

$$M_1 = E\left[\xi_\Upsilon^{1-1/\gamma}\right]$$

be finite and write:

$$y(W) = \left(\frac{W - \frac{A\gamma}{\alpha}e^{-r\Upsilon}}{\alpha^{1/\gamma-1}\gamma M_1}\right)^{-\gamma}.$$

It follows that:

$$W(\Upsilon) = \frac{W - \frac{A\gamma}{\alpha}e^{-r\Upsilon}}{M_1}\xi_\Upsilon^{-1/\gamma} + \frac{A\gamma}{\alpha}$$

and:

$$J(W,0) = \frac{\gamma}{1-\gamma}\left(\frac{\alpha}{\gamma}\right)^{1-\gamma} M_1^\gamma \left(W - \frac{A\gamma}{\alpha}e^{-r\Upsilon}\right)^{1-\gamma},$$

and more generally we have that:

$$J(W,t) = \frac{\gamma}{1-\gamma}\left(\frac{\alpha}{\gamma}\right)^{1-\gamma} E\left[\xi_\Upsilon^{1-1/\gamma}\,|\Im_t\right]^\gamma \left(W - \frac{A\gamma}{\alpha}e^{-r(\Upsilon-t)}\right)^{1-\gamma}.$$

We observe that for each t, J is differentiable in W and is in the *HARA* class. Furthermore, J is differentiable in t provided $E\left[\xi_\Upsilon^{1-1/\gamma}\,|\Im_t\right]$ is differentiable in t. This is a property of the specified measure change and can be checked for each application and we suppose it is valid. For the specific exposure design, we let the infinitessimal generator of the Markov wealth process under the measure P and control $w(\cdot,\cdot)$ be once again denoted by A^w. The generator now applies to functions that depend on both wealth and time and is given by:

$$\begin{aligned} A^w[\varphi](t,W) \equiv\; & \varphi_t + \left\{rW - \int_{-\infty}^{\infty} W\left[e^{w(x,t)}-1\right]k_Q(x)dx\right\}\varphi_W \\ & + \int_{-\infty}^{\infty}\left[\varphi(t,We^{w(x,t)}) - \varphi(t,W)\right]k_P(x)dx. \end{aligned} \tag{29}$$

In contrast to the infinite horizon problem, A^w does not depend on the consumption path, but does depend on t as does the wealth response function, $w(\cdot,\cdot)$. By Rishel [27], under the optimal control $w^*(x,t)$, we must have that:

$$A^{w^*}[J] = 0, \tag{30}$$

and the optimal control is given by:

$$w^* = \arg\max_w\{A^w[J]\}. \tag{31}$$

In addition, we must have that at the horizon date, the J function coincides with the terminal utility function:

$$J(W,T) = u[W]. \tag{32}$$

Substitution of the generator (29) into (31) defines the instantaneous optimization problem determining the optimal control w^* as:

$$\begin{aligned} w^* &= \arg\max_w \left\{ J_t + \left\{ rW - \int_{-\infty}^{\infty} W \left[e^{w(x,t)} - 1 \right] k_Q(x)dx \right\} J_W \right. \\ &\quad \left. + \int_{-\infty}^{\infty} \left[J(t, We^{w(x,t)}) - J(t,W) \right] k_P(x)dx \right\}. \end{aligned} \tag{33}$$

Differentiating with respect to w yields the first order condition:

$$J_W \left(t, We^{w^*(x,t)} \right) \frac{k_P(x)}{k_Q(x)} = J_W(t,W). \tag{34}$$

Once again investment is chosen so that the last dollar invested in each state increases expected utility by the same amount. Solving for $w^*(x,t)$ yields the optimal wealth response function:

$$w^*(t,x) = \ln \left\{ (J_W)^{-1} \left[J_W(t,W) \frac{k_Q(x)}{k_P(x)}, t \right] \right\} - \ln(W). \tag{35}$$

9 Optimal consumption and wealth response for HARA investors in VG economies

We now make specific assumptions on the statistical and risk-neutral probability measures with a view to obtaining closed form solutions to our consumption and exposure design problems. For our choice of statistical and risk-neutral processes, we restrict attention to the *VG* class. Since Clark [7], it has been well-known that the excess kurtosis observed in historical returns and in risk-neutral densities (butterfly spreads) can be generated from standard Brownian motion by randomizing its clock. When a gamma process is used as the subordinator for standard Brownian motion, the resulting stochastic process is known as a symmetric VG process. This process has no skewness which is consistent with the empirical evidence for historical returns as presented in Madan et al. [20] (henceforth MCC). Thus, we will assume that under the statistical measure P, the log of the price is driven by a symmetric VG process:

$$\ln S_t = \ln S_0 + \alpha t + sW(G(t;\kappa)), t > 0$$

where α is the drift in the log, s is the volatility, and $\{G(t;\kappa), t > 0\}$ is a gamma process with a mean rate of unity and a variance rate of $\kappa \geq 0.$. Recall from

(2) that α differs from the mean rate of return μ by $\int_{-\infty}^{\infty} (1 - e^x) k_P(x) dx$. For the symmetric VG process, $sW(G(t, \kappa))$, MCC show that the Lévy density is given by the symmetric function:

$$k_P(x) = \frac{1}{\kappa \mid x \mid} \exp\left(-\sqrt{\frac{2}{\kappa}} \frac{\mid x \mid}{s} \right). \tag{36}$$

Hence, the statistical price process is:

$$S_t = S_0 \exp\left[\mu t + \frac{t}{\kappa} \ln(1 - s^2\kappa/2) + sW(G(t;\kappa)) \right], \tag{37}$$

since:

$$t \int_{-\infty}^{\infty} (1 - e^x) k_P(x) dx = \frac{t}{\kappa} \ln(1 - s^2\kappa/2). \tag{38}$$

While historical returns display negligible skewness, the risk-neutral probability distributions implied by index option prices typically display appreciable negative skewness. To generate a skewed risk-neutral process for the log of the stock price, the driver can be amended to be Brownian motion *with drift*, evaluated under a gamma time change. MCC show that if the drift parameter θ is assumed to be negative, then the resulting (asymmetric) VG process will have negative skewness. Thus, suppose we assume that the accumulated jumps in the log of the price are given by $X_t = \theta G(t;\nu) + \sigma W(G(t;\nu))$. For the risk-neutral VG process, MCC show that the Lévy density generalizes to:

$$k_Q(x) = \frac{\exp(\theta x/\sigma^2)}{\nu \mid x \mid} \exp\left(-\sqrt{\frac{2}{\nu} + \frac{\theta^2}{\sigma^2}} \frac{\mid x \mid}{\sigma} \right). \tag{39}$$

The difference between the drift in the log and the risk-neutral expected stock return r simplifies to:

$$t \int_{-\infty}^{\infty} (1 - e^x) k_Q(x) dx = \frac{t}{\nu} \ln(1 - \theta\nu - \sigma^2\nu/2), \tag{40}$$

and so one may write the stock price process as:

$$S_t = S_0 \exp\left[rt + \frac{t}{\nu} \ln(1 - \theta\nu - \sigma^2\nu/2) + \theta G(t;\nu) + \sigma W(G(t;\nu)) \right]. \tag{41}$$

The three parameters σ, θ, and ν control the volatility, skewness, and kurtosis of the log price respectively. We note from the Lévy density (39) that when $\theta < 0$, the left tail is fatter than the right tail. Meanwhile, increases in ν symmetrically increase both tails.

We now assume that all options are priced by the closed form formulas obtained in MCC. Furthermore, the discounted stock and option prices are Q

martingales by construction. Under these further assumptions on the statistical and risk-neutral processes, we next solve for the optimal consumption and wealth response in both the infinite and finite horizon problems.

9.1 Solution of infinite horizon problem

We have observed that the function $J(W)$ has the form:

$$J(W) = \frac{\gamma}{1-\gamma}\left(\frac{\eta}{\gamma}W - B\right)^{1-\gamma}, \tag{42}$$

for constants η and B to be determined.

The first step is to solve for the optimal consumption levels and wealth response functions consistent with (27) and (28). Substituting the assumed form of the J function into the first order condition (25) yields the optimal consumption flow:

$$c^* = \gamma\left[\frac{A}{\alpha} - \left(\frac{\alpha}{\eta}\right)^{\frac{1}{\gamma}-1}\frac{B}{\eta}\right] + \left(\frac{\alpha}{\eta}\right)^{\frac{1}{\gamma}-1} W. \tag{43}$$

We observe that as in Merton, the optimal consumption is a linear function of the investor's wealth. For the optimal wealth response function, we first evaluate the measure change:

$$\frac{k_Q(x)}{k_P(x)} = \frac{\kappa}{\nu}\exp\left(\zeta x + \lambda \mid x \mid\right), \tag{44}$$

where:

$$\zeta \equiv \frac{\theta}{\sigma^2} \text{ and } \lambda \equiv \frac{\sqrt{\frac{2}{\kappa}}}{s} - \frac{\sqrt{\frac{2}{\nu} + \frac{\theta^2}{\sigma^2}}}{\sigma}. \tag{45}$$

Substituting the candidate for $J(W)$ into the first order condition (26) yields the optimal wealth response function:

$$w^*(x) = \ln\left[\frac{\gamma}{\eta}\frac{B}{W} + \left(1 - \frac{\gamma}{\eta}\frac{B}{W}\right)\left(\frac{\kappa}{\nu}\right)^{-\frac{1}{\gamma}}\exp\left(-\frac{\zeta}{\gamma}x - \frac{\lambda}{\gamma}\mid x \mid\right)\right]. \tag{46}$$

Exponentiating both sides implies that the optimal wealth *relative* is affine in a power of the stock price relative. We consider further the case where the investor and the market agree on the time change, i.e. $\kappa = \nu$. In this case, the optimal *return* can be written as:

$$e^{w^*(x)} - 1 = \left(1 - \frac{\gamma}{\eta}\frac{B}{W}\right)\left[\exp\left(-\frac{\zeta}{\gamma}x - \frac{\lambda}{\gamma}\mid x \mid\right) - 1\right]. \tag{47}$$

Having explicitly solved for c^* and w^*, we now need to solve for η and B. Substituting c^* and w^* into the i.d.e. (22) and simplifying yields:

$$\left(\frac{\eta}{\gamma}W - B\right)^{-\gamma} \quad \left\{\left[\frac{B\gamma}{1-\gamma}\left(\beta - \gamma\left(\frac{\alpha}{\eta}\right)^{\frac{1}{\gamma}-1}\right) - \frac{\eta\gamma A}{\alpha}\right] + \left[\frac{\eta}{1-\gamma}\left(r(1-\gamma) - \left(\beta - \gamma\left(\frac{\alpha}{\eta}\right)^{\frac{1}{\gamma}-1}\right)\right)\right]W\right\}$$

$$+\left(\frac{\eta}{\gamma}W - B\right)^{1-\gamma} \quad \left\{\frac{\gamma}{1-\gamma}\int_{-\infty}^{\infty}\left[\exp\left(-\frac{\zeta(1-\gamma)}{\gamma}x - \frac{\lambda(1-\gamma)}{\gamma} \mid x \mid\right) - 1\right]k_P(x)dx - \gamma\int_{-\infty}^{\infty}\left[\exp\left(-\frac{\zeta}{\gamma}x - \frac{\lambda}{\gamma} \mid x \mid\right) - 1\right]k_Q(x)dx\right\} = 0. \quad (48)$$

Both integrals can be done analytically:

$$c_1 \equiv \int_{-\infty}^{\infty}\left(\exp\left[-\frac{\zeta(1-\gamma)}{\gamma}x - \frac{\lambda(1-\gamma)}{\gamma} \mid x \mid\right) - 1\right]k_P(x)dx \quad (49)$$

$$= -\frac{1}{\kappa}\ln\left\{\left[1 + \frac{(1-\gamma)s(\lambda+\zeta)/\gamma}{\sqrt{2/\kappa}}\right]\left[1 + \frac{(1-\gamma)s(\lambda-\zeta)/\gamma}{\sqrt{2/\kappa}}\right]\right\}$$

$$c_2 \equiv \int_{-\infty}^{\infty}\left[\exp\left(-\frac{\zeta}{\gamma}x - \frac{\lambda}{\gamma} \mid x \mid\right) - 1\right]k_Q(x)dx \quad (50)$$

$$= -\frac{1}{\nu}\ln\left\{\left[1 + \frac{(\lambda+\zeta)/\gamma}{\sqrt{2/\nu + \theta^2/\sigma^2} - \theta/\sigma^2}\right]\left[1 + \frac{(\lambda-\zeta)/\gamma}{\sqrt{2/\nu + \theta^2/\sigma^2} + \theta/\sigma^2}\right]\right\}.$$

Substituting these expressions into (48) and simplifying yields:

$$0 = \left(\frac{\eta}{\gamma}W - B\right)^{-\gamma}\left\{\left[\frac{B\gamma}{1-\gamma}\left(\beta - \gamma\left(\frac{\alpha}{\eta}\right)^{\frac{1}{\gamma}-1}\right) - \frac{\eta\gamma A}{\alpha} - B\left(\frac{\gamma}{1-\gamma}c_1 - \gamma c_2\right)\right] + \left[\frac{\eta}{\gamma}\left(\frac{\gamma}{1-\gamma}c_1 - \gamma c_2\right) + \frac{\eta}{1-\gamma}\left(r(1-\gamma) - \left(\beta - \gamma\left(\frac{\alpha}{\eta}\right)^{\frac{1}{\gamma}-1}\right)\right)\right]W\right\}.$$

It follows that we must have the constant and linear terms in braces both equal to zero. These may be solved explicitly for η and B:

$$\eta = \alpha\left[\frac{1-\gamma}{\gamma}(c_2 - r) + \frac{\beta - c_1}{\gamma}\right]^{-\frac{\gamma}{1-\gamma}}, \quad (51)$$

and:

$$B = \frac{\eta\gamma A}{\alpha}\left\{\frac{\gamma}{1-\gamma}\left[\beta - \gamma\left(\frac{\alpha}{\eta}\right)^{\frac{1}{\gamma}-1}\right] - \frac{\gamma}{1-\gamma}c_1 + \gamma c_2\right\}^{-1}. \tag{52}$$

This completes the solution of program A for the case $\kappa = \nu$.

When κ differs from ν, we observe from (46) that at $x = 0$:

$$w^*(0) = \ln\left[\frac{\gamma}{\eta}\frac{B}{W} + \left(1 - \frac{\gamma}{\eta}\frac{B}{W}\right)\left(\frac{\kappa}{\nu}\right)^{-\frac{1}{\gamma}}\right], \tag{53}$$

which is positive for $\kappa < \nu$ and negative otherwise. Since the optimal wealth exposure design is continuous at zero and there are an infinite number of jumps whose absolute size is arbitrarily small, such payoffs violate the local integrability condition for wealth response functions. Thus, for this case we suppose that derivatives are available only to enable investors to alter positions for values of x where $\mid x \mid > a$ for some small value of a. Hence, we restrict the optimal wealth response function $w^*(x)$ to be of the form:

$$w^*(x) = b(x)1_{|x|>a}, \tag{54}$$

and we solve for $b(x)$. The first order condition (26) defining $w^*(x)$ is now applied to just the case $|x| > a$ and defines:

$$b(x) = \ln\left[\frac{\gamma}{\eta}\frac{B}{W} + \left(1 - \frac{\gamma}{\eta}\frac{B}{W}\right)\left(\frac{\kappa}{\nu}\right)^{-\frac{1}{\gamma}}\exp\left(-\frac{\zeta}{\gamma}x - \frac{\lambda}{\gamma}\mid x\mid\right)\right].$$

For $|x| < a$, $w(x) = 0$, and the terms involving integration over x with respect to k_P or k_Q in (22) are altered to have zero integrands for $|x| < a$, while for $|x| > a$, the integrands are altered to revise the definitions of c_1 and c_2 to:

$$c_1' \equiv \int_{|x|>a}\left[\left(\frac{\kappa}{\nu}\right)^{-\frac{1-\gamma}{\gamma}}\exp\left(-\frac{\zeta(1-\gamma)}{\gamma}x - \frac{\lambda(1-\gamma)}{\gamma}\mid x\mid\right) - 1\right]k_P(x)dx$$

and:

$$c_2' \equiv \int_{|x|>a}\left[\left(\frac{\kappa}{\nu}\right)^{-\frac{1}{\gamma}}\exp\left(-\frac{\zeta}{\gamma}x - \frac{\lambda}{\gamma}\mid x\mid\right) - 1\right]k_Q(x)dx$$

respectively. These integrations may be performed in terms of the exponential integral function and yield:

$$c_1' = \left(\frac{\kappa}{\nu}\right)^{-\frac{1-\gamma}{\gamma}}\left(d_{1p} + d_{1n}\right) - 2e_p \tag{55}$$

$$c_2' = \left(\frac{\kappa}{\nu}\right)^{-\frac{1}{\gamma}}\left(d_{2p} + d_{2n}\right) - \left(e_{2p} + e_{2n}\right), \tag{56}$$

where:

$$d_{1p} \equiv \frac{1}{\kappa}\text{ExpInt}\left[\left(\frac{(\zeta+\lambda)(1-\gamma)}{\gamma}+\frac{\sqrt{2}}{s\sqrt{\kappa}}\right)a\right],$$
$$d_{1n} \equiv \frac{1}{\kappa}\text{ExpInt}\left[\left(\frac{(\zeta-\lambda)(1-\gamma)}{\gamma}+\frac{\sqrt{2}}{s\sqrt{\kappa}}\right)a\right],$$

$$d_{2p} \equiv \frac{1}{\nu}ExpInt\left[\left(\frac{\zeta+\lambda}{\gamma}+\lambda_p\right)a\right], \qquad d_{2n} \equiv \frac{1}{\nu}\text{ExpInt}\left[\left(\frac{\zeta-\lambda}{\gamma}+\lambda_n\right)a\right],$$
$$e_{2p} \equiv \frac{1}{\nu}\text{ExpInt}\left(\lambda_p a\right), \qquad e_{2n} \equiv \frac{1}{\nu}\text{ExpInt}\left(\lambda_n a\right),$$
$$\lambda_p \equiv \sqrt{\frac{\theta^2}{\sigma^4}+\frac{2}{\sigma^2\nu}}-\frac{\theta}{\sigma^2}, \qquad \lambda_n \equiv \sqrt{\frac{\theta^2}{\sigma^4}+\frac{2}{\sigma^2\nu}}+\frac{\theta}{\sigma^2},$$
$$\text{and } e_p \equiv \frac{1}{\kappa}\text{ExpInt}\left(\frac{\sqrt{2}}{s\sqrt{\kappa}}a\right).$$

9.2 Solution of finite horizon problem

The trial solution for the J function which we will show provides a complete solution of Program B is of the form:

$$J(t,W) = \frac{\gamma}{1-\gamma}\left[\frac{\eta(t)}{\gamma}W - B(t)\right]^{1-\gamma}, \tag{57}$$

where $\eta(t)$ and $B(t)$ are functions of time to be determined subject to the boundary conditions:

$$\eta(T) = \alpha \text{ and } B(T) = A. \tag{58}$$

As in the infinite horizon case, the first order condition (34) may be solved for the optimal wealth response:

$$w^*(x,t) = \ln\left[\frac{\gamma}{\eta(t)}\frac{B(t)}{W}+\left(1-\frac{\gamma}{\eta(t)}\frac{B(t)}{W}\right)\left(\frac{\kappa}{\nu}\right)^{-\frac{1}{\gamma}}\exp\left(-\frac{\zeta}{\gamma}x-\frac{\lambda}{\gamma}\mid x\mid\right)\right],$$

and once again for the case $\kappa = \nu$, we have that the optimal return is:

$$e^{w^*(x,t)} - 1 = \left(1-\frac{\gamma}{\eta(t)}\frac{B(t)}{W}\right)\left[\exp\left(-\frac{\zeta}{\gamma}x-\frac{\lambda}{\gamma}\mid x\mid\right)-1\right]. \tag{59}$$

Substituting (59) into (30) yields an ordinary differential equation (o.d.e.) in J:

$$J_t = -\left[\frac{\eta(t)}{\gamma}W - B(t)\right]^{-\gamma}\left[\left(r-c_2+\frac{c_1}{1-\gamma}\right)\eta(t)W+\left(\gamma c_2-\frac{\gamma c_1}{1-\gamma}\right)B_t\right]. \tag{60}$$

Differentiating the trial solution (57) for the J function yields:

$$J_t = \left[\frac{\eta(t)}{\gamma}W - B(t)\right]^{-\gamma} \left[W\eta'(t) - \gamma B'(t)\right]. \tag{61}$$

Equating (61) and (60) yields two separate o.d.e.'s in η and B:

$$\eta'(t) = -\left(r - c_2 + \frac{c_1}{1-\gamma}\right)\eta(t) \tag{62}$$

$$B'(t) = \left(c_2 - \frac{c_1}{1-\gamma}\right)B(t). \tag{63}$$

Solving these o.d.e.'s subject to the boundary conditions yields,

$$\eta(t) = \alpha \exp\left[\left(r - c_2 + \frac{c_1}{1-\gamma}\right)(T-t)\right] \tag{64}$$

$$B(t) = A \exp\left[-\left(c_2 - \frac{c_1}{1-\gamma}\right)(T-t)\right]. \tag{65}$$

Finally, substituting (64) and (65) in (57) completes the description of the J function. This completes the solution of program B in the case $\kappa = \nu$.

For the case when $\kappa \neq \nu$, we follow the same strategy as in the infinite horizon case and define $w(x)$ to be zero for $|x| < a$, where a is a small positive number. It follows that the solution is similar to that obtained for the case $\kappa = \nu$, except that we replace c_1 and c_2 by c_1' and c_2' as defined by (55) and 56).

10 Discussion of solutions

For both an infinite and a finite horizon, the optimal wealth response is a function of only the price relative $R = e^x$, where x is the jump in the log of the stock price. The specific function of interest to a HARA investor is:

$$f(R) = 1_{R>e^a}\left[\left(\frac{\kappa}{\nu}\right)^{-\frac{1}{\gamma}} R^{-\frac{\zeta+\lambda}{\gamma}} - 1\right] + 1_{R<e^{-a}}\left[\left(\frac{\kappa}{\nu}\right)^{-\frac{1}{\gamma}} R^{-\frac{\zeta-\lambda}{\gamma}} - 1\right], \tag{66}$$

and the position taken in this derivative is at the level of the investor's risk capital defined by:

$$RC = W - \frac{\gamma}{\eta}B,$$

where the wealth is time-dependent as are B and η when we have a finite horizon. The infinite horizon problem has a fixed floor for wealth, with the excess over this floor being the invested risk capital. The finite horizon problem has a rising floor, that rises at the interest rate as may be observed from the solutions (64) and (65). The final level of the finite horizon floor is that of the terminal utility function A/α. The rise in the level of the floor reflects a decline in risk tolerance as we approach the terminal date. All capital is invested in the risky optimal derivative if the horizon is far away.

Consider now the optimal payoff (66). The general shape of the payoff desired in response to market jumps may be determined by restricting attention to the case $\kappa = \nu$. Consider first the case $s = \sigma$. In this case for $\theta < 0$ (negative skewness), we have that ζ and λ are both negative. For positive returns, we observe that risk-averse investors ($\gamma > -\zeta - \lambda$) would be positioned to have increasing but concave payoffs as functions of R. These investors buy at-the-money calls and write out-of-the-money calls. For low risk aversion ($\gamma < -\zeta - \lambda$), the payoff is convex in R, and these investors enhance their long stock position by writing at-the-money calls and buying out-of-the-money calls.

For negative returns, one may show that $\zeta - \lambda$ is negative and so payoffs decline with returns. Highly risk-averse investors ($\gamma > -\zeta + \lambda$) prefer a concave payoff. Thus, in addition to holding stocks, they buy at-the-money puts and then sell out-of-the-money puts to acheive the concavity. Investors with low risk aversion ($\gamma < -\zeta + \lambda$) prefer a convex payoff, acheived by selling at-the-money puts and buying out-of-the-money puts. Investors with intermediate risk aversion ($-\zeta + \lambda < \gamma < -\zeta - \lambda$) take convex positions on the upside and concave positions on the downside.

In summary, for $s = \sigma$, the most risk-averse investors buy at-the-money options of both types and sell out-of-the-money options, while the less risk-averse do the opposite. Investors with intermediate risk aversion take convex positions on the upside and concave positions on the downside.

There is considerable empirical evidence that historical volatilities are below their risk-neutral counterparts, i.e. $s < \sigma$. In this case, λ may be positive and we can have $-\zeta - \lambda < \gamma < -\zeta + \lambda$. For $s < \sigma$, investors with such intermediate risk aversion achieve convex payoffs on the downside by buying out-of-the-money puts and create concave payoffs on the upside by writing out-of-the-money calls. These collared payoffs have long been popular for options on price and this analysis suggests that collared payoffs linked to returns are optimal for investors with intermediate levels of risk aversion in markets with $s < \sigma$.

If investors differ in their ability to dynamically trade short-term options, one would expect that low cost traders such as investment houses would provide the optimal payoffs to others and would then hedge this liability using dynamic trading strategies in the stock and options. Thus, it is interesting to observe that derivative security payoffs tied to daily return levels are now emerging in certain over-the-counter markets.

11 Summary and extensions

We considered the problem of optimal investment in continuous time economies in which dynamic trading strategies in options allow investors to hedge jumps of all sizes. In particular, we studied this problem when the underlying asset price dynamics are given by a pure jump Lévy process with an infinite arrival rate. Both the infinite and finite horizon problems are considered for HARA utility and for price dynamics given by the VG process.

We found that investors in these economies are interested in derivatives written on future price relatives, rather than on future prices. The position of HARA investors in such a derivative varies with their wealth and their time horizon. Infinite horizon investors place the excess over a floor in the optimal derivative, while finite horizon investors raise the floor as they approach their horizon.

The optimal payoff for highly risk-averse investors is achieved by buying at-the-money options and selling out-of-the-money options, while low risk aversion investors do the opposite. For the typical case of statistical volatility lower than implied, we find that the optimal financial product is a collar structure on the price relative, with investors financing downside protection by sacrificing upside gain. The resulting position is concave with respect to market up jumps and convex with respect to market drops. We also note that market interest is emerging for precisely such a derivative product.

This research can be extended in a number of directions. For example, there may well be alternative restrictions on preferences, beliefs, and price processes which yield explicit solutions. A random horizon problem may be considered and stochastic labor income may be added. Finally, a difficult open problem concerns the existence of a general equilibrium in which the risk-neutral price process is a consequence of heterogeneous agents simultaneously optimizing their intertemporal consumption and investment decisions in an economy in which options of all strikes and maturities trade. In the interests of brevity, these questions are left for future research.

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