Hindawi
Discrete Dynamics in Nature and Society
Volume 2019, Article ID 2150878, 8 pages
https://doi.org/10.1155/2019/2150878

Research Article

Optimal Layer Reinsurance for Compound Fractional Poisson Model

Jiesong Zhang

School of Management, Huaibei Normal University, Huaibei 235000, China

Correspondence should be addressed to Jiesong Zhang; j_s_zhang@126.com

Received 27 November 2018; Accepted 23 January 2019; Published 7 February 2019

Academic Editor: Rigoberto Medina

In this paper, we study the optimal retentions for an insurer with a compound fractional Poisson surplus and a layer reinsurance treaty. Under the criterion of maximizing the adjustment coefficient, the closed form expressions of the optimal results are obtained. It is demonstrated that the optimal retention vector and the maximal adjustment coefficient are not only closely related to the parameter of the fractional Poisson process, but also dependent on the time and the claim intensity, which is different from the case in the classical compound Poisson process. Numerical examples are presented to show the impacts of the three parameters on the optimal results.

1. Introduction

In various geophysical applications, it is observed that the interarrival times between extreme events are power-law distributed, and the exponentially distributed interarrivals cannot be applied [1]. Musson et al. [2] studied the earthquake interarrival times for several regions in Japan and Greece and found that a lognormal distribution provided a good fit. Salim and Pawitan [3] investigated the hourly rainfall data in the southwest of Ireland by a generalized Bartlett–Lewis model with Pareto storm interarrival time. Stoynov et al. [4] proposed an approach for modeling the flood arrivals on Chinese rivers Yangtze and Huanghe by switch-time distributions, which can be considered as distributions of sums of random number exponentially distributed random variables.

Considering the importance of quantifying the stochastic behavior of extreme events in actuarial sciences, Beghin and Macci [5] deal with a fractional Poisson model for insurance, in which the interarrival times between claims are assumed to have Mittag-Leffler distribution instead of the exponential distribution as in the classical Poisson model. Inspired by this work and motivated by the use of the fractional Poisson process in modeling extreme events, such as earthquakes and storms, Biard and Saussereau [6] initiatively described surplus processes of insurance companies by compound fractional Poisson processes, and some results for ruin probabilities are also presented under various assumptions on the distribution of the claim sizes. Different from the case in the classical compound Poisson process (CPP), the compound fractional Poisson process (CFPP) becomes nonstationary [6] and is no longer Markovian [7]. The long-range dependence and the short-range dependence of the CFPP are studied in [6, 8], the estimation of parameters is given by [9], and the convergence of quadratic variation is investigated by [10]. To complete the review of the existing literature on the CFPP, we refer the reader to [11–18].

In this paper, we model the surplus process of an insurance company by the abovementioned CFPP proposed by [6], which can be expressed as

$$u + ct - \sum_{i=1}^{N_h(t)} X_i, \quad t \geq 0, \tag{1}$$

where u is the initial capital, c is the constant premium rate, and X_i, $i = 1, 2, 3, \cdots$ represents the size of the ith claim and the claim sizes are assumed to be independent and identically distributed nonnegative variables with a common distribution function F. The counting process $N_h(t)$ is the fractional Poisson process that was first defined in [11, 19] as a renewal process with Mittag-Leffler waiting time. Specifically, it has independent and identically distributed interarrival times (τ_i) between two claims with distribution given by

$$\Pr(\tau > t) = E_h\left(-\lambda t^h\right) \tag{2}$$

for $\lambda > 0$ and $0 < h \le 1$, where

$$E_h(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+hk)} \tag{3}$$

is the Mittag-Leffler function (Γ denotes the Euler gamma function) defined for any complex number z. With $T_n = \tau_1 + \tau_2 + \cdots \tau_n$, the time of the nth jump, the process $(N_h(t))_{t\ge0}$ defined by

$$N_h(t) = \max\{n \ge 0 : T_n \le t\} = \sum_{k\ge1} 1_{\{T_k \le t\}} \tag{4}$$

is the so-called fractional Poisson process of parameter h. It includes the usual Poisson process when $h = 1$.

This paper supposes the insurer reinsures his or her risk by a layer reinsurance treaty. As in [20, 21], we assume that the common distribution function $F(x)$ of X_i is such a continuous function that $F(0) = 0$, $0 < F(x) < 1$ for $0 < x < M$ and $F(x) = 1$ for $x \ge M$, here $M = \inf\{x : F(x) = 1$, and $0 < M \le +\infty$; that the moment generating function of $F(x)$, $M_X(r)$, exists for $r \in (-\infty, r_\infty)$ for some $0 < r_\infty \le \infty$; and that $\lim_{r\longrightarrow r_\infty} M_X(r) = +\infty$. Let μ be the expected value of X_i. Denote the decision variables representing the layer retention by d_1 and d_2. The ceded loss function is the layer reinsurance in the form of

$$\begin{aligned} g(x) &= \min\{(x-d_1)_+, d_2 - d_1\} \\ &= (x-d_1)_+ - (x-d_2)_+, \end{aligned} \tag{5}$$

where $\{x\}_+ = \max\{x, 0\}$, and $0 < d_1 \le d_2 \le M$. Thus, the insurer will retain from the ith claim

$$X_i(d_1, d_2) = (X_i \wedge d_1) + (X_i - d_2)_+, \quad i = 1, 2, \cdots, N_h(t). \tag{6}$$

Then $\{X_i(d_1, d_2)\}$ are i.i.d. strictly positive random variables and independent of the claim counting process $N_h(t)$.

Assume that the reinsurance premium is charged by the expected value principle, and denote the expected value of $X_i(d_1, d_2)$ by $\mu(d_1, d_2)$. Then the premium income rate becomes

$$\begin{aligned} c(d_1, d_2) &= \frac{\lambda h t^{h-1}}{\Gamma(1+h)} \mu(1+\theta) \\ &\quad - (1+\eta) \frac{\lambda h t^{h-1}}{\Gamma(1+h)} E(X_i - X_i(d_1, d_2)) \\ &= (\theta - \eta) \frac{\lambda h t^{h-1}}{\Gamma(1+h)} \mu \\ &\quad + (1+\eta) \frac{\lambda h t^{h-1}}{\Gamma(1+h)} \mu(d_1, d_2) \\ &= \frac{\lambda h t^{h-1}}{\Gamma(1+h)} ((\theta - \eta)\mu + (1+\eta)\mu(d_1, d_2)) \end{aligned} \tag{7}$$

where $\theta = (c/\mu)(\Gamma(1+h)/\lambda h t^{h-1}) - 1$ denotes the security loading of the insurer, and η is the security loading of the reinsurer. As usual, we assume that $\eta > \theta$. Note that the following inequality should be held,

$$(\theta - \eta)\mu + (1+\eta)\mu(d_1, d_2) > 0. \tag{8}$$

Otherwise, the insurance company faces ruin with probability one.

Thus, the reserve process of the insurer with the layer reinsurance policy can be represented by

$$U_t^{d_1,d_2} = u + c(d_1, d_2)t - \sum_{i=1}^{N_h(t)} X_i(d_1, d_2). \tag{9}$$

Now define the ruin time by

$$\tau^{d_1,d_2} = \inf\left\{t \ge 0 : U_t^{d_1,d_2} < 0\right\}, \tag{10}$$

and define the ruin probability by

$$\psi(u) = \psi^{d_1,d_2}(u) = \Pr\left\{\tau^{d_1,d_2} < \infty \mid U_0^{d_1,d_2} = u\right\}. \tag{11}$$

2. Optimal Results

In this section, we devote to get the explicit expressions for the optimal retentions in the layer reinsurance treaty. It is difficult to derive the explicit expression of the ruin probability in the CPP and even more difficult in CFPP. We consider the optimal retentions to maximize the adjustment coefficient, i.e., to maximize the coefficient R which satisfies the following inequality

$$\psi(u) \le e^{-Ru}. \tag{12}$$

Lemma 1. *If $R = R_c(d_1, d_2) \ge 0$ satisfies the equation*

$$\lambda \int_0^{\infty} e^{rx} dP_{X_i(d_1,d_2)}(x) = \lambda + (r \bullet c(d_1, d_2))^h, \tag{13}$$

which is an implicit equation with respect to r; then the inequality (12) follows.

Proof. Assume (13) holds; we prove the inequality (12) by mathematical induction (see [22] for the CPP case). Let $\psi_n(u)$ be the probability that ruin occurs on the nth claim or before with an initial surplus u. Clearly,

$$0 \le \psi_0(u) \le \psi_1(u) \le \cdots \le \cdots \psi_n(u) \le \cdots \tag{14}$$

and

$$\lim_{n\longrightarrow\infty} \psi_n(u) = \psi(u). \tag{15}$$

Furthermore, from

$$\psi_0(u) = \begin{cases} 1, & u < 0 \\ 0, & u \ge 0 \end{cases} \tag{16}$$

we have

$$\psi_0(u) \le e^{-Ru}. \tag{17}$$

To complete the nontrivial part of the mathematical induction, we apply the total probability formula with respect to the arrival time and the size of the first claim. Then, we obtain

$$\begin{aligned}\psi_n(u) &= \int_0^\infty \sum_{k=1}^\infty (-1)^{k-1} \\ &\cdot \frac{kh\lambda^k t^{hk-1}}{\Gamma(1+hk)} \int_0^\infty \psi_{n-1}(u + c(d_1,d_2)t - x)\, dP_{X_i(d_1,d_2)}(x)\, dt \\ &\le \int_0^\infty \sum_{k=1}^\infty (-1)^{k-1} \frac{kh\lambda^k t^{hk-1}}{\Gamma(1+hk)} e^{-Ru-R\bullet c(d_1,d_2)t} dt \\ &\cdot \int_0^\infty e^{Rx} dP_{X_i(d_1,d_2)}(x) = e^{-Ru} \bullet E\left[e^{-\tau\bullet R\bullet c(d_1,d_2)}\right] \\ &\bullet \int_0^\infty e^{Rx} dP_{X_i(d_1,d_2)}(x) = e^{-Ru} \bullet \frac{\lambda}{\lambda + (R\bullet c(d_1,d_2))^h} \\ &\bullet \int_0^\infty e^{Rx} dP_{X_i(d_1,d_2)}(x),\end{aligned} \tag{18}$$

where the last equation is obtained from equation (4.15) in [23]. Thus, the inequality (12) follows immediately from (13). □

Since $M_{X_i(d_1,d_2)}(r) = \int_0^\infty e^{rx} dP_{X_i(d_1,d_2)}(x)$, (13) is equivalent to

$$(r\bullet c(d_1,d_2))^h = \lambda\left(M_{X_i(d_1,d_2)}(r) - 1\right). \tag{19}$$

Substituting (7) into (19) yields

$$\begin{aligned}&\left(\frac{\lambda h t^{h-1}}{\Gamma(1+h)}((\theta-\eta)\mu + (1+\eta)\mu(d_1,d_2))\right)^h r^h \\ &\quad - \lambda\left(M_{X_i(d_1,d_2)}(r) - 1\right) = 0.\end{aligned} \tag{20}$$

Our goal is to maximize $R_c(d_1,d_2)$, i.e., to find the optimal retention (d_1^*, d_2^*), such that

$$R_C \coloneqq R_C(d_1^*, d_2^*) = \sup_{d_1,d_2} R_c(d_1,d_2). \tag{21}$$

Note that the left-hand side of (19) is a concave function and the right-hand side is a convex function, with respect to r. Therefore, there are at most two solutions to (19), and the left-hand side of (20) is nonpositive at $r = R_C$, i.e., R_C is the solution to

$$\begin{aligned}\sup_{d_1,d_2} &\left\{\left(\frac{\lambda h t^{h-1}}{\Gamma(1+h)}((\theta-\eta)\mu + (1+\eta)\mu(d_1,d_2))\right)^h r^h \right. \\ &\left. - \lambda\left(M_{X_i(d_1,d_2)}(r) - 1\right)\right\} = 0,\end{aligned} \tag{22}$$

or, equivalently,

$$\sup_{d_1,d_2}\{g(d_1,d_2)\} = 0, \tag{23}$$

where

$$\begin{aligned}g(d_1,d_2) &= \left(\frac{\lambda h t^{h-1}}{\Gamma(1+h)}((\theta-\eta)\mu \right. \\ &\quad \left. + (1+\eta)\mu(d_1,d_2))\right)^h r^h \\ &\quad - r\lambda\left(\int_0^{d_1}(1-F(x))e^{rx}dx \right. \\ &\quad \left. + \int_{d_2}^M (1-F(x))e^{(x+d_1-d_2)r}dx\right).\end{aligned} \tag{24}$$

Next we adopt the method used by [21] to determine the optimal retention level (d_1^*, d_2^*).

Lemma 2. *Denote the maximizer of $g(d_1,d_2)$ with d_1 and d_2 being $\overline{d}_1$ and $\overline{d}_2$, respectively. Then, $\overline{d}_1$ is the solution to the following equation with respect to d_1,*

$$\begin{aligned}&h(1+\eta)\left(\frac{h t^{h-1}}{\Gamma(1+h)}\right)^h \left(\lambda r\left((\theta-\eta)\mu \right.\right. \\ &\quad \left.\left. + (1+\eta)\int_0^{d_1}(1-F(x))dx\right)\right)^{h-1} = e^{rd_1},\end{aligned} \tag{25}$$

and $\overline{d}_2 = M$.

Proof. By differentiating $g(d_1,d_2)$ with respect to d_1, we have

$$\begin{aligned}\frac{\partial g(d_1,d_2)}{\partial d_1} &= h r^h (1+\eta)\left(\frac{\lambda h t^{h-1}}{\Gamma(1+h)}\right)^h \\ &\cdot (((\theta-\eta)\mu + (1+\eta)\mu(d_1,d_2)))^{h-1}(1 - F(d_1) \\ &- r\lambda\left((1-F(d_1))e^{rd_1} \right. \\ &\left. + r\int_{d_2}^M (1-F(x))e^{(x+d_1-d_2)r}dx\right),\end{aligned} \tag{26}$$

which means that

$$\begin{aligned}&\left(h(1+\eta)\left(\frac{h t^{h-1}}{\Gamma(1+h)}\right)^h \right. \\ &\quad \left. \cdot \left(\lambda r\left((\theta-\eta)\mu + (1+\eta)\mu(\overline{d}_1,d_2)\right)\right)^{h-1} - e^{r\overline{d}_1}\right) \\ &\quad \cdot \left(1 - F(\overline{d}_1)\right) = r\int_{d_2}^M (1-F(x)) \\ &\quad \cdot e^{(x+\overline{d}_1-d_2)r}dx,\end{aligned} \tag{27}$$

for any fixed d_2.

Then, differentiating $g(\overline{d}_1, d_2)$ with respect to d_2 and combining with (27), we obtain

$$\begin{aligned}\frac{\partial g(\overline{d}_1, d_2)}{\partial d_2} &= hr^h(1+\eta)\left(\frac{\lambda h t^{h-1}}{\Gamma(1+h)}\right)^h \\ &\cdot \left(((\theta-\eta)\mu + (1+\eta)\mu(\overline{d}_1, d_2))\right)^{h-1}(F(d_2)-1) \\ &- r\lambda\left(e^{r\overline{d}_1}(F(d_2)-1)\right. \\ &\left. - r\int_{d_2}^{M}(1-F(x))e^{(x+\overline{d}_1-d_2)r}dx\right) = r\lambda\left(h(1+\eta)\right. \\ &\cdot\left(\frac{ht^{h-1}}{\Gamma(1+h)}\right)^h \\ &\left.\cdot\left(r\lambda((\theta-\eta)\mu + (1+\eta)\mu(\overline{d}_1, d_2))\right)^{h-1} - e^{r\overline{d}_1}\right) \\ &\cdot (F(d_2)-1) + r^2\lambda\int_{d_2}^{M}(1-F(x)) \\ &\cdot e^{(x+\overline{d}_1-d_2)r}dx = r\lambda \\ &\cdot \frac{r\int_{d_2}^{M}(1-F(x))e^{(x+\overline{d}_1-d_2)r}dx}{1-F(d_1)}(F(d_2)-1) \\ &+ r^2\lambda\int_{d_2}^{M}(1-F(x))e^{(x+\overline{d}_1-d_2)r}dx \\ &= r^2\lambda\left(\frac{F(d_2)-1}{1-F(\overline{d}_1)}+1\right)\int_{d_2}^{M}(1-F(x)) \\ &\cdot e^{(x+\overline{d}_1-d_2)r}dx.\end{aligned} \tag{28}$$

Note that

$$\frac{F(d_2)-1}{1-F(d_1)} + 1 \geq 0 \tag{29}$$

holds for any $d_2 \geq d_1$, and we have

$$\begin{aligned}r^2\lambda\left(\frac{F(d_2)-1}{1-F(\overline{d}_1)}+1\right)\int_{d_2}^{M}(1-F(y))e^{(y+\overline{d}_1-d_2)r}dy \\ \geq 0,\end{aligned} \tag{30}$$

and thus

$$\overline{d}_2 = M. \tag{31}$$

By replacing $d_2 = M$ back into (27), we can derive

$$\begin{aligned}h(1+\eta)\left(\frac{ht^{h-1}}{\Gamma(1+h)}\right)^h\left(\lambda r\left((\theta-\eta)\mu\right.\right. \\ \left.\left. + (1+\eta)\int_0^{\overline{d}_1}(1-F(x))\,dx\right)\right)^{h-1} - e^{r\overline{d}_1} = 0,\end{aligned} \tag{32}$$

which completes the proof of Lemma 2. □

Since

$$\mu(d_1, d_2) = \int_0^{d_1}(1-F(x))\,\mathrm{d}x + \int_{d_2}^{M}(1-F(x))\,\mathrm{d}x \tag{33}$$

and $\overline{d}_2 = M$, by (8), we have

$$\mu(\overline{d}_1, M) = \int_0^{\overline{d}_1}(1-F(x))\,dx > \frac{(\eta-\theta)}{1+\eta}\mu > 0. \tag{34}$$

Denote $\underline{d}_1 = \inf\{\overline{d}_1 | \overline{d}_1 \text{ satisfies (34)}\}$.

According to Lemma 2, we know that to solve the optimization problem (23) is equivalent to solving the equation:

$$\begin{aligned}&\left(\frac{\lambda h t^{h-1}}{\Gamma(1+h)}\left((\theta-\eta)\mu\right.\right. \\ &\left.\left. + (1+\eta)\int_0^{\overline{d}_1}(1-F(x))\,dx\right)\right)^h r^h - r\lambda\int_0^{\overline{d}_1}(1 \\ &- F(x))e^{rx}dx = 0,\end{aligned} \tag{35}$$

or, alternatively,

$$\begin{aligned}G(\overline{d}_1) &:= \left(\frac{\lambda h t^{h-1}}{\Gamma(1+h)}\left((\theta-\eta)\mu\right.\right. \\ &\left.\left. + (1+\eta)\int_0^{\overline{d}_1}(1-F(x)\,dx)\right)\right)^h - \lambda \\ &\cdot \frac{\int_0^{\overline{d}_1}(1-F(x))e^{rx}dx}{r^{h-1}} = 0,\end{aligned} \tag{36}$$

where $r = r(\overline{d}_1)$ is a univariate function of $\overline{d}_1$ determined by (32). In fact, we have Lemmas 3–5.

Lemma 3. *Equation (32) has a unique positive root $r = R_C$ for any given $\overline{d}_1 \in (\underline{d}_1, M)$.*

Proof. For any given $\overline{d}_1 \in (\underline{d}_1, M)$, define the left-hand side of (32) by $H(r)$, i.e.,

$$\begin{aligned}H(r) = h(1+\eta)\left(\frac{ht^{h-1}}{\Gamma(1+h)}\right)^h\left(\lambda r\left((\theta-\eta)\mu\right.\right. \\ \left.\left. + (1+\eta)\int_0^{\overline{d}_1}(1-F(x))\,dx\right)\right)^{h-1} - e^{r\overline{d}_1}.\end{aligned} \tag{37}$$

It is not difficult to see that

$$\lim_{r\downarrow 0} H(r) = +\infty \quad \text{and} \lim_{r\uparrow\infty} H(r) = -\infty. \tag{38}$$

Moreover, note that $0 < h \leq 1$; we know that $H(r)$ is a strictly decreasing function in r. Thus, it completes the proof of Lemma 3. □

Lemma 4. *The function $r = r(\overline{d}_1)$ is strictly decreasing in d_1 and $\partial r/\partial\overline{d}_1 < 0$.*

Proof. Rewrite (32) as

$$h(1+\eta)\left(\frac{ht^{h-1}}{\Gamma(1+h)}\right)^h\left(\lambda\left((\theta-\eta)\mu + (1+\eta)\int_0^{\overline{d}_1}(1-F(x)\,dx)\right)\right)^{h-1} = \frac{e^{r\overline{d}_1}}{r^{h-1}}. \tag{39}$$

By differentiating both sides of (39) with respect to $\overline{d}_1$, we have

$$h(h-1)(1+\eta)^2\lambda^{h-1}\left(\frac{ht^{h-1}}{\Gamma(1+h)}\right)^h \cdot\left((\theta-\eta)\mu + (1+\eta)\int_0^{\overline{d}_1}(1-F(x)\,dx)\right)^{h-2} \cdot\left(1-F\left(\overline{d}_1\right)\right) - \frac{e^{r\overline{d}_1}}{r^{h-2}} = \frac{d_1e^{r\overline{d}_1} - e^{r\overline{d}_1}(h-1)r^{-1}}{r^{h-1}} \cdot\frac{\partial r}{\partial\overline{d}_1}. \tag{40}$$

Then, we find that

$$\frac{\partial r}{\partial\overline{d}_1} = -\left(h(1-h)(1+\eta)^2\lambda^{h-1}\left(\frac{ht^{h-1}}{\Gamma(1+h)}\right)^h \cdot\left((\theta-\eta)\mu + (1+\eta)\int_0^{\overline{d}_1}(1-F(x)\,dx)\right)^{h-2} \bullet (1-F(d_1))e^{-r\overline{d}_1} + \frac{1}{r^{h-2}}\right)\bullet\frac{r^h}{r\overline{d}_1 + (1-h)} < 0. \tag{41}$$

□

Lemma 5. *The equation $G(\overline{d}_1) = 0$ has a unique positive root $\overline{d}_{1C} \in (\underline{d}_1, M)$.*

Proof. Differentiating $G(\overline{d}_1)$ with respect to $\overline{d}_1$, by (39), we have

$$\begin{aligned}
G'\left(\overline{d}_1\right) &= h\left(\frac{\lambda ht^{h-1}}{\Gamma(1+h)}\right)^h\left((\theta-\eta)\mu + (1+\eta)\int_0^{\overline{d}_1}(1-F(x)\,dx)\right)^{h-1}(1+\eta)\left(1-F\left(\overline{d}_1\right)\right) - \lambda\frac{\partial r}{\partial\overline{d}_1} \\
&\quad\bullet\frac{\left(\int_0^{\overline{d}_1}x(1-F(x))e^{rx}dx + (1-F(d_1))e^{r\overline{d}_1}\right)r^{h-1} - (h-1)\left(\int_0^{\overline{d}_1}(1-F(x))e^{rx}dx\right)r^{h-2}}{r^{2h-2}} \\
&= h\left(\frac{\lambda ht^{h-1}}{\Gamma(1+h)}\right)^h\left((\theta-\eta)\mu + (1+\eta)\int_0^{\overline{d}_1}(1-F(x)\,dx)\right)^{h-1}(1+\eta)\left(1-F\left(\overline{d}_1\right)\right) \\
&\quad - \lambda\frac{\left(\int_0^{\overline{d}_1}x(1-F(x))e^{rx}dx\left(\partial r/\partial\overline{d}_1\right) + \left(1-F\left(\overline{d}_1\right)\right)e^{r\overline{d}_1}\right)}{r^{h-1}} \\
&\quad + \lambda\frac{(h-1)\left(\int_0^{\overline{d}_1}(1-F(x))e^{rx}dx\right)r^{-1}}{r^{h-1}}\frac{\partial r}{\partial\overline{d}_1} \\
&= h\left(\frac{\lambda ht^{h-1}}{\Gamma(1+h)}\right)^h\left((\theta-\eta)\mu + (1+\eta)\int_0^{\overline{d}_1}(1-F(x)\,dx)\right)^{h-1}(1+\eta)\left(1-F\left(\overline{d}_1\right)\right) \\
&\quad - \lambda\left(\frac{\int_0^{\overline{d}_1}x(1-F(x))e^{rx}dx}{r^{h-1}} - \frac{(h-1)\left(\int_0^{\overline{d}_1}(1-F(x))e^{rx}dx\right)r^{-1}}{r^{h-1}}\right)\frac{\partial r}{\partial\overline{d}_1}
\end{aligned}$$

$$
\begin{aligned}
&- \lambda \frac{\left(1-F\left(\bar{d}_1\right)\right) e^{r\bar{d}_1}}{r^{h-1}} \\
&= \lambda \left(1-F\left(\bar{d}_1\right)\right) \frac{e^{r\bar{d}_1}}{r^{h-1}} - \lambda \frac{\left(1-F\left(\bar{d}_1\right)\right) e^{r\bar{d}_1}}{r^{h-1}} \\
&\quad - \lambda \left(\frac{\int_0^{\bar{d}_1} x\left(1-F(x)\right) e^{rx} dx}{r^{h-1}} - \frac{(h-1)\left(\int_0^{\bar{d}_1} \left(1-F(x)\right) e^{rx} dx\right) r^{-1}}{r^{h-1}} \right) \frac{\partial r}{\partial \bar{d}_1} \\
&= -\lambda \left(\frac{\int_0^{\bar{d}_1} x\left(1-F(x)\right) e^{rx} dx}{r^{h-1}} + \frac{(1-h)\left(\int_0^{\bar{d}_1} \left(1-F(x)\right) e^{rx} dx\right) r^{-1}}{r^{h-1}} \right) \frac{\partial r}{\partial \bar{d}_1}.
\end{aligned} \tag{42}
$$

Hence, from Lemma 4 we know that $G'(\bar{d}_1) > 0$. Moreover, we have $\lim_{\bar{d}_1 \downarrow \underline{d}_1} r(\bar{d}_1) = +\infty$ and $\lim_{\bar{d}_1 \uparrow M} r(\bar{d}_1) = 0$, which can be seen from (39). Thus, for any $h \in (0,1)$, it is held that

$$
\lim_{\bar{d}_1 \downarrow \underline{d}_1} G\left(\bar{d}_1\right) \leq -\lim_{d_1 \downarrow \underline{d}_1} \lambda r^{1-h} \int_0^{\bar{d}_1} 1 - F(x)\, dx = -\infty, \tag{43}
$$

and

$$
\begin{aligned}
\lim_{\bar{d}_1 \uparrow M} G\left(\bar{d}_1\right) &= \lim_{\bar{d}_1 \uparrow M} \left[\left(\frac{\lambda h t^{h-1}}{\Gamma(1+h)} \left((\theta-\eta)\mu + (1+\eta) \int_0^{\bar{d}_1} \left(1-F(x)\,dx\right) \right) \right)^h - \lambda \frac{\int_0^{\bar{d}_1} \left(1-F(x)\right) e^{rx} dx}{r^{h-1}} \right] \\
&= \left(\frac{\lambda h t^{h-1}}{\Gamma(1+h)} \left((\theta-\eta)\mu + (1+\eta) E\left[X_1 \wedge M\right] \right) \right)^h - \lim_{d_1 \uparrow M} \left[\lambda r^{1-h} \int_0^M \left(1-F(x)\right) e^{rx} dx \right] \\
&= \left(\frac{\lambda h t^{h-1}}{\Gamma(1+h)} \left((\theta-\eta)\mu + (1+\eta)\mu \right) \right)^h > 0.
\end{aligned} \tag{44}
$$

If $h = 1$, it is easy to see that $\lim_{\bar{d}_1 \downarrow \underline{d}_1} G(\bar{d}_1) \leq -\lim_{d_1 \downarrow \underline{d}_1} \lambda \int_0^{\bar{d}_1} 1 - F(x)dx < 0$ and $\lim_{\bar{d}_1 \uparrow M} G(\bar{d}_1) = \lambda\theta\mu > 0$. Therefore, the proof of Lemma 5 is completed. □

Now, we can conclude the main result of this paper.

Theorem 6. *Let $\bar{d}_{1C}$ be the unique positive root of the equation $G(d_1) = 0$. Then the optimal layer reinsurance retention level of the compound fractional Poisson surplus (1) to maximize the adjustment coefficient is $(\bar{d}_{1C}, M)$, and the maximal adjustment coefficient R_C is the unique positive root of (32).*

Since the CFPP degenerates into the classical CPP when $h = 1$, we immediately obtain the following corollary from Theorem 6.

Corollary 7. *Let $\bar{d}_{1C}$ be the unique positive root of the equation*

$$
\begin{aligned}
&(\theta-\eta)\mu + (1+\eta) \int_0^{\bar{d}_1} \left(1-F(x)\,dx\right) \\
&\quad - \int_0^{\bar{d}_1} \left(1-F(x)\right) e^{rx} dx = 0.
\end{aligned} \tag{45}
$$

Then the optimal layer reinsurance retention level of the classical compound Poisson surplus to maximize the adjustment coefficient is $(\bar{d}_{1C}, M)$, and the maximal adjustment coefficient is the unique positive root of (32), i.e.,

$$
R_C = \frac{\ln(1+\eta)}{\bar{d}_{1C}}. \tag{46}
$$

Remark. It is not difficult to see that Corollary 7 is in fact Theorem 4.3 of [21]. By comparing the obtained Theorem 6 in this paper for CFPP with those in [21] for CPP, we find that the optimal retention level $(\bar{d}_{1C}, M)$ and the maximal adjustment coefficient R_C here not only depend on the parameter h of the fractional Poisson process, but also depend on the claim intensity λ and are both relevant to time t, which should be more realistic. In fact, the claim intensity is a very important parameter for estimating the ruin probability and by the dynamic reinsurance strategy the change of the insurer's best risk position is reflected with respect to time.

To illustrate the impact of replacing the exponential distributed interarrivals by the general Mittag-Leffler distributed interarrivals, as well as the claim intensity λ and the time

Table 1: Optimal retention levels and the upper bounds with different parameters.

t=1 ***λ=2*** ***h***	$\bar{d}_{1C}$	R_C	e^{-uR_C}	***h=0.5*** ***λ=2*** ***t***	$\bar{d}_{1C}$	R_C	e^{-uR_C}	***h=0.5*** ***t=1*** ***λ***	$\bar{d}_{1C}$	R_C	e^{-uR_C}
0.1	0.1489	1.8234	0.0001	0.6	0.2508	0.8241	0.0162	1.6	0.2514	0.8021	0.0166
0.2	0.1659	1.3195	0.0014	0.7	0.2521	0.7728	0.0210	1.7	0.2523	0.7629	0.0220
0.3	0.1878	1.0144	0.0063	0.8	0.2531	0.7308	0.0259	1.8	0.2532	0.7273	0.0263
0.4	0.2163	0.8099	0.0174	0.9	0.2540	0.6955	0.0309	1.9	0.2540	0.6946	0.0310
0.5	0.2548	0.6650	0.0360	1.0	0.2548	0.6649	0.0360	2.0	0.2548	0.6649	0.0360
0.6	0.3084	0.5606	0.0606	1.1	0.2555	0.6383	0.0411	2.1	0.2555	0.6377	0.0412
0.7	0.3861	0.4870	0.0876	1.2	0.2561	0.6150	0.0462	2.2	0.2562	0.6128	0.0467
0.8	0.5002	0.4436	0.1088	1.3	0.2567	0.5940	0.0513	2.3	0.2569	0.5895	0.0525
0.9	0.6557	0.4399	0.1109	1.4	0.2572	0.5752	0.0564	2.4	0.2574	0.5682	0.0584
1	0.8053	0.5035	0.0807	1.5	0.2576	0.5583	0.0613	2.5	0.2579	0.5486	0.0644

t, on the optimal results, we give some numerical examples and compare the optimal retention levels and the maximal adjustment coefficient with different parameters h, λ, and t.

3. Examples

Assume that the insurer has an initial capital $u = 5$, that the claim size X_i has a uniform distribution on the interval $[0, 4]$, and that $\theta = 0.4$, $\eta = 0.5$. We compute the values of $\bar{d}_{1C}$, R_C and the upper bound of ruin probability with different parameter values of h, λ, and t. To this end, we need to solve the following equations:

$$\begin{gathered}1.5h\left(\frac{ht^{h-1}}{\Gamma(1+h)}\right)^{h}\\ \cdot\left(\lambda R_C\left(-0.2+1.5\left(\bar{d}_{1C}-\frac{\bar{d}_{1C}^{2}}{8}\right)\right)\right)^{h-1}\\ -e^{R_C\bullet\bar{d}_{1C}}=0,\\ \left(\frac{\lambda ht^{h-1}}{\Gamma(1+h)}\left(-0.2+1.5\left(\bar{d}_{1C}-\frac{\bar{d}_{1C}^{2}}{8}\right)\right)\right)^{h}-\lambda\\ \cdot\frac{-1-4R_C+e^{R_C\cdot\bar{d}_{1C}}\left(4R_C-R_C\bullet\bar{d}_{1C}+1\right)}{4R_C^{1+h}}=0,\end{gathered} \tag{47}$$

for different given (h, λ, t) with $\bar{d}_{1C} \in (\underline{d}_1, M) = (0.1356, 4.0000)$ and $R_C > 0$.

By applying numerical method, the results for different cases are given in Table 1.

From Table 1, it is not difficult to see that the impacts of the parameter h on the optimal retention level and the upper bound of ruin probability are significant, and the impacts of the parameters t and λ are also obvious. Specifically, if the risk process of the insurance company obeys the compound fractional Poisson model and the compound Poisson model is used, then the insurer may take more risk and the ruin probability is overestimated or underestimated. Even with the CFPP, the optimal strategy should vary timely and according to the change of the claim intensity.

4. Conclusion

To characterize and to disperse the extreme event risk that the insurer may face in practice, this paper models the underwriting risk as a compound fractional Poisson process and studies the optimal retentions with a layer reinsurance treaty. At first, the equation that the adjustment coefficient of the compound fractional Poisson process should satisfy is given and proved. Secondly, to overcome the difficulties caused by the newly adopted model, some lemmas are given, and the closed form expressions of the optimal retention levels are obtained. It is found that the optimal retention level and the maximal adjustment coefficient here relate to the parameter h of the fractional Poisson process, the time t, and the claim intensity λ, which are all absent in the optimal results for the classical compound Poisson process. Finally, numerical examples demonstrate the impacts of the three parameters on the optimal results, respectively. The obtained results in this paper may help the insurers, especially the ones who underwrite extreme risk, to make more appropriate decisions in reinsurance contracts.

Data Availability

There is a numerical example in this article; the parameter data used to support the findings of this study are included within the article. For the software (Matlab) code data to obtain the numerical results in the example, it will be available upon request by contact with the corresponding author.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work is supported by Anhui Provincial Natural Science Foundation (No. 1608085QG169) and Humanity and Social Science Youth Foundation of Ministry of Education (No. 17YJC630212).

References

[1] D. A. Benson, R. Schumer, and M. M. Meerschaert, "Recurrence of extreme events with power-law interarrival times," *Geophysical Research Letters*, vol. 34, no. 16, Article ID L16404, 2007.

[2] R. M. W. Musson, T. Tsapanos, and C. T. Nakas, "A power-law function for earthquake interarrival time and magnitude," *Bulletin of the Seismological Society of America*, vol. 92, no. 5, pp. 1783–1794, 2002.

[3] A. Salim and Y. Pawitan, "Extensions of the Bartlett-Lewis model for rainfall processes," *Statistical Modelling. An International Journal*, vol. 3, no. 2, pp. 79–98, 2003.

[4] P. Stoynov, P. Zlateva, D. Velev et al., "Modelling of major flood arrivals on Chinese rivers by switch-time processes," in *Proceedings of the IOP Conference Series: Earth and Environmental Science*, vol. 68, IOP Publishing, May 2017.

[5] L. Beghin and C. Macci, "Large deviations for fractional Poisson processes," *Statistics & Probability Letters*, vol. 83, no. 4, pp. 1193–1202, 2013.

[6] R. Biard and B. Saussereau, "Fractional Poisson process: long-range dependence and applications in ruin theory," *Journal of Applied Probability*, vol. 51, no. 3, pp. 727–740, 2014.

[7] E. Scalas, "A class of CTRWs: compound fractional Poisson processes," in *Fractional dynamics*, pp. 353–374, World Sci. Publ., Hackensack, NJ, USA, 2012.

[8] A. Maheshwari and P. Vellaisamy, "On the long-range dependence of fractional Poisson and negative binomial processes," *Journal of Applied Probability*, vol. 53, no. 4, pp. 989–1000, 2016.

[9] Y. Wang, D. Wang, and F. Zhu, "Estimation of parameters in the fractional compound Poisson process," *Communications in Nonlinear Science and Numerical Simulation*, vol. 19, no. 10, pp. 3425–3430, 2014.

[10] E. Scalas and N. Viles, "On the convergence of quadratic variation for compound fractional Poisson processes," *Fractional Calculus and Applied Analysis An International Journal for Theory and Applications*, vol. 15, no. 2, pp. 314–331, 2012.

[11] O. N. Repin and A. I. Saichev, "Fractional Poisson law," *Radiophysics and Quantum Electronics*, vol. 43, no. 9, pp. 738–741, 2000.

[12] M. M. Meerschaert, D. A. Benson, H.-P. Scheffler, and B. Baeumer, "Stochastic solution of space-time fractional diffusion equations," *Physical Review E: Statistical, Nonlinear, and Soft Matter Physics*, vol. 65, no. 4, part 1, Article ID 041103, 2002.

[13] M. M. Meerschaert and H.-P. Scheffler, "Limit theorems for continuous-time random walks with infinite mean waiting times," *Journal of Applied Probability*, vol. 41, no. 3, pp. 623–638, 2004.

[14] M. M. Meerschaert and H.-P. Scheffler, "Triangular array limits for continuous time random walks," *Stochastic Processes and Their Applications*, vol. 118, no. 9, pp. 1606–1633, 2008.

[15] B. Baeumer, M. M. Meerschaert, and E. Nane, "Space-time duality for fractional diffusion," *Journal of Applied Probability*, vol. 46, no. 4, pp. 1100–1115, 2009.

[16] M. M. Meerschaert, E. Nane, and P. Vellaisamy, "The fractional Poisson process and the inverse stable subordinator," *Electronic Journal of Probability*, vol. 16, pp. no. 59, 1600–1620, 2011.

[17] L. Beghin and C. Macci, "Alternative forms of compound fractional poisson processes," *Abstract and Applied Analysis*, vol. 2012, Article ID 747503, 30 pages, 2012.

[18] L. Beghin and C. Macci, "Fractional discrete processes: compound and mixed Poisson representations," *Journal of Applied Probability*, vol. 51, no. 1, pp. 19–36, 2014.

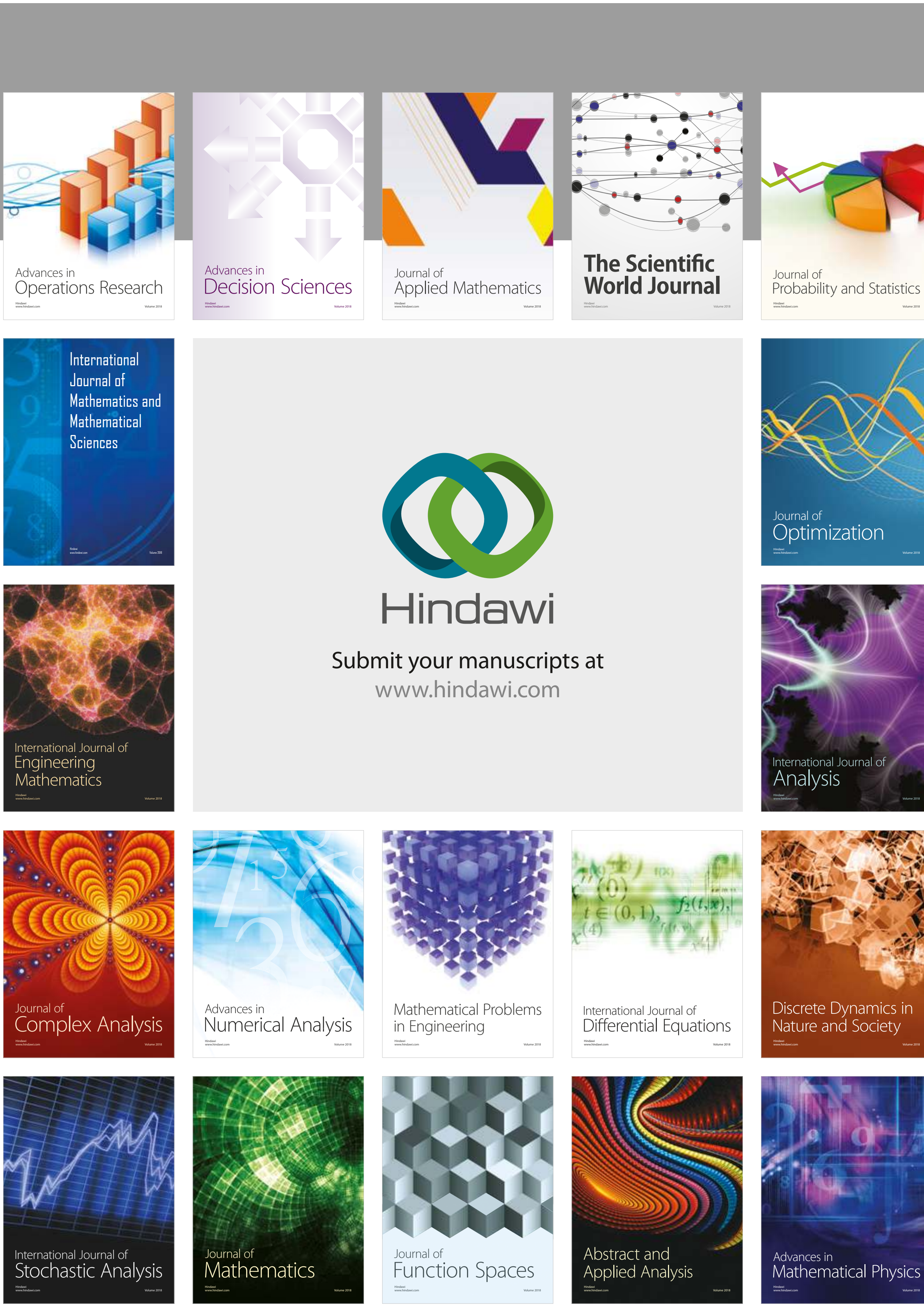
[19] F. Mainardi, R. Gorenflo, and E. Scalas, "A fractional generalization of the Poisson processes," *Vietnam Journal of Mathematics*, vol. 32, no. Special Issue, pp. 53–64, 2004.

[20] Z. B. Liang and J. Y. Guo, "Ruin probabilities under optimal combining quota-share and excess-of-loss reinsurance," *Acta Mathematica Sinica*, vol. 53, no. 5, pp. 857–870, 2010.

[21] X. Zhang and Z. Liang, "Optimal layer reinsurance on the maximization of the adjustment coefficient," *Numerical Algebra, Control and Optimization*, vol. 6, no. 1, pp. 21–34, 2016.

[22] H. U. Gerber, *An Introduction to Mathematical Risk Theory*, College of Insurance, New York, NY, USA, 1979.

[23] L. Beghin and E. Orsingher, "Fractional Poisson processes and related planar random motions," *Electronic Journal of Probability*, vol. 14, pp. no. 61, 1790–1827, 2009.


Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
The Scientific World Journal
Journal of Probability and Statistics
International Journal of Mathematics and Mathematical Sciences
Hindawi
Submit your manuscripts at
www.hindawi.com
Journal of Optimization
International Journal of Engineering Mathematics
International Journal of Analysis
Journal of Complex Analysis
Advances in Numerical Analysis
Mathematical Problems in Engineering
International Journal of Differential Equations
Discrete Dynamics in Nature and Society
International Journal of Stochastic Analysis
Journal of Mathematics
Journal of Function Spaces
Abstract and Applied Analysis
Advances in Mathematical Physics