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Abstract

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Disciplines

Finance and Financial Management

Optimal Investment under Uncertainty

By ANDREW B. ABEL*

This paper examines the effect of output price uncertainty on the investment decision of a risk-neutral competitive firm which faces convex costs of adjustment.¹ This issue has been analyzed by Richard Hartman (1972) and by Robert Pindyck (1982), but they reached dramatically different results. Hartman showed that with a linearly homogeneous production function, increased output price uncertainty leads the competitive firm to increase its investment. However, Pindyck found increased output price uncertainty leads to increased investment only if the marginal adjustment cost function is convex; but, if the marginal adjustment cost function is concave, then increased uncertainty will reduce the rate of investment. Pindyck argues that his results differ from Hartman's results because of a different stochastic specification of the price of output. In Hartman's discrete-time model, price is random in each period including the current period, whereas in Pindyck's continuous-time model, the current price is known but the future evolution of prices is stochastic. In this paper, I demonstrate that Hartman's results continue to hold using Pindyck's stochastic specification and that Pindyck's analysis applies to a so-called "target" rate of investment, which in general is not optimal.

The model developed herein, which is a special case of Pindyck's model, is used because it can be solved explicitly, unlike

Pindyck's more general model. Since Pindyck did not derive an expression for the optimal rate of investment, he used a phase diagram to determine the target capital stock. This target capital stock is determined by the intersection of a locus for which the rate of change of the capital stock is zero, and a locus for which the expected change in the rate of investment is zero. A problem with this stochastic phase diagram approach is that in general there is no reason for the firm to be on the locus with zero expected change in investment, even in the long run. Indeed, in the particular model in this paper, optimal behavior is such that the expected proportional rate of change of investment is (in general, a nonzero) constant over time.

I. The Model of the Firm

Since the model presented below is a special case of Pindyck's model, the description of it will be brief. The competitive firm uses labor, L_t , and capital, K_t to produce output according to a Cobb-Douglas production function. The firm hires labor at a fixed wage rate w and undertakes gross investment I_t , by incurring an increasing convex cost of adjustment $c(I_t)$. It is assumed that the cost of adjustment function has constant elasticity $\beta > 1$. Therefore, the firm's cash flow at time t is $p_t L_t^\alpha K_t^{1-\alpha} - wL_t - \gamma I_t^\beta$ where p_t is the price of output. Suppose that the firm is risk neutral and maximizes the expected present value of its cash flow subject to the capital accumulation equation

$$(1) \quad dK_t = (I_t - \delta K_t) dt,$$

and the equation which describes the behavior of the price of output

$$(2) \quad dp_t/p_t = \sigma dz,$$

where dz is a Wiener process with mean zero and unit variance. Equation (1) simply states

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¹Cost of adjustment models were introduced by Robert Eisner and Robert Strotz (1963), Robert Lucas (1967), John Gould (1968) and Arthur Treadway (1969). More recently, Michael Mussa (1977), my (1979, 1981, 1982) studies, Hiroshi Yoshikawa (1980), and Fumio Hayashi (1982) have used cost of adjustment models to provide a more rigorous foundation for James Tobin's (1969) q theory of investment.

that net investment is equal to gross investment less depreciation where δ is the constant proportional rate of physical depreciation. The price process described by (2) has the properties² that $E_t(p_s) = p_t, s \geq t$, and the variance of p_s , conditional on p_t , is $(s - t)\sigma^2$. The value of the firm is the maximized expected present value of cash flow. Assuming that the discount rate r is constant, we can write the value of the firm as

$$(3) \quad V(K_t, p_t) = \max_{I_s, L_s} E_t \int_t^\infty [p_s L_s^\alpha K_s^{1-\alpha} - wL_s - \gamma I_s^\beta] \exp(-r(s-t)) ds,$$

where the maximization is subject to the constraints in (1) and (2).

The value function in (3) must obey the following optimality condition

$$(4) \quad rV(K_t, p_t) dt = \max_{L_t, I_t} [p_t L_t^\alpha K_t^{1-\alpha} - wL_t - \gamma I_t^\beta] dt + E_t(dV).$$

The optimality condition in (4) has a straightforward economic interpretation. If the owners of the firm require a mean rate of return r , then the left-hand side of (4) is the total mean return required by the owners of the firm over the time interval dt . The right-hand side of (4) is the total return expected by the owners of the firm. It consists of the cash flow plus the expected capital gain or loss $E_t(dV)$. Optimality requires that the expected return equals the required mean return.

To calculate the capital gain or loss, dV , we recognize that the value of the firm is a function of the two state variables K_t and p_t and then apply Ito's Lemma to obtain

$$(5) \quad dV = V_K dK + V_p dp + (1/2)V_{KK}(dK)^2 + (1/2)V_{pp}(dp)^2 + V_{pK}(dp)(dK).$$

²For good discussions of stochastic calculus set in an economic context, the reader is referred to William Brock, Gregory Chow (1981), Stanley Fischer (1975), and Robert Merton (1971). The solution to a more general form of the stochastic differential equation in (2) is presented in Fischer, equation (13A).

Substituting (1) and (2) into (5), and recognizing that $E_t(dz) = (dt)^2 = (dt)(dz) = 0$, we obtain the expected change in the value of the firm over the time interval dt :

$$(6) \quad E_t(dV) = [(I_t - \delta K_t)V_K + (1/2)p_t^2 \sigma^2 V_{pp}] dt.$$

Substituting (6) into (4) yields

$$(7) \quad rV(K_t, p_t) = \max_{L_t, I_t} \{ p_t L_t^\alpha K_t^{1-\alpha} - wL_t - \gamma I_t^\beta + (I_t - \delta K_t)V_K + \frac{1}{2} p_t^2 \sigma^2 V_{pp} \}.$$

It is easily shown that

$$(8) \quad \max_{L_t} \{ p_t L_t^\alpha K_t^{1-\alpha} - wL_t \} = hp_t^{1/(1-\alpha)} K_t,$$

where $h = (1 - \alpha)(\alpha/w)^{\alpha/(1-\alpha)}$.

Observe that $hp_t^{1/(1-\alpha)}$ is the marginal revenue product of capital.

Differentiating the right-hand side of (7) with respect to I_t , we obtain

$$(9) \quad \gamma \beta I_t^{\beta-1} = V_K.$$

According to (9), the optimal rate of investment is such that the marginal cost of investment is equal to the marginal valuation of capital V_K . Substituting (8) and (9) into (7) yields

$$(10) \quad rV(K_t, p_t) = hp_t^{1/(1-\alpha)} K_t + (\beta - 1)\gamma I_t^\beta - \delta K_t V_K + (1/2)p_t^2 \sigma^2 V_{pp}.$$

Equations (9) and (10) together can be expressed as a nonlinear second-order partial differential equation. In general, such equations cannot be solved explicitly, as noted by Pindyck. However, I have imposed enough structure on this problem to obtain an explicit solution. It can be verified that the

equations below satisfy (9) and (10).

(11a)

$$V(K_t, p_t) = q_t K_t + \frac{(\beta - 1)\gamma(q_t/\beta\gamma)^{\beta/(\beta-1)}}{r - \frac{\beta(1-\alpha + \alpha\beta)\sigma^2}{2(1-\alpha)^2(\beta-1)^2}}$$

where

$$(11b) \quad q_t = \frac{hp_t^{1/(1-\alpha)}}{r + \delta - \frac{\alpha\sigma^2}{2(1-\alpha)^2}}$$

and

$$(12) \quad I_t = (q_t/\beta\gamma)^{1/(\beta-1)}.$$

Several results follow immediately from equations (11a), (11b), and (12). First we observe that the value of the firm is a linear function of the capital stock, since the slope of the value function, q_t , is independent of the capital stock.³ As shown in Section II, q_t is equal to the present value of expected marginal revenue products of capital. Since, for a competitive firm with a constant returns to scale production function, the marginal product of capital depends only on the real wage rate, and thus is independent of the level of the capital stock, it follows that q_t is independent of K_t . According to (12), the optimal rate of investment is an increasing function of q_t . Moreover, I_t depends only on q_t and is independent of K_t .

II. The Effect of Uncertainty of Investment

Since the optimal rate of investment is an increasing function of q_t , and depends only on q_t , we can determine the qualitative effect of uncertainty on investment simply by analyzing the effect of uncertainty on q_t . It follows immediately from (11b) that for a given level of the current price of output p_t , an increase in uncertainty, as measured by σ^2 , will lead to an increase in the optimal rate of investment. Contrary to the results of

³Mussa showed that for a linearly homogeneous production function $F(K, L)$, the value of the firm under certainty is linear in K_t .

Pindyck, this result holds whether the marginal adjustment function is convex ($\beta > 2$), concave ($\beta < 2$) or linear ($\beta = 2$).

To explain the positive effect of uncertainty on investment, I will first show that q_t is the expected present value of marginal revenue products accruing to the undepreciated portion of capital from time t onward. Since the marginal revenue product of capital, $p_t F_{K_t}$, is equal to $hp_t^{1/(1-\alpha)}$, it can be shown that, for the price process in (2),⁴

$$(13) \quad E_t(p_s F_{K_s}) = hE_t(p_s^{1/(1-\alpha)}) \\ = hp_t^{1/(1-\alpha)} \exp\left[\alpha\sigma^2(s-t)/2(1-\alpha)^2\right].$$

Using (13), the expected present value of marginal revenue products of capital is

$$(14) \quad \int_t^\infty E_t(p_s F_{K_s}) \exp[-(r+\delta)(s-t)] ds \\ = \int_t^\infty hp_t^{1/(1-\alpha)} \exp\left[\left(\alpha\sigma^2(s-t) / 2(1-\alpha)^2\right) - (r+\delta)(s-t)\right] ds.$$

The integral on the right-hand side of (14) can be evaluated by inspection and is obviously equal to q_t in (11b). Thus q_t is indeed the expected present value of marginal products of capital. Note from equation (13) that increased uncertainty tends to increase the expected value of future marginal revenue products of capital and hence increases q_t and investment. Although equation (13) applies only for a Cobb-Douglas production function, the reasoning applies more generally to competitive firms with linearly homogeneous production functions. As long as the marginal revenue product of capital is a strictly convex function of the price of output, then increased uncertainty about the future price of output tends to increase the

⁴Given p_t , the \log of the price of output at some future date s is normally distributed with $E_t(\ln p_s) = \ln p_t - (1/2)\sigma^2(s-t)$ and $\text{var}_t(\ln p_s) = \sigma^2(s-t)$ (see Fischer's Appendix). Using the fact that if $\ln x$ is normally distributed, then $E(x) = \exp[E(\ln x) + (1/2)\text{var}(\ln x)]$, we can derive my equation (13).

expected future marginal revenue product, and hence increases both q_t and investment.⁵

Contrary to the results presented above, Pindyck finds that the effect of uncertainty on investment depends on the curvature of the marginal adjustment cost function. His results are derived under the assumption that (eventually) the expected rate of change of investment, $E_t(dI_t)/dt$, is equal to zero. However, the optimal rate of investment does not, in general, obey this assumption.

To examine the dynamic behavior of investment, I first apply Ito's Lemma to (11a) to obtain

$$(15) \quad \frac{dq_t}{q_t} = \frac{1}{1-\alpha} \frac{dp_t}{p_t} + \frac{\alpha}{2(1-\alpha)^2} \left(\frac{dp_t}{p_t} \right)^2,$$

which implies

$$(16) \quad (1/dt)E_t(dq_t/q_t) = \frac{\alpha\sigma^2}{2(1-\alpha)^2}.$$

Substituting (16) into (11b), we obtain

$$(17) \quad q_t = hp_t^{1/(1-\alpha)} / (r + \delta - E_t(dq_t/q_t)/dt).$$

Interpreting q_t as the shadow price of capital, the user cost of capital is $[r + \delta - (1/dt)E_t(dq_t/q_t)]q_t$. Therefore, equation (17) merely expresses the equality of the marginal revenue product of capital and the user cost of capital.

Now to analyze the dynamic behavior of investment, let us apply Ito's Lemma to (12) to obtain

$$(18) \quad \frac{dI_t}{I_t} = \frac{1}{\beta-1} \frac{dq_t}{q_t} + \frac{2-\beta}{2(\beta-1)^2} \left(\frac{dq_t}{q_t} \right)^2.$$

Taking expectations on both sides of (18), and using (15) to calculate $(dq_t/q_t)^2$, we

⁵This line of argument was developed by Richard Hartman (1972).

obtain

$$(19) \quad \frac{1}{dt}E_t\left(\frac{dI_t}{I_t}\right) = \frac{1}{(\beta-1)} \frac{1}{dt}E_t\left(\frac{dq_t}{q_t}\right) + \frac{(2-\beta)\sigma^2}{2(\beta-1)^2(1-\alpha)^2}.$$

Now substituting (16) into (19) yields

$$(20) \quad \frac{1}{dt}E_t\left(\frac{dI_t}{I_t}\right) = \frac{1}{2(\beta-1)(1-\alpha)^2} \left(\alpha + \frac{2-\beta}{\beta-1} \right) \sigma^2.$$

From equation (20), we observe that the expected proportional growth rate of investment is independent of the state variables and is constant over time. Although this constant growth rate is zero under certainty ($\sigma^2 = 0$), we find that in the presence of uncertainty, the expected growth rate of investment is not equal to zero in general, nor does it tend toward zero. Thus Pindyck's analysis, which assumes that $E_t(dI_t) = 0$, is inappropriate to the analysis of the behavior of the optimal rate of investment.⁶

III. Concluding Comments

Pindyck has emphasized the curvature of the marginal adjustment cost function in determining the effect of uncertainty on investment. Although I have shown that, given the current price of output, higher uncertainty leads to a higher current rate of investment regardless of the curvature of the marginal adjustment cost function, this curvature does

⁶In order for Pindyck's analysis to apply to optimal investment behavior, the expression on the right-hand side of (20) must equal zero. This expression is zero if either (a) there is no uncertainty ($\sigma = 0$) or (b) the parameters of technology happen to be such that $\alpha = (\beta - 2)/(\beta - 1)$. More generally, if the price of output evolves according to $dp_t/p_t = \pi dt + \sigma dz$, where π is the expected rate of inflation, it can be shown that the expected rate of change of optimal investment is zero if and only if $\pi = [\alpha + (2 - \beta)/(\beta - 1)]\sigma^2/2(1 - \alpha)$. (See my 1981 paper.) Pindyck's results apply only to situations in which this condition holds.

have an important implication for the relation between the expected growth rate of investment and the expected growth rate of the marginal valuation of capital, q_t . Under certainty, the growth rate of investment is equal to the growth rate of q_t multiplied by the elasticity of investment with respect to q_t , $1/(\beta - 1)$, as may be verified from (19). However, under uncertainty, this relation holds only if the marginal adjustment cost function is linear. If the marginal adjustment cost is convex (concave), then, under uncertainty, the expected growth rate of investment is less (greater) than the expected growth rate of q_t multiplied by the elasticity of investment with respect to q_t .

The analysis of this paper is easily extended to allow for uncertainty in the wage rate, w , and uncertainty in γ , which enters multiplicatively into the adjustment cost function. In this extended framework, the value function is again linear in the capital stock. Investment is an increasing function of only q_t/γ_t , where q_t is the slope of the value function.⁷ Uncertainty affects investment only to the extent that it affects the variance of the logarithm of the real wage rate. Specifically, increased variance in the real wage rate leads to an increase in the optimal rate of investment.

Finally, note that, according to (16), the marginal valuation of capital q_t is expected to grow without bound as we look further and further into the future. This disquieting feature of the model is a consequence of the assumption in (2) that p_t evolves according to a random walk. Therefore, given today's price p_t , the variance of the future price of output, p_s , grows without bound as s grows without bound. Since the marginal revenue product of capital is a convex function of the price of

output, the expected value of this marginal revenue product is an increasing function of the variance of the price. Therefore, the expected marginal revenue product grows without bound over time. This feature of the model could be removed by assuming that the price of output evolves according to a process for which the forecast variance is bounded. However, in the present context, the easy interpretations of the explicit solutions made possible by the random walk assumption seem to be worth the cost.

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⁷If $dp_t/p_t = \pi_p dt + \sigma_p dz_p$, $dw_t/w_t = \pi_w dt + \sigma_w dz_w$ and $d\gamma_t/\gamma_t = \pi_\gamma dt + \sigma_\gamma dz_\gamma$ where dz_p , dz_w , and dz_γ are Wiener processes with mean zero and unit variance, then the optimal rate of investment is proportional to $(q_t/\gamma_t)^{1/(\beta-1)}$ where

$$q_t = \frac{hp_t^{1/(1-\alpha)}}{r + \delta - \frac{1}{1-\alpha}[\pi_p - \alpha\pi_w] - \frac{1}{2} \frac{\alpha}{(1-\alpha)^2} \text{var}(p-w)}$$

See my 1981 paper for details.

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