Optimal modal Fourier transform wave-front Control

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We want fast and smart algorithms

- Large systems and high frame rates require fast algorithms:
  - *Matrix-based methods, which apply the control matrix more quickly*
    - **Local control, sparse matrix methods, conjugate gradient & multigrid**
  - *Fourier transform reconstruction (FTR), which filters the slopes*
- Our controller should use knowledge about the phase aberration and noise to its advantage:
  - *Can use a priori model of atmosphere and noise*
    - **MVU, weighted least-squares**
  - *Can incorporate real-time information about the phase aberration and noise level as determined from telemetry of AO system*

What is modal control?

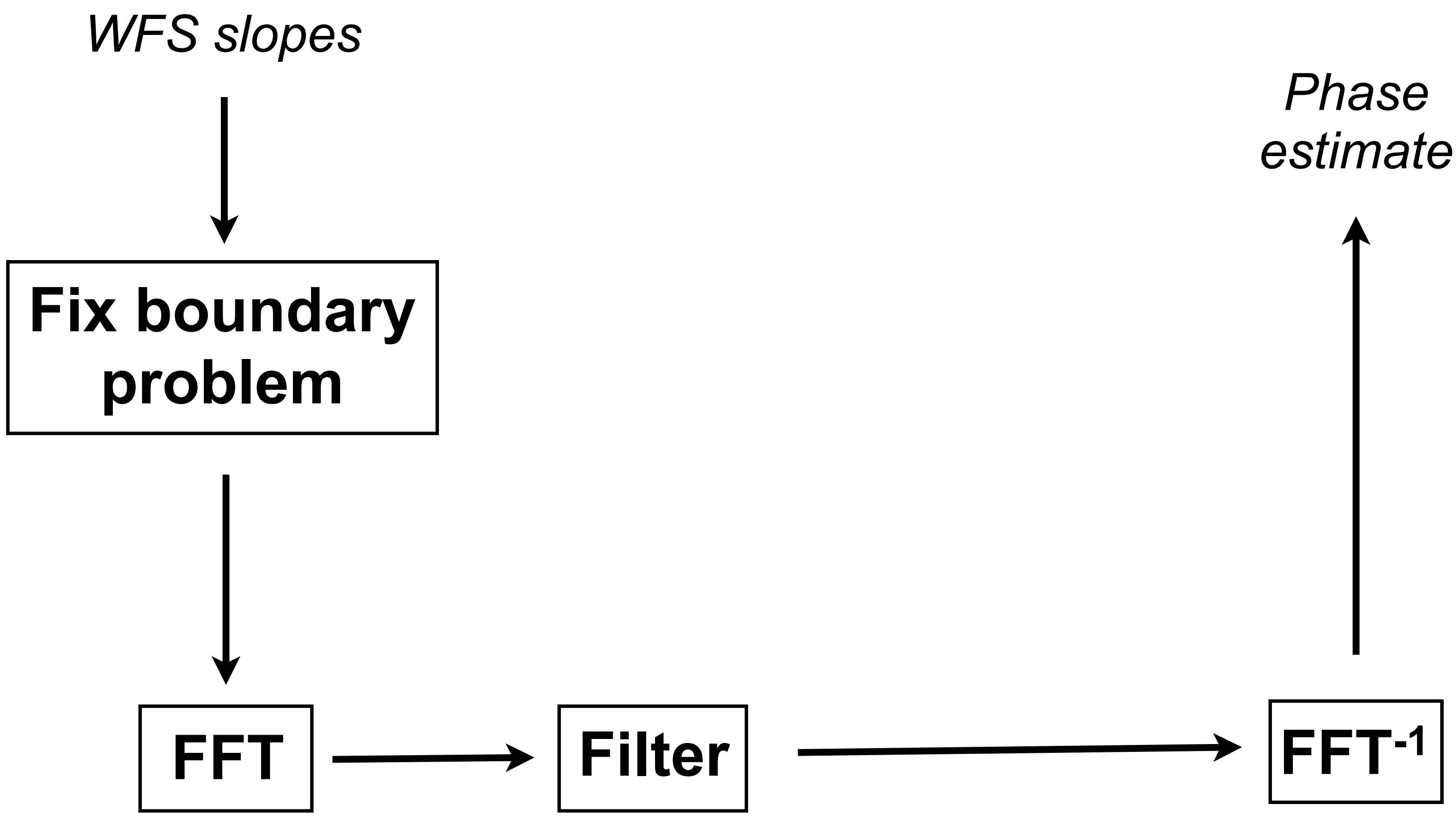
- For n actuators, create an orthonormal basis of n functions
  - *these basis functions are normally chosen to concentrate signal power in few modes*
- Instead of controlling the phase value at each actuator, we control the amount of each mode present
  - *this is expressed as the modal coefficient*
- The AO control problem simplifies to finding the optimal control law for each mode, independently of all the others

- Explicit modal control with optimization is used in Altair and ESO's NAOS

Optimal Fourier Control is our solution

- Our modal set is the Fourier basis. This works even on an **arbitrary aperture**.
- Reconstruction at each time step is with FTR.
- Closed-loop modal coefficients are used to estimate optimal gains for control law for each mode. Gains are implemented as a filter.
- Computationally feasible for 64x64 ExAO right now.
- Extra benefits include
  - *Modal coefficients are available for free, unlike matrix-based modal control, which requires extra computation.*
  - *There is a natural relationship between filter structure and PSF structure.*

FTR works by filtering the slopes



The Fourier basis is the modal set

- FTR uses the DFT on N x N real signals. This leads to a real cosine and sine ONB with N2 modes:
  - *For [k,l] either [0,0], [0,N/2], [N/2,0] or [N/2,N/2], there is only a cosine mode*

$$\mathcal{C}_{k,l}[m,n] = \frac{1}{N}\cos\left(\frac{2\pi}{N}[km+ln]\right)$$

  - *All other frequencies have a sine and cosine*

$$\mathcal{C}_{k,l}[m,n] = \frac{\sqrt{2}}{N}\cos\left(\frac{2\pi}{N}[km+ln]\right)$$

$$\mathcal{S}_{k,l}[m,n] = \frac{\sqrt{2}}{N}\sin\left(\frac{2\pi}{N}[km+ln]\right)$$

Use the DFT to get modal coefficients

- FTR uses the DFT

$$X[k,l] = \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} x[m,n] \exp\left(\frac{-j2\pi(km+ln)}{N} \right)$$

- Modal coefficients are obtainable directly from the DFT

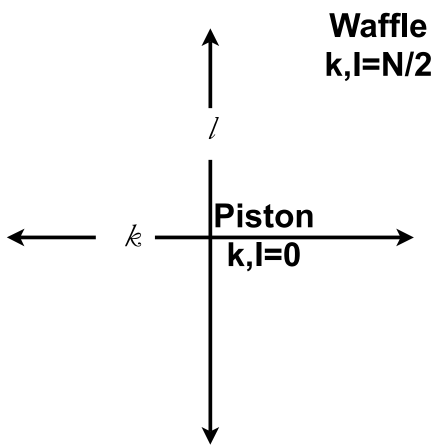
$$< x[m,n], \mathcal{C}_{k,l}[m,n] >= \frac{1}{D_{k,l}} \mathrm{Re}\left\{ X[k,l] \right\}$$

$$< x[m,n], \mathcal{S}_{k,l}[m,n] >= \frac{-1}{D_{k,l}} \mathrm{Im}\left\{ X[k,l] \right\}$$

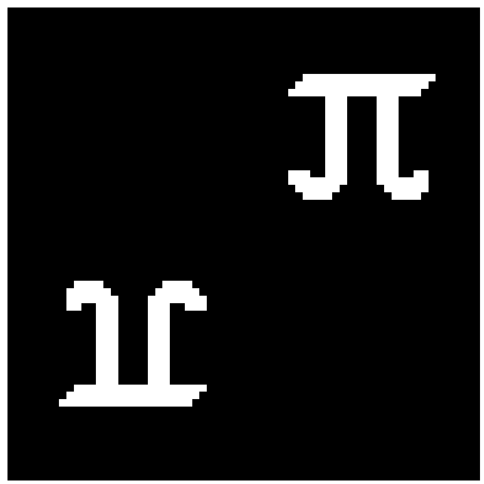
Modes correspond to PSF locations

§ Each Fourier mode lives at a specific spatial frequency pair [k,l]

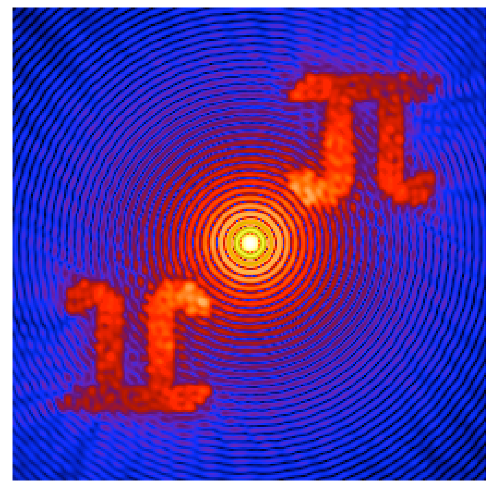
§ Because the PSF is approximately the PSD of the residual phase (to second order), each Fourier mode appears at a specific location in the PSF



Frequency domain



PSD/power at each mode



PSF

Modes are eigenfunctions

- Fourier modes are eigenfunctions of linear, shift-invariant (LSI) systems
  - *The modes for the slopes (on a square aperture) are the same as the modes for the phase*
  - *A cosine of phase at frequency [k,l] produces x- and y-slopes only at the cosine and sine of that frequency [k,l]*
  - *Where Mx[k,l] describes the filter which measures the x-slopes from actuator commands*

phase	**x-slope**
$\mathcal{C}_{k,l}[m,n]$	$A\mathcal{C}_{k,l}[m,n] + B\mathcal{S}_{k,l}[m,n]$
$S_{k,l}[m,n]$	$-B\mathcal{C}_{k,l}[m,n] + A\mathcal{S}_{k,l}[m,n]$

$$A = \mathrm{Re}\{M_x[k,l]\},\ B = -\mathrm{Im}\{M_x[k,l]\}$$

Filter comes from modal responses

§ We simply pseudo-invert the measurement matrix

x-slope cosine and sine
y-slope cosine and sine

$$= \begin{bmatrix} A & -B \\ B & A \\ C & -D \\ D & C \end{bmatrix}$$

phase cosine and sine

§ And obtain the reconstruction matrix for the phase modes as:

$$\frac{1}{A^2+B^2+C^2+D^2}\begin{bmatrix} A & B & C & D \\ -B & A & -D & C \end{bmatrix}$$

§ This produces an extremely sparse modal control matrix

Filter inverts the measurement process

- Filter derived from modal coefficients...

$$\hat{P}[k,l] = \frac{(A+jB)X[k,l] + (C+jD)Y[k,l]}{|(A+jB)|^2 + |(C+jD)|^2}$$

- ... is exactly the same as if we knew the measurement filters

$$\hat{P}[k,l] = \frac{M_x^*[k,l]X[k,l] + M_y^*[k,l]Y[k,l]}{|M_x[k,l]|^2 + |M_y[k,l]|^2}$$

Many filtering options now available

- Best point-based model filter is Modified-Hudgin
- Assuming ideal continuous models for WFS and DM, can derived the Ideal filter
  - *Ideal filter and Mod-Hud very similar*
- Given any LSI AO system or simulation, we can **measure** the coefficients that describe the modal responses and produce a Custom filter
  - *captures influence function response of DM*
- Each filter **exactly reconstructs** given the assumed model, except for invisible modes of piston and sometimes waffle.

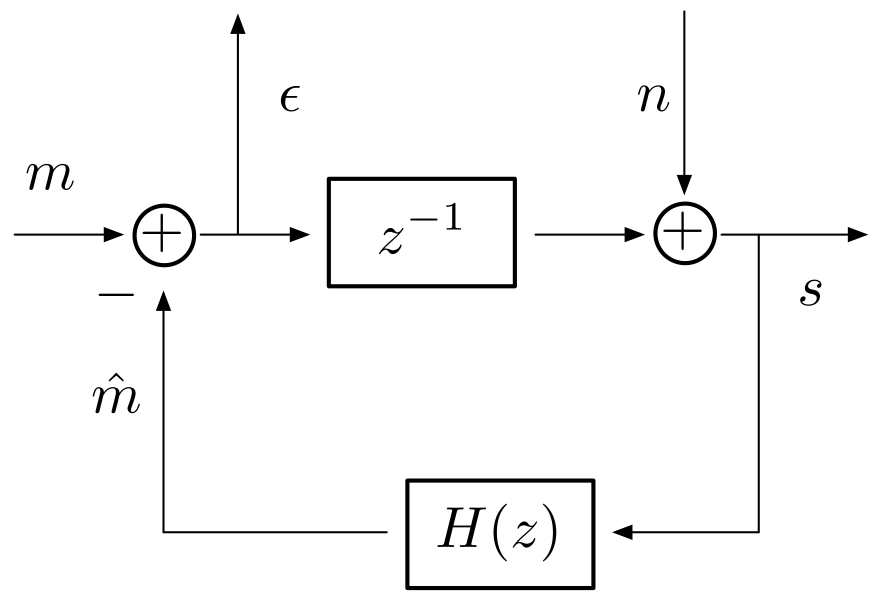
We can do all this with an aperture

- Fourier basis in an arbitrary aperture is a tight Frame that allows analysis and synthesis like an ONB.
- If we window the data, we can use a fast DFT to get modal coefficients.
- New method of slope management called edge correction ensures high-quality coefficient estimation by making outside region of phase flat.

- Result - we get the modal coefficients for free at each time step in the FTR process.

Optimal modal control scheme

- We follow Altair's implementation and assume an approximate model of control system (exact in simulation case) for each of the independent modes.
- We control a mode with feedback in the presence of noise.



Block diagram of control loop for a modal coefficient

Optimize the squared-residual error

- Since the noise at any step is independent of past errors, if we minimize on the measurement s, we minimize on the residual error.
- If we had perfect knowledge we would minimize

$$\mathcal{J} = \int \left| \frac{1}{1 + \exp(-j\omega) H(\omega)} \right|^2 [M(\omega) + N(\omega)] \; d\omega$$

- But we don't... so we have to estimate the open-loop PSD from the closed-loop measurements using

$$\hat{M}(\omega) + \hat{N}(\omega) = |1 + \exp(-j\omega) H_0(\omega)|^2 \, \hat{S}(\omega)$$

Gain estimation for FTR (1)

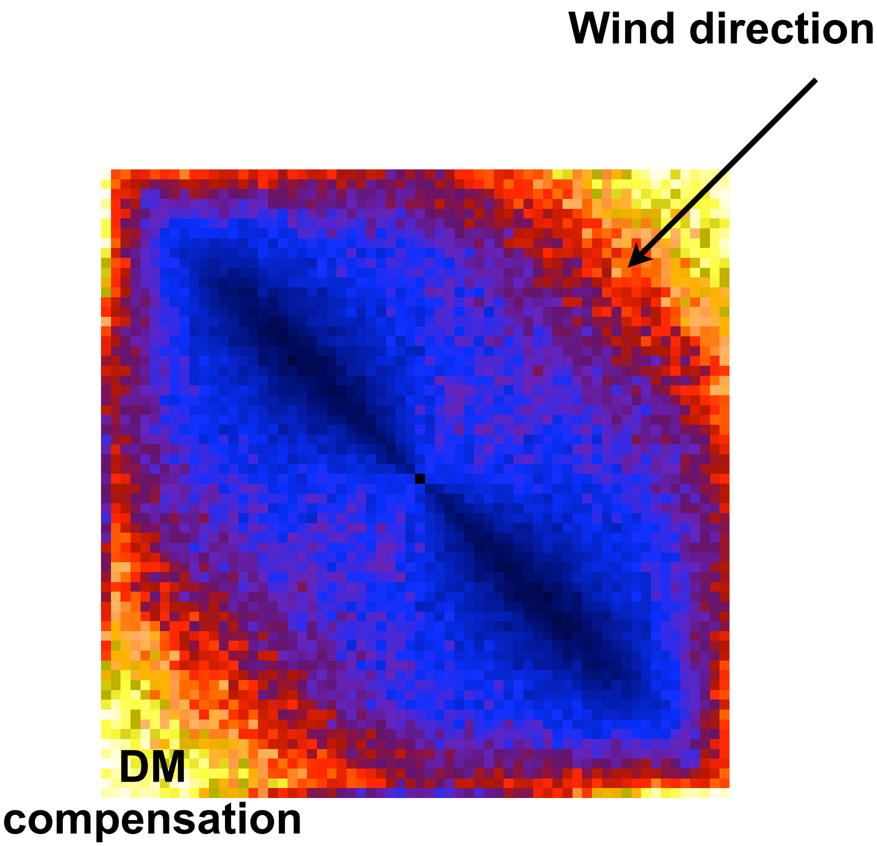
- From closed-loop telemetry, we estimate the closed-loop measurement PSDs
- Convert these to open-loop PSD estimates
- Find the control law which minimizes the error for the sine and cosine modes together

$$\operatorname{argmin} H(z) \left\{ \int \left| \frac{1}{1+\exp(-j\omega)H(\omega)} \right|^2 |1+\exp(-j\omega)H_0(\omega)|^2 \left[\hat{S}_S(\omega) + \hat{S}_C(\omega) \right] \, d\omega \right\}$$

- Where our control law is simple: $H(z) = \dfrac{g}{1 - cz^{-1}}$

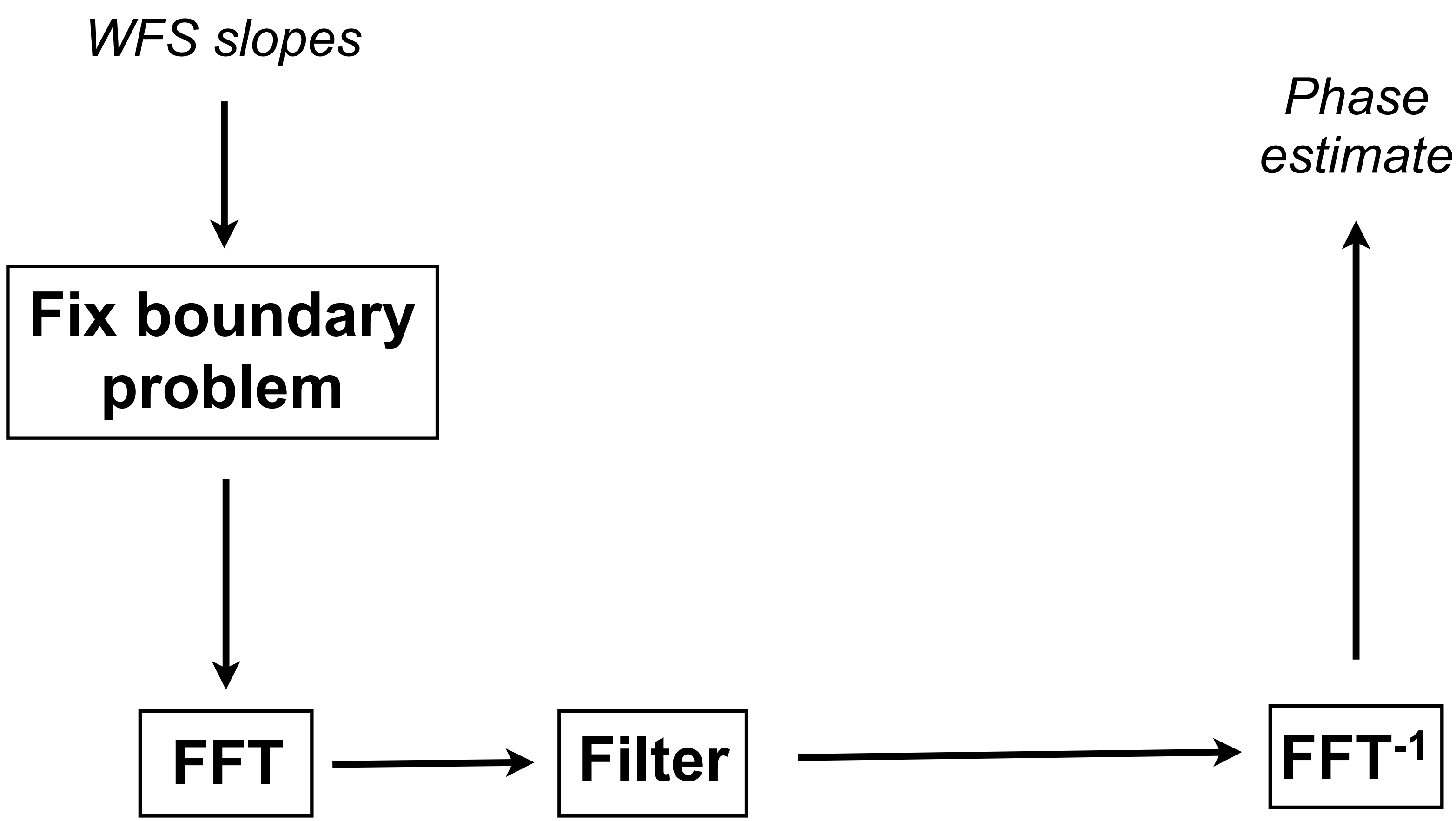
Gain estimation for FTR (2)

- For a single variable (gain g) we can solve the optimization problem efficiently.
- At each frequency [k,l] we have a gain - we construct the filter of these gains using Hermitian symmetry. This filter in then incorporated into the reconstruction filter.



Example filter, N=64

Gains are incorporated into filter



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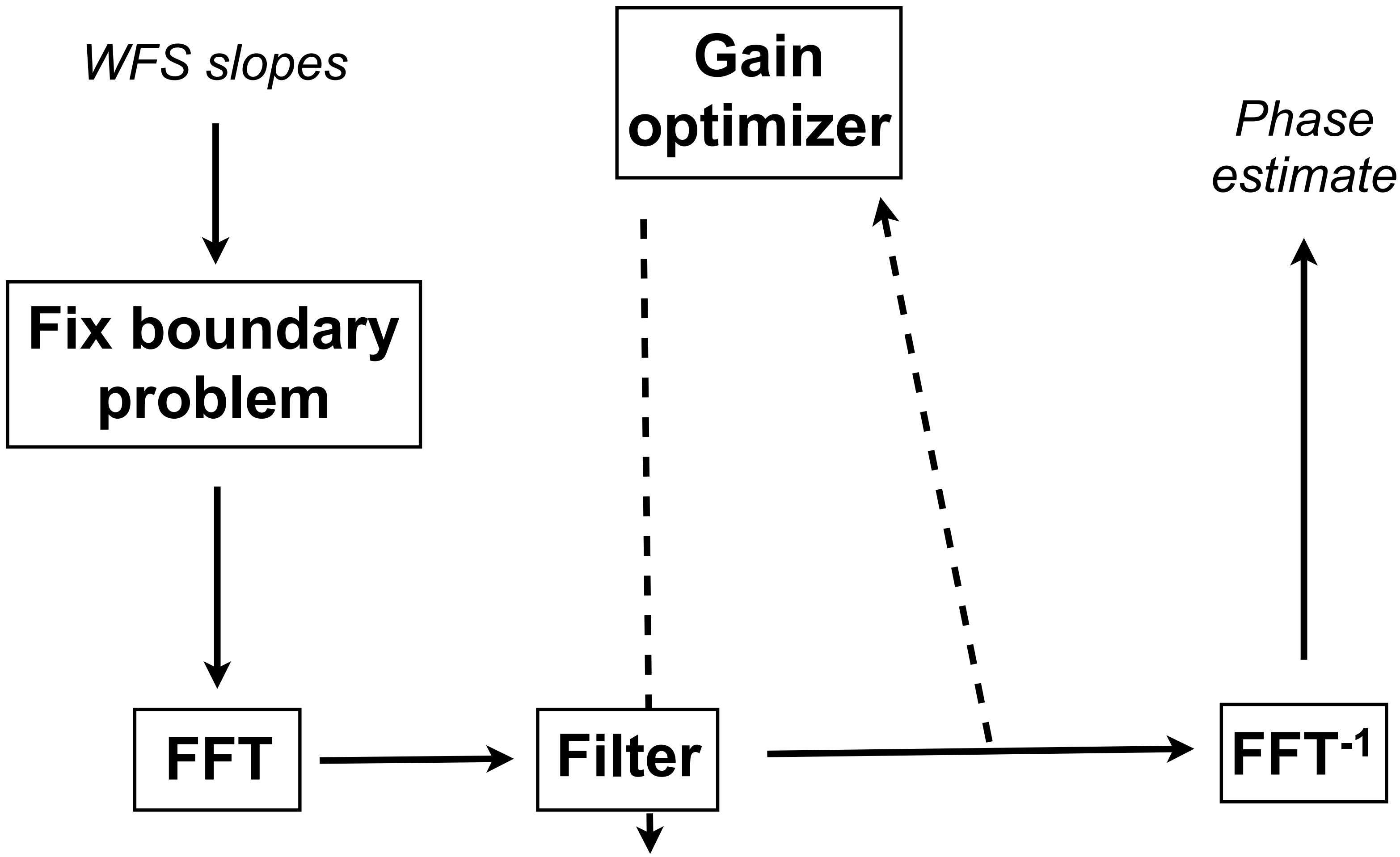
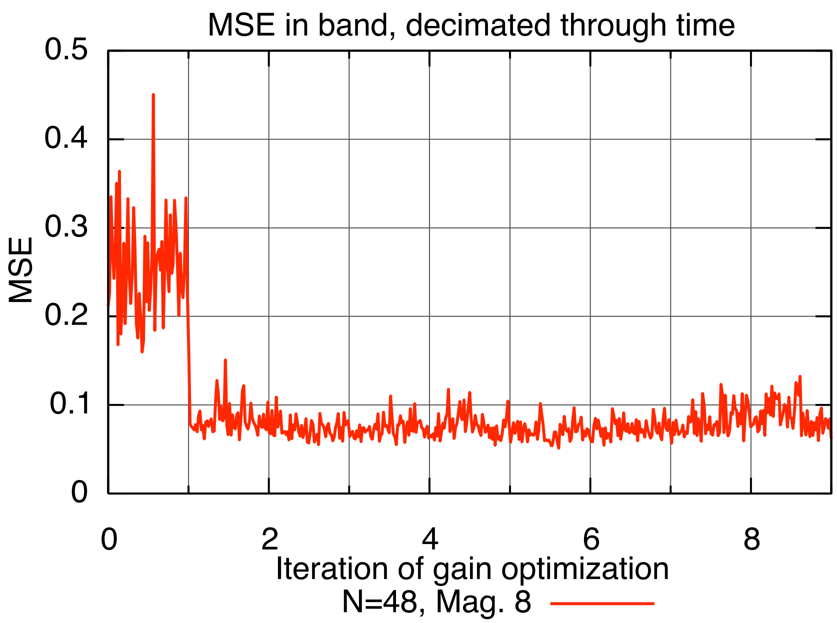


Details of end-to-end ExAOC simulation

- Features of ExAOC simulation include:
  - *Fourier Optics for Spatially-filtered WFS onto CCD, quadcell config*
  - *Altair-based DM model using influence functions*
  - *Input phase aberration is a very long screen shifted at 10m/s*
  - *FTR reconstruction*
  - *Modal coefficients obtained in reconstruction stage*
  - *Gain optimization as describe above*
  - *Full diagnostics including long-exposure PSDs and PSFs from the residual wavefront and instantaneous residual error in different spatial frequency bands*
  - *Run either single long case to watch optimization or many short cases with a specific filter to analyze general case performance*

Significant reduction in residual error

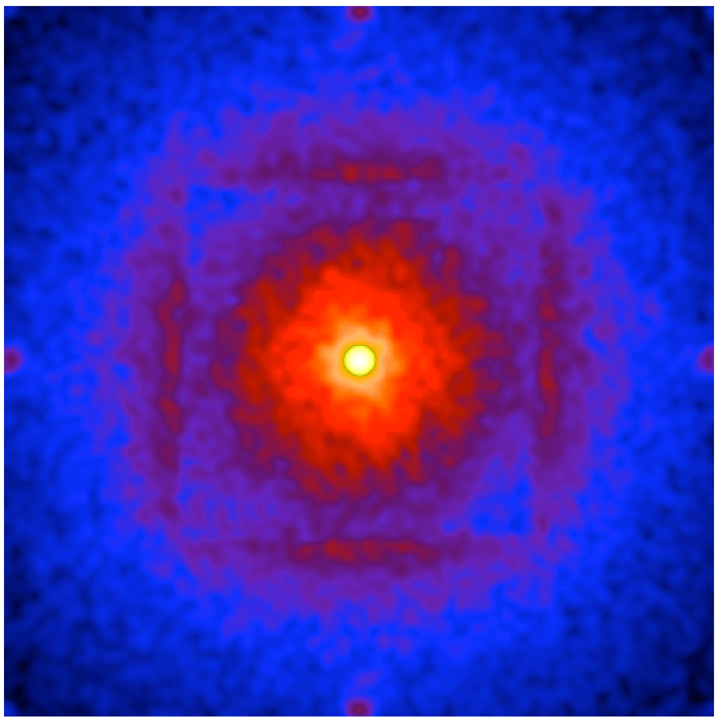


N=48, NGS Mag 8 example for 8 iterations of gain optimization

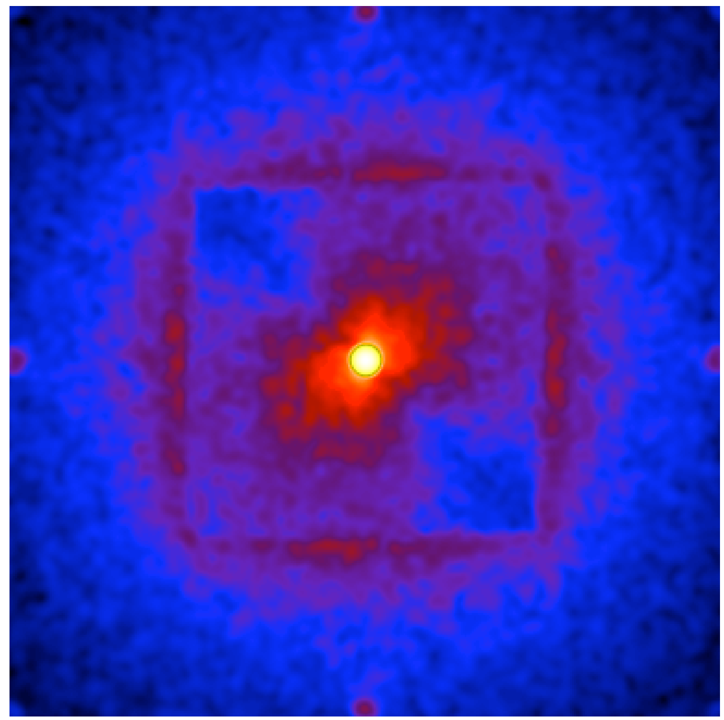
- Use of optimal gains improves performance
  - *significant reduction in residual MSE at each timestep*
  - *less variation in MSE at each timestep*

Comparison shows improvement

- N=48 case with WFS SNR of 2.16
- Strehl increased from 0.75 to 0.87 (+12%)
- MSE in band reduced from 0.224 to 0.074 (3 times less)

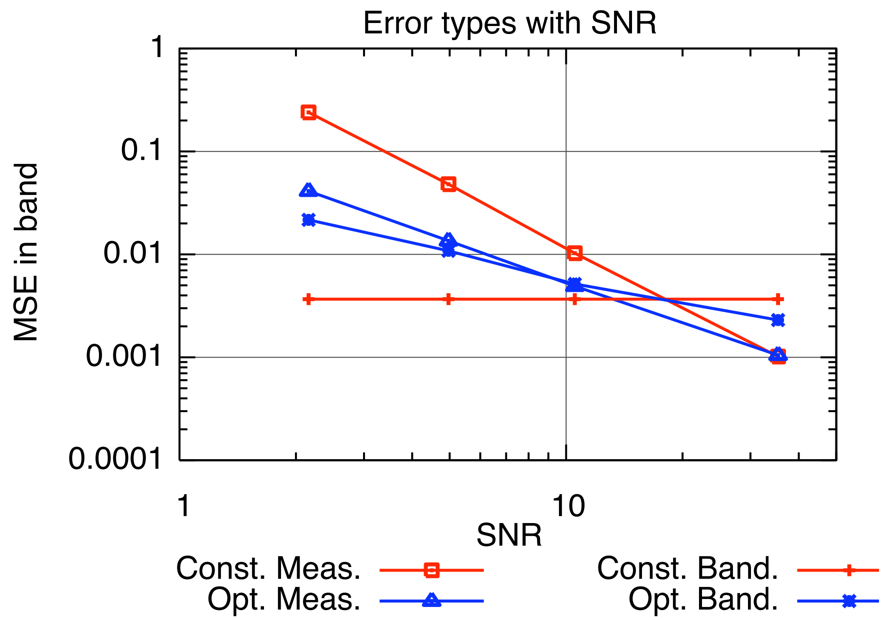


Before



After

Trade bandwidth and sensor errors

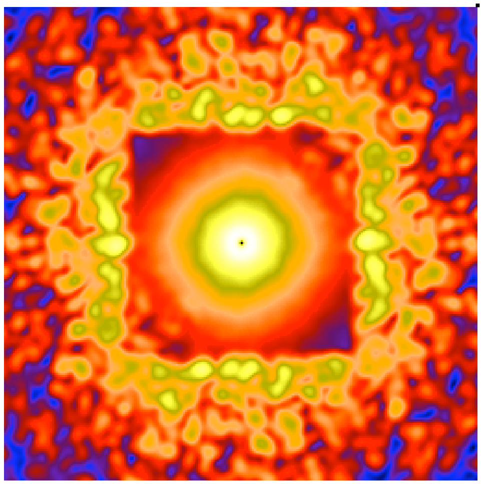


Data for N=48, median over a set of 25 random phase screens

- At high SNRs, optimal gains produce equivalent or more measurement error but less temporal error than before
- At low SNRs, optimal gains produce less measurement error but more temporal error than before

We can control modes independently

- Given an optimal gain profile, we compare three filters
  - *(1) constant gain of 0.6 for all modes*
  - *(2) optimized gains*
  - *(3) constant gains with optimized for a smaller region of filter*
- PSD of case (3) is almost exactly the combination of parts of the other two responses in the right places



Constant **Mix** **Optimal**

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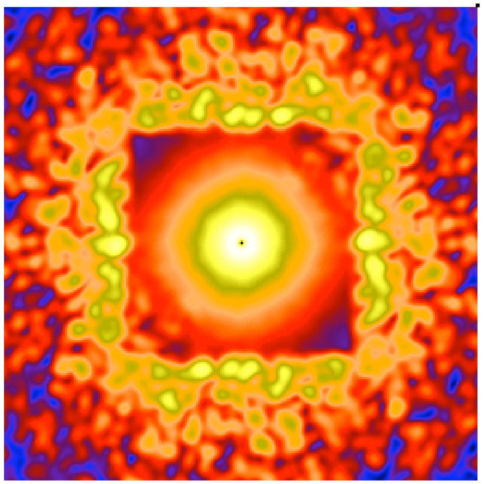
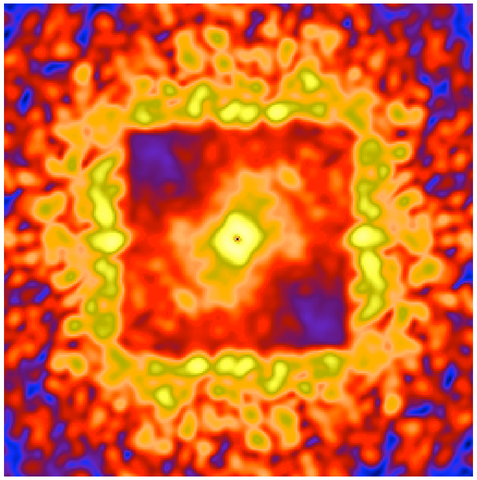


Constant

Mix



Optimal

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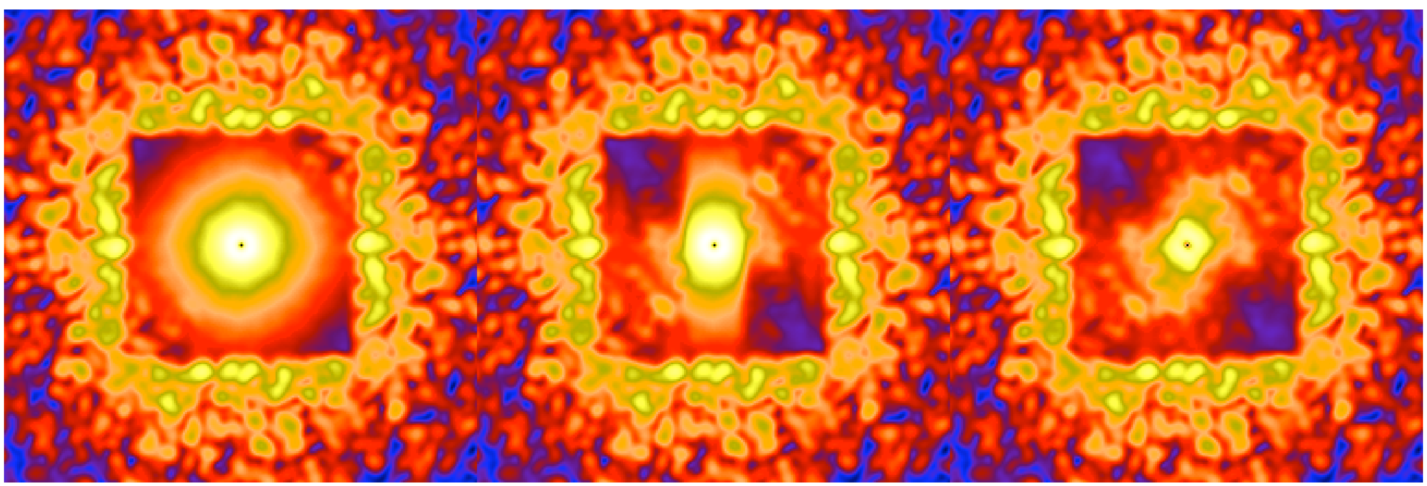
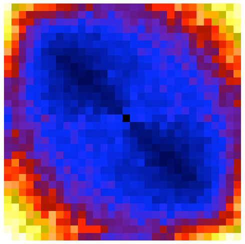


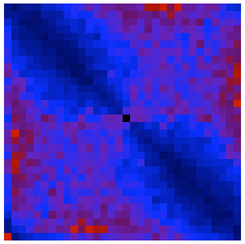
Constant **Mix** **Optimal**

Optimal gains compensate for DM

- Test: run Mod-Hud and Custom filter on same aberration and determine steady-state optimal gains

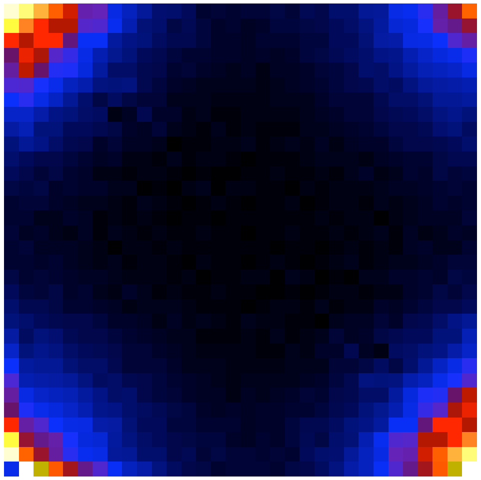


No DM knowledge



DM knowledge

- Ratio of these gains is almost exactly the inversion of the DM response, as determined empirically from AO system models



Inversion of DM influence function

Computational load is satisfiable today

- FTR each timestep: $15N^2 \lg N + 20N^2$
- Estimating periodograms for t steps of telemetry:

$$N^2(5 + 2.5 \lg t)$$

- Averaging the periodograms and finding the optimal gain (k is for evaluations in root-finding):

$$N^2(1 + k) + 4k$$

- Assuming k = 10 (using fast method), a 64x64 system at 2.5k kHz has a maximum load of 1.43 GFLOPs/sec.

Plans for Optimal Fourier Control

- Short term continuation of theory:
  - *Explore complex gain filters and higher-order control laws*
  - *Contribute to ExAOC system design with performance predictions*
  - *Verify system measurement procedure (custom filters) at LAO ExAO testbed*
- Long term experimental verification of performance:
  - *Implement at LAO testbed in ExAOC control system*
- Paper preprint (PDF) available