Optimal Management and Sizing of Energy Storage under Dynamic Pricing for the Efficient Integration of Renewable Energy

Pavithra Harsha, Munther Dahleh

***Abstract*—We address the optimal energy storage management and sizing problem in the presence of renewable energy and dynamic pricing associated with electricity from the grid. We formulate the problem as a stochastic dynamic program that aims to minimize the long-run average cost of electricity used and investment in storage, if any, while satisfying all the demand. We model storage with ramp constraints, conversion losses, dissipation losses and an investment cost. We prove the existence of an optimal storage management policy under mild assumptions and show that it has a dual threshold structure. Under this policy, we derive structural results, which indicate that the marginal value from storage decreases with its size and that the optimal storage size can be computed efficiently. We prove a rather surprising result, as we characterize the maximum value of storage under constant prices and i.i.d. net-demand processes: if the storage is a profitable investment then the ratio of the amortized cost of storage to the constant price is less than $\frac{1}{4}$. We further perform sensitivity analysis on the size of optimal storage and its gain via a case study. Finally, with a computational study on real data we demonstrate significant savings with energy storage.**

***Index Terms*—Energy storage, operations, management, investment, pricing, dynamic programming/optimal control, infinite horizon, renewable energy**

I. Introduction

Fossil-fuel based electricity generation is one of the largest sources of greenhouse gas emissions [1]. This coupled with the increasing demand for electricity has motivated the need to integrate a vast amount of renewable energy such as wind and solar with the electric grid. In the US, the Department of Energy mandates that by 2030 wind energy should contribute to 20% of the electric power consumption [2]. Similar aggressive renewable energy integration targets have been set across the world for different forms of renewable energy.

Renewable energy sources are non-dispatchable sources that are both variable and uncertain in nature. At large penetration levels this variability can pose significant challenges in the operation of the power grid. This is because renewables introduce large ramps that increase the need for reserves and can lead to grid stability issues amidst other concerns such as the need for costly upgrades in the transmission network [3]. Energy storage technologies can address all these concerns and facilitate power balancing as they decouple the time of generation and consumption. In addition, they improve the quality of power and its reliability, defer and/or eliminate costly upgrades in the transmission network and increase the value of distributed renewable energy sources. However, storage devices are very expensive and their cost has been the main barrier in their deployment. Owing to their benefits, governments and industries have been investing significantly in the research and development of newer and cheaper storage technologies with the hope that storage will be an integral part of the future smart grid.

This work broadly focuses on the interplay between renewables and energy storage from a power balancing perspective in the presence of dynamic pricing associated with electricity from the grid. For example, consider settings such as industries, game parks, smart cities, microgrids (not in an island mode), utilities or homes that own renewable generators in form of solar panels and/or wind turbines. The purpose of the renewable generators is to satisfy a local, potentially price-sensitive demand using energy storage devices and electricity from the grid. The main questions in such applications are the following: what is the value of storage, what is the tradeoff between the value of storage and its capital cost? how should it be managed and finally how different factors influence the above?

To answer the above questions we consider a setting where a price-sensitive demand is satisfied at all times using electricity from the grid and/or renewable generation. Electricity from the grid has a real-time but exanté price associated with it whereas renewable generation is assumed to have zero marginal cost and is hence free. The goal is to identify the optimal storage management policy and in turn, the optimal size of energy storage to invest in to minimize the average cost of electricity used and investment in storage, if any. We refer to these problems together as the *optimal energy storage management and sizing problem* and interchangeably refer to storage sizing as storage investment.

In this paper, we formulate this problem as a discrete time *average cost stochastic dynamic program* over an infinite horizon. In modeling the storage device, we take into account ramp constraints, energy losses during charging and discharging, dissipation losses, and an amortized capital cost of investment. The main contributions of the paper are as follows:

P. Harsha is with Business Analytics and Mathematical Sciences, IBM Watson Research Center, Yorktown Heights, NY 10598. Corresponding author: pharsha@us.ibm.com, pavithra@mit.edu

M. Dahleh is with the Laboratory for Information and Decision Systems, Massachusetts Institute of Technology

1) We prove the *existence of the optimal storage management policy for the average cost infinite horizon criterion*. The proof of the existence is non-trivial because we allow for continuous state and action spaces in the underlying Markov Decision Problem (MDP). Using the vanishing discount method, we establish the existence of an optimal stationary policy and show that it can be computed using the infinite horizon average cost optimality equation.
2) We show that *the optimal management policy has a dual-threshold structure*. The optimal policy is characterized by two threshold functions as follows. If there is excess generation then store the excess and possibly buy (and store) to reach the lower threshold. Alternatively, if there is excess demand it is one of three possibilities: buy and store to reach the lower threshold if there is insufficient energy in the storage device, do nothing or extract, as needed, but only down to the higher threshold if there is sufficient energy in the storage device. In special cases we are able to characterize the nature of the threshold functions and hence the optimal policy. For example, at a constant price or at the highest price for electricity, the optimal policy is a simple *greedy* policy, and at zero price, it is to fill the storage entirely. Moreover, when prices and the net-demand are i.i.d. processes, we show that the thresholds are decreasing functions of price.
3) We show that the *optimal storage management policy has a non-increasing and convex average cost of electricity as a function of the storage size*. This implies that the marginal value from storage (without considering investment cost) is decreasing with storage size and that the optimal storage size under the optimal policy can be computed efficiently.
4) In the special case of a constant price for electricity, we quantify the maximum value from a given storage (savings in electricity cost with storage) under any i.i.d. stochastic process for net demand (demand minus generation). This characterization allows us to prove a rather surprising result: *if storage is a profitable investment then ratio of the amortized cost of storage to the constant price of electricity is less than* $\frac{1}{4}$.
5) Through a simple case study, *we understand how different factors such as the uncertainty of the exogenous distributions, differential pricing, price elasticity of demand, losses, and ramp constraints impact the size of optimal storage and its gain*. We derive closed form solution for the size of optimal storage in a special case and use it as a baseline to study other trends (i.e., perform sensitivity analysis).
6) Finally, with a *computational study* using the Pacific Northwest GridWise Testbed Demonstration Projects [4] and the western wind integration study at NREL [5], we show that a 2.2% (100kWh) and 4.4% (200kWh) level of storage penetration (size of storage relative to the total annual load) results in 2.6% and a 4.4% savings in electricity cost for a group faces a constant price and a 7.3% and a 12.9% savings for a group that faces time-of-use pricing.

Related Work: Interaction between renewable energy and storage has been the subject of many papers. Previous work has mostly focused on the setting where bulk renewable generators are used for participating in conventional electricity markets to maximize revenue using energy storage (see [6], [7] and references therein). The goal in these papers is to make renewable energy commitments in day-ahead markets that have associated penalties if the contracts are breached in real-time. Our setting differs from these in that we consider renewable generators that directly face demand and use energy storage devices to reduce the total cost of electricity used. In such a setting, [8] show that energy storage can be used to smooth peak consumption under time-varying deterministic, as opposed to stochastic, variations of demand, price and wind power. In order to handle the uncertainty of renewables and demand, the papers [9]–[11] adopt a MDP based approach to solve the storage management while [12] adopts a scenario tree based method.

In our earlier work [9] (a preliminary conference version of this paper), we study the optimal sizing and management problem when the prices are restricted to at most two levels and formulate it as an average cost infinite horizon problem. We showed that a greedy policy turns out to be optimal under a constant price and that the marginal value of storage is non-increasing and convex as a function of the storage size. The optimality of the greedy policy was also observed in [11]. In [10], the authors show that under a constant price and no losses the value of storage is increasing in storage size and stabilizes at a certain level. The results we present in this paper extends the work in the above papers to the general case of time-varying stochastic dynamic pricing. This work was conducted simultaneously and independently of the recent works in [13] and [14]. We include some results from [9] for the completeness of this paper and they will be clearly cited.

The commodity trading literature [15], [16] is also related to our work. This literature focuses on an arbitrage setting with bounds on the trades and a limited warehouse capacity. This work has been extended in the context of energy storage recently by [17] who provide an explicit formula for optimal thresholds under i.i.d. prices with a storage that has a perfect roundtrip efficiency. In [18] the authors numerically observe that the optimal storage management strategy has a threshold structure where the number of thresholds depends on the number of piecewise-linear components in its convex cost function. Similar to the results in this literature, we show that the optimal policy has two threshold functions, one for injection and one for withdrawal. Though unlike the arbitrage setting where the lower threshold arises because of the price for injection, the lower threshold in our setting is because of the efficiency losses in the storage device. Due to the nature of the setting, the trading literature has focused primarily on finite horizon applications while the methodology that we provide in this paper extends that analysis to discounted and average cost infinite horizon settings as well.

Amongst papers that study the structural properties of average cost infinite horizon problems, the paper that is closely related to our work is the inventory control problem with supply risks [19]. Unlike the inventory setting, the presence of price uncertainties, storage ramp constraints, and storage dissipation and conversion losses contributes to the difference in the structure of the optimal policy and the related analysis.

Organization: The rest of the paper is organized as follows. In Section II, we formalize our models. In Section III, we study the optimal storage management problem followed by the optimal storage investment problem in Section IV. In Section V, we derive a theoretical bound on the storage cost. We present the sensitivity analysis in Section VI, the computational results on real data in Section VII and finally, conclude in Section VIII. All the proofs are included in the appendix of the paper.

II. Model

Consider a renewable generator the purpose of which is to satisfy some (local) demand. Any excess generation is assumed to be lost or pumped into the grid at zero price unless stored in an energy storage device. Any excess demand that is not satisfied from renewable generation is supported by other generators connected to the electric grid at prices which are revealed prior to the consumption. Our goal is to identify the optimal storage size and the optimal policy to manage the storage so as to minimize the long term average cost of electricity used along with the cost associated with storage investment.

To simplify the analysis we address this problem in the absence of any network constraints (as a single bus) and model only the flow of real power. We also discretize time into intervals of length τ with constant power in each interval ignoring any variations in generation or load within an interval.

Energy storage model. Any storage can be characterized using the following parameters:

- *Energy rating* is the net capacity or size of storage represented by S.
- *Power rating* specifies the rate at which storage can be charged or discharged. This can be the same or different for charging and discharging cycles and denoted by $\hat{R}_i$ and $\hat{R}_o$ respectively.
- *Efficiency* accounts for conversion losses and denoted by ρ. This is commonly referred to as the roundtrip efficiency because it is the product of two conversion loss efficiencies: converting renewable energy to its stored form, ρ_i, and the reverse, ρ_o. If $\rho = 1$, the storage is known to have a perfect roundtrip efficiency.
- *Dissipation losses* refers to the losses that occur due to leakages and are accounted for by the constant η.
- *Total ownership cost* refers to the investment cost in storage. We model this cost as a simple amortized per unit cost of capital that we denote by c. One method to estimate this amortized cost from the investment cost is discussed in Appendix A.

All the above parameters are assumed to be inputs to our problem except for S which we treat as a decision variable.

Note 1. *For the simplicity of the mathematical expressions in this paper, we scale all the parameters of the problem so that we can work with the* useful *component of the energy in the storage device where conversion losses are already accounted for. That is, we redefine S to be the maximum amount useful energy that can be stored in the storage device. Therefore, the energy rating is $\frac{S}{\rho_o}$. Similarly, the actual power ratings are $\frac{\hat{R}_i}{\rho}$ and $\frac{\hat{R}_o}{\rho_o}$ for charging and discharging respectively where $\hat{R}_i$, $\hat{R}_o$ reflect the constraints on the useful power. Finally, c is the amortized cost per (useful) unit of the storage device while $c\rho_o$ is the true cost.*

Renewable generation, price, demand. We assume that renewable generation, price and demand are known exogenous stochastic processes and denote them by W_t, p_t and D_t respectively. The support for each of these processes is assumed to be non-negative and bounded. We allow for possible correlation in these processes and in turn account for price-sensitive (i.e., elastic) demand and dependence of prices on wind levels. We will soon see in our formulation that at any time, we are only interested in the net difference between demand and renewable generation. We denote it by $Y_t = D_t - W_t$ and refer to it as the net load with units in energy.

These exogenous processes are typically governed by prediction models that we assume are characterized using known Markovian function as follows:

$$\{p_t, Y_t\} = g\left(\mathbf{p}_t^H, \mathbf{Y}_t^H, \epsilon_t\right), \tag{1}$$

where the vector $\epsilon_\mathbf{t}$ is assumed to be independently and identically distributed (i.i.d) random process that is known for each t. The bolded terms with superscript H refers to the history of the respective quantities required for predicting the current state. We often use the notation $Q_t \in \mathbb{Q}$ and $\mathbf{Q}_t^H \in \mathbb{Q}^H$ to compactly denote the tuples $\{p_t, Y_t\}$ and $\{\mathbf{p}_t^H, \mathbf{Y}_t^H\}$ and their supports.

Problem formulation. Let X_t denote the level of useful energy in the storage at time t. By definition, $X_t \in [0, \eta S]$ where η accounts for the dissipation losses. The tuple $\{X_t, \mathbf{Q}_t^H\}$ forms the state of the system.

We assume the following sequence of events in each period: at the beginning of period t, we are revealed exanté the price, p_t and the net load, Y_t. Next, the decision, u_t, the amount of useful energy to store is made. We allow this to be negative as we can extract energy from storage as well. Note that u_t is an ex-post decision. So, we define u_t for every $Q_t = \{p_t, Y_t\}$. By definition, u_t is restricted by the size of storage and the ramp constraints as follows:

$$-\min\{X_t, R_o\} \leq u_t \leq \min\{(S - X_t), R_i\}, \tag{2}$$

where $R_i = \tau\hat{R}_i$ and $R_o = \tau\hat{R}_o$. The state update equations

$\forall\ X_t \in [0, \eta S]$ and $\mathbf{Q}_t^H \in \mathbb{Q}^H$ are as follows:

$$X_{t+1} = \eta \left[X_t + u_t\right], \text{ and} \tag{3}$$
$$\mathbf{Q}_{t+1}^H = f\left(Q_t, \mathbf{Q}_t^H\right). \tag{4}$$

Eq. (3) increments the current storage level by the amount of useful energy that is stored depending on the value of $Q_t = \{p_t, Y_t\}$ in period t and then discounts it by the dissipation losses, η, to arrive at the storage level in the next time period. Eq. (4) is the state update equation for p_t and Y_t where a known function $f(.)$ updates the exogenous processes by the most recent history.

We now formulate the optimal storage management and investment problem as a discrete-time average-cost infinite horizon stochastic dynamic program. We assume that the granularity of the discretizations (e.g., hourly) are relatively small compared to the life-cycle of storage devices (e.g., a few years) and hence choose an infinite horizon metric.

$$\mathbf{I}: \min_{S\geq 0, \pi} \lim_{T\to\infty} \sup \frac{1}{T} E_{p_t, Y_t} \sum_{t=1}^{T} p_t \left[Y_t + \frac{u_t}{\beta_t}\right]^+ + cS \tag{5}$$
$$s.t. \quad (2), (3), (4)\ \forall\ t \in \{1, ..., T\}$$
$$\text{where } \beta_t = \begin{cases} \rho & \text{if } u_t \geq 0 \\ 1 & \text{otherwise.} \end{cases}$$

The first term in the objective is the average cost of conventional generation that is drawn from the grid in order to meet demand. We divide u_t by β_t to reflect the actual energy stored as u_t is just the useful part of the energy stored. The second term is the amortized per-period investment cost in storage. For a fixed S, we refer to the problem as the optimal storage management problem. The goal here is to identify the optimal policy, π^*, amongst feasible policies denoted by $\pi = \{u_t | t = 1, 2, \dots\}$ which is a sequence of feasible actions u_t. Each u_t depends on $(X_t, Q_t, \mathbf{Q}_t^H)$ i.e., $u_t = u_t(X_t, Q_t, \mathbf{Q}_t^H)$ with a slight abuse of notation. A policy π is considered stationary if u_t is a time independent function, i.e., $u_t = u(X_t, Q_t, \mathbf{Q}_t^H)\ \forall\ t$. When S is also a decision variable, we refer to the problem as the optimal storage management and sizing problem.

Under the optimal policy, we refer to the decrease in the cost of electricity with storage (without investment cost) as the value of storage and the decrease in total cost of problem **I** (i.e., with investment cost) as the gain from storage. An investment in storage is considered profitable if the gain is non-negative. The goal in the sizing problem is to identify the most profitable storage to invest in.

III. Optimal storage management policy

In this section, we prove the existence, identify the structure and discuss some computational aspects of the optimal stationary storage management policy, $u^*(X_t, Q_t, \mathbf{Q}_t^H)$, for the problem **I** with a known storage of size S. We first discuss the case of a constant price and then the more general case of time varying prices.

A. Constant price: Balancing/greedy control [9]

Consider the following stationary greedy policy that we refer to as the *balancing control*: if there is excess demand (over renewable generation), we extract as needed from storage and alternatively, if there is excess generation, we store it. Let

$$h^B(X_t, Y_t) = (X_t - \gamma_t Y_t)^+, \tag{6}$$

where γ_t is ρ if $Y_t \leq 0$ and 1 otherwise. Then, the balancing control $u^*(X_t, Y_t) =$

$$\min\left\{R_i, (S - X_t), \max\{h_t^B(X_t, Y_t) - X_t, -R_o\}\right\}. \tag{7}$$

The superscript B refers to the balancing control. This stationary policy is optimal at a constant price because there is no gain in satisfying demand in a future period and not satisfying it in the current period because the energy from the grid has a constant price. In particular, in the presence of conversion and dissipation losses this action can result in losses. This simple management scheme is optimal for both finite and infinite horizon problems and does not depend on the stationarity assumptions and/or the Markovian nature of the exogenous stochastic processes.

B. Time varying prices: Dual-threshold control

We now consider the case of time-varying prices. To identify the structure of the optimal policy for the average cost infinite horizon storage management problem, we first prove that the corresponding Bellman equation is satisfied and that there exists a stationary policy that satisfies it. The existence is no surprise for finite horizon problems from dynamic programming and for finite state problems with an average cost metric using linear programming. But in general, it need not be true for infinite horizon problems when there is a continuous state or action space (see examples in section 4.6 in [20]). In this section, we allow the storage level X_t to be continuous in $[0, \eta S]$. Note that the main structural results that we prove in this section for the optimal policy and the optimal cost-to-go functions (except for the continuity property in X_t) continue to hold even if X_t is discrete.

We first study the discounted finite horizon problem. We prove certain structural results for the cost-to-go functions in the Bellman equation. Using this we show that the optimal policy has a *dual-threshold* structure. In particular, above the highest threshold, a modified greedy control that we refer to as the *threshold-dependent balancing control* is optimal. We then prove the existence of a stationary policy for the discounted infinite horizon problem and that it satisfies its corresponding Bellman equation. We then extend the structural results by characterizing the asymptotic behavior of the cost-to-go function with time. Finally, we extend the results to the average cost problem using the *vanishing discount approach* (see [21]) that characterizes the asymptotic behavior of the cost as the discount factor approaches one. A key property that is established is that the relative difference between the infinite horizon discounted cost from any starting state and some reference state is always bounded by a constant at all discount levels.

1) Discounted finite horizon problem: Consider the discounted finite horizon storage management problem with a storage of finite size S. Using dynamic programming, the problem can be rewritten as follows where $V_{\alpha,t}(X_t, \mathbf{Q}_t^H)$ is the cost-to-go of the finite horizon problem at period t with state $\{X_t, \mathbf{Q}_t^H\}$ and discount factor α. For all $t = 1, ..., T-1$, $X_t \in [0, \eta S]$ and $\mathbf{Q}_t^H \in \mathbb{Q}^H$,

$$V_{\alpha,t}(X_t, \mathbf{Q}_t^H) = E_{Q_t/\mathbf{Q}_t^H} \; J_{\alpha,t}(X_t, Q_t, \mathbf{Q}_t^H) \tag{8}$$

$$\begin{aligned} &\text{where } J_{\alpha,t}(X_t, Q_t, \mathbf{Q}_t^H) = \\ &\min_{u_t} \; p_t \left[Y_t + \frac{u_t}{\beta_t} \right]^+ + \alpha V_{\alpha,t+1}\big(\eta(X_t + u_t), f\left(Q_t, \mathbf{Q}_t^H\right)\big) \\ &\quad s.t. \; -\min\{X_t, R_o\} \le u_t \le \min\{(S - X_t), R_i\}, \\ &\text{and } V_{\alpha,T}(X_T, \mathbf{Q}_t^H) = 0. \end{aligned} \tag{9}$$

Theorem 2. *$V_{\alpha,t}(X_t, \mathbf{Q}_t^H)$ is a non-increasing, continuous convex function in $X_t \in [0, \eta S]$ for all $\mathbf{Q}_t^H \in \mathbb{Q}^H$ and $t = 1, ..., T$.*

Consider the optimization problem without ramp constraints in period t with a change of variables from u_t to $z_t = X_t + u_t$:

$$\min_{z_t \in [0,S]} \quad p_t \left[Y_t + \frac{z_t - X_t}{\beta_t} \right]^+ + \alpha V_{\alpha,t+1}(\eta z_t, \mathbf{Q}_{t+1}^H). \tag{10}$$

We make two observations that simplifies the first term: suppose $z_t < X_t$ then $\beta_t = 1$ and the first term is zero when $z_t \le X_t - Y_t$; suppose $z_t \ge X_t$, then $\beta_t = \rho$ and the first term is zero when $z_t \le X_t - \rho Y_t$. So, depending on whether z_t is less than or greater than X_t, we separate the problem into two subproblems and impose that constraint on z_t accordingly. In each of the problems, incorporating the interesting cases of Y_t, we can state without loss of generality that the first term is 0 for all $z_t \le X_t - \gamma_t Y_t$ where $\gamma_t = \rho$ if $Y_t \le 0$ and 1 otherwise. Note that the second term is a decreasing function of z_t by Theorem 2. So, the optimum is always higher than $X_t - \gamma_t Y_t$. We impose this as an additional constraint and further simplify the first term. Also, note that because the objective is a convex function in z_t (first term is convex, second term by Theorem 2 and hence the sum), the two sub-problems can be solved as unconstrained problems after which these constraints are imposed sequentially. With these steps, we arrive at the following dynamic programming algorithm for the finite horizon storage management problem. This is illustrated when Q_t is i.i.d in nature (i.e., the state is just X_t) for the simplicity of exposition. In the general case, all the threshold and control functions below are also functions of the current state $\mathbf{Q}_t^H$ and the current realization Q_t.

1) Solve for the two thresholds under the cases when $z_t < X_t$ and $z_t \ge X_t$.

$$h_t^+(p_t) = argmin_{z_t \in [0,S]} \; p_t z_t + \alpha V_{\alpha,t+1}(\eta z_t), \text{and} \tag{11}$$

$$h_t^-(p_t) = argmin_{z_t \in [0,S]} \; \frac{p_t}{\rho} z_t + \alpha V_{\alpha,t+1}(\eta z_t). \tag{12}$$

Note that $h_t^-(p_t) \le h_t^+(p_t)$. If $\rho = 1$, $h_t^-(p_t) = h_t^+(p_t)$.

2) Impose constraints $z_t < X_t$ and $z_t \ge X_t$ to arrive at an intermediate threshold,

$$h_t(X_t, p_t) = \begin{cases} h_t^-(p_t) & \text{if } X_t \le h_t^-(p_t) \\ X_t & \text{if } h_t^-(p_t) < X_t \le h_t^+(p_t) \\ h_t^+(p_t) & \text{o.w.} \end{cases} \tag{13}$$

3) Impose constraint $z_t \le X_t - \gamma_t Y_t$ to arrive at the optimal threshold,

$$h_t^*(X_t, Q_t) = h_t(X_t, p_t) + [X_t - \gamma_t Y_t - h_t(X_t, p_t)]^+, \tag{14}$$

$$\text{where } \gamma_t = \begin{cases} \rho & \text{if } Y_t \le 0 \\ 1 & \text{otherwise.} \end{cases}$$

4) Incorporate ramp and storage limit constraints to arrive at the optimal decision, $u_t^*(X_t, Q_t) =$

$$\min\{R_i, (S - X_t), \max\{h_t^*(X_t, Q_t) - X_t, -R_o\}\}. \tag{15}$$

Observe that the optimal policy involves computing two threshold functions $\{h_t^-(p_t), h_t^+(p_t)\}$ for the i.i.d case and in general, $\{h_t^-(Q_t, \mathbf{Q}_t^H), h_t^+(Q_t, \mathbf{Q}_t^H)\}$. One is tempted to compare these two thresholds to the base stock $[s, S]$ policy in inventory theory [22]. The major difference is that the lower threshold in the $[s, S]$ policy is because of the presence of fixed ordering costs and in the storage management problem is because of the presence of efficiency losses. In particular, in the absence of ordering costs, $s = 0$ but in the absence of efficiency losses ($\rho = 1$), $h_t^-(p_t) = h_t^+(p_t)$.

In a nutshell, the optimal policy is as follows: if there is excess generation store all the excess and buy, if necessary, at least up to $h_t^-(p_t)$; alternatively, if there is excess demand, it is one of three possibilities: (1) extract as much as is needed but only down to $h_t^+(p_t)$ if there is sufficient energy ($> h_t^+(p_t)$) in the storage device; (2) buy and store up to $h_t^-(p_t)$ if there is insufficient energy in storage (i.e., $< h_t^-(p_t)$); or, (3) do nothing if $h_t^-(p_t) \le X_t \le h_t^+(p_t)$. The exact amount to store or extract depend on the level of energy in the storage device, the price-dependent threshold functions, the size of storage and the ramp constraints as described in the above equations. We refer to the policy whenever $X_t \ge h_t^+(p_t)$ as a a threshold-dependent balancing control or a modified greedy policy as it reduces to the balancing/greedy policy when $h_t^+(p_t) = 0$.

Corollary 3. *The thresholds, $h_t^+(p_t)$ and $h_t^-(p_t)$, and in general, $h_t^+(Q_t, \mathbf{Q}_t^H)$ and $h_t^-(Q_t, \mathbf{Q}_t^H)$), are 0 at the highest price and S at zero price. This implies that the optimal control policy at the highest price is the balancing policy and at zero price is to fill the storage completely.*

Corollary 4. *Suppose $Q_t = \{p_t, Y_t)\}$ is an i.i.d. process then the thresholds $\{h_t^+(p_t), h_t^-(p_t)\}$ and the optimal control $u_t^*(X_t, Q_t)$ are all non-increasing in price p_t for all t.*

2) Discounted cost infinite horizon: In this section, we are interested in proving the existence of the stationary policy for the discounted cost infinite horizon problem:

$$V^*_\alpha(X_0,\mathbf{Q}^H_0)=\inf_{\pi\in\Pi}E_{p_t,Y_t}\left\{\sum_{t=0}^{\infty}\alpha^t p_t\left[Y_t+\frac{u_t}{\beta_t}\right]^+\right\},\quad (16)$$

where Π is the class of all feasible strategies. The reason we study this is because the management policy for the average cost problem is derived from the asymptotic behavior of this cost-to-go function as the discount factor goes to one. We will show that $V^*_\alpha(.)$ satisfies the following optimality equation. For all $X\in[0,\eta S]$ and $\mathbf{Q}^H\in\mathbb{Q}^H$,

$$V_\alpha(X,\mathbf{Q}^H)=E_{Q/\mathbf{Q}^H}\ J(X,Q,\mathbf{Q}^H)\quad (17)$$

where $J(X,Q,\mathbf{Q}^H)=$

$$\min_u\ p\left[Y+\frac{u}{\beta_u}\right]^+ +\alpha V_\alpha\left(\eta(X+u),f\left(Q,\mathbf{Q}^H\right)\right)$$
$$s.t.\ \ \min\{X,R_o\}\le u\le\min\{(S-X),R_i\}.$$

In the following when we refer to the finite horizon problem, $V_{\alpha,t}(X,\mathbf{Q}^H)$, we use backward numbering i.e., t periods left till the end of the horizon with $V_{\alpha,0}(.)\overset{\text{def}}{=}0$.

Theorem 5. *For all $X\in[0,\eta S]$ and $\mathbf{Q}^H\in\mathbb{Q}^H$,*

1) $0=V_{\alpha,0}(X,\mathbf{Q}^H)\le V_{\alpha,1}(X,\mathbf{Q}^H)\le\ ....$ *and* $V_{\alpha,\infty}(X,\mathbf{Q}^H)\overset{def}{=}\lim_{t\to\infty}V_{\alpha,t}(X,\mathbf{Q}^H)$ *exists and is finite.*
2) $V_{\alpha,\infty}(X,\mathbf{Q}^H)$ *satisfies Eq. (17), the optimality equation for discounted cost infinite horizon problem.*
3) $V_{\alpha,\infty}(X,\mathbf{Q}^H)=V^*_\alpha(X,\mathbf{Q}^H)$.
4) $V^*_\alpha(X,\mathbf{Q}^H)$ *is a non-increasing continuous convex function in X.*

The above theorem proves the existence of a stationary policy $u^*(X,Q,\mathbf{Q}^H)$ that is optimal and that it satisfies the optimality condition with $V_\alpha(X,\mathbf{Q}^H)=V_{\alpha,\infty}(X,\mathbf{Q}^H)=V^*_\alpha(X,\mathbf{Q}^H)$. Although the optimal policy of this problem is not the focus of this paper, it is easy to see the solution of the Bellman equation (17) has a stationary dual threshold structure similar to the finite horizon problem where we replace $V_{\alpha,t}(X,\mathbf{Q}^H)$ by $V^*_\alpha(X,\mathbf{Q}^H)$. Similarly, Corollaries 3-4 can be easily extended to this case.

3) Average cost: In this section, we are interested in characterizing the optimal policy for average cost infinite horizon problem as follows:

$$F^*(X_0,\mathbf{Q}^H_0)=\min_{\pi\in\Pi}\lim_{T\to\infty}\sup\frac{1}{T}E_{p_t,Y_t}\sum_{t=1}^{T}p_t\left[Y_t+\frac{u_t}{\beta_t}\right]^+,\quad (18)$$

where Π is the class of all feasible strategies. We will first show that there exists a constant g^* such that $F^*(X_0,\mathbf{Q}^H_0)=g^*$ for all $\{X_0,\mathbf{Q}^H_0\}$. This constant, g^*, and the optimal policy can be identified using the infinite horizon average cost optimality equation given below. For all $X\in[0,\eta S]$ and $\mathbf{Q}^H\in\mathbb{Q}^H$,

$$w(X,\mathbf{Q}^H)+g^*=E_{Q/\mathbf{Q}^H}\ v(X,Q,\mathbf{Q}^H),\qquad\text{where}\quad (19)$$
$$v(X,Q,\mathbf{Q}^H)=\min_u p\left[Y+\frac{u}{\beta_u}\right]^+ +w\left(\eta(X+u),f(Q,\mathbf{Q}^H)\right)$$
$$s.t.\ -\min\{X,R_o\}\le u\le\min\{(S-X),R_i\}.$$

Here w is referred to as the *relative cost* function. In constructing the solution, we adopt the vanishing discount factor approach. Here, we start with a reference state $\{\widetilde{X},\widetilde{\mathbf{Q}}^H\}$. We chose $\widetilde{X}$ to be the zero storage level value and $\widetilde{\mathbf{Q}}^H$ to be some nominal value. We define $\bar{V}_\alpha(X,\mathbf{Q}^H)=V^*_\alpha(X,\mathbf{Q}^H)-V^*_\alpha(0,\widetilde{\mathbf{Q}}^H)$ and show that that there exists a sequence α_n that converges to 1 such that $w(X,\mathbf{Q}^H)=\lim_{n\to\infty}\bar{V}_{\alpha_n}(X,\mathbf{Q}^H)$ exists and is a solution to the optimality condition.

We make the following assumptions in order to prove the main result.

Assumption 6. *There is a strictly positive probability that the net load, Y_t is strictly positive i.e., $P[Y_t>0]>0$ and strictly negative i.e., $P[Y_t<0]>0$.*

Assumption 7. *The exogenous stochastic process $Q_t=\{p_t,Y_t\}$ has a discrete support in general and can have a continuous support if i.i.d.*

The above assumptions aid in bounding the term $\bar{V}_\alpha(X,\mathbf{Q}^H)$ independent of the discount factor α in the following theorem. The bounding procedure involves two steps: ensuring that the reference state is reached with probability 1 from an arbitrary state (and vice versa) and that the expected cost taken to get there is bounded. The first assumption is related to the endogenous state X and allows a specific choice of actions to reach and get from any arbitrary state to the reference state. The second assumption relates to the exogenous state $\mathbf{Q}^H$ and simplifies the proof when it is discrete and hence the assumption.

Theorem 8.

1) *There exists a constant g^* and a continuous function $w(X,\mathbf{Q}^H)$ in the storage level X which satisfies Eq. (19), the infinite horizon average cost optimality equation.*
2) $g^*=F^*(X,\mathbf{Q}^H)$ *for all $X\in[0,\eta S]$ and $\mathbf{Q}^H\in\mathbb{Q}^H$.*
3) *Any stationary policy that achieves the minimum to the optimality equation (19) is also optimal to the average cost criterion.*
4) $w(X,\mathbf{Q}^H)$ *is a non-increasing continuous convex function in X for all $\mathbf{Q}^H\in\mathbb{Q}^H$.*

The above theorem proves the existence of a stationary policy $u^*(X,Q,\mathbf{Q}^H)$ that is optimal and satisfies the optimality conditions. Using ideas similar to those discussed for the finite horizon problem, it is easy to show that the optimal policy has a dual threshold structure with stationary thresholds $\{h^-(p_t),h^+(p_t)\}$ when Q_t is i.i.d and $\{h^-(Q,\mathbf{Q}^H),h^+(Q,\mathbf{Q}^H)\}$ in general. We illustrate the details of deriving the thresholds below when Q_t is i.i.d (i.e., X_t is the only state of the system). In general, all the threshold and control functions below are also functions of the current state $\mathbf{Q}_t^H$ and the current realization Q_t.

1) Solve for $\{h^-(p_t), h^+(p_t)\}$ as follows:

$$h^+(p_t) = argmin_{z\in[0,S]}\ p_t z + w(\eta z), \text{and} \quad (20)$$

$$h^-(p_t) = argmin_{z\in[0,S]}\ \frac{p_t}{\rho} z + w(\eta z). \quad (21)$$

Note that $h^-(p_t) \leq h^+(p_t)$. If $\rho = 1$, $h^-(p_t) = h^+(p_t)$.

2) Arrive at the next intermediate threshold:

$$h(X_t, p_t) = \begin{cases} h^-(p_t) & \text{if } X_t \leq h^-(p_t) \\ X_t & \text{if } h^-(p_t) < X_t \leq h^+(p_t) \\ h^+(p_t) & \text{o.w.} \end{cases} \quad (22)$$

3) Optimal threshold,

$$h^*(X_t, Q_t) = h(X_t, p_t) + [X_t - \gamma Y_t - h(X_t, p_t)]^+, \quad (23)$$

$$\text{where } \gamma_t = \begin{cases} \rho & \text{if } Y_t \leq 0 \\ 1 & \text{otherwise.} \end{cases}$$

4) Incorporate ramp and storage limit constraints to arrive at the optimal decision, $u^*(X_t, Q_t) =$

$$\min\{R_i, (S - X_t), \max\{h^*(X_t, Q_t) - X_t, -R_o\}\}. \quad (24)$$

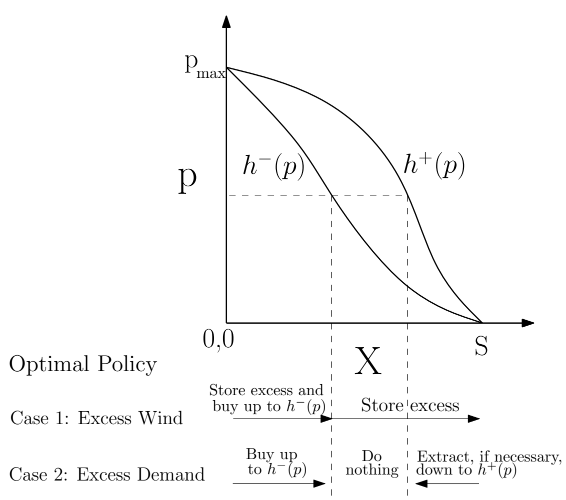

Fig. 1: Structure of the optimal storage management policy when $Q_t = \{p_t, Y_t\}$ is an i.i.d. process.

Corollaries 3-4 can be easily extended to the average cost infinite horizon problem and we state them as corollaries without proof.

Corollary 9. *The thresholds, $h^+(p)$ and $h^-(p)$, and in general, $h^+(Q, \mathbf{Q}^H)$ and $h^-(Q, \mathbf{Q}^H)$, are 0 at the highest price and S at zero price. This implies that the optimal control policy at the highest price is the balancing policy and at zero price is to fill the storage completely.*

Corollary 10. *Suppose $Q_t = \{p_t, Y_t(p_t)\}$ is an i.i.d. process then the thresholds $\{h^-(p), h^+(p)\}$ and the optimal control $u^*(X, Q)$ are non-increasing in price p.*

Fig. 1 provides a pictorial representation of the threshold functions and the optimal stationary storage management policy under different scenarios when $Q_t = \{p_t, Y_t(p_t)\}$ is an i.i.d. process. Observe that the threshold-dependent balancing control is optimal above $h^+(p_t)$.

C. Some remarks on the computation of the optimal policy

The optimal policy and the cost-to-go functions can be computationally evaluated using standard dynamic programming techniques such as value/policy iteration or solving a linear program [20]. The run times of these methods can be enhanced by hard-coding the results of Corollaries (9–10) whenever applicable. The optimal policy derived from these methods encode the dual-threshold policy (e.g., output of Eq. (24)). The dual-thresholds themselves can be derived by solving Eqs. (20–21) with those cost-to-go functions.

An interesting property about the dual-threshold policy structure is that even if the cost-to-go functions are approximately evaluated, as long as they are non-increasing and convex in X for all $\mathbf{Q}^H$, the policy evaluated from these cost-to-go functions (even though approximate) are dual-threshold. The reasoning is exactly along the lines of the discussion below Eq. (10). Note that this property holds even when X takes only discrete values.

The above property raises the following question: can the optimal dual thresholds be obtained with a *restricted* policy iteration method where one restricts the search in the policy improvement step in the policy iteration algorithm to the set of dual threshold policies only. Clearly, such an algorithm will lead to significant computational speed-up over standard methods. We observe in several instances that an arbitrary dual threshold policy (i.e., not the optimal) results in cost-to-go functions that are not convex in X for some $\mathbf{Q}^H$. These cost-to-go functions are obtained by solving the steady state equations Eq. (19) for the chosen policy. This implies a restricted policy iteration method may converge to a local optima. Further research is needed to address the computational aspects of the energy storage management problem in the presence of large state space.

IV. Optimal Storage Sizing Under Optimal Management Scheme

In this section, we are interested to find the optimal investment in storage, S^*, in problem **I** under the optimal stationary storage management policy derived in Section III-B3.

Let $g(S)$ correspond to the average cost of energy drawn from the grid under the optimal management policy for a storage of size S. Suppose the optimal stationary dual thresholds are $h^+(Q_t, \mathbf{Q}_t^H), h^-(Q_t, \mathbf{Q}_t^H)$, denoted in short by h^+, h^- respectively. Then the storage level state update equation under this policy is

$$X_{t+1} = \eta \min\left\{R_i + X_t, S, \hat{h}(X_t, Q_t, \mathbf{Q}_t^H)\right\} \tag{25}$$

where $\hat{h}(X_t, Q_t, \mathbf{Q}_t^H) =$

$$\begin{cases} \max\{h^-, X_t - \rho Y_t\} & \text{if } X_t \leq h^- \\ X_t + [-\rho Y_t]^+ & \text{if } h^- \leq X_t \leq h^+ \\ \max\{h^+, X_t - \gamma_t Y_t, X_t - R_o\} & \text{o.w.} \end{cases}$$

and $\gamma_t = \begin{cases} \rho & \text{if } Y_t \leq 0 \\ 1 & \text{otherwise.} \end{cases}$

Let $L(X_t, Q_t, \mathbf{Q}_t^H)$ be the *optimal* one period cost for a storage of size S. Then,

$$L(X_t, Q_t, \mathbf{Q}_t^H) = p_t \left[Y_t + \frac{X_{t+1} - X_t}{\beta_{(X_{t+1} - X_t)}}\right]^+, \tag{26}$$

and the average cost is

$$g(S) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} E_{Q_t/\mathbf{Q}_t^H} L(X_t, Q_t, \mathbf{Q}_t^H). \tag{27}$$

Theorem 11. *$g(S)$ is non-increasing in S with ramps (i.e., R_i^S, R_o^S) that are non-decreasing with size S .*

Theorem 12. *$g(S)$ is convex in S with ramps (i.e., R_i^S, R_o^S) that satisfy the convexity property with size S.*

The above structural results imply that the marginal value from storage is a decreasing function of the storage size. With a strictly-increasing linear amortized capital cost function for storage, it is easy to see that the objective of the optimal sizing problem under the optimal management policy is convex. Thus there exists an optimum S^* (not necessarily unique) and this can be evaluated efficiently using any gradient descent method. These result holds even when investment cost functions are strictly increasing and convex in S.

V. Value of storage and the fundamental limit on the cost of storage under a constant price and an i.i.d net demand process

Theorem 13. *Under a constant price, the i.i.d. process for Y_t that provides maximum value from a storage of size S has equal weights at $\frac{-S}{\rho}$ and S. This distribution results in a value of $\frac{pS}{4}$ over zero storage.*

Corollary 14. *Under a constant price and an i.i.d. Y_t process, if storage is a profitable investment then $\frac{c}{p} \leq \frac{1}{4}$.*

Note that the statement and proofs of the above results have been modified from their earlier versions that we provided in [9] for the purpose of correctness and completeness.

The above theorem and corollary together show that any investment in storage is profitable only if the ratio of the amortized capital cost of storage to the constant price of energy is less than $\frac{1}{4}$ under i.i.d. processes of net-demand, Y_t. To the best of our knowledge this is the first theoretical (tight) upper bound on the cost of storage. An i.i.d. assumption on net-demand Y_t can be viewed as the random prediction error on net-load after balancing that is managed with storage operations. Note that $\frac{1}{4}$ is an upper bound and that for many i.i.d. distributions of the net load the $\frac{c}{p}$ bound may be even smaller (see observation 15).

VI. Sensitivity analysis for optimal storage size and its gain through a simple case study

In this section, we study the tradeoffs between the different parameters/settings of the problem and the optimal size of storage and its corresponding gain. Recall that we defined gain as the decrease in the total cost of problem **I** by investing in storage. For characterizing our results, we consider a setting where $Y_t = \widetilde{Y}_t - \epsilon(p_t - \widetilde{p})$ where $\widetilde{p}$ is some nominal price, $\widetilde{Y}_t$ is a uniform i.i.d. process with mean m and width u (i.e., variance is $\frac{u^2}{12}$) and ϵ is the price elasticity of demand.

We first consider the case of a constant price with no losses or ramp constraints. We derive the optimal storage size and its corresponding gain in closed form and study the impact of the mean and standard deviation of the net load on them. This is our baseline case which we then extend to two cases: when there are (a) ramp constraints and losses and (b) two price levels. In both these extensions, it is not easy to derive the optimal storage size in closed form. We therefore compute it numerically and present the results. In particular, we discretize the net load Y_t and the level in storage X_t and solve a linear program to identify the optimal policy and the associated infinite horizon average cost. We use the fact that the balancing policy is optimal for the baseline case, case (a) and the high price in case (b). The optimal policy for the lower price in case (b) are deduced from the solution of the linear program where it is verified to be dual threshold. The thresholds themselves are deduced when the demand is high and the storage is empty or full (as ramp constraints are absent in (b)).

A. Explicit expression for optimal storage under a constant price, no losses and no ramp constraints

Consider the case of a constant price for electricity, denoted by p, no losses ($\rho, \eta = 1$) and no ramp constraints ($R_i, R_o \geq S$) with zero price elasticity ($\epsilon = 0$). In this setting, we proved that the balancing policy is optimal. We use this to derive the steady state distribution, $f_X(x)$, for storage of size S using the following equation:

$$\int_0^S f_X(y)\left[(1 - F_Y(y))\delta(x) + f_Y(y - x)\mathbb{I}(0 < x < S) + F_Y(y - S)\delta(x - S)\right] dy = f_X(x), \tag{28}$$

where $\delta(x)$ is a dirac-delta function that is 1 when the $x = 0$ and 0 otherwise and $\mathbb{I}(0 < x < S)$ is the unit function which is 1 if $0 < x < S$ and 0 otherwise. Since Y has a uniform distribution, $F_Y(y) = \frac{y - m + \frac{u}{2}}{u}\ \forall\ m - \frac{u}{2} \leq y \leq m + \frac{u}{2}$. Substituting $x = 0, x = S$ and x s.t $0 < x < S$, in the

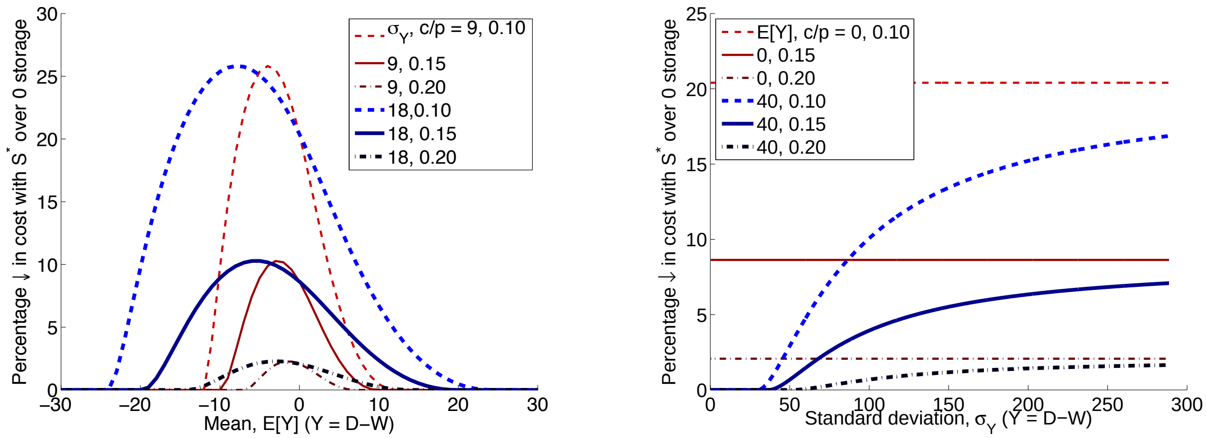

Fig. 2: Percentage gain from optimal storage investment, S^* with mean and standard deviation for uniform net load Y_t under a constant price and no losses.

integral equation, we get $f_X(x)$ as follows:

$$f_X(x) = \frac{m + \frac{u}{2} - E[X]}{u}\delta(x) + \frac{1}{u}\mathbb{I}(0 < x < S) + \frac{E[X] - (S + m - \frac{u}{2})}{u}\delta(x - S), \quad (29)$$

where $E[X] = \frac{-S(S+2m-u)}{2(u-S)}$. We can now estimate the objective function $Z(S) = g(S) + cS$ where $g(.)$ is the cost associated with electricity from the grid and cS refers to the investment cost as follows:

$$Z(S) = cS + p\int_Y \int_X (y - x)^+ f_Y(y) f_X(x)\, dx\, dy. \quad (30)$$

Substituting the density functions, $Z(S)$ takes the form

$$\frac{p}{4u^2}\left[-\frac{S^3}{3} - uS(u - S) + \frac{4m^2uS}{u - S} + (2m + u)^2\frac{u}{2}\right] + cS.$$

Taking derivatives and picking the solution such that $Z'(S) = 0$ and $Z''(S) < 0$ gives the optimal storage as follows:

$$S^* = \max\left\{0, u\left[1 - \sqrt{\frac{2c}{p}\left(1 + \sqrt{1 + \frac{m^2p^2}{u^2c^2}}\right)}\right]\right\}. \quad (31)$$

Observation 15. *$S^* > 0$ if and only if $\frac{c}{p} < \frac{1}{4} - \left(\frac{m}{u}\right)^2$.*

This observation strengthens the result of Theorem 14 for the case of the uniform distribution in the following ways: (a) the uniform distribution with 0 mean is a class of distribution for which the $\frac{1}{4}$ bound is tight; (b) the bound is smaller for larger $\frac{m}{u}$ values; and, (c) it proves converse of the theorem as well. A simple application of the central limit theorem extends this observation when $m = 0$ to the case where Y_t has a Gaussian distribution with 0 mean i.e., $S^*_{G(0,\sigma^2)} > 0$ if and only if $\frac{c}{p} < \frac{1}{4}$.

Observation 16. *For a given u and $\frac{c}{p}$, S^* is maximum at 0 mean decreases in the $O(\sqrt{m})$ symmetrically around 0.*

Observation 17. *For a given m and u with $m << u$, S^* decreases in the $O\left(\sqrt{\frac{c}{p}}\right)$.*

Observation 18. *For a given m and $\frac{c}{p}$ ratio with $m << u$, S^* increases linearly with respect to u, the standard deviation whenever $S^* > 0$.*

In the above observations we restrict to the region when $m << u$, i.e., m is much smaller than u. In many situations this is exactly the region of interest as the net load, Y_t, can be viewed as the net load after balancing which usually has a small mean and a large standard deviation.

To understand the variations in the gain from an optimal investment in storage, S^*, we plot the percentage decrease in the total cost, $\frac{Z(0)-Z(S^*)}{Z(0)}$ with respect to m and u for different values of $\frac{c}{p}$ in Fig. 2. With respect to m and $\frac{c}{p}$, we observe that the percentage gain from storage decreases rapidly (has a square effect) for larger values of $|m|$ and $\frac{c}{p}$. Note that the absolute gain peaks at 0 mean but the slight skew is because we are plotting the percentage gain. The percentage gain is non-decreasing in the standard deviation and asymptotically approaches the percentage gain at 0 mean.

B. Impact of losses and ramp constraints

In Fig. 3, we plot the variations in optimal storage size and its corresponding percentage gain (equivalently gain as $Z(0)$ is fixed) as we vary each of the following: conversion losses (ρ), dissipation losses (η) and ramp constraints ($R_i = R_o = R$). *Observe that the optimal storage size as well as its corresponding gain, not surprisingly, increases with ρ, η and R i.e., decrease in the magnitude of losses and tightness of the ramp constraints with other factors kept the same (such as cost of the storage).* In particular, we observe that the decrease in gain with the increase in dissipation losses (decrease in η) is at

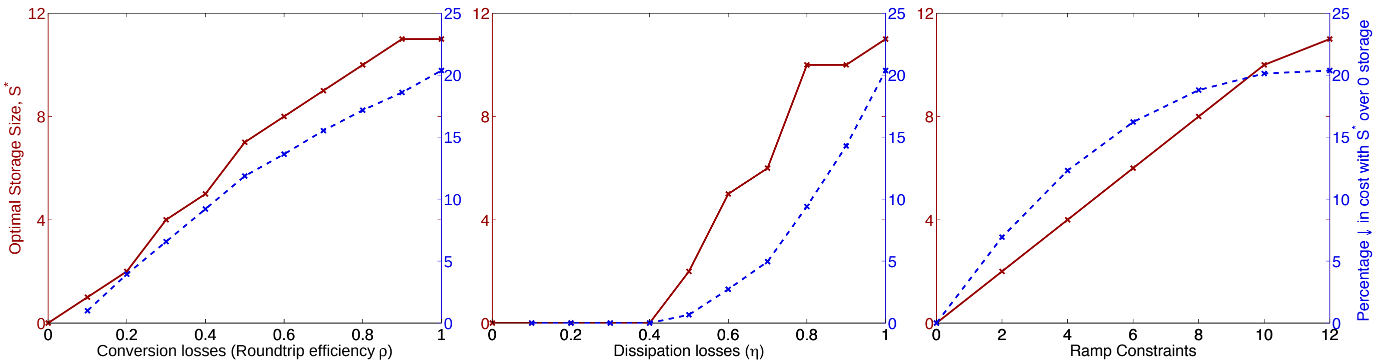

Fig. 3: Variation in the optimal storage size (solid line) and its corresponding percentage gain (dashed line) with conversion losses, dissipation losses and ramp constraints. Here, $E[Y_t] = 0$, $\sigma_{Y_t} = 8.9$, $\frac{c}{p} = 0.1$.

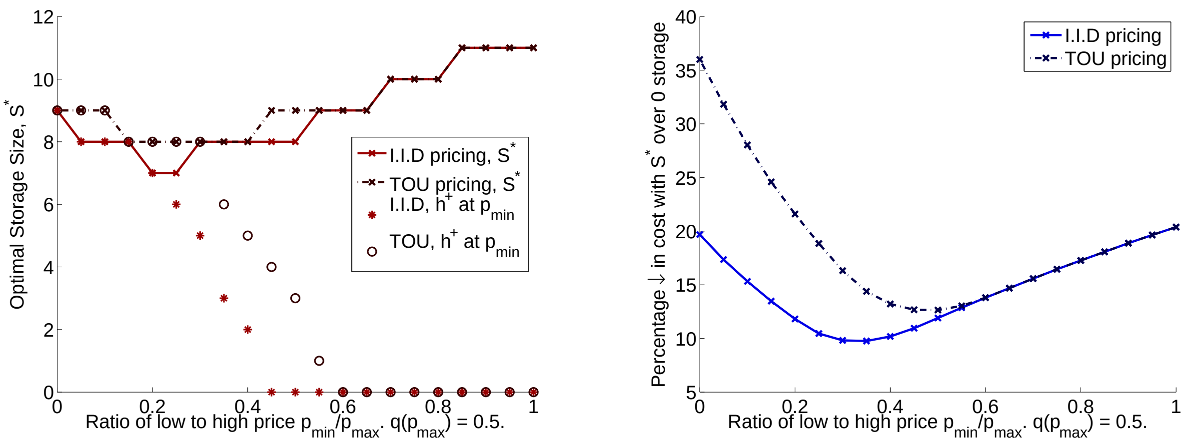

Fig. 4: Variation in the optimal storage size, the optimal threshold at p_{min} and the corresponding percentage gain as a function of $\frac{p_{\min}}{p_{\max}}$ ratio for different pricing schemes: i.i.d. and TOU. Here, $\epsilon = 0$, $E[Y_t] = 0$, $\sigma_{Y_t} = 8.9$, $q(p_{\max}) = 0.5$ and $\frac{c}{p_{\max}} = 0.1$.

a much faster rate than with conversion losses (decrease in ρ). This is because one can obtain up to S units of energy from the storage device after conversion losses have been accounted for but only up to ηS units after dissipation losses are account for. This difference at all levels of ρ and η causes the higher rate of decrease. On the other hand, with ramp constraints, the marginal gain from investing in optimal storage by relaxing the ramp constraints (increasing R) is decreasing till the point there no gain, This points to the fact that the marginal value from a storage is also decreasing as a function of R similar to storage size as shown in Theorem 12.

C. Impact of differential pricing

In this section, we study how differential pricing and the uncertainty in differential pricing impacts the optimal storage size and its corresponding gain. To answer these questions, we consider two simple differential pricing schemes: (1) i.i.d. pricing, a proxy for real-time pricing (RTP) where the peak price, $p_{\max}$, occurs with probability q and off-peak price $p_{\min}$ occurs with probability $(1-q)$; (2) time-of-use pricing (TOU) where the peak price is always followed by the off-peak price. These schemes differ in the information about sequence of future prices. We normalize $p_{\max}$ to be 1 and choose the nominal price $\widetilde{p} = 0.5$. We also assume that there are no losses or ramp constraints.

Fig. 4 shows how the optimal storage and its percentage gain change as a function of $\frac{p_{\min}}{p_{\max}}$ ratio for the i.i.d. and TOU pricing schemes. For an appropriate comparison between the two schemes, we chose the q in the i.i.d. scheme to be 0.5. We also plot the change in the optimal threshold, $h^+ (= h^-$ because $\rho = 1)$ at $p_{\min}$. Observe that the storage size follows discrete increments because we approximate Y_t with a discrete distribution. We make the following two observations from this plot.

First *the optimal storage size and the percentage gain first decreases and then increases. In fact, after a certain point, the values are the same for both the schemes. The former is because at low* $\frac{p_{\min}}{p_{\max}}$ *ratio, the optimal storage management*

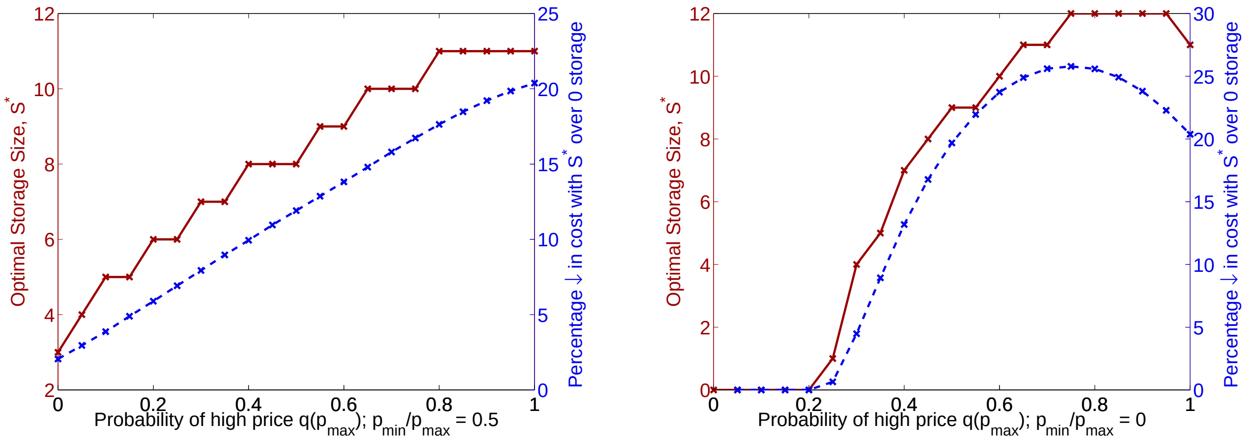

Fig. 5: Impact of uncertainty, $q(p_{\max})$, in i.i.d. pricing scheme on the optimal storage size (solid line) and its corresponding percentage gain (dashed line) for two different $\frac{p_{\min}}{p_{\max}}$ ratios. Here, $\epsilon = 0$, $E[Y_t] = 0$, $\sigma_{Y_t} = 8.9$ and $\frac{c}{p_{\max}} = 0.1$.

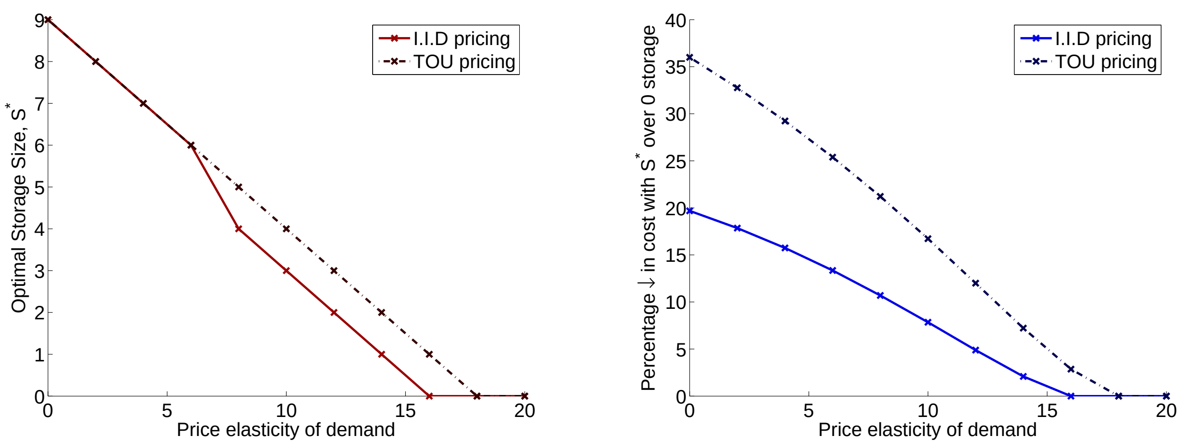

Fig. 6: Variation in the optimal storage size and its corresponding percentage gain with price elasticity of demand, ϵ, due to load shifting in the i.i.d. and TOU pricing schemes. Here, $\frac{\widetilde{p}}{p_{\max}} = 0.5$, $\frac{p_{\min}}{p_{\max}} = 0$, $E[\widetilde{Y}_t] = 0$, $\sigma_{\widetilde{Y}_t} = 8.9$ and $\frac{c}{p_{\max}} = 0.1$.

goal in both schemes is to primarily store for the uncertainty in net load at the peak price. But this becomes expensive as the $\frac{p_{\min}}{p_{\max}}$ ratio increases. When the $\frac{p_{\min}}{p_{\max}}$ ratio is sufficiently high, it is beneficial to start storing for the net load uncertainty at the off-peak price itself. This happens when the threshold h^+ at $p_{\min}$ decreases and is equal to 0. And from that point on, as we increase the $\frac{p_{\min}}{p_{\max}}$ ratio, the optimal storage size and percentage gain increases. This happens for both the pricing schemes at some point (not necessarily the same). Because $q = 0.5$, the two schemes under the balancing policy at both the price levels are identical. Therefore, the optimal storage sizes and their gains (hence the percentage gains as well) are the same for both the schemes after a certain point.

Second *the optimal storage size and its percentage gain tends to be higher for TOU pricing compared to the i.i.d. pricing scheme*. One can attribute this to the riskier nature of the objective with price uncertainty for the same level of price variability. Note the opposite trend compared to the uncertainty in Y_t (see observation 18 and Fig. 2) where it is beneficial to invest in a larger storage with larger net load uncertainty.

In Fig. 5 we plot the change in optimal storage size and the percentage gain for the i.i.d. pricing scheme with respect to the frequency of the high price, $q(p_{\max})$ for two different $\frac{p_{\min}}{p_{\max}}$ ratios. *Comparing $q(p_{\max}) = 1$ (constant pricing case) with any other value of q in that plot, we observe that i.i.d. differential pricing can sometimes increase the percentage gain from storage but not always.* This is because of the tradeoff between how much energy is actually needed at the peak price to support excess demand but also how often it can be obtained at the lower price (and the corresponding value of the lower price).

We next study the effect of varying price elasticity of demand on the optimal storage size and gain for the two pricing schemes (while setting $q(p_{\max}) = 0.5$). To make the experiment interesting, we choose $\frac{p_{\min}}{p_{\max}} = 0$ because $\frac{\widetilde{p}}{p_{\max}} = 0.5$. By doing so we are essentially simulating load shifting from a high to low price because $E[Y_t^{p_{\max}} + Y_t^{p_{\min}}] = 2E[\widetilde{Y}_t] = 0$. *In Fig. 6, we observe that higher price elasticity tends to decrease the optimal storage size and its percentage gain.* This is expected because both demand response (load shifting in this context) and storage are different ways to manage uncertainty in net load and one can compensate the other depending on the degree of demand elasticity. The relationship between the two pricing schemes observed here is similar to Fig. 4.

VII. Computational Study

In this section we present results of a computational case study to illustrate the potential savings using energy storage on a realistic scenario. We used the Pacific Northwest GridWise Testbed Demonstration Project [4] data as a source for the demand and price data and the western wind integration study at NREL [5] for the wind data. Below we explain the details of the data sets, the models used, the calibration techniques and finally, discuss our results.

a) Demand and price data, models: We use the Olympic Peninsula field demonstration data gathered from 112 residential households on the Olympic Peninsula. The demonstration data consists of electricity prices and the demand by user aggregated every 15 min for a period of one year from 1 April, 2006 to 31 March, 2007. In our experiments, we focus on two groups of households (roughly 25-30 households in each) that observed fixed prices and time-of-use prices. The number of households varied by a very small number through the year. So, we first normalize the consumption data to consumption per household in each group and scaled it up by 100 households per group to reflect the approximate size of the demonstration. The prices for the fixed price control group was 8.1¢ through the entire year. The prices for the TOU group differed based on the month of the year and hour of the day and as in Table I.

Season	Period	Times	Price ¢/kWh
Spring, Fall/Winter (Apr-Jul; Oct-Mar)	off-peak	9a-6p; 9p-6a	4.119
	peak	6a-9p; 6p-9a	12.15
Summer (Jul-Sep)	off-peak	9p-3p	5
	peak	3p-9p	13.5

TABLE I: TOU pricing in the demonstration.

Fig. 7 provides an example of the aggregate estimated demand by hour of day for a typical day in the month of March. Even though the consumption pattern differed by group, the net consumption of the two groups differed in total across all months by less than 0.5% implying that the TOU consumers primarily shifted their consumption from peak to off-peak hours. Because prices are pre-announced in these groups, we assume that the aggregate load can be perfectly forecasted by hour-of-the-day differing by month to account for changes in seasonality.

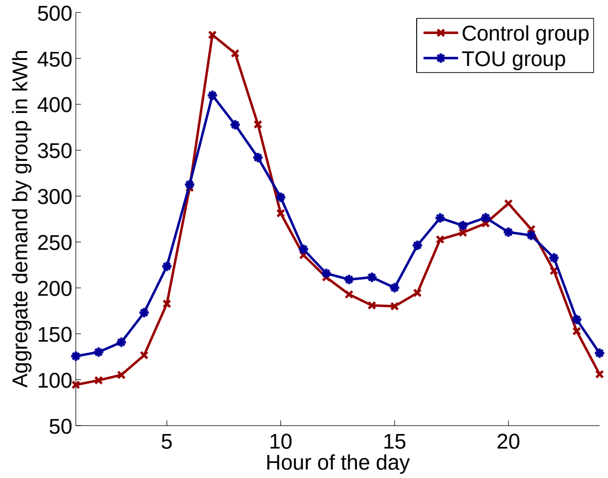

Fig. 7: Aggregate estimated demand by hour of day for a typical day in March by pricing group.

b) Wind data, model and calibration: We use the NREL simulated western wind dataset for years 2005 and 2006. The NREL data was generated using numerical weather prediction models for several locations in the western United States by recreating the historical weather data. For our study we chose a wind farm closest to Port Angeles that consists of 10 wind turbines of 3 MW capacity each. The data set consists of wind production level sampled every 10 mins as a time series. Due to a small residential sample size we scale the output of the turbines to a 1MW capacity (a typical size of one small wind turbine). For each year, this resulted in roughly 40% wind penetration. We use an hourly discretization in our experiments and hence aggregate the time series by hour.

Researchers have shown that wind speed and energy can be modeled effectively using autoregressive (AR) processes that have time varying parameters to account for seasonal characteristics of wind throughout the year [23], [24]. We model wind energy using an autoregressive process of order one, AR(1) as follows. In order to account for seasonality, we have a different model for each month:

$$W_{m,t} = \alpha_m W_{m,t-1} + \beta_m + \epsilon_m, \quad (32)$$

where $W_{m,t}$ is the time series for wind in month m, α_m, β_m are the calibrated constants of the AR(1) model for month m and ϵ_m is the i.i.d error term for month m. Among AR models, our choice of an order one lag was based on the mean absolute error (MAE) metric. An AR(2) model revealed that it did not improve the MAE metric any further than an AR(1) model. We use ordinary least squares to calibrate the parameters of the model. The error distribution was chosen to be the empirical distribution of the residuals. We use the data

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
α_m	0.93	0.96	0.93	0.88	0.88	0.93	0.89	0.90	0.91	0.92	0.91	0.92
β_m	23.2	8.1	16.0	27.7	24.8	17.0	15.5	11.0	11.9	18.8	28.9	29.5
MAE 2005	74.9	39.4	64.8	78.5	70.6	58.0	46.1	38.2	41.3	66.8	82.7	78.4
MAE 2006	93.4	77.1	67.5	54.7	61.8	52.1	53.0	41.4	45.1	60.6	112.8	85.2

TABLE II: AR(1) model by month calibrated from 2005 data set with corresponding MAE in kWh for the 2005 and 2006.

for year 2005 as the training data set and the data for year 2006 for testing purposes. Table II provides the calibrated constants for the AR(1) model and the corresponding MAEs for the 2005 training and 2006 test data sets.

c) Experimental setup: Consider a storage with an efficiency constant $\rho = 0.85$, a dissipation constant $\eta = 0.95$ and ramp constraints $\hat{R}_i = \hat{R}_o = \frac{150}{\eta}$kW, an investment cost $K = \$1500/\text{kWh}$ and a lifetime of 15 years. Assume that the annual interest rate is 8%. First we numerically compute the long run average value of storage for different storage sizes and next identify the optimal storage size.

We use 2005 wind training and 2006 residential demand data as an input to our models. We note that the residential demand data for the months January through March are available for the year 2007 (and not 2006). We assume that the 2007 demand data is representative of the 2006 load as well. We solve 12 daily long-term average cost problems, one for each month, with an hourly discretization to arrive at the average per period (hour) value of storage. Note that this is an approximation because we are ignoring the boundary effects that connects the problems between consecutive months. We make this approximation to maintain a small state space.

For the long term average cost problem each month, the state space is (h_t, X_t, W_{t-1}) where h_t is the hour of the day, X_t is the level of energy in the storage device and W_{t-1} wind energy in the previous hour. We discretize demand and wind at a $\delta = 50$kWh granularity and choose a $\frac{\delta}{\eta}$ granularity for the energy level X_t in storage.

We use linear programming (LP) to solve each average cost problem. We implemented the balancing policy at the high price to reduce the size of the problem and verify that at the lower prices, we observe a dual-threshold policy. On a laptop with 2.2GHz Intel Core i7 processor and an 8GB memory, the time to solve the average cost problem each month for the constant price case varied from 0.1 to 9 minutes depending on the size of storage (2.8 minutes on average) and similarly varied from 0.3 minutes to 4 hours (45 minutes on average) for the TOU case. This drastic difference for larger storage sizes is because at lower prices for the TOU case the LP is searching to find the optimal policy whereas for the case with a constant price the structure of the policy is encoded.

d) Results - savings in electricity cost with storage: In Fig. 8, we plot the percentage savings in the cost of electricity from the grid over zero storage (without including the investment cost) for the two residential groups for different storage sizes. The metric that is plotted is $\frac{g(0)-g(S)}{g(0)}$ where $g(.)$ is the cost associated with electricity from the grid.

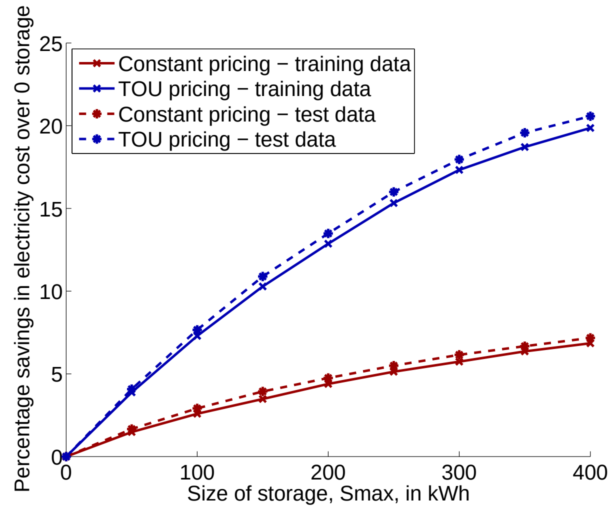

Fig. 8: Predicted (2005) and actual (2006) percentage savings in the cost of electricity over zero storage as a function of storage size for the constant and time-of-use pricing groups.

For each group, we compute the value based on the optimal dual-threshold policy on the training and test data sets. First we observe that the savings increases as a function of the storage size and the marginal savings decreases as proved in Theorem 11-12.

Suppose we refer to size of storage relative to the total annual load as *storage penetration* then our study predicts from the 2005 training data that a 2.2% (100kWh) and 4.4% (200kWh) level of storage penetration results in an savings of about 2.6% and a 4.4% for the constant price group and more than double for the TOU group of about 7.3% and a 12.9% respectively. We note that the actual savings computed from the 2006 test data are relatively close to those in the training data set data.

Observe that the percentage savings for the TOU group is much higher than the group with a constant price. This may not be surprising but it certainly depends on the choice of prices. The savings of the TOU pricing group over the group with a constant price is purely the savings from differential pricing and this is observed to increase in the size of storage.

In Table III, we provide results for the 2006 test data on actual energy consumption in addition to the savings for the two residential groups for different scenarios (with and without wind energy and with and without storage for two storage sizes). Not surprisingly, the scenario without wind indicates

No wind	No storage		With storage			
			100 kWh storage		200kWh storage	
	C	TOU	C	TOU	C	TOU
Energy from the grid (MWh)	1806	1827	1806	1844	1806	1866
% savings in cost	-	-	-	4.1%	-	7.8%
With wind	**No storage**		**With storage**			
			100 kWh storage		**200kWh storage**	
	C	TOU	C	TOU	C	TOU
Energy from the grid (MWh)	832.2	825.5	808.5	810.6	792.5	807.4
% of wind generation used	43.5%	44.8%	45.6%	47.3%	47.7%	49.0%
% savings in cost	-	-	2.8%	7.6%	4.8%	13.2%

TABLE III: Energy consumption, percentage wind penetration and savings in cost of electricity for different scenarios based on the 2006 test data. C and TOU refer to the constant and time-of-use pricing groups.

that storage provides value only in the presence of differential pricing. In this data, because wind penetration is more than 40%, the savings from wind penetration is much higher than the savings from storage penetration. But storage further aids in the increase in wind penetration for constant pricing group and much more for TOU pricing group resulting in significant savings for both groups.

e) Results - optimal storage sizing: For the specifications of the storage that we are considering, the hourly amortized investment cost using Eq. (A) is $c = 2$¢/kWh every hour. The optimal sizing result in this section is aimed at being a proof of concept and can be tuned to a more robust estimate with additional years of data. Under constant pricing, the optimal storage size is 100kWh and under TOU pricing, it is 350kWh. They contribute to 0.5% and 10% in total savings respectively when we include the investment cost in storage based on the 2005 training data set.

VIII. Conclusions

In this paper, we study the optimal energy storage management and sizing problem in the presence of renewable energy and dynamic pricing associated with electricity from the grid. We formulate the problem as an infinite horizon average cost stochastic dynamic program and obtain various structural results. These results show that (a) the optimal storage management policy has a simple dual threshold structure, and (b) the marginal value of storage is decreasing with storage size and therefore the optimal size under the optimal management policy can be computed efficiently. Through detailed computational experiments, we demonstrate that energy storage can provide significant value and savings by integrating renewable energy and decreasing the use of electricity from the grid. Both of these metrics are enhanced in the presence of well-designed dynamic pricing schemes.

Further theoretical analysis and simulation studies, will be of considerable interest for the development storage management and sizing strategies that extend the stationary analysis developed in this paper for structured non-stationary distributions (for example, quasi-stationary distributions using a receding horizon control). Interesting directions for future research include the incorporation of battery degradation and lifecycle effects into the analysis, and the use of robust optimization methods for storage management that would be adaptive to limited forecast information available about renewable energy.

Acknowledgements

This project was conducted as a part of Masdar Institute and MIT collaboration. The authors would like to thank IBM for the data from the Olympic Peninsula demonstration project as well as NREL for the western wind data set. The first author would also like to thank Devavrat Shah for discussions related to earlier version of the proof of Theorem 12. We also thank the anonymous reviewers for their valuable comments that helped improve the paper.

References

[1] "Inventory of u.s. greenhouse gas emissions and sinks: 1990-2009," EPA, Tech. Rep., 2011.
[2] "20% wind energy by 2030," Department of Energy, Tech. Rep., 2008.
[3] EnerNex Corporation, "Eastern wind integration and transmission study," National Renewable Energy Laboratory, Tech. Rep., 2010.
[4] "Pacific northwest gridwise testbed demonstration projects: Part I. olympic peninsula project," Pacific Northwest National Laboritory, Tech. Rep., 2007.
[5] GE Energy, "Western wind and solar integration study," National Renewable Energy Laboratory, Tech. Rep., 2010.
[6] J. H. Kim and W. B. Powell, "Optimal energy commitments with storage and intermittent supply," *Operations Research*, vol. 59, no. 6, pp. 1347–1360, 2011.
[7] E. Bitar, R. Rajagopal, P. Khargonekar, K. Poolla, and P. Varaiya, "Bringing wind energy to the market," *IEEE Transactions on Power Systems*, vol. 27, no. 3, pp. 1225–1235, 2012.
[8] P. C. D. Granado, S. W. Wallace, and Z. Pang, "The value of electricity storage in domestic homes: A smart grid perspective," 2012, submitted.
[9] P. Harsha and M. A. Dahleh, "Optimal sizing of energy storage for efficient integration of renewable energy," in *50th IEEE Conference on Decision and Control and European Control Conference*, 2011.
[10] Y. Ru, J. Kleissl, and S. Martinez, "Storage size determination for grid-connected photovoltaic systems," 2011, working paper.
[11] H.-I. Su and A. E. Gamal, "Modeling and analysis of the role of fast-response energy storage in the smart grid," in *Allerton Conference on Communication, Control, and Computing*, 2011.
[12] P. D. Brown, J. A. P. Lopes, and M. A. Matos, "Optimization of pumped storage capacity in an isolated power system with large renewable penetration," *IEEE Transactions on Power Systems*, vol. 23, no. 2, pp. 523–531, May 2008.

[13] Y. Zhou, A. Scheller-Wolf, and N. Secomandi, "Managing wind-based electricity generation with storage and transmission capacity," 2011, tepper Working Paper 2011-E36.
[14] P. M. van de Ven, N. Hegde, and L. M. and Theodoros Salonidis, "Optimal control of end-user energy storage," 2012, submitted.
[15] R. Rempala, "Optimal strategy in a trading problem with stochastic prices," in *System Modelling and Optimization*, ser. Lecture Notes in Control and Information Sciences, 1994, vol. 197, pp. 560–566.
[16] N. Secomandi, "Optimal commodity trading with a capacitated storage asset," *Management Science*, vol. 56, no. 3, pp. 449–467, 2010.
[17] A. Faghih, M. Roozbehani, and M. A. Dahleh, "Optimal utilization of storage and the induced price elasticity of demand in the presence of dynamic pricing," in *50th IEEE Conference on Decision and Control and European Control Conference*, 2011.
[18] C. Codemo, T. Erseghe, and A. Zanella, "Energy storage optimization strategies for smart grids," in *Communications (ICC), 2013 IEEE International Conference on*, June 2013, pp. 4089–4093.
[19] A. Federgruen and N. Yang, "Infinite horizon strategies for replenishment systems with a general pool of suppliers," 2010, submitted.
[20] D. P. Bertsekas, *Dynamic Programming and Optimal Control - Volume 2*. Athena Scientific, 2007.
[21] O. Hernandez-Lerma and J. B. Lasserre, *Discrete-Time Markov Control Processes: Basic Optimality Criteria*. Springer Verlag, 1996, vol. 30.
[22] P. H. Zipkin, *Foundations of Inventory Management*. McGraw-Hill, 2000.
[23] Z. Huang and Z. Chalabi, "Use of time-series analysis to model and forecast wind speed," *Journal of Wind Engineering and Industrial Aerodynamics*, vol. 56, no. 2-3, pp. 311–322, 1995.
[24] F. Ettoumi, H. Sauvageot, and A.-E.-H. Adane, "Statistical bivariate modelling of wind using first-order markov chain and weibull distribution," *Renewable Energy*, vol. 28, no. 11, pp. 1787–1802, 2003.
[25] D. P. Heyman and M. J. Sobel, *Stochastic Models in Operations Research, Volume II: Stochastic Optimization*. McGraw-Hill, 1984.
[26] D. P. Bertsekas and S. E. Shreve, *Stochastic Optimal Control: The Discrete Time Case*. Athena Scientific, 1978.
[27] R. Durrett, *Probability: Theory and Examples*. Duxbury Press, 2005, ch. Markov Chains.

APPENDIX A
A METHOD TO ESTIMATE THE AMORTIZED COST, c

One method to estimate the amortized per unit cost, c, from the per unit capital cost, K is as follows:

$$c = Kr\frac{(1+r)^n}{(1+r)^n - 1}\frac{1}{N}, \tag{33}$$

where r is the fixed annual interest rate, n is the lifetime of the storage device and N is the number of discretization periods considered in a year. Typically, n is in the order of 10-20 years and N is in the order of hours, say roughly 365*24 = 8760 hours. As can be noted, we amortize over a finite but sufficiently large horizon which we approximate as an infinite horizon in this paper.

APPENDIX B
PROOF OF THEOREM 2

Proof. We will prove the statement using backward induction. Because $V_{\alpha,T}(X_T, \mathbf{Q}_t^H) = 0$, the statement is clearly true at T. For the induction hypothesis, we assume that the statement is true at some $t+1$ and prove it at t. If we show that $J_{\alpha,t}(X_t, Q_t, \mathbf{Q}_t^H)$ is a non-increasing continuous convex function in X_t then the result also holds for $V_{\alpha,t}(X_t, \mathbf{Q}_t^H)$. This is because $V_{\alpha,t}(X_t, \mathbf{Q}_t^H) = E_{Q_t/\mathbf{Q}_t^H} J_{\alpha,t}(X_t, Q_t, \mathbf{Q}_t^H)$ and Q_t has an distribution that is independent of X_t. So, using the fact that sum of convex functions is convex, the result holds.

Continuous convex function: Consider the optimization problem related to $J_{\alpha,t}(X_t, Q_t, \mathbf{Q}_t^H)$. Here, $J_{\alpha,t}(X_t, Q_t, \mathbf{Q}_t^H) =$

$$\min_{u_t} \; p_t\left[Y_t + \frac{u_t}{\beta_t}\right]^+ + \alpha V_{\alpha,t+1}\left(\eta(X_t+u_t), f\left(Q_t, \mathbf{Q}_t^H\right)\right)$$
$$s.t. \; -\min\{X_t, R_o\} \le u_t \le \min\{(S - X_t), R_i\}.$$

The first term in the objective is continuous and convex in u_t and hence jointly convex in (u_t, X_t). The second term is also a continuous and convex function in $X_{t+1} = \eta(X_t + u_t)$ because of the induction hypothesis. This implies the second term is also jointly convex in (u_t, X_t) because it is convex composition of an affine function. Therefore the objective is jointly convex in (u_t, X_t). This implies u_t^* exists (may not be unique) because the objective is continuous and we are optimizing on a compact set.

Note that the optimization problem has a jointly convex objective in (u_t, X_t) with affine constraints on (u_t, X_t). Therefore, the resulting function $J_{\alpha,t}(X_t, Q_t, \mathbf{Q}_t^H)$ is also continuous and convex in X_t. (Proof sketch: Join the two independent optimization problems for any two points X_t^1 and X_t^2; Say the corresponding variables are u_t^1 and u_t^2 respectively; Use the fact the objective is convex and get a joint objective in $(\bar{X}, \bar{u}) = (\lambda X_t^1 + (1-\lambda)X_t^2, \lambda u_t^1 + (1-\lambda)u_t^2)$ and independent set of constraints in (X_t^1, u_t^1) and (X_t^2, u_t^2) for any $\lambda \in [0,1]$; relax the problem by combining the affine constraints in a weighted average fashion to retrieve the formulation purely in variables $(\bar{X}, \bar{u})$; This is the optimization problem for $\bar{X} = \lambda X_t^1 + (1-\lambda)X_t^2$ and hence proved).

Non-increasing function: To show that $J_{\alpha,t}(X_t, Q_t, \mathbf{Q}_t^H)$ is non-increasing in X_t, it suffices to show that every feasible solution to the optimization problem related to $J_{\alpha,t}(X_t, Q_t, \mathbf{Q}_t^H)$ has a cost that is greater than or equal to some feasible solution to optimization problem related to $J_{\alpha,t}(X_t+\delta, Q_t, \mathbf{Q}_t^H)$ where $0 \le X_t < X_t + \delta \le \eta S$. Let u_t be any feasible to the optimization problem related to $J_{\alpha,t}(X_t, Q_t, \mathbf{Q}_t^H)$. We now consider two cases:

Case 1: Suppose u_t is also feasible to $J_{\alpha,t}(X_t+\delta, Q_t, \mathbf{Q}_t^H)$.

$$p_t\left[Y_t + \frac{u_t}{\rho}\right]^+ + \alpha V_{\alpha,t+1}(\eta(X_t+u_t), \mathbf{Q}_{t+1}^H)$$
$$\ge p_t\left[Y_t + \frac{u_t}{\rho}\right]^+ + \alpha V_{\alpha,t+1}(\eta(X_t+\delta+u_t), \mathbf{Q}_{t+1}^H) \tag{34}$$

(from induction hypothesis).

Case 2: Suppose u_t is infeasible to problem $J_{\alpha,t}(X_t+\delta, Q_t, \mathbf{Q}_t^H)$. This implies $(S - X_t - \delta) < u_t \le \min\{(S - X_t), R_i\}$.

$$p_t\left[Y_t + \frac{u_t}{\rho}\right]^+ + \alpha V_{\alpha,t+1}(\eta(X_t+u_t), \mathbf{Q}_{t+1}^H)$$
$$\ge p_t\left[Y_t + \frac{S - X_t - \delta}{\rho}\right]^+ + \alpha V_{\alpha,t+1}(\eta(X_t+u_t), \mathbf{Q}_{t+1}^H)$$
$$\ge p_t\left[Y_t + \frac{S - X_t - \delta}{\rho}\right]^+ + \alpha V_{\alpha,t+1}(\eta S, \mathbf{Q}_{t+1}^H) \tag{35}$$

(from induction hypothesis).

Combining (34–35) completes the proof that $J_{\alpha,t}(X_t, Q_t, \mathbf{Q}_t^H)$ and hence $V_{\alpha,t}(X_t, \mathbf{Q}_t^H)$ are non-increasing in X_t. □

APPENDIX C
PROOF OF COROLLARY 3

Proof. Using a simple backward induction argument over time, it is easy to see that the maximum slope of the function, $V_{\alpha,t}(X_t, \mathbf{Q}_t^H)$ is always less than the largest price, $p_{\max}$. This follows from Eq. (10) and Eq. (15) where the optimal control decision u_t^* is an affine function of X_t with the absolute value of the slope being less than 1. This means the objective of the problem corresponding to the $h_t^+(Q_t, \mathbf{Q}_t^H)|_{p_t=p_{\max}}$ is increasing resulting in $h_t^+(Q_t, \mathbf{Q}_t^H)|_{p_t=p_{\max}} = 0 \;\forall\; t$. Because $0 \le h_t^-(Q_t, \mathbf{Q}_t^H)|_{p_t=p_{\max}} \le h_t^+(Q_t, \mathbf{Q}_t^H)|_{p_t=p_{\max}}$, $h_t^-(Q_t, \mathbf{Q}_t^H)|_{p_t=p_{\max}} = 0 \;\forall\; t$ as well. This implies that the optimal policy at the highest price is to just perform the balancing policy.

An analogous proof holds when the price is 0. At this price, the objective is decreasing and the minimum is always at S. Therefore,

$h_t^-(Q_t, \mathbf{Q}_t^H)|_{p_t=0} = h_t^+(Q_t, \mathbf{Q}_t^H)|_{p_t=0} = S \ \forall\ t$. This implies the optimal control policy to fill the storage device completely. □

Appendix D
Proof of Corollary 4

Proof. The second term of the optimization problems corresponding to $h_t^-(p_t)$ and $h_t^+(p_t)$ is non-increasing in the storage level z_t and independent of p_t under the i.i.d assumption of the stochastic process Q_t. So together with the first term that is increasing linearly in z_t with a constant proportional to p_t, the solutions to the optimization problems (i.e., the thresholds $h_t^-(p_t)$ and $h_t^+(p_t)$) are non-increasing in p_t. This implies that $h_t(X_t, p_t), h_t^*(X_t, Q_t)$ and hence $u_t^*(X_t, Q_t)$ are all non-increasing in p_t. □

Appendix E
Proof of Theorem 5

Proof. 1) $V_{\alpha,t}(X_t, \mathbf{Q}_t^H)$ is less than or equal to the expected cost when nothing is stored from period t to the end of the horizon, i.e., $t-1, ..., 0$.

$$V_{\alpha,t}(X_t, \mathbf{Q}_t^H) \le \sum_{m=0}^{t} \alpha^{t-m} p_t Y_t^+ \le \frac{1}{1-\alpha} p_{\max} Y_{\max}^+, \qquad (36)$$

where $p_{\max}$ and $Y_{\max}$ are the maximum values of the respective parameters. Note that in practice prices, renewable generation and demand are all bounded quantities. Using induction, we now establish monotonicity. This together with the bound, will establish that the limit exists. $0 \le V_{\alpha,0}(X, \mathbf{Q}^H) \le V_{\alpha,1}(X, \mathbf{Q}^H)$ is immediate from the description. Assume that $V_{\alpha,t-1}(X, \mathbf{Q}^H) \le V_{\alpha,t}(X, \mathbf{Q}^H)$ for all X and $\mathbf{Q}^H$. For any u such that $-\min\{X, R_o\} \le u \le \min\{(S - X), R_i\}$,

$$p\left[Y + \frac{u}{\beta_u}\right]^+ + \alpha V_{\alpha,t-1}\left(\eta(X+u), f(Q_t, \mathbf{Q}^H)\right) \le p\left[Y + \frac{u}{\beta_u}\right]^+ + \alpha V_{\alpha,t}\left(\eta(X+u),, f(Q_t, \mathbf{Q}^H)\right). \qquad (37)$$

Taking the minimum over all u on both sides, we get $J_{\alpha,t}(X, Q, \mathbf{Q}^H) \le J_{\alpha,t+1}(X, Q, \mathbf{Q}^H)$ and hence $V_{\alpha,t}(X, \mathbf{Q}^H) \le V_{\alpha,t+1}(X, \mathbf{Q}^H)$ as well.

2) This result directly follows from Theorem 8-14 in [25]. Here we discuss and verify the four conditions of the theorem. The first condition requires $\lim_{t\to\infty} V_{\alpha,t}(X, \mathbf{Q}^H)$ to exist for every $\{X, \mathbf{Q}^H\}$. This is satisfied by part (1) of this theorem. The second condition requires the one period cost to be non-negative or non-positive for all periods and all state-action tuples. This is true because the one period cost is non-negative for all periods and all $(u, X, Q, \mathbf{Q}^H)$. The third condition requires the action space to be compact which is true. Finally, for the fourth condition requires that $K_{\alpha,t}(u, X, Q, \mathbf{Q}^H)$ be continuous in u for all feasible states $\{X, Q, \mathbf{Q}^H\}$ where $K_{\alpha,t}(u, X, Q, \mathbf{Q}^H) =$

$$p\left[Y + \frac{u}{\beta_u}\right]^+ + \alpha V_{\alpha,t}\left(\eta(X+u), f(Q, \mathbf{Q}^H)\right).$$

This is true from theorem 2.

3) This is immediate from proposition 5.8 in [26]. We discuss and verify the three conditions of the proposition. The first condition is the uniform increase assumption that requires the sequence $\{V_{\alpha,t}(.)\}$ to be non-decreasing which is true from part (1) of this theorem. The second condition requires that for any non-decreasing sequence of functions $\{V_{\alpha,t}(.)\}$, with $V_{\alpha,\infty}(X, \mathbf{Q}^H) = \lim_{t\to\infty} V_{\alpha,t}(X, \mathbf{Q}^H)$,

$$E_{Q/\mathbf{Q}^H} p\left[Y + \frac{u^*}{\beta_{u^*}}\right]^+ + \alpha \lim_{t\to\infty} E_{Q/\mathbf{Q}^H} V_{\alpha,t}\left(\eta(X+u^*), f(Q, \mathbf{Q}^H)\right)$$

$$= E_{Q/\mathbf{Q}^H} p\left[Y + \frac{u^*}{\beta_{u^*}}\right]^+ + \alpha V_{\alpha,\infty}\left(\eta(X+u^*), f(Q, \mathbf{Q}^H)\right)$$

where $u^* = u(X, Q, \mathbf{Q}^H)$. The interchange of limit and expectation follows from the Monotone Convergence Theorem. This implies that the second condition is also satisfied. Finally, the last condition requires that there exists a scalar $A > 0$ such that for all scalars $r > 0$ and $V_{\alpha,0}(X, \mathbf{Q}^H) \le V_{\alpha,t}(X, \mathbf{Q}^H)$,

$$\begin{aligned} K_{\alpha,t}(u, X, Q, \mathbf{Q}^H) &\le K_{\alpha,t}(u, X, Q, \mathbf{Q}^H) + \alpha r \\ &\le K_{\alpha,t}(u, X, Q, \mathbf{Q}^H) + Ar. \end{aligned}$$

This is trivially satisfied when $A = \alpha$.

4) The limit of non-increasing continuous convex functions is a non-increasing continuous convex function. □

Appendix F
Proof of Theorem 8

Proof. The proof of parts 1, 2 and 3 of theorem follows from theorem 5.5.4 in book by Hernández- Lerma and Lasserre [21]. Here, we discuss and verify the conditions of that theorem. The proof of that theorem is constructive and follows the steps of the vanishing discount approach. All the references that we use in this proof refer to that in book [21] except stated otherwise.

Assumption 4.2.1, the first condition in the theorem, can be replaced with condition 3.3.2 as stated in section 5.1 in the book because of theorem 3.3.5. Condition 3.3.2 requires the following to hold which we verify as well: (a) the control set should be compact which holds; (b) the one step cost function should be non-negative as well as lower semi-continuous in the control u for all $X \in [0, \eta S]$ and $\mathbf{Q}^H \in \mathbb{Q}^H$ which is true; and, (c) one of the conditions 3.3.2(c1) or 3.3.2(c2) hold and we appeal to both the assumptions in this proof because of the hybrid nature of the state of the system i.e., X is continuous but $\mathbf{Q}^H$ is discrete. The condition requires that for every bounded measurable function $\beta(.,.)$ on $\{X, \mathbf{Q}^H\}$ so that $\beta(., \mathbf{Q}^H)$ is bounded and continuous on X, $E_{Q/\mathbf{Q}^H}\left[\beta\left(\eta(X+u), f(Q, \mathbf{Q}^H)\right)\right]$ is l.s.c in u for any $\{X, \mathbf{Q}^H\}$. This is true because the expectation is over a discrete sum and $\beta(., \mathbf{Q}^H)$ is continuous in X.

Assumption 5.5.1, the second assumption in the theorem, is satisfied because the following:

- $(1-\alpha)V_\alpha^*(X, \mathbf{Q}^H)$ should be bounded from above at the reference state for all $\alpha \in (0,1)$. This is true as shown in part (1) of theorem 5 for every state and in particular the reference state $\{0, \widetilde{\mathbf{Q}}^H\}$.
- There should be a constant $K \ge 0$ and a non-negative measurable function $b(.)$ such that $-K \le V_\alpha^*(X, \mathbf{Q}^H) - V_\alpha^*(0, \widetilde{\mathbf{Q}}^H) \le b(X, \mathbf{Q}^H)$ for all $\{X, \mathbf{Q}^H\}$ and $\alpha \in (0,1)$.
  We show below that there exists a $K < \infty$ such that $|V_\alpha^*(X, \mathbf{Q}^H) - V_\alpha^*(0, \widetilde{\mathbf{Q}}^H)| < K$ for all X and $\alpha \in (0,1)$. We show the upper bound. The proof for the lower bound is analogous.
  Starting at state $\{X, \mathbf{Q}^H\}$, we implement a (history-dependent) policy $\widetilde{\pi}$ that sets the action u_t to the smallest feasible value in each period until the reference state is achieved. This means to set u_t to $-\min\{Y_t^+, X_t, R_o\}$ during this period. The period is divided and analyzed in two parts: first is to hit the zero reference storage state and then we do nothing till the reference state $\widetilde{\mathbf{Q}}^H$ is achieved. Note that due to assumption 6 the event of hitting the zero storage reference state happens with probability 1 as the net load has a non-zero probability of being positive. In the event that Q_t is not i.i.d, due to assumption 7, the reference state $\widetilde{\mathbf{Q}}^H$ is reached with probability 1 because of the stationarity assumption of the stochastic process $\mathbf{Q}_t^H$.
  Once this matching to state $\{0, \widetilde{\mathbf{Q}}^H\}$ is achieved, the policy $\widetilde{\pi}$ continues forward with the optimal policy of $V_\alpha^*(0, \widetilde{\mathbf{Q}}^H)$.

To prove the upper bound, it suffices to bound the expected cost till the matching is reached. Therefore, $V_\alpha^*(X,\mathbf{Q}^H) - V_\alpha^*(0,\widetilde{\mathbf{Q}}^H) \leq p_{\max} Y_{\max}^+ (E[\tau_X]+E[\tau_Q])$ where $\tau_X = \min\{k|\sum_{t=1}^k \min\{Y_t^+, R_o\} \geq X\}$ and τ_Q is the time to reach the reference state $\widetilde{\mathbf{Q}}^H$ from any state $\mathbf{Q}^H$. Because of assumption 6 $E[\tau_X]$ is finite. This is because τ_X refers to the number of renewals of a Markovian system (see the initial part of the proof of theorem 4.2 in section 3.4 of [27]). Also, $E[\tau_Q]$ is finite because we are focused on a discrete Markov chain for the exogenous distributions when Q_t is not i.i.d because of assumption 7. Therefore K can be chosen to be greater than $p_{\max} Y_{\max}^+ (E[\tau_X]+E[\tau_Q])$.

- Finally, because the cost-to-go function $V_\alpha(X)$ is convex in X, the sequence $\{\bar{V}_{\alpha_n}(X)\}$ is equicontinuous in X (see remark 5.5.3 in [21]).

$w(X)$ is continuous from part (1) of this theorem. It is also a non-increasing convex function as it is the limit of a sequence of non-increasing convex function. This proves part (4) of this theorem. □

Appendix G
Proof of Theorem 11

Proof. Consider two storage facilities of sizes S and S' where $S \geq S'$. At time 0, we assume without loss of generality that the storage facilities are both empty. Let ω be any instance of the sequence $Q_t = \{p_t, Y_t\}$ for $t = 1, 2, \ldots$. Suppose we implement the optimal storage management policy for S' denoted by $u_t^{S'}$ even for storage with size S. Note that with induction it is easy to see that $X_t^S = X_t^{S'}$ $\forall t$ and that this policy is feasible for the storage of size S because $S \geq S'$, $R_i^S \geq R_i^{S'}$, $R_o^S \geq R_o^{S'}$. Therefore,

$$L_t^{S'}(X_t^{S'}, Q_t, \mathbf{Q}_t^H) = \widetilde{L}_t^S(X_t^S, Q_t, \mathbf{Q}_t^H). \tag{38}$$

We place a tilde on L when storage is S to remember that we are considering a feasible but possibly suboptimal policy for this storage. This implies from Eq. (27) that $g(S') = \widetilde{g}(S) \geq g(S)$. The first (in)equality is because the limits exists under a stationary policy and it is independent of the initial state. The second inequality holds because we switch from a feasible, possibly suboptimal, policy to an optimal policy for the storage with size S. □

Appendix H
Proof of Theorem 12

Proof. Consider three storage facilities of sizes S_1, S_3 and $S_2 = \lambda S_1 + (1-\lambda)S_3$ respectively where $S_1 \geq S_3$ and $\lambda \in [0,1]$.

Let ω be any instance of the sequence $Q_t = \{p_t, Y_t\}$ for $t = 1, 2, \ldots$. We assume without loss of generality that the storage facilities are all empty to begin with. We implement the optimal management strategy for storages of size S_1 and S_3. For S_2 we use a superposition of the two schemes i.e., $u_t^{S_2} = \lambda u_t^{S_1} + (1-\lambda)u_t^{S_3}$ $\forall t$. Using induction it is easy to see that (a) $X_t^{S_2} = \lambda X_t^{S_1} + (1-\lambda)X_t^{S_3}$ $\forall\, t$ and (b) that the proposed control is feasible for a storage of size S_2 because $S_2 = \lambda S_1 + (1-\lambda)S_3$, $R_i^{S_2} = \lambda R_i^{S_1} + (1-\lambda)R_i^{S_3}$ and $R_o^{S_2} = \lambda R_o^{S_1} + (1-\lambda)R_o^{S_3}$. This implies for any t,

$$\lambda \left[Y_t + \frac{u_t^{S_1}}{\beta_t^1}\right]^+ + (1-\lambda)\left[Y_t + \frac{u_t^{S_3}}{\beta_t^3}\right]^+ \geq \left[Y_t + \frac{u_t^{S_2}}{\beta_t^2}\right]^+, \tag{39}$$

where β_t^i equals ρ if the control corresponding to that storage S_i is positive and 1 otherwise. The statement is obvious when $u_t^{S_1}$ and $u_t^{S_3}$ have the same sign and hence $u_t^{S_2}$. On the other hand when $u_t^{S_1} \geq 0$ and $u_t^{S_3} < 0$ or the vice-versa it is not obvious and the reasoning is as follows. Consider the case when $u_t^{S_1} \geq 0$ and $u_t^{S_3} < 0$ (an analogous argument holds in the other case). Here, $\frac{u_t^{S_1}}{\rho} \geq u_t^{S_1}$ and $u_t^{S_3} \geq \frac{u_t^{S_3}}{\rho}$. Therefore, $\lambda\frac{u_t^{S_1}}{\rho} + (1-\lambda)u_t^{S_3} \geq \frac{\lambda u_t^{S_1} + (1-\lambda)u_t^{S_3}}{\rho} = \frac{u_t^{S_2}}{\rho}$ and $\lambda\frac{u_t^{S_1}}{\rho} + (1-\lambda)u_t^{S_3} \geq \lambda u_t^{S_1} + (1-\lambda)u_t^{S_3} = u_t^{S_2}$. Hence, Eq. (39) is true. Substituting in Eq. (26), we get

$$\lambda L_t^{S_1}(X_t^{S_1}, Q_t, \mathbf{Q}_t^H) + (1-\lambda)L_t^{S_3}(X_t^{S_3}, Q_t, \mathbf{Q}_t^H) \geq \widetilde{L}_t^{S_2}(X_t^{S_2}, Q_t, \mathbf{Q}_t^H). \tag{40}$$

We use a tilde for L when the storage size is S_2 to denote the fact that we have used a feasible, possibly suboptimal, policy for S_2. This implies from Eq. (27) that $\lambda g(S_1) + (1-\lambda)g(S_3) \geq \widetilde{g}(S_2) \geq g(S_2)$. The first inequality is because the limits exists under a stationary policy and it is independent of the initial state. The second inequality holds because we switch from a feasible, possibly suboptimal, policy to an optimal policy for the storage with size S_2. □

Appendix I
Proof of Theorem 13

Proof. Let $V_{\mathcal{D}}(S)$ refer to the optimal cost of the storage management problem for a storage of size S for some i.i.d. process $\mathcal{D}$ of net demand under a constant price. The value of storage is defined as $V_{\mathcal{D}}(0) - V_{\mathcal{D}}(S)$. Suppose $\mathcal{D}^* = argmax_{\mathcal{D}}\ V_{\mathcal{D}}(0) - V_{\mathcal{D}}(S)$. We first show that $\mathcal{D}^*$ has the following structure: $P\left(Y_t = \frac{-S}{\rho}\right) = \alpha_{-S},\ \ P(Y_t = S) = \alpha_S$ where $\alpha_{-S} + \alpha_S = 1$. Then, in particular, we show that the maximum value from storage is when $\alpha_{-S} = \alpha_S = \frac{1}{2}$.

We prove this result by contradiction. Suppose there exists one state j different from S or $\frac{-S}{\rho}$ such that $P(Y_t = y) = 1 - \alpha_{-S} - \alpha_S > 0$. Note that without loss of generality it suffices to study the case when $\eta = 1$ because dissipation losses just scale the steady states of the storage level by η. Similarly, it suffices to study the problem in the absence of the ramp constraints as j can be appropriately increased (or decreased when $j > 0$) depended the ramp constraints. *Case 1:* Suppose $y < 0$ and $j = \rho|y|$. Here the interesting states of storage level that will have non-zero probability under the balancing policy are $0, j, 2j, \ldots, nj, S$ where $(n+1)j > S$. It is easy to see that the transition matrix, P has the following structure where $\beta = 1 - \alpha_S - \alpha_{-S}$.

$$P = \begin{pmatrix} \alpha_S & \beta & 0 & \ldots & 0 & \alpha_{-S} \\ \alpha_S & 0 & \beta & \ldots & 0 & \alpha_{-S} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_S & 0 & 0 & \ldots & \beta & \alpha_{-S} \\ \alpha_S & 0 & 0 & \ldots & 0 & 1-\alpha_S \\ \alpha_S & 0 & 0 & \ldots & 0 & 1-\alpha_S \end{pmatrix} \tag{41}$$

Therefore the steady state distribution of storage level when solving $\pi P = \pi$ and $\sum_i \pi_i = 1$ is as follows:

$$\pi_0 = \alpha_S, \quad \pi_S = 1 - \alpha_S\left(\sum_{k=0}^n \beta^k\right), \quad \pi_{kj} = \beta^k \alpha_S$$

where $k \in \{1, \ldots, n\}$.

$$V_{\mathcal{D}}(0) - V_{\mathcal{D}}(S)$$
$$= p\int_Y f_Y(y)\int_{X=0}^S \left(y^+ - (y-x)^+\right) f_X(x)\ dx\ dy \tag{42}$$
$$= p\ P(Y_t = S)\ \int_{X=0}^S x f_X(x)\ dx \tag{43}$$
$$= p\alpha_S\left[\sum_{k=1}^n \beta^k \alpha_S kj + S\left(1 - \alpha_S\left(\sum_{k=0}^n \beta^k\right)\right)\right] \tag{44}$$

$$= p\alpha_S \left[S(1-\alpha_S) + \alpha_S \left(\sum_{k=1}^{n} \beta^k (kj - S) \right) \right] \tag{45}$$

Since $kj \le S \ \forall k$, $\max_{\mathcal{D}} V_{\mathcal{D}}(0) - V_{\mathcal{D}}(S)$ occurs when $\beta = 0$ and in particular, when $\alpha_S = \alpha_{-S} = \frac{1}{2}$.

Case 2: Suppose $j > 0$. Here the interesting states of storage level that will have non-zero probability are $0, S-nj, ..., S-2j, S-j, S$ where $(n+1)j > S$. It is easy to see that the transition matrix, P has the following structure where $\beta = 1 - \alpha_S - \alpha_{-S}$.

$$P = \begin{pmatrix} 1-\alpha_{-S} & 0 & ... & 0 & 0 & \alpha_{-S} \\ 1-\alpha_{-S} & 0 & ... & 0 & 0 & \alpha_{-S} \\ \alpha_S & \beta & ... & 0 & 0 & \alpha_{-S} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \alpha_S & 0 & ... & \beta & 0 & \alpha_{-S} \\ \alpha_S & 0 & ... & 0 & \beta & \alpha_{-S} \end{pmatrix} \tag{46}$$

Therefore the steady state distribution of storage level when solving $\pi P = \pi$ and $\sum_i \pi_i = 1$ is as follows:

$$\pi_S = \alpha_{-S}, \quad \pi_0 = 1 - \alpha_{-S} \left(\sum_{k=0}^{n} \beta^k \right), \ \pi_{S-kj} = \beta^k \alpha_{-S}$$

where $k = \in \{1, ..., n\}$.

$$V_{\mathcal{D}}(0) - V_{\mathcal{D}}(S) = p\beta\alpha_{-S} \left[(S - nj)\beta^n + j \sum_{k=0}^{n-1} \beta^k \right]$$

$$+ p\alpha_S\alpha_{-S} \left[\sum_{k=1}^{n} \beta^k (S - kj) + S \right] \tag{47}$$

$$= p\alpha_{-S} \left[S(1-\alpha_{-S}) - \alpha_{-S} \sum_{k=1}^{n} \beta^k (S - kj) \right] \tag{48}$$

Since $kj \le S \ \forall k$, $\max_{\mathcal{D}} Z_{\mathcal{D}}(0) - Z_{\mathcal{D}}(S)$ occurs when $\beta = 0$ and in particular, when $\alpha_S = \alpha_{-S} = \frac{1}{2}$. □

APPENDIX J
PROOF OF COROLLARY 14

Proof. Consider a storage of size S. Let $Z_{\mathcal{D}}(S)$ refer to the total optimal cost for a storage of size S (i.e., $V_{\mathcal{D}}(S) + cS$) for some distribution $\mathcal{D}$ of the exogenous parameters. Storage is a profitable investment only if $Z_{\mathcal{D}}(0) - Z_{\mathcal{D}}(S) \ge 0$. For the latter to be true, it has to be the case that $\max_{\mathcal{D}} Z_{\mathcal{D}}(0) - Z_{\mathcal{D}}(S) \ge 0 \implies \max_{\mathcal{D}} V_{\mathcal{D}}(0) - V_{\mathcal{D}}(S) - cS \ge 0 \implies \frac{pS}{4} - cS \ge 0$ (from theorem 13) $\implies \frac{c}{p} \le \frac{1}{4}$. □