Optimal Investment Strategies in the presence of a Minimum Guarantee.

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Abstract: In a continuous-time framework, we consider the problem of a Defined Contribution Pension Fund in the presence of a minimum guarantee. The problem of the fund manager is to invest the initial wealth and the (stochastic) contribution flow into the financial market, in order to maximize the expected utility function of the terminal wealth under the constraint that the terminal wealth must exceed the minimum guarantee. We assume that the stochastic interest rates follow the affine dynamics, including the CIR (Cox, Ingersoll and Ross 1985) model and the Vasiček model. The optimal investment strategies are obtained by assuming the completeness of financial markets and a CRRA utility function. Explicit formulae for the optimal investment strategies are included for different examples of guarantees and contributions.

Key words: consumption-investment strategy, pension funds, stochastic interest rates, stochastic optimization.

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1 Introduction

In a continuous-time framework, we consider the problem of defined contribution pension funds contracts in the presence of a minimum guarantee. The problem of the fund manager is to invest the initial wealth and the (stochastic) contribution flow into the financial market, in order to maximize the expected utility function of the terminal wealth, which should exceed the minimum guarantee. In exchange, the fund manager keeps back a percentage of the surplus. Since we are interested in a long-term investment problem of typically 30 or 40 years, it is crucial to allow for a stochastic term structure for the interest rates. We investigate the case in which interest rates follow the affine dynamics of Duffie & Kan (1996), which includes as special cases the CIR (Cox, Ingersoll and Ross 1985) model and the Vasiček (1977) model.

Optimal consumption-investment problems are studied in the literature since a long time (see e.g. Merton 1971 where the interest rates are constant). Afterwards, many authors have introduced a stochastic term structure for the interest rates.

On one hand, some papers like Karatzas, Lehoczky and Shreve (1987), Karatzas (1989) or El Karoui and Jeanblanc-Picqué (1998) do not specify the stochastic process which leads to very general results. However, the general feature of the interest rates does not permit to test the results of those papers by comparing them with reality, since their solutions are not explicit.

On the other hand, papers by Bajeux-Besnainou, Jordan and Portait (1998, 1999) or Lioui and Poncet (2000) choose a Vasiček specification of the term structure. This choice permits to obtain a closed-form solution, and to analyse its behaviour.

In Deelstra, Grasselli and Koehl (2000), we investigated the case where interest rates follow the Cox-Ingersoll-Ross dynamics. Assuming completeness of the markets and power utility function, we obtained by the use of the Cox-Huang methodology closed-form solutions for a utility maximization problem of terminal wealth, without considering contributions or a positive guarantee.

In this paper, we stress the pension fund problem and we therefore include a contribution flow and a minimum guarantee. Moreover, we concentrate on the affine term structure model in order to include both the CIR model and the Vasicek model. To the opposite of the Vasiček one, the CIR specification does not permit negative interest rates. This is the main reason why Rogers (1995) recommends to choose the CIR rather than the Vasiček specification.

However, the literature on pension funds or long-term investment problems only considers the Vasiček model.

Boulier, Huang and Taillard (2001) study the optimal management of a defined contribution plan where the guarantee depends on the level of interest rates at the fixed retirement date. Jensen and Sørensen (1999) measure the effect of a minimum interest rate guarantee constraint through the wealth equivalent in case of no constraints and show numerically that guarantees may induce a significant utility loss for relatively risk tolerant investors. Both the papers by Boulier, Huang and Taillard (2001) and Jensen and Sørensen (2000) choose a Vasiček specification of the term structure in the spirit of Bajeux-Besnainou, Jordan and Portait (1998, 1999).

The paper is organized as follows: in Section 2, we define the market structure and introduce the optimization problem under consideration. We show also how this problem is related to the pension fund management. In Section 3, we transform the initial problem into an equivalent one, which we solve explicitly in the power utility case. In Section 4, we come back to the solution of the initial problem and we find it explicitly by specifying the form of the contribution process in some interesting cases. Section 5 concludes the paper.

2   The model

In this section, we present:

(i) the financial market, that is the assets available and their equilibrium dynamics, given exogenously,

(ii) the optimization program, that is the characteristics of the agent and his optimization criteria.

We interpret also this modelization in terms of our pension fund problem.

2.1   The financial market

Randomness is described by a 2-dimensional Brownian motion

$$\underline{z}(t) = \left\{ (z(t), z_r(t))' ; \ t \in [0, +\infty[\right\}$$

defined on a complete probability space $(\Omega, \mathcal{F}, P)$, where P is the real world probability. The filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$, represents the information struc-

ture generated by the Brownian motion and is assumed to satisfy the usual conditions.

Hereafter $\mathbb{E}_t$ stands for $\mathbb{E}(. \mid \mathcal{F}_t)$, the conditional expected value under the real world probability.

The market is composed of three financial assets, that the agent can buy or sell continuously without incurring any restriction as short sales constraints or any trading cost.

The first asset is the riskles asset (i.e. the cash). Its price, denoted by $S_0(t), t \geq 0$, evolves according to:

$$\frac{dS_0(t)}{S_0(t)} = r_t dt, \quad S_0(0) = 1,$$

where the dynamics of the short rate process r_t are described by the following stochastic differential equation:

$$dr_t = (a - br_t)dt - \sqrt{\eta_1 r_t + \eta_2} dz_r(t), \ t \geq 0, \tag{1}$$

r_0, a, b, η_1 and η_2 being positive constants.

These dynamics have been studied by Duffie and Kan (1996). Their paper shows that, under these dynamics, the term structure of the interest rates is affine. Moreover, the converse is true under a regularity hypothesis.

Note that these dynamics recover, as special cases, the Vasiček (1977) (resp. Cox-Ingersoll-Ross 1985) dynamics, when η_1 (resp. η_2) is equal to zero.

The second asset is the stock, whose price is denoted by $S(t), t \geq 0$. The dynamics of $S(t)$ are given by:

$$\frac{dS(t)}{S(t)} = r_t dt + \sigma_1 \left(dz(t) + \lambda_1 dt \right) + \sigma_2 \sqrt{\eta_1 r_t + \eta_2} \left(dz_r(t) + \lambda_2 \sqrt{\eta_1 r_t + \eta_2} dt \right),$$
$$\tag{2}$$

with $S(0) = 1$ and λ_1, λ_2 (resp. σ_1, σ_2) being constant (resp. positive constants).

Notice that Merton (1971) considered in a constant interest rate framework only the first volatility term in the risky asset. As the stock prices also will be influenced by the stochastic interest rates, we introduce the term with the Brownian motion of the short-term interest rates with the corresponding affine market price of risk (see Deelstra et al. 2000 in a Cox-Ingersoll-Ross framework). This is equivalent to assuming that the market price of risk vector equals $\underline{\lambda} = (\lambda_1, \lambda_2 \sqrt{\eta_1 r_t + \eta_2})'$.

The third asset is a zero-coupon bond with maturity T, whose price at time t is denoted by $B(t,T)$, $t \geq 0$.

The following proposition fixes the dynamics of the zero-coupon bond price.

Proposition 1 *Let us denote $B(t,T)$ the price at date t of the zero coupon bond maturing at date T. Then:*

$$\frac{dB(t,T)}{B(t,T)} = r_t dt + \sigma_B(T-t, r_t)\left(dz_r(t) + \lambda_2\sqrt{\eta_1 r_t + \eta_2}dt\right), \qquad B(T,T) = 1$$
(3)

where

$$\sigma_B(T-t, r_t) = h(T-t)\sqrt{\eta_1 r_t + \eta_2}$$

with

$$h(t) = \frac{2(e^{\delta t} - 1)}{\delta - (b - \eta_1\lambda_2) + e^{\delta t}(\delta + b - \eta_1\lambda_2)}, \qquad t \geq 0,$$
(4)

$$\delta = \sqrt{(b - \eta_1\lambda_2)^2 + 2\eta_1}.$$

The proof of this proposition is based upon the following Lemma, which will be crucial in the sequel:

Lemma 2 *If the interest rates follow (1), then there exist two deterministic functions $K_1^{\eta_1}(\alpha, \beta, T-t)$, $K_2^{\eta_1}(\alpha, \beta, T-t)$ such that for $\alpha \in R, \beta > 0$*

$$\mathbb{E}_t\left[e^{-\alpha r_T - \beta \int_t^T r_s ds}\right] = K_1^{\eta_1}(\alpha, \beta, T-t)e^{-r_t K_2^{\eta_1}(\alpha, \beta, T-t)}.$$
(5)

Proof See Appendix. ∎

Proof of the proposition

It is well-known that the dynamics of the bond follow a SDE of the form (3), where $(dz_r(t) + \lambda_2\sqrt{\eta_1 r_t + \eta_2}dt)$ forms a Brownian motion under the risk-neutral measure and where we have to specify the volatility term. Stressing the Brownian motion under the risk-neutral measure, we see that the dynamics of the short term interest rates are given by

$$dr_t = (a + \lambda_2\eta_2 - (b - \lambda_2\eta_1)r_t)\,dt - \sqrt{\eta_1 r_t + \eta_2}\left(dz_r(t) + \lambda_2\sqrt{\eta_1 r_t + \eta_2}dt\right).$$

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Noticing that the bond price is given by

$$B(t,T) = \mathbb{E}_t^Q \left[e^{-\int_t^T r_s ds} \right] = K_1^{\eta_1}(0,1,T-t)e^{-r_t K_2^{\eta_1}(0,1,T-t)},$$

we find that

$$\frac{dB(t,T)}{B(t,T)} = r_t dt + K_2^{\eta_1}(0,1,T-t)\sqrt{\eta_1 r_t + \eta_2}\left(dz_r(t) + \lambda_2 \sqrt{\eta_1 r_t + \eta_2}dt\right),$$

where $K_2^{\eta_1}(0,1,t) = \frac{2(e^{\delta t}-1)}{\delta - (b-\eta_1\lambda_2)+e^{\delta t}(\delta + b - \eta_1\lambda_2)} = h(t)$ with $\delta = \sqrt{(b-\eta_1\lambda_2)^2 + 2\eta_1}$.
∎

At last, we assume that the parameters are such that the financial market is arbitrage-free and complete. Then, for any $t \geq 0$, we can define the deflator price process

$$H(t) = \exp\left\{ -\int_0^t r_s ds - \int_0^t \underline{\lambda}'(s)d\underline{z}(s) - \frac{1}{2}\int_0^t |\underline{\lambda}(s)|^2 ds \right\}, \qquad (6)$$

with $\underline{\lambda}'(s) = (\lambda_1, \lambda_2\sqrt{\eta_1 r_s + \eta_2})$.

2.2 The optimization program

We consider a financial agent who is the manager of (and is confounded with) a pension fund.

On one hand, the pension fund is endowed with a strictly positive initial wealth W_0, and receives a non-negative, progressive measurable and square-integrable process, at a rate denoted by $c(t), t \geq 0$, which represents the contributions elarged at any time by participants to the pension fund.

On the other hand, the pension fund must in any case (and thus ignoring death) provide at date T, at least the minimum guarantee G_T, a strictly positive square-integrable $\mathcal{F}_T-$measurable random variable; it can be the value at time T of a benchmark portfolio or an annuity.

We denote by $W(t)$ the wealth of the fund at date $t \in [0,T]$.

From now on, we assume that

$$\mathbb{E}\left[H(T)G_T\right] < W_0 + \mathbb{E}\left[\int_0^T H(t)c_t dt\right], \qquad (7)$$

which means that the manager can always choose a strategy such that, at date T, the surplus $W(T) - G_T$ is (strictly) positive.

We assume moreover that the manager is controlled by a regulator and will effectively choose a strategy such that $W(T) - G_T \geq 0$ almost surely. The surplus is then shared according to the following rule:

- the manager takes $\beta(W(T) - G_T)$, where $\beta(.)$ is a strictly increasing concave function such that $\beta(0) = 0$,
- the contributors receive $W(T) - G_T - \beta(W(T) - G_T)$.

The manager's preferences are described by a CRRA utility function defined by:

$$\nu(y) = \frac{y^\gamma}{\gamma}, \quad \gamma \in (-\infty, 1)\backslash\{0\}, \tag{8}$$

The program of the manager is then to maximize the expected utility of his terminal wealth under feasibility constraints, namely:

$$\max_{(\underline{u}_t)_{t\in[0,T]}\in\mathcal{A}} \mathbb{E}U\left(W(T) - G_T\right) \tag{9}$$

where:

- $U := \nu \circ \beta$ is strictly increasing, strictly concave and satisfies the Inada conditions $U'(+\infty) = 0$ and $U'(0) = +\infty$,
- the wealth process $\{W(t)\}_{t\geq 0}$ is defined by the following dynamics (remember that $(c_t)_{t\geq 0}$ is the contribution process):

$$
\begin{aligned}
dW(t) &= W(t)\underline{u}_t' diag\left[\underline{S}(t)\right]^{-1} d\underline{S}(t) + c_t dt \tag{10}\\
&= W(t)(1 - u_t^B - u_t^S)\frac{dS_0(t)}{S_0(t)} + W(t)u_t^B\frac{dB(t,T)}{B(t,T)} + W(t)u_t^S\frac{dS(t)}{S(t)} + c_t dt,\\
W(0) &= W_0 > 0,
\end{aligned}
$$

with $\underline{u}_t = \left((1 - u_t^B - u_t^S), u_t^B, u_t^S\right)'$ and $\underline{S}(t) = (S_0(t), B(t,T), S(t))'$,
- $\mathcal{A}$ is the set of admissible controls, that is

$$\mathcal{A} = \left\{(\underline{u}_t)_{t\in[0,T]} : \underline{u}_t \in \mathcal{F}_t, W(t)\underline{u}_t \text{ is square integrable, and } W(T) - G_T \geq 0 \text{ a.e.}\right\}. \tag{11}$$

From equation (7), it is easy to see that $\mathcal{A} \neq \emptyset$.

The quantity $\left(1 - u_t^B - u_t^S\right)$, (resp. u_t^B, resp. u_t^S) denotes the proportion of wealth invested into the riskless asset, (resp. the bond, resp. the stock).

3 Transformation of the initial problem

Since the wealth process (10) is not a self-financing process, the optimization program (9) has no standard solution method. In this section, we introduce an auxiliary process in order to obtain an equivalent program which turns out to be easier. Indeed, we define the surplus process and we prove that it is self-financing.

Definition 3 *The surplus process $Y(t)$, $t \geq 0$ is defined by:*

$$Y(t) = W(t) + D(t) - G(t), \tag{12}$$

where

$$D(t) = \mathbb{E}_t \int_t^T \frac{H(s)}{H(t)} c_s ds, \;\; G(t) = \mathbb{E}_t \left[\frac{H(T)}{H(t)} G_T \right].$$

This process can be interpreted as a surplus process, in the sense that, at date t, it is equal to:

- the value of the portfolio $W(t)$

- plus the discounted value of the future engagements coming from the contributor $D(t)$,

- minus the discounted value of the pension fund future engagement (that is the guarantee) $G(t)$.

Note also that the value of the process at date T is equal to the surplus $W(T) - G_T$, while

$$Y(0) = W_0 + \mathbb{E} \int_0^T H(s) c_s ds - \mathbb{E} \left[H(T) G_T \right] = Y_0 > 0. \tag{13}$$

Proposition 4 *i) The surplus process is self-financing, that is there exists a progressive measurable random process $(\underline{y_t})_{t \in [0,T]} = \left((1 - y_t^B - y_t^S), y_t^B, y_t^S \right)$ denoting the proportions of $Y(t)$ invested into resp. $(S_0(t), B(t,T), S(t))$, such that*

$$\begin{aligned} dY(t) &= Y(t) \underline{y_t}' diag[\underline{S}(t)]^{-1} d\underline{S}(t) \tag{14} \\ Y(0) &= Y_0. \end{aligned}$$

ii) Let $\mathcal{A}^Y = \{(\underline{y}_t)_{t\in[0,T]} : \underline{y}_t \in \mathcal{F}_t, Y(t)\underline{y}_t$ is square integrable and (14) holds almost surely$\}$ denotes the set of admissible controls of the problem

$$\max_{(\underline{y}_t)_{t\in[0,T]}\in\mathcal{A}^Y} \mathbb{E}U\left(Y(T)\right) \tag{15}$$

Then the problem (9) is equivalent to (15).

Proof. i) For a given process K_t let denote $\widetilde{K}_t := H_t K_t$. Then:

$$d\widetilde{Y}_t = d\widetilde{W}_t + d\widetilde{D}_t - d\widetilde{G}_t$$

From (2), (6), and (10), easy computations lead to:

$$d\left(\widetilde{W}_t\right) = \widetilde{W}_t\left(\widehat{\underline{u}}_t'\sigma(t,r_t) - \underline{\lambda}'(t)\right)d\underline{z}_t + \widetilde{c}_t dt$$

where $\widehat{\underline{u}}_t' = \left(u_t^B, u_t^S\right)$ and $\sigma(t,r_t) = \begin{pmatrix} 0 & \sigma_B(T-t,r_t) \\ \sigma_1 & \sigma_2\sqrt{\eta_1 r_t + \eta_2} \end{pmatrix}$.

Using the martingale representation theorem for the Brownian motion, (Karatzas and Shreve 1990), it turns out that there exists a unique square integrable process $(\underline{\zeta}_t)_{t\in[0,T]}$ with $\underline{\zeta}_t = (\zeta_t, \zeta_t^r)'$, satisfying

$$\int_0^T |\underline{\zeta}_t|^2 \, dt < +\infty \; P-a.e. \tag{16}$$

such that

$$d\left(\widetilde{D}_t\right) = -\widetilde{c}_t dt + \underline{\zeta}_t' d\underline{z}_t \tag{17}$$

Analogously, there exists a unique square integrable process $(\underline{\rho}_t)_{t\in[0,T]}$ with $\underline{\rho}_t = (\rho_t, \rho_t^r)'$, satisfying

$$\int_0^T |\underline{\rho}_t|^2 \, dt < +\infty \; P-a.e. \tag{18}$$

such that

$$d\left(\widetilde{G}_t\right) := d\left(\mathbb{E}_t\left[\widetilde{G}_T\right]\right) = \underline{\rho}_t' d\underline{z}(t)$$

Finally, we get:

$$d\widetilde{Y}(t) = \left(\widetilde{W}(t)\left(\widehat{\underline{u}}_t'\sigma(t,r_t) - \underline{\lambda}'(t)\right) + \underline{\zeta}_t' - \underline{\rho}_t'\right)d\underline{z}(t)$$

9

and therefore the process $Y(t)$ is self-financing. Indeed, in order to prove (14), it suffices to define $\widehat{\underline{y}}_t' = \left(y_t^B, y_t^S\right)$ as follows:

$$Y(t)\widehat{\underline{y}}_t = W(t)\widehat{\underline{u}}_t + (D_t - G_t)\left[\sigma(t, r_t)'\right]^{-1}\underline{\lambda}(t) + H^{-1}(t)\left[\sigma(t, r_t)'\right]^{-1}\left(\underline{\zeta_t} - \underline{\rho_t}\right) \tag{19}$$

which ends the proof of i).

ii) Since all terms in the right hand of (19) are square integrable, it follows that the strategy $(\underline{y_t})_{t\in[0,T]}$ (resp. $(\underline{u_t})_{t\in[0,T]}$) defined in (19) is admissible for (15) (resp. for 9). This in turns implies that the optimal values of (9) and (15) are equal. $\blacksquare$

If we explicit the expression (14), we obtain

$$\begin{aligned}
\frac{dY(t)}{Y(t)} &= r_t dt + y_t^B \sigma_B(T - t, r_t)\left(dz_r(t) + \lambda_2\sqrt{\eta_1 r_t + \eta_2}dt\right) + \\
&\quad y_t^S\left[\sigma_1\left(dz(t) + \lambda_1 dt\right) + \sigma_2\sqrt{\eta_1 r_t + \eta_2}\left(dz_r(t) + \lambda_2\sqrt{\eta_1 r_t + \eta_2}dt\right)\right], \\
Y(0) &= Y_0 > 0, \tag{20}
\end{aligned}$$

where $(\underline{y_t})_{t\in[0,T]} = \left((1 - y_t^B - y_t^S), y_t^B, y_t^S\right)$ is linked by (19) with $(\underline{u_t})_{t\in[0,T]}$.

4   Explicit solution in the power utility case

We noticed in equation (19) that the optimal strategies $(\underline{u_t})_{t\in[0,T]}$ of the initial optimization progam (9) are linked with the controls $(\underline{y_t})_{t\in[0,T]}$ of program (15). In this section, we therefore derive the explicit expressions of $(\underline{y_t})_{t\in[0,T]}$ of program (15) with a CRRA utility function U as defined in (8). In the following section, we will fully determine the optimal strategies $(\underline{u_t})_{t\in[0,T]}$ for different choices of contributions and guarantees.

In order to determine the solution $(\underline{y_t})_{t\in[0,T]}$ of program (15), we first need the explicit expression of the following quantity:

$$\mathbb{E}_t\left[\left(\frac{H(t)}{H(T)}\right)^{\frac{\gamma}{1-\gamma}}\right],$$

which will be done by using Lemma 5.

Lemma 5 *Suppose that c is a real number such that $c\eta_1\left(1 + \frac{\lambda_2^2\eta_1}{2} - \lambda_2 b\right) \leq 0$, then there exist two deterministic functions $k_1(t, c)$, $k_2(t, c)$ such that*

$$\mathbb{E}_t\left[\left(\frac{H(t)}{H(T)}\right)^c\right] = k_1(T - t, c)\exp\left\{-r_t k_2(T - t, c)\right\}. \tag{21}$$

10

Proof. We adapt the reasoning of Deelstra et al. (2000). For $c = 0$ the statement is obvious, so we concentrate on $c \neq 0$. From (6) it turns out that

$$
\begin{aligned}
\mathbb{E}_t \left[\left(\frac{H(t)}{H(T)} \right)^c \right] &= \mathbb{E}_t \left[\exp \left\{ c \int_t^T \left(r_s + \frac{1}{2} \left(\lambda_1^2 + \lambda_2^2 (\eta_1 r_s + \eta_2) \right) \right) ds \right. \right. \\
&\qquad \left. \left. + c \int_t^T \lambda_1 dz(s) + c \int_t^T \lambda_2 \sqrt{\eta_1 r_s + \eta_2} dz_r(s) \right\} \right] \\
&= f(T - t, r_t, c) \, \mathbb{E}_t \left[\exp -\alpha r_T - \beta \int_t^T r_s ds \right], \qquad (22)
\end{aligned}
$$

where

$$
\begin{aligned}
f(T - t, r_t, c) &= \exp c \left\{ (T - t) \left(c \frac{\lambda_1^2}{2} + \frac{\lambda_1^2}{2} + \lambda_2 a + \frac{\lambda_2^2 \eta_2}{2} \right) + \lambda_2 r_t \right\} \quad (23) \\
\beta &= -c \left(1 + \frac{\lambda_2^2 \eta_1}{2} - \lambda_2 b \right) > 0, \qquad (24) \\
\alpha &= c \lambda_2. \qquad (25)
\end{aligned}
$$

We now apply Lemma 2 to (22) and obtain:

$$
\mathbb{E}_t \left[\left(\frac{H(t)}{H(T)} \right)^c \right] = f(T - t, r_t, c) K_1^{\eta_1} (\alpha, \beta, T - t) \exp -r_t K_2^{\eta_1} (\alpha, \beta, T - t).
$$
(26)

From (26) and (23) we obtain the result, with

$$
k_1(t, c) = K_1^{\eta_1} (\alpha, \beta, t) \exp c \left\{ \left(c \frac{\lambda_1^2}{2} + \frac{\lambda_1^2}{2} + \lambda_2 a + \frac{\lambda_2^2 \eta_2}{2} \right) t \right\}, \qquad (27)
$$

$$
k_2(t, c) = -\lambda_2 c + K_2^{\eta_1} (\alpha, \beta, t). \qquad (28)
$$

∎

Notice that with $c = -1$ we obtain the bond price. In the sequel we will use that indeed

$$
k_2(t, -1) = h(t),
$$

with $h(t)$ as defined in (4).

Equipped with (21), we are now able to solve the purely investment problem (15).

Proposition 6 *Under the hypotheses of Lemma 5, the trading strategy which solves (15) is given by*

$$
\begin{aligned}
y_t^{S_0} &= 1 - y_t^B - y_t^S, \\
y_t^B &= \frac{k_2\left(T-t,\frac{\gamma}{1-\gamma}\right)}{h(T-t)} + \frac{1}{1-\gamma}\frac{\sigma_1\lambda_2 - \sigma_2\lambda_1}{\sigma_1 h(T-t)}, \\
y_t^S &= \frac{1}{1-\gamma}\frac{\lambda_1}{\sigma_1}.
\end{aligned}
\tag{29}
$$

Proof. Following Deelstra et al. (2000), the optimal surplus process is given by

$$
Y(t) = \mathbb{E}_t\left[\frac{H(T)}{H(t)}I\left(\theta H(T)\right)\right],
$$

where $I(x) = (U')^{-1}(x) = x^{\frac{1}{\gamma-1}}$, while θ is determined by $Y_0 = \mathbb{E}\left[H(T)I\left(\theta H(T)\right)\right]$. By applying the previous Lemma, we obtain

$$
\begin{aligned}
Y(t) &= \theta^{\frac{1}{\gamma-1}}H(t)^{-1}\mathbb{E}_t\left[H(T)^{\frac{\gamma}{\gamma-1}}\right] \\
&= \theta^{\frac{1}{\gamma-1}}H(t)^{\frac{1}{\gamma-1}}\mathbb{E}_t\left[\left(\frac{H(t)}{H(T)}\right)^{\frac{\gamma}{1-\gamma}}\right] \\
&= (\theta H(t))^{-\frac{1}{1-\gamma}}k_1\left(T-t,\frac{\gamma}{1-\gamma}\right)\exp\left\{-r_t k_2\left(T-t,\frac{\gamma}{1-\gamma}\right)\right\}.
\end{aligned}
$$

Now we differentiate both sides and by grouping the locally deterministic factors into $[.]dt$, we have

$$
\begin{aligned}
\frac{dY(t)}{Y(t)} &= \frac{1}{1-\gamma}\frac{dH(t)^{-1}}{H(t)^{-1}} - k_2\left(T-t,\frac{\gamma}{1-\gamma}\right)dr_t + [.]dt \\
&= \frac{1}{1-\gamma}\frac{dH(t)^{-1}}{H(t)^{-1}} + k_2\left(T-t,\frac{\gamma}{1-\gamma}\right)\sqrt{\eta_1 r_t + \eta_2}dz_r(t) + [.]dt \\
&= \frac{1}{1-\gamma}\frac{dH(t)^{-1}}{H(t)^{-1}} + k_2\left(T-t,\frac{\gamma}{1-\gamma}\right)\frac{\sigma_B(T-t,r_t)}{h(T-t)}dz_r(t) + [.]dt \\
&= \frac{1}{1-\gamma}\frac{dH(t)^{-1}}{H(t)^{-1}} + \frac{k_2\left(T-t,\frac{\gamma}{1-\gamma}\right)}{h(T-t)}\frac{dB(t,T)}{B(t,T)} + [.]dt,
\end{aligned}
\tag{30}
$$

which says that there exists a (dynamic) combination of the processes $H(t)^{-1}$, $B(t,T)$ and $S_0(t)$ with weights resp. $\frac{1}{1-\gamma}$, $\frac{k_2\left(T-t,\frac{\gamma}{1-\gamma}\right)}{h(T-t)}$ and $1-\frac{1}{1-\gamma}-\frac{k_2\left(T-t,\frac{\gamma}{1-\gamma}\right)}{h(T-t)}$,

which allows to replicate $Y(t)$ $P-a.e.$ In fact, in (30), since the diffusion terms are equal, then also the drifts are, for arbitrage arguments. Finally, it is easy to see that the strategy which replicates the process $H(t)^{-1}$ is given by $\left(\left(1-\frac{\lambda_1}{\sigma_1}-\frac{\sigma_1\lambda_2-\sigma_2\lambda_1}{\sigma_1 h(T-t)}\right),\frac{\lambda_1}{\sigma_1},\frac{\sigma_1\lambda_2-\sigma_2\lambda_1}{\sigma_1 h(T-t)}\right)$, and we obtain the statement (29). ∎

5 Solution of the initial problem: examples

The solution $(\underline{y}_t)_{t\in[0,T]} = \left((1-y_t^B-y_t^S),y_t^B,y_t^S\right)$ of the auxiliary problem (15), given by (29), is linked with the solution $(\underline{u}_t)_{t\in[0,T]}$ of the initial problem (9) by (19).

The aim in this section is to come back to the solution $(\underline{u}_t)_{t\in[0,T]}$, given by

$$W(t)\widehat{\underline{u}}_t = Y(t)\widehat{\underline{y}}_t - (D_t-G_t)\left[\sigma(t,r_t)'\right]^{-1}\underline{\lambda}(t) - H(t)\left[\sigma(t,r_t)'\right]^{-1}(\underline{\zeta}_t - \underline{\rho}_t),\quad(31)$$

and to show a methodology for finding explicitly the solution of the pension fund problem $(\widehat{\underline{u}}_t)_{t\in[0,T)} = \left(u_t^B,u_t^S\right)_{t\in[0,T)}$, when the contribution process c_t and the guarantee G_T assume some interesting *stochastic* features. This will permit the fund manager to quantify the impacts on his investment strategy due to variations of the contribution policy and guarantee.

In order to find $(\widehat{\underline{u}}_t)_{t\in[0,T)} = \left(u_t^B,u_t^S\right)_{t\in[0,T)}$, it suffices to find the processes $\left(D_t,(\underline{\zeta}_t)\right)_{t\in[0,T]}$ and $\left(G_t,(\underline{\rho}_t)\right)_{t\in[0,T]}$.

Let us consider the following quite general (stochastic) contribution process:

$$c_t = c_0\exp\left\{\int_0^t\left(\alpha_1(s)+\alpha_2 r_s\right)ds + \int_0^t\alpha_3 dz(s) + \int_0^t\alpha_4\sqrt{\eta_1 r_t+\eta_2}dz_r(s)\right\},\quad(32)$$

with $\alpha_1(.)$ being a deterministic function and $\alpha_2,\alpha_3,\alpha_4$ being real constants. Notice that the specific form of the contributions is similar to the one of the risky asset: this is quite natural, since in this complete market model contributions must be generated by the market. That is, we make the hypothesis that the contributions at time t can depend on the entire past salary history.

Moreover, let us consider an *interest rate guarantee* (see e.g. Jensen and Sørensen 2000): the pension fund assures a deterministic positive interest rate $(g_t)_{t\in[0,T]}$, so that the guarantee G_T becomes

$$G_T = W_0\exp\left\{\int_0^T g_t dt\right\} + \int_0^T c_t\exp\left\{\int_t^T g_s ds\right\}dt.\quad(33)$$

Obviously, there must be some admissibility constraint on $(g_t)_{t\in[0,T]}$ in order to avoid arbitrage opportunities: for notational reasons we will show this condition at the end of the section.

First, let us consider the problem of finding $\left(D_t, (\underline{\zeta}_t)\right)_{t\in[0,T]}$.

Proposition 7 *Suppose that the contribution process is given by (32) and that the following relation holds:*

$$\eta_1\left(1 + \frac{1}{2}\lambda_2^2\eta_1 - \alpha_2 + (\alpha_4 - \lambda_2)b\right) \geq 0. \tag{34}$$

Then there exist two deterministic functions $A_1(t,s), A_2(t,s)$ such that

$$
\begin{aligned}
D_t &= \frac{1}{H(t)}\mathbb{E}_t\int_t^T H(s)c_s ds \\
&= c_t\int_t^T A_1(t,s)\exp\{-A_2(t,s)r_t\}\,ds. \tag{35}
\end{aligned}
$$

Proof. From (6), by independence of $z(t)$ and $(r_t, z_r(t))$it results that

$$
\begin{aligned}
D_t &= c_t\int_t^T \mathbb{E}_t\left[\exp\left\{\int_t^s\left(\alpha_1(u) - \frac{1}{2}\lambda_1^2 - \frac{1}{2}\lambda_2^2\eta_2 + r_u\left(\alpha_2 - 1 - \frac{\lambda_2^2\eta_1}{2}\right)\right)du\right.\right. \\
&\qquad\left.\left. + \int_t^s(\alpha_3 - \lambda_1)dz(u) + \int_t^s(\alpha_4 - \lambda_2)\sqrt{\eta_1 r_u + \eta_2}dz_r(u)\right\}\right]ds \\
&= c_t\exp\left\{(\alpha_4 - \lambda_2)r_t\right\} \\
&\qquad \cdot\int_t^T\left\{\exp\left\{\int_t^s\left(\alpha_1(u) - \frac{1}{2}\lambda_1^2 - \frac{1}{2}\lambda_2^2\eta_2 + (\alpha_4 - \lambda_2)a + \frac{(\alpha_3 - \lambda_1)^2}{2}\right)du\right\}\right. \\
&\qquad\left. \cdot\mathbb{E}_t\left[\exp\left\{-(\alpha_4 - \lambda_2)r_s - \left(1 + \frac{\lambda_2^2\eta_1}{2} + b(\alpha_4 - \lambda_2) - \alpha_2\right)\int_t^s r_u du\right\}\right]\right\}ds.
\end{aligned}
$$

We apply now (5) and we obtain the result, with

$$
\begin{aligned}
A_1(t,s) &= K_1^{\eta_1}(\alpha^c, \beta^c, s - t) \tag{36} \\
&\qquad \exp\left\{\int_t^s\left(\alpha_1(u) - \frac{1}{2}\lambda_1^2 - \frac{1}{2}\lambda_2^2\eta_2 + (\alpha_4 - \lambda_2)a + \frac{(\alpha_3 - \lambda_1)^2}{2}\right)du\right\}, \\
A_2(t,s) &= K_2^{\eta_1}(\alpha^c, \beta^c, s - t) - (\alpha_4 - \lambda_2), \tag{37} \\
\beta^c &= 1 + \frac{1}{2}\lambda_2^2\eta_1 + (\alpha_4 - \lambda_2)b - \alpha_2, \tag{38} \\
\alpha^c &= \alpha_4 - \lambda_2, \tag{39}
\end{aligned}
$$

and $\xi = \sqrt{b^2 + 2\beta^c}$. $\blacksquare$

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Proposition 8 *Under the hypotheses of the previous proposition, the process* $(\underline{\zeta}_t)_{t\in[0,T]} = (\zeta_t, \zeta_t^r)'$ *is given by*

$$\zeta_t = H(t)D_t(\alpha_3 - \lambda_1),$$

$$\zeta_t^r = H(t)\sqrt{\eta_1 r_t + \eta_2}\left\{D_t(\alpha_4 - \lambda_2) + c_t\left(\int_t^T A_1(t,s)A_2(t,s)\exp\{-A_2(t,s)r_t\}ds\right)\right\}.$$

Proof. From (35) it turns out that ζ_t (resp. ζ_t^r) is the coefficient of $dz(t)$ (resp. $dz_r(t)$) in the development of $d\widetilde{D}_t = d(H(t)D_t)$, so that we can group the (locally) deterministic factors into $[.]dt$ and focus on the others:

$$\begin{aligned} d\widetilde{D}_t &= [.]dt + H(t)dD_t + D_t dH(t) \\ &= [.]dt + H(t)dD_t - H(t)D_t\underline{\lambda}'(t,r_t)d\underline{z}(t) \end{aligned}$$

Now we have

$$dD_t = [.]dt + D_t\frac{dc_t}{c_t} + c_t\left(\int_t^T A_1(t,s)A_2(t,s)\exp\{-A_2(t,s)r_t\}ds\right)\sqrt{\eta_1 r_t + \eta_2}dz_r(t),$$

so from (32), we obtain

$$\begin{aligned} dD_t &= [.]dt + D_t\alpha_3 dz(t) \\ &\quad + \left[D_t\alpha_4 + c_t\left(\int_t^T A_1(t,s)A_2(t,s)\exp\{-A_2(t,s)r_t\}ds\right)\right]\sqrt{\eta_1 r_t + \eta_2}dz_r(t). \end{aligned}$$

Finally,

$$\begin{aligned} d\widetilde{D}_t &= [.]dt + H(t)D_t(\alpha_3 - \lambda_1)dz(t) + H(t)D_t(\alpha_4 - \lambda_2)\sqrt{\eta_1 r_t + \eta_2}dz_r(t) \\ &\quad + H(t)c_t\left(\int_t^T A_1(t,s)A_2(t,s)\exp\{-A_2(t,s)r_t\}ds\right)\sqrt{\eta_1 r_t + \eta_2}dz_r(t) \end{aligned}$$

which proves the result. ∎

By the same methodology, we can determine the processes $\left(G_t, (\underline{\rho}_t)\right)_{t\in[0,T]}$ for the guarantee G_T given by (33):

Proposition 9 *Suppose that the guarantee G_T is defined by (33) and (34) holds. Then there exist two deterministic functions $\widetilde{A}_1(t,s,T)$ and $\widetilde{A}_2(t,s)$ such that*

$$\begin{aligned} G_t &= \left[W_0\exp\left\{\int_0^T g_t dt\right\} + \int_0^t c_s\exp\left\{\int_s^T g_u du\right\}ds\right]B(t,T) \\ &\quad + c_t\int_t^T \widetilde{A}_1(t,s,T)\exp\left\{\int_s^T g_u du - \widetilde{A}_2(t,s)r_t\right\}ds, \end{aligned} \tag{40}$$

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while the process $(\underline{\rho}_t)_{t\in[0,T]}$ *is given by*

$$\rho_t = H(t)c_t\alpha_3 \int_t^T \widetilde{A}_1(t,s,T)\exp\left\{\int_s^T g_u du - \widetilde{A}_2(t,s)r_t\right\}ds - H(t)G_t\lambda_1,$$

$$\begin{aligned}
\rho_t^r &= \sqrt{\eta_1 r_t + \eta_2}H(t)\left\{B(t,T)\left[W_0\exp\left\{\int_0^T g_t dt\right\} + \int_0^t c_s\exp\left\{\int_s^T g_u du\right\}ds\right]h(T-t)\right. \\
&\quad +c_t\alpha_4\left(\int_t^T \widetilde{A}_1(t,s,T)\exp\left\{\int_s^T g_u du - \widetilde{A}_2(t,s)r_t\right\}ds\right) \\
&\quad \left.+c_t\left(\int_t^T \widetilde{A}_2(t,s)\widetilde{A}_1(t,s,T)\exp\left\{\int_s^T g_u du - \widetilde{A}_2(t,s)r_t\right\}ds\right) - \lambda_2 G_t\right\}.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
G_t &= \mathbb{E}_t\left[\frac{H(T)}{H(t)}G_T\right] \\
&= \left[W_0\exp\left\{\int_0^T g_t dt\right\} + \int_0^t c_s\exp\left\{\int_s^T g_u du\right\}ds\right]B(t,T) \\
&\quad +\int_t^T \exp\left\{\int_s^T g_u du\right\}\mathbb{E}_t\left[\frac{H(T)}{H(t)}c_s\right]ds.
\end{aligned}$$

Now,

$$\begin{aligned}
\mathbb{E}_t\left[\frac{H(T)}{H(t)}c_s\right] &= \mathbb{E}_t\left[\frac{H(s)}{H(t)}c_s B(s,T)\right] \\
&= c_t\mathbb{E}_t K_1^{\eta_1}(0,1,T-s)\exp\left\{-\int_t^s\left(r_u + \frac{1}{2}|\underline{\lambda}(u)|^2\right)du - \int_t^s \underline{\lambda}'(u)d\underline{z}(u)\right. \\
&\quad +\int_t^s(\alpha_1(u) + \alpha_2(\eta_1 r_u + \eta_2))du + \int_t^s \alpha_3 dz(u) + \int_t^s \alpha_4\sqrt{\eta_1 r_u + \eta_2}dz_r(u) \\
&\quad \left.-r_s h(T-s)\right\} \\
&= c_t K_1^{\eta_1}(0,1,T-s)\exp\left\{\int_t^s\left(\alpha_1(u) - \frac{1}{2}\lambda_1^2 - \frac{1}{2}\lambda_2^2\eta_2 + \frac{(\alpha_3 - \lambda_1)^2}{2}\right)du\right\} \\
&\quad .\exp\left\{a\left(\alpha_4 - \lambda_2\right)(s-t) + (\alpha_4 - \lambda_2)r_t\right\} \\
&\quad .\mathbb{E}_t\exp\left\{-\widetilde{\alpha}^c r_s - \beta^c\int_t^s r_u du\right\},
\end{aligned}$$

with

$$\begin{aligned}
\widetilde{\alpha}^c &= \alpha^c + h(T-s) \\
&= \alpha_4 - \lambda_2 + h(T-s),
\end{aligned}$$

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where β^c is given by (38). We apply (5) and we obtain the result, with $\widetilde{A}_1(t, s, T)$ and $\widetilde{A}_2(t, s)$ defined by analogy with (36) and (37) with α^c replaced by $\widetilde{\alpha}^c$:

$$
\begin{aligned}
\widetilde{A}_1(t, s, T) &= K_1^{\eta_1}(\widetilde{\alpha}^c, \beta^c, s - t)\, K_1^{\eta_1}(0, 1, T - s) \\
&\quad \exp\left\{ \int_t^s \left(\alpha_1(u) - \frac{1}{2}\lambda_1^2 - \frac{1}{2}\lambda_2^2\eta_2 + (\alpha_4 - \lambda_2)a + \frac{(\alpha_3 - \lambda_1)^2}{2} \right) du \right\}, \\
\widetilde{A}_2(t, s) &= K_2^{\eta_1}(\widetilde{\alpha}^c, \beta^c, s - t) - (\alpha_4 - \lambda_2).
\end{aligned}
$$

Finally, in order to find the process $(\underline{\rho}_t)_{t \in [0,T]}$, it is enough to differenziate the process $H(t)G_t$. ∎

Remark 1 *For the particular case of deterministic contributions (i.e. with $\alpha_2 = \alpha_3 = \alpha_4 = 0$), it turns out that:*

$$
dD_t = -c_t dt + \int_t^T c_s\, (dB(t, s))\, ds
$$

and that the process $(\underline{\zeta}_t)_{t \in [0,T]} = (\zeta_t, \zeta_t^r)'$ is given by

$$
\begin{aligned}
\zeta_t &= -H(t)D_t\lambda_1, \\
\zeta_t^r &= \int_t^T c_s B(t, s)\sigma_B(s - t, r_t)ds - H(t)D_t\lambda_2\sqrt{\eta_1 r_t + \eta_2},
\end{aligned}
$$

while, when the guarantee G_T is a strictly positive constant, it is easy to check that the process $(\underline{\rho}_t)_{t \in [0,T]} = (\rho_t, \rho_t^r)'$ is given by

$$
\begin{aligned}
\rho_t &= -H(t)G_T B(t, T)\lambda_1, \\
\rho_t^r &= \left(\sigma_B(T - t, r_t) - \lambda_2\sqrt{\eta_1 r_t + \eta_2} \right) H(t)G_T B(t, T).
\end{aligned}
$$

As mentioned above, we end this section by showing the admissibility condition on the interest rate g_t: from (7) it follows

$$
W_0 e^{\int_0^T g_t dt} B(0, T) + \int_0^T e^{\int_t^T g_s ds} \mathbb{E}\left[c_t H(t)B(t, T) \right] dt < W_0 + D_0,
$$

then

$$
W_0 e^{\int_0^T g_t dt} B(0, T) + c_0 \int_0^T \widetilde{A}_1(0, t, T)e^{\int_t^T g_s ds - \widetilde{A}_2(0,t)r_0} dt < W_0 + D_0,
$$

which defines an upper bound for the possible values of the deterministic process $(g_t)_{t \in [0,T]}$.

6   Conclusion

We considered a model for a Defined Contribution Pension Fund: we studied the problem of a fund manager to invest some given initial wealth and a contribution flow into the financial market in such a way that the expected utility of the terminal wealth is maximized and a minimum guarantee is satisfied at the final date. We investigated the case of affine interest rates and we supposed the markets to be complete. By introducing an auxiliary process, called *surplus process*, we reduced to a purely investment-problem. This problem has been explicitly solved under the assumption that the utility function of the fund manager belongs to the CRRA family. Finally we came back to the solution of the initial problem by specifying the contribution process and the guarantee.

There are several directions for future research.

First, it would be interesting to extend our approach to the case of incomplete markets, since the contribution process is not necessarily generated by the market.

We further notice that the constraint on the surplus does not imply that the wealth at date t, W_t, is positive almost surely. In fact, closed-form solution can be negative with a strictly positive probability. It would be interesting to study also the constrained case ($W(t) \geq 0$, $\forall t \geq 0$). In a consumption-investment framework such problem has been solved by El Karoui and Jeanblanc-Picqué (1998). In a pension fund context, however, the fund manager could be more interested in constraints on the *surplus process* or he could be interested in ways to diversify the risk that the *global wealth* becomes negative over the fund population. It would therefore be interesting to find some actuarial hypotheses on the fund population under which this implication is valid for the global fund wealth.

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7 Appendix

In this appendix, we give the proof of Lemma 2. For the reader's convenience, we recall Lemma 2:

Lemma 2 *If the interest rates follow*

$$dr_t = (a - br_t)dt - \sqrt{\eta_1 r_t + \eta_2}dz_r(t),$$

with $\eta_1 \geq 0, \eta_2 \geq 0$, then there exist two deterministic functions

$$K_1^{\eta_1}(\alpha, \beta, T - t) \text{ and } K_2^{\eta_1}(\alpha, \beta, T - t)$$

such that for $\alpha \in R, \beta > 0$

$$\mathbb{E}_t \left[e^{-\alpha r_T - \beta \int_t^T r_s ds} \right] = K_1^{\eta_1}(\alpha, \beta, T - t)e^{-r_t K_2^{\eta_1}(\alpha, \beta, T - t)}.$$

Proof. Let us consider first the case $\eta_1 = 0$, i.e. the Vasiček dynamics

$$dr_t = (a - br_t)dt - \sqrt{\eta_2}dz_r(t).$$

Since $I(\alpha, \beta, T - t) := -\alpha r_T - \beta \int_t^T r_s ds$ is Gaussian, it follows that

$$\mathbb{E}_t\left[e^{-\alpha r_T - \beta \int_t^T r_s ds}\right] = e^{\mathbb{E}_t[I(\alpha,\beta,T-t)] + \frac{1}{2}\mathbb{V}_t[I(\alpha,\beta,T-t)]}.$$

For $s \geq t$,

$$r_s = e^{-b(s-t)}r_t + a\int_t^s e^{-b(s-u)}du - \sqrt{\eta_2}\int_t^s e^{-b(s-u)}dz_r(u),$$

then

$$\mathbb{E}_t[r_T] = r_t e^{-b(T-t)} + \frac{a}{b}\left(1 - e^{-b(T-t)}\right),$$

$$\mathbb{V}_t[r_T] = \frac{\eta_2}{2b}\left(1 - e^{-2b(T-t)}\right),$$

while from

$$\begin{aligned}
\int_t^T r_s ds &= r_t\int_t^T e^{-b(s-t)}ds + a\int_t^T\int_t^s e^{-b(s-u)}duds - \sqrt{\eta_2}\int_t^T\int_t^s e^{-b(s-u)}dz_r(u)ds \\
&= \frac{1 - e^{-b(T-t)}}{b}r_t + \frac{a}{b}(T-t) - \frac{a}{b^2}\left(1 - e^{-b(T-t)}\right) - \sqrt{\eta_2}\int_t^T\frac{1 - e^{-b(T-u)}}{b}dz_r(u),
\end{aligned}$$

we obtain

$$\mathbb{E}_t\left[\int_t^T r_s ds\right] = \frac{1 - e^{-b(T-t)}}{b}r_t + \frac{a}{b}(T-t) - \frac{a}{b^2}\left(1 - e^{-b(T-t)}\right),$$

$$\mathbb{V}_t\left[\int_t^T r_s ds\right] = \eta_2\int_t^T\left(\frac{1 - e^{-b(T-u)}}{b}\right)^2 du,$$

$$Cov_t\left(r_T, \int_t^T r_s ds\right) = \eta_2\int_t^T e^{-b(T-u)}\left(\frac{1 - e^{-b(T-u)}}{b}\right)du.$$

Finally

$$\begin{aligned}
\mathbb{E}_t\left[e^{-\alpha r_T - \beta \int_t^T r_s ds}\right] &= e^{-\alpha\mathbb{E}_t[r_T] - \beta\mathbb{E}_t\left[\int_t^T r_s ds\right] + \frac{\alpha^2}{2}\mathbb{V}_t[r_T] + \frac{\beta^2}{2}\mathbb{V}_t\left[\int_t^T r_s ds\right] + \alpha\beta Cov_t\left(r_T, \int_t^T r_s ds\right)} \\
&= K_1^0(\alpha, \beta, T-t)e^{-r_t K_2^0(\alpha,\beta,T-t)},
\end{aligned}$$

with

$$K_1^0(\alpha, \beta, T - t) = \exp\left\{\frac{a}{b}\left(\frac{\beta}{b} - \alpha\right)\left(1 - e^{-b(T-t)}\right) - \beta\frac{a}{b}(T - t)\right.$$

$$+\frac{\alpha^2\eta_2}{4b}\left(1 - e^{-2b(T-t)}\right) + \frac{\beta^2\eta_2}{2}\int_t^T\left(\frac{1 - e^{-b(T-u)}}{b}\right)^2 du$$

$$\left. +\alpha\beta\eta_2\int_t^T e^{-b(T-u)}\left(\frac{1 - e^{-b(T-u)}}{b}\right) du\right\},$$

$$K_2^0(\alpha, \beta, T - t) = \alpha e^{-b(T-t)} + \frac{\beta}{b}\left(1 - e^{-b(T-t)}\right).$$

On the other hand, for $\eta_1 > 0$, let us consider the process

$$R_t = \eta_1 r_t + \eta_2,$$

with

$$dR_t = (a\eta_1 + b\eta_2 - bR_t)dt - \eta_1\sqrt{R_t}dz_r(t).$$

From Pitman and Yor (1982) (see also Lamberton and Lapeyre 1991), it follows

$$\mathbb{E}_t\left[e^{-\alpha r_T - \beta\int_t^T r_s ds}\right] = e^{\frac{\eta_2}{\eta_1}(\alpha + \beta(T-t))}\mathbb{E}_t\left[e^{-\frac{\alpha}{\eta_1}R_T - \frac{\beta}{\eta_1}\int_t^T R_s ds}\right]$$

$$= K_1^{\eta_1}(\alpha, \beta, T - t)e^{-r_t K_2^{\eta_1}(\alpha,\beta,T-t)},$$

with

$$K_1^{\eta_1}(\alpha, \beta, T - t) = \exp\left\{\frac{\eta_2}{\eta_1}(\alpha + \beta(T - t)) - (a\eta_1 + b\eta_2)\Phi_{\frac{\alpha}{\eta_1},\frac{\beta}{\eta_1}}(T - t) - \eta_2\Psi_{\frac{\alpha}{\eta_1},\frac{\beta}{\eta_1}}(T - t)\right\},$$

$$K_2^{\eta_1}(\alpha, \beta, T - t) = \eta_1\Psi_{\frac{\alpha}{\eta_1},\frac{\beta}{\eta_1}}(T - t),$$

where

$$\Phi_{\lambda,\mu}(u) = \frac{2}{\eta_1^2}\ln\left(\frac{2\gamma e^{\frac{\gamma+b}{2}u}}{\eta_1^2\lambda(e^{\gamma u} - 1) + \gamma - b + e^{\gamma u}(\gamma + b)}\right)$$

$$\Psi_{\lambda,\mu}(u) = \frac{\lambda(\gamma + b + e^{\gamma u}(\gamma - b)) + 2\mu(e^{\gamma u} - 1)}{\eta_1^2\lambda(e^{\gamma u} - 1) + \gamma - b + e^{\gamma u}(\gamma + b)},$$

$$\gamma = \sqrt{b^2 + 2\eta_1^2\mu}.$$

■