Iranian Journal of Mathematical Sciences and Informatics
Vol. 10, No. 1 (2015), pp 11-22
DOI: 10.7508/ijmsi.2015.01.002

Optimal Linear Codes Over $GF(7)$ and $GF(11)$ with Dimension 3

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Abstract. Let $n_q(k, d)$ denote the smallest value of n for which there exists a linear $[n, k, d]$-code over the Galois field $GF(q)$. An $[n, k, d]$-code whose length is equal to $n_q(k, d)$ is called *optimal*. In this paper we present some matrix generators for the family of optimal $[n, 3, d]$ codes over $GF(7)$ and $GF(11)$. Most of our given codes in $GF(7)$ are non-isomorphic with the codes presented before. Our given codes in $GF(11)$ are all new.

Keywords: Linear codes, Optimal codes, Griesmer bound.

2010 Mathematics Subject Classification: 68P30, 94A29.

1. Introduction

Let $V_n(q)$ be the vector space of all ordered n-tuples over $GF(q)$ (Galois field of q elements). Each subspace of $V_n(q)$ is called a linear code. By an $[n, k, d]$-code of length n and dimension k over $GF(q)$ we mean a k-dimensional subspace of $V_n(q)$ with minimum Hamming distance d. Sometimes we use the term $[n, k]$-code if the minimum distance d is not under consideration. Optimizing any one of the parameters n, k and d, when the other two values are given is one of the main problems in coding theory. These problems over a

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Received 31 January 2013; Accepted 13 September 2014

fixed $GF(q)$ can be characterized as follows [7,9,11]:

1. What is the maximum value of d (denoted by $d_q(n, k)$) for which there exists an $[n, k, d]$-code?

2. What is the minimum value of n (denoted by $n_q(k, d)$) for which there exists an $[n, k, d]$-code?

3. What is the maximum value of k (denoted by $k_q(n, d)$) for which there exists an $[n, k, d]$-code?

For a literature backlog of this topic when $q = 2, 3, 5$ and 7 one is refereed to [2,3,4,5,7,9,14,15,17,18,21,22,23]. In this paper we consider the problem for the case of codes over $GF(7)$ and $GF(11)$. In section 2 we give a brief review of our method. In Section 3 we give some basic necessary preliminaries. In Section 4 and 5 we study the value of functions $n_7(k, d)$ and $n_{11}(k, d)$ for $k \leq 3$ and some values of d.

2. A Brief Review of Our Method

In all parts of this study we followed a random process to obtain generator matrices for specified codes. For a given k and d, the length n of the optimal code can be obtained from the Griesmer bound and the existence of a possible $[n, k, d]$-code can be investigated by the MacWilliams identities. In case of nonexistence, we try to produce an $[n + 1, k, d]$-code. In case of existence, the weight distributions given by MacWilliams identities help us to produce the code. We generate matrices of size $k \times n$ by a random process (based on a computer programming), we then test each of these matrices to be a generator matrix for a specific $[n, k, d]$-code. Since the parameters are small, in case of necessity we may produce all code words to find the weight distributions of the code to see whether the weight distributions satisfy the MacWilliams identities.

In quasi-cyclic codes we studied only the cases where n is a multiple of k. Now since $n = ks$, for some positive integer "s", we produced the first row of each of the s circulant matrices G_i of size $k \times k$, by the same random process, where $G = [G_1|G_2| \cdots |G_s]$ by the notations given in the corresponding section, is the generating matrix of a code. The remaining rows of each G_i would fill cyclically. In last step the weight distributions should be tested.

As an example the $QC[32, 4, 25]$-code is consist of 8 circulant matrices of size 4×4, which is built as above. Gulliver method, as cited in Grassl code table [6], gives a different construction for this code which is based on two polynomials of degree 15 to produce the first rows of two matrices of sizes 4×16.

Also, it is important to note that in [4] as cited in Grassl table [6], the study uses the following method: given the parameters n and k, the optimality of code is focused on d (minimum distance), whereas in our papers, we employ a

[DOI: 10.7508/ijmsi.2015.01.002]

different method: given the parameters k and d, the optimality is focused on n. Sure both methods reach to a unique optimal code, while the methods are completely different. In quasi cyclic codes we tried to find the generating matrices, where their weight distributions satisfy MacWilliams identities, meanwhile the other references such as [4] have a different approach.

For more emphasis on the originality of our methods and our results we tested some of our generator matrices, for any possible isomorphism, compared with the generator matrices given in Grassl table [6]. So we computed the weight distributions of some of the codes given in Grassl tables by a computer programming and found that they are all completely different.

For example our $QC[20,4,15]$ and [32,4,25]-codes are both non-isomorphic with the same $QC[20,4,15]$ and [32,4,25] codes given in Grassl table, as their weight distributions are different. One of the [28,4,21]-codes and two of the [24,4,18]-codes given in Grassl table, are all non-isomorphic from our corresponding codes (their weight distributions are different).

3. Preliminaries

let $w(x)$ denotes the Hamming weight of a vector x. That is the number of nonzero entries in x. For a linear code, the minimum distance d is equal to the smallest value of $w(x)$ when x range over all nonzero codewords. Let C be an $[n,k]$-code and let A_i and B_i be the number of codewords of weight i in C and in dual code $C^{\perp}$, respectively [9]. Now note that:

Theorem 3.1. *(The MacWilliams identities [19]). Let C be an $[n,k]$-code over $GF(q)$. Then the A_i's and B_i's satisfy*

$$\sum_{j=0}^{n-t} \binom{n-j}{t} A_j = q^{k-t} \sum_{j=0}^{t} \binom{n-j}{n-t} B_j. \tag{3.1}$$

for $t = 0, 1, \ldots, n$.

Lemma 3.2. [9]. *For an $[n,k,d]$-code over $GF(q)$, $B_i = 0$ for each value of i (where $1 \leq i \leq k$) such that there does not exist an $[n-i, k-i+1, d]$-code.*

Lemma 3.3. [10] *: Let C be an $[n,k,d]$-code over $GF(q)$ with $k \geq 2$, and with weight enumerator $\sum_{i=0}^{n} A_i z^i$. Then*

(i) if x and y are a linearly independent pair of codeword of C ,

$$w(x) + w(y) \leq qn - qd + d, \tag{3.2}$$

(ii) $A_i = 0$ for $i > q(n-d)$.

Corollary 3.4. *(I) Let C be an $[n,k,d]$-code over $GF(7)$ with $k \geq 2$. Then :*

(i) if x and y are a linearly independent pair of codewords of C, then $w(x) + w(y) \leq 7n - 6d$,

(ii) $A_i = 0$ for $i > 7(n-d)$,

[DOI: 10.7508/ijmsi.2015.01.002]

(iii) $A_i = 0$ *or* 6 *for* $i > 1/2(7n - 6d)$,
(iv) if $A_i > 0$, *then* $A_j = 0$ *for* $j > 7n - 6d - i$ *and* $i \neq j$;
(II) *If* C *be an* $[n, k, d]$*-code over* $GF(11)$ *with* $k \geq 2$, *then :*
(i) if x *and* y *are a linearly independent pair of codewords of* C, *then* $w(x) + w(y) \leq 11n - 10d$,
(ii) $A_i = 0$ *for* $i > 11(n - d)$,
(iii) $A_i = 0$ *or* 10 *for* $i > 1/2(11n - 10d)$,
(iv) if $A_i > 0$, *then* $A_j = 0$ *for* $j > 11n - 10d - i$ *and* $i \neq j$.

Proof. (I) (i) and (ii) are immediate from Lemma 2.
(iii) Suppose $i > 1/2(7n - 6d)$. By part (i), there cannot be two linearly independent codeword of weight i. So there are either no codeword of weight i or just six (x, $2x$, $3x$, $4x$, $5x$ and $6x$ for some $x \in C$).
(iv) By part (i), there cannot exist codeword of weight i and j, with $i \neq j$, satisfying $i + j > 7n - 6d$.
(II) The same way of (I). □

Definition 3.5. Let C be an $[n, k, d]$-code over $GF(q)$. If we delete a given coordinate from all codewords of C then we have a *punctured code* of C. This code is an $[n - 1, k, d - 1]$-code. The set of all codewords of C having zero in a given coordinate position and then deleting that coordinate is a code called a *shortened code* of C. This code is an $[n - 1, k - 1, d]$-code, provided not the given position in all code words C is zero [9].

Lemma 3.6. [9] *(i)* $n_q(k, d) \leq n_q(k, d + 1) - 1$,
(ii) $n_q(k, d) \geq n_q(k, d - 1) + 1$,
(iii) $n_q(k, d) \leq n_q(k + 1, d) - 1$,
(iv) $n_q(k, d) \geq n_q(k - 1, d) + 1$.

Definition 3.7. Let G be the generator matrix of a linear $[n, k, d]$-code C over $GF(q)$. Then the *residual code* of C with respect to a codeword c, denoted be $\text{Res}(C, c)$, is the code generated by the restriction of G to the columns where c has a zero entry [9].

Lemma 3.8. [9] *Suppose* C *is an* $[n, k, d]$*-code over* $GF(q)$ *and suppose* $c \in C$ *has weight* w, *where* $d > w(q - 1)/q$. *Then* $Res(C, c)$ *is an* $[n - w, k - 1, d^0]$*-code with* $d^0 \geq d - w + \lceil w/q \rceil$.
($\lceil x \rceil$ denotes the smallest integer greater than or equal to x.)

Corollary 3.9. *Suppose* C *is an* $[n, k, d]$*-code over* $GF(q)$, *and let* c *be a codeword of weight* d. *Then* $Res(C, c)$ *is an* $[n - d, k - 1, \lceil d/q \rceil]$*-code [9].*

Theorem 3.10. *(The Griesmer bound). Let* $g_q(k, d)$ *denote the sum expression* $\sum_{i=0}^{k-1} \lceil d/q^i \rceil$. *Then* $n_q(k, d) \geq g_q(k, d)$.

The class of codes which satisfy the Griesmer bound is addressed as *codes of type BV*. Such codes can be produced by certain puncturings of concatenations of simplex codes and one can show that, for given q and k, the Griesmer bound is attained for all sufficiently large d. The following theorem gives a necessary and sufficient condition for the existence of a code of type BV [7,11,16].

Theorem 3.11. *For given q, k and d, write $d = sq^{k-1} - \sum_{i=1}^{p} q^{u_i - 1}$, where $s = \lceil d/q^{k-1} \rceil$, $k > u_1 \geq u_2 \geq \ldots \geq u_p \geq 1$, and at most $q-1$ of u_i's take any given value. Then there exists a $[g_q(k,d), k, d]$-code of type BV if and only if $\sum_{i=1}^{min(s+1,p)} u_i \leq sk$.*

4. Optimal Codes with $q = 7, 11$ of Dimension ≤ 3

For $k \leq 2$, it follows from Theorem 11 that $n_7(k,d) = g_7(k,d)$ for all d. Thus $n_7(1,d) = d$ and $n_7(2,d) = d + \lceil d/7 \rceil$ for all d.
For $k = 3$, Theorem 11 implies that $n_7(3,d) = g_7(3,d)$ for $d \geq 36$. The remaining values of d are listed in Table 1.

Table 1 : value of $n_7(3,d)$

d	$g_7(3,d)$	$n_7(3,d)$
1	3	3
2	4	4
3	5	5
4	6	6
5	7	7
6	8	8
7	9	10
8	11	11
9	12	12
10	13	13
11	14	14
12	15	15
13	16	17
14	17	18
15	19	19
16	20	20
17	21	21
18	22	22
19	23	24
20	24	25
21	25	26
22	27	27
23	28	28
24	29	29
25	30	31
26	31	32
27	32	33
28	33	34
29	35	35
30	36	36
31	37	38
32	38	39
33	39	40
34	40	41
35	41	42

Theorem 4.1. *(i) $n_7(3,5) \leq 7$, (ii) $n_7(3,6) \leq 8$, (iii) $n_7(3,11) \leq 14$.*

[DOI: 10.7508/ijmsi.2015.01.002]

Proof. (i) The matrix

$$\begin{pmatrix} 0 & 4 & 3 & 2 & 0 & 3 & 1 \\ 6 & 0 & 0 & 3 & 2 & 1 & 3 \\ 5 & 2 & 4 & 2 & 1 & 5 & 1 \end{pmatrix}$$

generates $[7, 3, 5]$-code over $GF(7)$.
(ii) it is shown in [11] that $[q+1, 3, q-1]$-code exists over $GF(q)$. In particular, there exists $[8, 3, 6]$-code over $GF(7)$. Its generator matrix is

$$\begin{pmatrix} 4 & 4 & 3 & 2 & 1 & 0 & 3 & 0 \\ 4 & 0 & 1 & 5 & 4 & 3 & 6 & 0 \\ 5 & 6 & 3 & 5 & 0 & 0 & 5 & 1 \end{pmatrix}$$

(iii) The matrix

$$\begin{pmatrix} 1 & 3 & 2 & 1 & 1 & 1 & 2 & 3 & 0 & 1 & 3 & 1 & 3 & 3 \\ 5 & 0 & 4 & 1 & 5 & 2 & 5 & 5 & 0 & 1 & 2 & 6 & 2 & 5 \\ 2 & 1 & 2 & 6 & 3 & 4 & 6 & 4 & 4 & 1 & 1 & 6 & 2 & 3 \end{pmatrix}$$

generates $[14, 3, 11]$-code over $GF(7)$ and its weight distribution is $A_{11} = 162$, $A_{12} = 60$, $A_{13} = 66$ and $A_{14} = 54$. □

Theorem 4.2. $n_7(3, 10) \leq 13$.

Proof. The matrix

$$\begin{pmatrix} 0 & 6 & 1 & 6 & 2 & 5 & 1 & 0 & 1 & 5 & 0 & 6 & 2 \\ 2 & 4 & 6 & 3 & 1 & 6 & 1 & 6 & 1 & 2 & 2 & 1 & 4 \\ 2 & 1 & 4 & 3 & 2 & 4 & 2 & 1 & 3 & 2 & 0 & 6 & 3 \end{pmatrix}$$

generates $[13, 3, 10]$-code over $GF(7)$ and its weight distribution is $A_{10} = 126$, $A_{11} = 90$, $A_{12} = 66$ and $A_{13} = 60$. □

Theorem 4.3. *(i)* $n_7(3, 7) > 9$, *(ii)* $n_7(3, 35) > 41$.

Proof. (i) If q is odd, an $[q+k-1, k]$-code MDS does not exist [1]. Then $[9, 3, 7]$-code does not exist. This matrix generates $[10, 3, 7]$-code with weight distribution $A_7 = 54$, $A_8 = 108$, $A_9 = 102$ and $A_{10} = 78$

$$\begin{pmatrix} 3 & 1 & 1 & 3 & 1 & 2 & 0 & 1 & 1 & 4 \\ 2 & 6 & 3 & 6 & 5 & 0 & 1 & 6 & 5 & 6 \\ 1 & 4 & 1 & 6 & 6 & 0 & 0 & 1 & 0 & 6 \end{pmatrix}$$

(ii) For $d = (k-2)q^{k-1} - (k-1)q^{k-2}$, $n_q(k, d) > g_q(k, d)$ holds for $q \geq k$, $k = 3, 4, 5$ [12]. Then we have for $k = 3$ and $q = 7$, $[41, 3, 35]$-code dose not exist. □

Theorem 4.4. $n_7(3, 13) > 16$.

Proof. Suppose, for a contradiction, that there exist a $[16, 3, 13]$-code C over $GF(7)$. Since there do not exist codes over $GF(7)$ with parameters $[15, 3, 13]$ and $[14, 2, 13]$, it follows from Lemma 2 that $B_1 = B_2 = 0$. The first three MacWilliams identities (Theorem 1) become,
$A_{13} + A_{14} + A_{15} + A_{16} = 342$,
$A_{14} + 2A_{15} + 3A_{16} = 258$,

[DOI: 10.7508/ijmsi.2015.01.002]

$A_{15} + 3A_{16} = 210$.
By Lemma 8, the residual code of C with respect to a codeword of weight 15 would be a $[1, 2, 1]$-code, which dose not exist. so $A_{15} = 0$. Bearing in mind that each A_i must be a nonnegative integer multiple of 6 (because if x is a nonzero codeword, then so also are $2x, 3x, 4x, 5x$ and $6x$ of the same weight). The last equation gives $A_{16} = 70$ that is not divisible by 6.
This matrix generates $[17, 3, 13]$-code

$$\begin{pmatrix} 4 & 6 & 0 & 4 & 5 & 5 & 5 & 5 & 1 & 6 & 6 & 6 & 4 & 1 & 0 & 6 & 5 \\ 2 & 6 & 4 & 5 & 0 & 2 & 5 & 6 & 0 & 5 & 5 & 1 & 2 & 1 & 2 & 3 & 3 \\ 6 & 1 & 5 & 3 & 3 & 6 & 1 & 2 & 6 & 3 & 2 & 5 & 2 & 5 & 4 & 5 & 3 \end{pmatrix}$$

and its weight distribution is $A_{13} = 60$, $A_{14} = 126$, $A_{15} = 78$, $A_{16} = 42$ and $A_{17} = 36$. □

Theorem 4.5. $n_7(3, 21) > 25$.

Proof. Suppose there exist an $[25, 3, 21]$-code C over $GF(7)$.By Lemma 2, $B_1 = B_2 = 0$. The MacWilliams identities become,
(a) $A_{21} + A_{22} + A_{23} + A_{24} + A_{25} = 342$,
(b) $A_{22} + 2A_{23} + 3A_{24} + 4A_{25} = 168$,
(c) $A_{23} + 3A_{24} + 6A_{25} = 252$.
By Lemma 8, $A_{22} = A_{23} = A_{24} = 0$. By Corollary 4(iii), $A_{25} = 0$ or 6, this contradicts (c).
This matrix generates $[26, 3, 21]$-code and its weight distribution is $A_{21} = 108$, $A_{22} = 108$, $A_{23} = 60$, $A_{24} = 42$, $A_{25} = 12$ and $A_{26} = 12$.

$$\begin{vmatrix} 0 & 1 & 2 & 3 & 6 & 3 & 2 & 6 & 3 & 5 & 4 & 5 & 4 & 5 & 5 & 1 & 5 & 2 & 4 & 3 & 0 & 0 & 2 & 3 & 0 & 4 \\ 4 & 2 & 2 & 4 & 1 & 2 & 5 & 4 & 6 & 5 & 0 & 2 & 5 & 4 & 0 & 3 & 2 & 4 & 1 & 1 & 6 & 6 & 4 & 3 & 3 & 4 \\ 4 & 5 & 1 & 4 & 3 & 5 & 0 & 6 & 2 & 2 & 1 & 1 & 6 & 0 & 0 & 0 & 3 & 5 & 4 & 3 & 1 & 0 & 1 & 3 & 1 & 1 \end{vmatrix}$$

□

Theorem 4.6. $n_7(3, 28) > 33$.

Proof. Suppose there exist an $[33, 3, 28]$-code C over $GF(7)$.By Lemma 2, $B_1 = B_2 = 0$. The MacWilliams identities become,
(a) $A_{28} + A_{29} + A_{30} + A_{31} + A_{32} + A_{33} = 342$,
(b) $A_{29} + 2A_{30} + 3A_{31} + 4A_{32} + 5A_{33} = 126$,
(c) $A_{30} + 3A_{31} + 6A_{32} + 10A_{33} = 252$,
By Lemma 8, $A_{29} = A_{30} = A_{31} = A_{32} = 0$. By Corollary 4(iii), $A_{33} = 0$ or 6, which contradicts (c). □

Theorem 4.7. *(i)* $n_{11}(3, 5) \le 7$*, (ii)* $n_{11}(3, 7) \le 9$*, (iii)* $n_{11}(3, 13) \le 16$*, (iv)* $n_{11}(3, 14) \le 17$.

Proof. (i) This matrix generates $[7, 3, 5]$-code and its weight distribution is $A_5 = 210$, $A_6 = 420$ and $A_7 = 700$.

$$\begin{pmatrix} 3 & 9 & 10 & 0 & 9 & 3 & 4 \\ 6 & 9 & 5 & 3 & 8 & 10 & 9 \\ 7 & 1 & 6 & 4 & 8 & 9 & 4 \end{pmatrix}$$

[DOI: 10.7508/ijmsi.2015.01.002]

(ii) This matrix generates $[9,3,7]$-code and its weight distribution is $A_7 = 360$, $A_8 = 360$ and $A_9 = 610$. Another generator matrix is given in section 4.

$$\begin{pmatrix} 10 & 4 & 5 & 1 & 1 & 1 & 1 & 3 & 8 \\ 1 & 9 & 7 & 1 & 1 & 0 & 8 & 0 & 7 \\ 2 & 1 & 10 & 2 & 4 & 6 & 9 & 9 & 0 \end{pmatrix}$$

(iii) The matrix

$$\begin{pmatrix} 0 & 2 & 1 & 10 & 9 & 1 & 10 & 6 & 4 & 5 & 2 & 3 & 0 & 5 & 10 & 4 \\ 1 & 4 & 10 & 1 & 7 & 2 & 4 & 6 & 0 & 3 & 2 & 3 & 5 & 2 & 6 & 6 \\ 4 & 7 & 5 & 3 & 9 & 5 & 9 & 2 & 10 & 1 & 6 & 0 & 7 & 6 & 4 & 9 \end{pmatrix}$$

generates $[16,3,13]$-code over $GF(11)$ and its weight distribution is $A_{13} = 300$, $A_{14} = 300$, $A_{15} = 420$ and $A_{16} = 310$.
(iv) The matrix

$$\begin{pmatrix} 3 & 6 & 0 & 1 & 8 & 10 & 8 & 3 & 3 & 2 & 5 & 4 & 4 & 5 & 6 & 3 & 3 \\ 6 & 6 & 7 & 8 & 4 & 5 & 2 & 8 & 0 & 8 & 10 & 10 & 1 & 9 & 2 & 7 & 9 \\ 3 & 4 & 9 & 1 & 8 & 1 & 3 & 2 & 5 & 8 & 7 & 1 & 0 & 10 & 7 & 6 & 6 \end{pmatrix}$$

generates $[17,3,14]$-code over $GF(11)$ and its weight distribution is $A_{14} = 340$, $A_{15} = 340$, $A_{16} = 340$ and $A_{17} = 310$. □

5. Quasi-Cyclic Codes

QC codes are a generalization of cyclic codes whereby a cyclic shift of a codeword by p positions results in another codewords. It can be shown that p must be divisor of n [8]. Therefore, cyclic codes are QC codes with $p = 1$. With a suitable permutation of coordinate, many QC codes can be characterized in terms of $m \times m$ circulant matrices, so the blocklength, n, is a multiple of m, $n = mp$. The generator matrix can then be represented as $G = [C_0, C_1, C_2, \ldots, C_{p-1}]$. C_i is an $m \times m$ circulant matrix of the form

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \ldots & c_{m-1} \\ c_{m-1} & c_0 & c_1 & \ldots & c_{m-2} \\ c_{m-2} & c_{m-1} & c_0 & \ldots & c_{m-3} \\ \vdots & \vdots & \vdots & & \vdots \\ c_1 & c_2 & c_3 & \ldots & c_0 \end{pmatrix}$$

where each successive row is a right cyclic shift of the previous one. These codes are a subclass of the more general 1-generator QC codes [20], which is in turn a subclass of all QC codes.
This has been confined mainly to the case $m = k$. An s-QC $[sk, k]$-codes has a generator matrix of the form $G = [G_1 \mid G_2 \mid \ldots \mid G_s]$, where each G_i is a $k \times k$ circulant matrix. The matrix G_1 is usually taken to be the identity matrix I [8]. In this section we produce generator matrix of QC $[sk, k]$-codes with $k = 3$ over $GF(7)$ and $GF(11)$.

Theorem 5.1. *(i)* $n_7(3,4) = 6$, *(ii)* $n_7(3,9) = 12$, *(iii)* $n_7(3,12) = 15$, *(iv)* $n_7(3,14) = 18$.

[DOI: 10.7508/ijmsi.2015.01.002]

Proof. There exist codes with parameters $[6, 3, 4]$, $[12, 3, 9]$, $[15, 3, 12]$ and $[18, 3, 14]$-codes. The weight distributions and generators matrices are :
$[6, 3, 4]$ $A_0 = 1$, $A_4 = 90$, $A_5 = 108$ and $A_6 = 144$;
(422 | 533);

$[12, 3, 9]$ $A_0 = 1$, $A_9 = 102$, $A_{10} = 90$, $A_{11} = 90$ and $A_{12} = 60$;
(322 | 633 | 022 | 046);

$[12, 3, 9]$ $A_0 = 1$, $A_9 = 96$, $A_{10} = 108$, $A_{11} = 72$ and $A_{12} = 66$;
(532 | 066 | 212 | 040);

$[15, 3, 12]$ $A_0 = 1$, $A_{12} = 180$, $A_{13} = 90$, $A_{14} = 0$ and $A_{15} = 72$;
(205 | 021 | 642 | 054 | 346);

$[18, 3, 14]$ - $A_0 = 1$, $A_{14} = 90$, $A_{15} = 90$, $A_{16} = 108$, $A_{17} = 18$ and $A_{18} = 36$;
(322 | 633 | 022 | 046 | 450 | 244);

$[18, 3, 14]$ - $A_0 = 1$, $A_{14} = 90$, $A_{15} = 108$, $A_{16} = 54$, $A_{17} = 72$ and $A_{18} = 18$;
(515 | 212 | 343 | 531 | 630 | 500). □

Theorem 5.2. *(i)* $n_7(3, 17) = 21$*, (ii)* $n_7(3, 19) = 24$*, (iii)* $n_7(3, 22) = 27$ *and (iv)* $n_7(3, 27) = 33$.

Proof. There exist codes with parameters $[21, 3, 17]$, $[24, 3, 19]$, $[27, 3, 22]$ and $[33, 3, 27]$ codes. The weight distributions and generators matrices are :
$[21, 3, 17]$ - $A_0 = 1$, $A_{17} = 126$, $A_{18} = 168$, $A_{19} = 0$, $A_{20} = 0$ and $A_{21} = 48$;
(002 | 521 | 321 | 640 | 434 | 633 | 624);

$[24, 3, 19]$ - $A_0 = 1$, $A_{19} = 72$, $A_{20} = 108$, $A_{21} = 90$, $A_{22} = 18$, $A_{23} = 54$ and $A_{24} = 0$;
(255 | 520 | 452 | 045 | 423 | 544 | 546 | 202);

$[24, 3, 19]$ - $A_0 = 1$, $A_{19} = 108$, $A_{20} = 36$, $A_{21} = 84$, $A_{22} = 108$, $A_{23} = 0$ and $A_{24} = 6$;
(164 | 566 | 346 | 114 | 110 | 311 | 552 | 034);

$[27, 3, 22]$ - $A_0 = 1$, $A_{22} = 126$, $A_{23} = 90$, $A_{24} = 102$, $A_{25} = 0$, $A_{26} = 0$ and $A_{27} = 24$;
(646 | 623 | 565 | 101 | 354 | 136 | 605 | 130 | 043);

$[33, 3, 27]$ - $A_0 = 1$, $A_{27} = 96$, $A_{28} = 126$, $A_{29} = 54$, $A_{30} = 42$, $A_{31} = 18$, $A_{32} = 0$ and $A_{33} = 6$;
(245 | 664 | 023 | 432 | 513 | 404 | 056 | 104 | 543 | 051 | 246);

$[33, 3, 27]$ - $A_0 = 1$, $A_{27} = 54$, $A_{28} = 216$, $A_{29} = 18$, $A_{30} = 24$, $A_{31} = 18$, $A_{32} = 0$ and $A_{33} = 12$;
(426 | 210 | 060 | 533 | 461 | 105 | 453 | 552 | 242 | 305 | 022). □

Theorem 5.3. *(i)* $n_7(3, 30) = 36$*, (ii)* $n_7(3, 32) = 39$ *and (iii)* $n_7(3, 35) = 42$.

Proof. There exist codes with parameters $[36, 3, 30]$, $[39, 3, 32]$ and $[42, 3, 35]$ codes. The weight distributions and generators matrices are :
$[36, 3, 30]$ - $A_0 = 1$, $A_{30} = 168$, $A_{31} = 90$, $A_{32} = 54$, $A_{33} = 6$, $A_{34} = 18$, $A_{35} = 0$ and $A_{36} = 6$;
(515 | 146 | 605 | 153 | 130 | 200 | 012 | 253 | 302 | 251 | 411 | 434);

[DOI: 10.7508/ijmsi.2015.01.002]

$[36, 3, 30]$ - $A_0 = 1$, $A_{30} = 168$, $A_{31} = 72$, $A_{32} = 90$, $A_{33} = 0$, $A_{34} = 0$, $A_{35} = 0$ and $A_{36} = 12$;
(503 | 166 | 402 | 265 | 351 | 022 | 352 | 565 | 510 | 535 | 312 | 631);

$[39, 3, 32]$ - $A_0 = 1$, $A_{32} = 90$, $A_{33} = 108$, $A_{34} = 90$, $A_{35} = 18$, $A_{36} = 12$, $A_{37} = 18$, $A_{38} = 0$ and $A_{39} = 6$;
(601 | 164 | 426 | 210 | 060 | 533 | 461 | 105 | 453 | 552 | 242 | 305 | 022);

$[39, 3, 32]$ - $A_0 = 1$, $A_{32} = 90$, $A_{33} = 114$, $A_{34} = 72$, $A_{35} = 36$, $A_{36} = 6$, $A_{37} = 18$, $A_{38} = 0$ and $A_{39} = 6$;
(414 | 660 | 442 | 120 | 406 | 553 | 030 | 543 | 056 | 106 | 541 | 561 | 661);

$[42, 3, 35]$ - $A_0 = 1$, $A_{35} = 162$, $A_{36} = 78$, $A_{37} = 54$, $A_{38} = 18$, $A_{39} = 24$, $A_{40} = 0$, $A_{41} = 0$ and $A_{42} = 6$;
(400 | 423 | 320 | 224 | 463 | 606 | 145 | 221 | 312 | 334 | 020 | 155 | 402 | 612);

$[42, 3, 35]$ - $A_0 = 1$, $A_{35} = 162$, $A_{36} = 84$, $A_{37} = 126$, $A_{38} = 0$, $A_{39} = 0$, $A_{40} = 0$, $A_{41} = 0$ and $A_{42} = 6$;
(350 | 120 | 212 | 051 | 256 | 144 | 463 | 066 | 564 | 545 | 432 | 340 | 233 | 005). □

Generator matrices of two QC-codes that meet the Griesmer bound are given in the following,
$[45, 3, 38]$ - $A_0 = 1$, $A_{38} = 180$, $A_{39} = 120$, $A_{40} = 36$, $A_{41} = 0$, $A_{42} = 0$, $A_{43} = 0$, $A_{44} = 0$, $A_{45} = 6$.
(500 | 524 | 042 | 043 | 545 | 252 | 513 | 206 | 636 | 153 | 163 | 036 | 440 | 121 | 454).

$[48, 3, 41]$ - $A_{41} = 288$, $A_{42} = 48$, $A_{43} = 0$, $A_{44} = 0$, $A_{45} = 0$, $A_{46} = 0$, $A_{47} = 0$, $A_{48} = 6$
(643 | 406 | 332 | 216 | 533 | 525 | 003 | 263 | 521 | 660 | 103 | 240 | 415 | 014 | 136 | 121).

Theorem 5.4. *(i)* $n_{11}(3, 4) = 6$*, (ii)* $n_{11}(3, 7) = 9$ *and (iii)* $n_{11}(3, 10) = 12$.

Proof. There exist codes with parameters $[6, 3, 4]$, $[9, 3, 7]$ and $[12, 3, 10]$ codes. The weight distributions and generators matrices are :
$[6, 3, 4]$ - $A_0 = 1$, $A_4 = 150$, $A_5 = 420$ and $A_6 = 760$;
(5 2 7 | 10 3 4);

$[9, 3, 7]$ - $A_0 = 1$, $A_7 = 360$, $A_8 = 360$ and $A_9 = 610$;
(9 1 8 | 4 1 3 | 5 6 5);

$[12, 3, 10]$ - $A_0 = 1$, $A_{10} = 660$, $A_{11} = 120$ and $A_{12} = 550$;
(3 6 10 | 6 6 9 | 4 6 8 | 5 5 10).

□

Theorem 5.5. *(i)* $n_{11}(3, 12) = 15$*, (ii)* $n_{11}(3, 15) = 18$ *and (iii)* $n_{11}(3, 18) = 21$.

Proof. There exist codes with parameters $[15, 3, 12]$, $[18, 3, 15]$ and $[21, 3, 18]$ codes. The weight distributions and generators matrices are :
$[15, 3, 12]$ - $A_0 = 1$, $A_{12} = 210$, $A_{13} = 420$, $A_{14} = 330$ and $A_{15} = 370$;
(7 10 4 | 6 4 10 | 0 10 4 | 2 6 5 | 6 7 6);

[DOI: 10.7508/ijmsi.2015.01.002]

$[18, 3, 15]$ - $A_0 = 1$, $A_{15} = 400$, $A_{16} = 330$, $A_{17} = 300$ and $A_{18} = 300$;
(0 4 8 | 0 3 3 | 3 9 9 | 1 8 2 | 8 2 5 | 1 6 1);

$[21, 3, 18]$ - $A_0 = 1$, $A_{18} = 630$, $A_{19} = 210$, $A_{20} = 210$ and $A_{21} = 280$;
(6 2 1 | 1 1 2 | 1 5 4 | 8 10 10 | 6 4 9 | 7 1 10 | 2 2 9).

□

Theorem 5.6. *(i)* $n_{11}(3, 27) = 23$*, (ii)* $n_{11}(3, 26) = 30$ *and (iii)* $n_{11}(3, 34) = 39$.

Proof. There exist codes with parameters $[27, 3, 23]$, $[30, 3, 26]$ and $[39, 3, 34]$ codes. The weight distributions and generators matrices are :
$[27, 3, 23]$ - $A_0 = 1$, $A_{23} = 390$, $A_{24} = 280$, $A_{25} = 330$, $A_{26} = 180$ and $A_{27} = 150$;
(10 5 1 | 0 5 3 | 2 1 7 | 0 10 4 | 2 0 9 | 2 7 9 | 2 5 7 | 2 6 1 | 9 10 6);

$[30, 3, 26]$ - $A_0 = 1$, $A_{26} = 540$, $A_{27} = 300$, $A_{28} = 210$, $A_{29} =$ and $A_{30} = 120$;
(9 4 4 | 2 4 9 | 10 0 8 | 3 3 0 | 8 4 3 | 0 9 6 | 7 6 4 | 2 3 7 | 10 2 3 | 8 6 1);

$[39, 3, 34]$ - $A_0 = 1$, $A_{34} = 450$, $A_{35} = 360$, $A_{36} = 220$, $A_{37} = 90$, $A_{38} = 150$ and $A_{39} = 60$;
(5 3 6 | 4 4 0 | 2 9 3 | 2 6 1 | 4 5 3 | 1 1 4 | 5 9 2 | 9 7 9 | 6 10 8 | 3 7 10 | 7 1 3 | 3 5 7 | 3 8 6). □

Acknowledgments

The authors would like to give their best thanks to the anonymous referee for his/her valuable comments to promote this work.

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[DOI: 10.7508/ijmsi.2015.01.002]