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Optimal Network Topology Design in Multi-Agent Systems for Efficient Average Consensus

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Abstract— **The problem considered in the present article is optimal design of network topologies in multi-agent systems in order to make communication on the network as efficient as possible for the continuous-time average-consensus protocol. The network design problem can be posed in two different ways. (1) Assuming that the maximum communication cost, i.e. the maximum number of communication links, is known, the goal is to find the network topology which results in the fastest convergence to the consensus (in presence of communication time delays on the links). (2) If a minimum performance of the protocol is required, the design problem can be posed as finding the network with lowest possible communication cost which fulfills the required performance. In both approaches, we formulate the problem of finding the optimal communication graph among a class of directed graphs, *strongly balanced* digraphs, as a *Mixed Integer Semidefinite Program* (MISDP). By solving this MISDP, the optimal graph and the weights on communication links are obtained.**

I. INTRODUCTION

Distributed consensus algorithms have received a lot of attention among researchers in the last few years. This is mainly because of their application in various multi-agent systems, including formation control in robotic systems [9], [15] and flocking [13], cooperation in networked multi-agent systems [14]- [16], distributed sensor fusion and estimation [1]-[6], [8] among other examples.

The average-consensus algorithm mainly developed in [5] has received tremendous attention in recent years due to its applicability and elegance. Authors in [5] have studied the convergence and performance of the average-consensus algorithm under different conditions. It is shown that the convergence speed of the average-consensus protocol is determined by the second smallest eigenvalue of the *Laplacian matrix* of the *mirror graph* of the network which is itself determined by the topology of the network. A natural question that arises is how to choose the weights on communication links, the entries of the *adjacency matrix* of the graph, in order to achieve the best performance. In most cases, the network designer can also determine which agents communicate with each other. Therefore, one can go even further and investigate the best communication topology, i.e. the communication graph as well as the weights on the links, so that the performance of the consensus algorithm is optimized.

Since the development of the average-consensus algorithm, extensive efforts have been made to improve the performance of the algorithm. Authors in [19] consider the problem of fast distributed linear averaging in discrete time. They define the *asymptotic convergence factor* which they maximize by taking the weights on the links as decision variables. They formulate this maximization problem for networks with undirected graphs as a semidefinite program. Authors in [2] show that the convergence speed of the average-consensus protocol may be increased dramatically by adding a few long-range communication links. Authors in [18] use *genetic algorithm* (GA) methods to optimize the long-range link configuration to obtain a small-world network with a faster consensus. In [11] assuming that the weights for a link between two nodes is a function of the distance between the nodes, the authors find the best positional configuration of the nodes in order to maximize the convergence speed of the average-consensus protocol.

In the present article, we consider the network design problem for fast consensus in a general setting. We introduce two approaches to design an efficient network. In the first approach, with a given number of agents, the goal is to find the network topology such that the *communication cost* of the network is less than a given value and the average-consensus protocol converges as fast as possible in presence of communication time-delays on the links. In the second approach, a minimum performance is required for the protocol and the goal is to find the most efficient communication topology, which is a topology with the lowest communication cost, which fulfills the required performance condition. Here, by communication topology, we mean the configuration of the communication graph as well as the weights on the communication links. We formulate both forms of the problem as a *Mixed Integer Semidefinite Program* (MISDP). The resulted MISDP can be used as a powerful network design tool in other ways as well. For instance, consider a case where a network has already been designed. The method presented here can be used to investigate how much the performance of the network can be improved if a number of communication links are added. Depending on the extent of the improvement, the designer may consider slight increase of the number of communication links. Similarly, as will be seen in examples in the last section of this article, in some cases, eliminating some of the communication links does not affect the convergence speed of the protocol. In such cases, the communication cost of the network can be reduced without degrading the performance of the network.

The more general setting of directed graphs is considered. In fact, the case of undirected graphs will be included as a special case. Also, a more interesting case which is the

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case of networks with undirected graphs but non-symmetric weights is included in this setting. To this end, we first generalize a few theorems that are previously proven for undirected graphs to the case of directed graphs in section III; to our knowledge the extension of these results for undirected graphs have not previously been proven in the literature.

The rest of this article is organized as follows: Some background on average-consensus problem is summarized in section II. Specific theorems on directed graphs are proven in section III as extensions of known results. In section IV, we describe two different approaches of the network design problem and formulate the problem as a MISDP. We provide a few numerical examples in section V. Finally, we conclude the paper in section VI.

II. BACKGROUND

We consider a network of n agents with underlying communication graph G. Let $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted directed graph with $\mathcal{V} = \{v_1, \cdots, v_n\}$ the set of n nodes of the graph, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ the set of edges of the graph, and $\mathcal{A} = [a_{ij}]$ the *weighted adjacency matrix* of the graph. If agent i does not communicate with agent j, i.e. $e_{ij} \notin \mathcal{E}$, $a_{ij} = 0$, otherwise a_{ij} is a positive number. An edge of G can be denoted by $e_{ij} = (v_i, v_j)$. The set of indices of the nodes is denoted by $\mathcal{I} = \{1, 2, \cdots, n\}$. The set of *neighbors* of a node v_i is defined as $N_i = \{v_j \in \mathcal{V} | (v_i, v_j) \in \mathcal{E}\}$.

Let $x_i \in \mathbb{R}$ be the state of node v_i which might represent a physical quantity, e.g. position, velocity or the heading angle of agents. A *network* is defined as $G_x = (G, x)$ with $x = [x_1, \cdots, x_n]^T$ where G is called the *topology* of the network and x is called the state (value) of the network. The nodes v_i and v_j are said to *agree* if $x_i = x_j$. We say that a *consensus* has been reached among the nodes of a network if $x_i = x_j$ for all $i, j \in \mathcal{I}, i \neq j$ in which case the common value of all nodes is called the *group decision value*.

A *dynamic network* is a dynamical system with state (G, x) where the value of x evolves in time according to the network dynamics $\dot{x} = F(x, u) = [f(x_1, u_1), \cdots, f(x_1, u_1)]^T$. The *average-consensus problem* is the problem of calculating $\frac{1}{n}\sum_{i=1}^{n} x_i(0)$ in a distributed way, meaning that the input of each node u_i only depends on the states of the node and its neighbors. The state feedback $u_i = k_i(x_{j_1}, \cdots, x_{j_{m_i}})$ is called a *protocol* if we have $\{v_{j_1}, \cdots, v_{j_{m_i}}\} \subseteq \{v_i\} \cup N_i$. A protocol is said to asymptotically solve the average-consensus problem if $\frac{1}{n}\sum_{i=1}^{n} x_i(0)\mathbf{1}$ with $\mathbf{1}$ a 1-by-n vector of ones is an asymptotically stable equilibrium of $\dot{x} = F(x, k(x))$.

For a weighted digraph, the in-degree and the out-degree of node v_i is defined as follows:

$$\deg_{\text{in}}(v_i) = \sum_{j=1}^{n} a_{ji}, \qquad \deg_{\text{out}}(v_i) = \sum_{j=1}^{n} a_{ij} \tag{1}$$

The *degree matrix* of a graph is defined as $\Delta = \text{diag}(\mathcal{A}\mathbf{1})$ or equivalently as a diagonal matrix with the i^{th} diagonal entry being equal to the out-degree of the i^{th} node v_i.

A few definitions and previously-proven theorems that are used in the subsequent sections are stated here. We refer the reader to [5], [3] and [4], for further details.

Definition 1: A directed graph (digraph) is said to be *strongly connected* if there exists a path between any two distinct nodes of the graph.

Definition 2: A digraph is said to be *balanced* if the in-degree of each node is equal to its out-degree.

Consider the following protocol

$$\dot{x}(t) = -Lx(t) \tag{2}$$

where L is the *Laplacian* of the graph G which is defined as $L = \Delta - \mathcal{A}$, or equivalently

$$\dot{x}(t) = u(t) \tag{3}$$

with

$$u_i(t) = \sum_{v_j \in N_i} a_{ij}(x_j(t) - x_i(t)) \tag{4}$$

Theorem 2.1: Let G be a digraph with Laplacian matrix L. Denoting the maximum node out-degree of G by $d_{\max}(G)$, all eigenvalues of L lie in the following disk in the complex plane:

$$\mathcal{D} = \{z \in \mathbb{C} \mid |z - d_{\max}(G)| \leq d_{\max}(G)\} \tag{5}$$

Theorem 2.2: In a network with a directed graph, the above protocol globally asymptotically solves a consensus problem if the graph is strongly connected. With this assumption, this protocol globally asymptotically solves the average-consensus problem if and only if the graph is balanced.

Definition 3: Let $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted digraph. The *mirror* of G denoted by $\hat{G}$ is the underlying undirected graph of G with the adjacency matrix $\hat{\mathcal{A}} = [\hat{a}_{ij}]$ where

$$\hat{a}_{ij} = \hat{a}_{ji} = \frac{a_{ij} + a_{ji}}{2} \tag{6}$$

Theorem 2.3: Let G be a digraph with Laplacian L. Then $L_s = \text{Sym}(L) = (L + L^T)/2$ is the Laplacian matrix of the mirror of G, $\hat{G}$, if and only if G is balanced.

Theorem 2.4: In a network of integrators with a balanced strongly connected digraph, the protocol (2) solves the average-consensus problem globally asymptotically with a speed equal to $\lambda_2(\hat{G})$, the Fiedler eigenvalue of the mirror graph of G. $\lambda_2(\hat{G})$ is also called the *algebraic connectivity* of $\hat{G}$.

Assuming that there is time-delays on communication links, protocol (4) changes to

$$u_i(t) = \sum_{v_j \in N_i} a_{ij}(x_j(t - \tau_{ij}) - x_i(t - \tau_{ij})) \tag{7}$$

where τ_{ij} is the delay on the communication link between v_i and v_j.

Theorem 2.5: In a network of integrators with a fixed, undirected and connected graph and equal time-delay $\tau > 0$ on all links, protocol (7) globally asymptotically solves the average-consensus problem if and only if $\tau < \pi/2\lambda_{\max}(L)$, where $\lambda_{\max}(L)$ is the maximum eigenvalue of L.

Theorem 2.6: If a directed graph G is strongly connected, then $\text{rank}(L) = n-1$ where L is the Laplacian of G and n is the number of nodes of the graph. For the case of undirected graphs, this is a necessary and sufficient condition, i.e. an undirected graph G is connected if and only if $\text{rank}(L) = n-1$.

Definition 4: The *communication cost* of a network is defined as the number of its communication links, i.e.

$$C = \sum_{i,j=1}^{n} \text{sgn}(a_{ij}) \tag{8}$$

sgn denotes the sign function which takes negative numbers to -1, 0 to 0 and positive numbers to 1.

III. Preliminary results

The *in-valency* and *out-valency* of a node in a directed graph is defined as the number of edges ending on the node and starting from the node, respectively. A *weak path* in a directed graph is a sequence of distinct nodes $v_{i_1}, \cdots, v_{i_m}$ such that either (v_{i_k}, v_{i_k+1}) or (v_{i_k+1}, v_{i_k}) belong to the set of edges of the graph for $i = 1, \cdots, m$. A digraph is called *weakly connected* if any two distinct nodes of the graph can be connected by a weak path.

Definition 5: We call a weighted directed graph *strongly balanced* if the graph is balanced and the in-valency of each node of the graph is equal to its out-valency.

The following theorem states that strongly-connectedness is equivalent to weakly-connectedness for the case of strongly balanced graphs. Note that it is necessary that the in-valency and out-valency of each node are equal for the following result to hold. This condition is not satisfied for any balanced graph; in balanced graphs the in-degree and out-degree of each node are equal.

Theorem 3.1: Let G be a digraph such that the in-valency of each node is equal to its out-valency. Then G is strongly connected if and only if it is weakly connected.

Proof: We refer the reader to any book on algebraic graph theory, e.g. [10], for a proof. ■

The following theorem generalizes Theorem 2.6 to the case of directed graphs.

Theorem 3.2: Let G be a strongly balanced digraph. Then G is strongly connected if and only if $\text{rank}(\hat{L}) = n-1$ where $\hat{L}$ is the Laplacian of the mirror graph of G and n is the number of nodes of G.

Proof: It is quite easy to see that a digraph is weakly connected if and only if its mirror graph $\hat{G}$ is connected. From Theorem 2.6, we know that $\hat{G}$ is connected if and only if $\text{rank}(\hat{L}) = n-1$. Finally, according to Theorem 3.1, G is strongly connected if and only if it is weakly connected. ■

The following theorem provides a sufficent condition for convergence of protocol (7) for the case of directed graphs.

Theorem 3.3: In a network of integrators with a fixed, strongly connected and balanced digraph and equal time-delay $\tau > 0$ on all links, protocol (7) globally asymptotically solves the average-consensus problem if $\tau \leq \dfrac{1}{2\text{d}_{\max}(G)}$ where $d_{\max}(G)$ is the maximum degree of the graph G.

Proof: The first part of the proof follows directly from the proof of Theorem 2.5 [5]. Since the time delay on all links are assumed to be equal, we have $\sum_{i=1}^{n} u_i = 0$ which implies that $\text{Ave}(x)$ is invariant under protocol (7). Therefore, we just need to prove that (7) is stable. Since $X(s) = (sI_n + e^{-\tau s}L)^{-1}x(0)$, it suffices to show that all zeros of $Z_\tau(s) = (sI_n + e^{-\tau s}L)$, lie in the left half plan or at the origin for a graph with the prescribed properties. First, note that any eigenvector v of $Z_\tau(s)$ is an eigenvector of L and vice versa. Since the graph is assumed to be strongly connected, L has a simple eigenvalue at the origin. In fact, $s = 0$ in the dirction of v_0 is a zero of $Z_\tau(s)$ where v_0 is an eigenvector corresponding to the eigenvalue of L at the origin. Let us denote any nonzero eigenvalue of L by λ and its corresponding eigenvector by v. For $s \neq 0$ being a zero of $Z_\tau(s)$ in the direction of v, i.e. $Z_\tau(s)v = 0$, we must have

$$\frac{1}{\lambda} + \frac{e^{-\tau s}}{s} = 0 \tag{9}$$

Thus, using the *Nyquist stability criterion*, protocol (7) is stable if the net encirclement of the Nyquist plot of $\Omega(s) = \dfrac{e^{-\tau s}}{s}$ around $-\dfrac{1}{\lambda}$ is zero.

We have

$$\Omega(jv) = \frac{e^{-jv\tau}}{jv} = -\frac{\sin(v\tau)}{v} - j\frac{\cos(v\tau)}{v} \tag{10}$$

Note that $\text{Re}(\Omega(jv)) \geq -\tau$ meaning that the Nyquist plot of $\Omega(s)$ is entirely on the right side of $-\tau$.

According to Theorem 2.1, we know that all eigenvalues of L lie on or inside the disk $\mathcal{D} = \{z = z_r + jz_{im} \in \mathbb{C} \,|\, (z_r - d_{\max}(G))^2 + z_{im}^2 \leq d_{\max}(G)^2\}$ where $d_{\max}(G)$ is the maximum out-degree of the graph G. For any $z = z_r + jz_{im} \in \mathcal{D} \setminus \{0\}$, we have

$$(z_r - d_{\max}(G))^2 + z_{im}^2 \leq d_{\max}(G)^2 \tag{11}$$

$$\iff \quad z_r^2 + z_{im}^2 \leq 2z_r d_{\max}(G) \tag{12}$$

$$\iff \quad \frac{-z_r}{z_r^2 + z_{im}^2} \leq \frac{-1}{2d_{\max}(G)} \tag{13}$$

This implies that the map $f(z) = -\dfrac{1}{z}$ transforms $\mathcal{D} \setminus \{0 + j0\}$ into the half space $\mathcal{H} = \{z = z_r + jz_{im} \in \mathbb{C} \,|\, z_r \leq \dfrac{-1}{2d_{\max}(G)}\}$. Therefore, Re $\left(-\dfrac{1}{\lambda}\right)$ lies in $\mathcal{H}$ for all nonzero eigenvalues λ of L. As a result, if $\tau \leq \dfrac{1}{2\text{d}_{\max}(G)}$, the net encirclement of the Nyquist plot of $\Omega(s) = \dfrac{e^{-\tau s}}{s}$ around $-\dfrac{1}{\lambda}$ is zero and protocol (7) is stable. ■

IV. Optimal network design

A. Network design for fast consensus

Our goal is to design a communication topology for a network with a given number of nodes such that the convergence speed of protocol (7) is maximized while the communication cost of the network is below a predefined value $C_{\max}$ and an equal non-zero time-delay which is less than $\tau_{\max}$ exists on all communication links.

According to Theorems 2.4, 3.2 and 3.3, the network design problem can be formulated as the following optimization program:

$$\max_{L\in\mathbb{R}^{n\times n}} \lambda_2(\hat{L}) \tag{14}$$

$$\text{s.t.} \quad \hat{L} = \frac{L+L^T}{2} \tag{15}$$

$$\hat{L} \succeq 0 \tag{16}$$

$$l_{ij} \le 0 \quad \forall i,j \in \mathcal{I}, i \neq j \tag{17}$$

$$L\mathbf{1} = 0 \tag{18}$$

$$\mathbf{1}^T L = 0 \tag{19}$$

The underlying graph of L is strongly connected. (20)

$$d_{\max}(G) \le \frac{1}{2\tau_{\max}} \tag{21}$$

$$C = \sum_{i,j=1,i\neq j}^{n} \text{sgn}(-l_{ij}) \le C_{\max} \tag{22}$$

where l_{ij} is the entry of L located at the i^{th} row and j^{th} column of L.

In the above optimization program, equation (15) expresses $\hat{L}$ in terms of L, constraints (17) and (18) ensure that L is a legitimate Laplacian matrix. Constraint (19) is imposed to make L correspond to a balanced graph. According to Theorem 3.3, constraints (19), (20) and (21) are needed to ensure the convergence of the average-consensus protocol. Finally, constraint (22) imposes the bound on the communication cost of the network.

Our goal is to transform the above optimization program to a standard convex optimization program which can be solved efficiently using the available solvers. As will be seen later, we transform the above problem to a *Mixed Integer Semidefinite Program* (MISDP).

We start with the constraints. We shall transform all constraints to affine equalities or *Linear Matrix Inequalities* (LMI). In fact, constraints (15), (17), (18) and (19) are already in the desired form. We deal with constraint (20) after constraints (21) and (22). Constraint (21) may be formulated as

$$l_{ii} \le \frac{1}{2\tau_{\max}} \quad \forall i \in \mathcal{I}. \tag{23}$$

In order to transform constraint (22), we prove the following theorem

Theorem 4.1: For each $l_{ij} \le 0, i,j = 1\cdots,n, i\neq j$ introduce a binary variable $\gamma_{ij} \in \{0,1\}$. Assuming an arbitrarily small lower bound $M < 0$ on l_{ij} for $i,j = 1\cdots,n, i\neq j$, the following two sets of expressions are equivalent:

1)

$$l_{ij} \le 0 \quad i,j = 1,\cdots,n, i\neq j \tag{24}$$

$$\sum_{i,j=1,i\neq j}^{n} \text{sgn}(-l_{ij}) \le C_{\max} \tag{25}$$

2)

$$\gamma_{ij} \in \{0,1\} \quad i,j = 1,\cdots,n, i\neq j \tag{26}$$

$$l_{ij} \le 0 \quad i,j = 1,\cdots,n, i\neq j \tag{27}$$

$$l_{ij} < 1-\gamma_{ij} \quad i,j = 1,\cdots,n, i\neq j \tag{28}$$

$$l_{ij} \ge \gamma_{ij} M \quad i,j = 1,\cdots,n, i\neq j \tag{29}$$

$$\sum_{i,j=1,i\neq j}^{n} \gamma_{ij} \le C_{\max} \tag{30}$$

Proof: We first show that equations (26), (27), (28) and (29) state that $l_{ij} = 0$ if and only if $\gamma_{ij} = 0$ for $i,j = 1,\cdots,n, i\neq j$. Suppose $l_{ij} = 0$, then equation (29) implies that $\gamma_{ij} = 0$, and all other inequalities remain valid. If $\gamma_{ij} = 0$, then equations (27) and (29) imply that $l_{ij} = 0$ and other inequalities are valid. Equivalently, we have $l_{ij} < 0 \Leftrightarrow \gamma_{ij} = 1$ for $i,j = 1,\cdots,n, i\neq j$ and this completes the proof. ■

If we restrict the graph to strongly balanced digraphs, we can write constraint (20) in the desired form. We use the boolean variables defined in the above theorem to enforce the condition that the in-valency and out-valency of all nodes of the underlying graph of L are equal. This can be written as

$$\Gamma\mathbf{1} = \Gamma^T\mathbf{1} \tag{31}$$

where $\Gamma \in \{0,1\}^{n\times n}$ is a binary matrix with $\Gamma_{ij} = \gamma_{ij}$ for $i,j = 1,\cdots,n, i\neq j$ and $\Gamma_{ii} = 0$ for $i = 1,\cdots,n$.

By using Theorem 3.2, we can replace constraint (20) by the following condition

$$\text{rank}(\hat{L}) = n-1 \tag{32}$$

Since $\hat{L}$ is a positive semidefinite matrix with one eigenvalue equal to zero, we have the following equivalence:

$$\text{rank}(\hat{L}) = n-1 \Longleftrightarrow \lambda_2(\hat{L}) > 0 \tag{33}$$

Consequently, constraints (20) and (16) can be replaced by conditions (31) and (33).

Performing all transformations so far, the optimization program (14) can be written as the following mixed integer program:

$$\max_{L\in\mathbb{R}^{n\times n},\Gamma\in\{0,1\}^{n\times n}} \lambda_2(\hat{L}) \tag{34}$$

$$\text{s.t.} \quad \hat{L} = \frac{L+L^T}{2} \tag{35}$$

$$l_{ij} \le 0 \quad \forall i,j \in \mathcal{I}, i\neq j \tag{36}$$

$$L\mathbf{1} = 0 \tag{37}$$

$$\mathbf{1}^T L = 0 \tag{38}$$

$$l_{ii} \le \frac{1}{2\tau_{\max}} \quad \forall i \in \mathcal{I} \tag{39}$$
$$\gamma_{ij} \in \{0,1\}\ i,j = 1,\cdots,n, i \ne j \tag{40}$$
$$\gamma_{ii} = 0\ i = 1,\cdots,n \tag{41}$$
$$l_{ij} < 1 - \gamma_{ij}\ i,j = 1,\cdots,n, i \ne j \tag{42}$$
$$l_{ij} \ge \gamma_{ij} M\ i,j = 1,\cdots,n, i \ne j \tag{43}$$
$$\sum_{i,j=1, i\ne j}^{n} \gamma_{ij} \le C_{\max} \tag{44}$$
$$\lambda_2(\hat{L}) > 0 \tag{45}$$
$$\Gamma \mathbf{1} = \Gamma^T \mathbf{1} \tag{46}$$

We are now left with the cost function and constraint (45) which need to be transformed to LMIs.

For the second smallest eigenvalue of the Laplacian of a graph, we can write [10]

$$\lambda_2(\hat{L}) = \min_{z \ne 0} \frac{z^T \hat{L} z}{\|z\|^2} \quad : \ \mathbf{1}^T z = 0 \tag{47}$$

By defining $W = zz^T$, we can write

$$\lambda_2(\hat{L}) = \min_{W \in S^n} \langle W, \hat{L} \rangle \tag{48}$$
$$s.t. \quad W \succeq 0 \tag{49}$$
$$\mathbf{Tr} W = 1 \tag{50}$$
$$W\mathbf{1} = [0]_{n\times 1} \tag{51}$$
$$\text{rank}(W) = 1 \tag{52}$$

where the inner product between two matrices, W and $\hat{L}$ is defined as $\langle W, \hat{L} \rangle = \text{Tr}(\hat{L}W)$. We first relax the rank constraint and form the dual. Then we show that the rank relaxation is exact. After eliminating the rank constraint, i.e. constraint (52), the dual problem can be written as follows

$$\begin{aligned}\mathcal{L}(W, V, \nu_1, \nu_2) =& \langle W, \hat{L} \rangle - \langle W, V \rangle \\ &+ \nu_1(1 - \mathbf{Tr}\, W) + \nu_2^T W \mathbf{1}\end{aligned} \tag{53}$$

where $V \in S_+^n, \nu_1 \in \mathbb{R}, \nu_2 \in \mathbb{R}^n$.

$$g(V, \nu_1, \nu_2) = \min_{W \in S^n} \langle W, \hat{L} - V - \nu_1 I + \nu_2 \mathbf{1}^T \rangle + \nu_1 \tag{54}$$
$$= \begin{cases} \nu_1 & if \quad \hat{L} - V - \nu_1 I + \nu_2 \mathbf{1}^T = 0 \\ -\infty & otherwise \end{cases} \tag{55}$$

Therefore,

$$d^* = \max_{V \in S_+^n, \nu_1 \in \mathbb{R}, \nu_2 \in \mathbb{R}^n} g(V, \nu_1, \nu_2) \tag{56}$$

$$= \max_{\nu_1 \in \mathbb{R}, \nu_2 \in \mathbb{R}^n} \quad \nu_1 \tag{57}$$
$$s.t. \quad (\nu_2)_1 = \cdots = (\nu_2)_n = \alpha \tag{58}$$
$$\nu_2 \mathbf{1}^T - \nu_1 I + \hat{L} \succeq 0 \tag{59}$$

$$= \max_{\nu_1, \alpha \in \mathbb{R}} \quad \nu_1 \tag{60}$$
$$s.t. \quad \alpha \mathbf{1}\mathbf{1}^T - \nu_1 I + \hat{L} \succeq 0 \tag{61}$$

We now show that the rank relaxation, i.e. relaxation of constraint (52), is exact. In other words, eliminating constraint (52) does not change the optimal value.

The primal and the dual problems are both strictly feasible. Hence, according to the Slater's theorem, strong duality holds and the primal and dual problems are both attained by some primal-dual triplet (W^*, ν_1^*, α^*). Therefore, the *Karush-Kuhn-Tucker* (KKT) conditions are necessary and sufficient for the optimal solutions [7]. These conditions are as follows

- Primal feasibility: $W^* \succeq 0$, $\mathbf{Tr}\, W^* = 1$, $W^* \mathbf{1} = [0]_{n\times 1}$
- Dual feasibility: $\alpha^* \mathbf{1}\mathbf{1}^T - \nu_1^* I + \hat{L} \succeq 0$
- Complementary slackness: $(\alpha^* \mathbf{1}\mathbf{1}^T - \nu_1^* I + \hat{L}) W^* = 0$

Suppose W^* is a primal optimal solution. The last KKT condition proves that $(\alpha^* \mathbf{1}\mathbf{1}^T - \nu_1^* I + \hat{L}) W^* = 0$, therefore for any non-zero column w^* of W^* we have: $(\alpha^* \mathbf{1}\mathbf{1}^T - \nu_1^* I + \hat{L}) w^* = 0$. After normalizing w^* we have: $w^*(w^*)^T \succeq 0$, $\mathbf{Tr}\, w^*(w^*)^T = 1$, $w^*(w^*)^T \mathbf{1} = [0]_{n\times 1}$. Hence, $w^*(w^*)^T$ is a primal optimal solution whose rank is equal to 1. This proves the exactness of the rank relaxation.

Finally, the optimization program (34) can be formulated as the following MISDP

Optimization Program.I:

$$\max_{L \in \mathbb{R}^{n\times n}, \Gamma \in \{0,1\}^{n\times n}, \nu, \alpha \in \mathbb{R}} \nu \tag{62}$$
$$\text{s.t.} \quad \hat{L} = \frac{L + L^T}{2} \tag{63}$$
$$l_{ij} \le 0 \quad \forall i,j \in \mathcal{I}, i \ne j \tag{64}$$
$$L\mathbf{1} = 0 \tag{65}$$
$$\mathbf{1}^T L = 0 \tag{66}$$
$$\nu > 0 \tag{67}$$
$$\alpha \mathbf{1}\mathbf{1}^T - \nu I + \hat{L} \succeq 0 \tag{68}$$
$$l_{ii} \le \frac{1}{2\tau_{\max}} \quad \forall i \in \mathcal{I} \tag{69}$$
$$\gamma_{ii} = 0\ i = 1,\cdots,n \tag{70}$$
$$l_{ij} < 1 - \gamma_{ij}\ i,j = 1,\cdots,n, i \ne j \tag{71}$$
$$l_{ij} \ge \gamma_{ij} M\ i,j = 1,\cdots,n, i \ne j \tag{72}$$
$$\Gamma \mathbf{1} = \Gamma^T \mathbf{1} \tag{73}$$
$$\mathbf{1}^T \Gamma \mathbf{1} \le C_{\max} \tag{74}$$

Note that constraint (67) is imposed as a replacement for constraint (45).

A few remarks are in order:

Remark 1: For cases in which some constraints exist on communications between agents due to geometric configuration, hardware capabilities, etc., appropriate constraints may be imposed in the above program. For instance, if agent i may not communicate with agent j because of long distance between them, the constraint $l_{ij} = 0$ can be added to the constraints of the above program to incorporate this condition.

Remark 2: If the communication topology of the network is fixed *a priori*, the weights on the links may be chosen optimally by solving a SDP obtained from Optimization Program I after omitting constraints (70), (71), (73) and (74).

B. Network design for low communication cost

In this section, we adapt the previous results to another situation, in which a minimum performance in terms of the convergence speed of the consensus protocol (7) is desired and the goal is to design a network with the lowest possible communication cost which fulfills the performance requirement.

We assume that the desired performance is given in terms of the Fiedler eigenvalue of the mirror graph, i.e. a network is desired such that $\lambda_2(\hat{L}) \geq \lambda_2^{\min}$ where $\hat{L}$ is the Laplacian of the mirror graph. Note that this assumption is valid since $\lambda_2(\hat{L})$ determines the speed of convergence of the group disagreement.

According to the results of the previous section, this design problem can be formulated as the following mixed integer semidefinite program:

Optimization Program.II:

$$\min_{L\in\mathbb{R}^{n\times n},\Gamma\in\{0,1\}^{n\times n},\nu,\alpha\in\mathbb{R}} \mathbf{1}^T\Gamma\mathbf{1} \tag{75}$$

$$\text{s.t.} \quad \text{Equations} \quad (63)-(66)$$

$$\text{Equations} \quad (68)-(73)$$

$$\nu > \lambda_2^{\min} \tag{76}$$

Note that constraints (68) and (76) guarantee that $\lambda_2(\hat{L}) > \lambda_2^{\min}$.

V. Numerical examples and simulation results

In this section, we provide some numerical examples and simulation results. The optimization programs are solved using YALMIP [12] with SeDuMi [17] as the SDP solver. In all of the following examples, the communication time-delay upper bound is taken as .1 second, i.e. $\tau_{\max} = 0.1$.

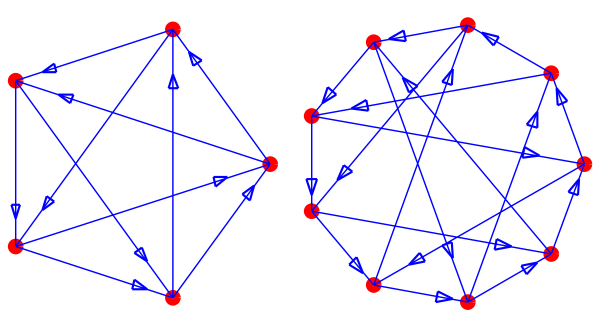

Fig. 1: Optimal graphs, $n = 5$ (left), $n = 9$ (right), corresponding to Example 1, obtained by solving Program.I.

Example 1: In this example, we implement Optimization Program I for two cases, a network of 5 agents with $C_{\max} = 14$, and a network of 9 agents with $C_{\max} = 20$. Figure 1, shows the optimal graphs in both cases. The eigenvalue distribution of the Laplacian matrices of the optimal mirror graphs are shown in Tables I and II. As can be seen in Table I, for the first network, $n = 5$, all eigenvalues of the Laplacian of the optimal mirror graph, except the smallest one which has to be 0, are equal to 6.2500. Furthermore, by solving Optimization Program I for larger $C_{\max}$, we realize that increasing the maximum allowable communication cost does not change the optimal solution. This means that this communication topology provides the best possible performance that one could possibly achieve for the average-consensus algorithm in terms of the convergence speed. On the other hand, note that the optimal graph has only 10 links. This means that adding any number of communication links to the network will not have any effect on the convergence speed of the average-consensus protocol.

In the second network, it can be seen in Figure 1 that the optimal graph has only 18 links although $C_{\max}$ was set to 20. This is due to the fact that we are restricting the feasible set to the case of strongly-balanced graphs for which the in-valency of each node must be equal to its out-valency.

TABLE I: Eigenvalues of the Laplacian of the optimal mirror graph for the first network in Example 1.

$\lambda_1(\hat{L})$	$\lambda_2(\hat{L})$	$\lambda_3(\hat{L})$	$\lambda_4(\hat{L})$	$\lambda_5(\hat{L})$
0.0000	6.2500	6.2500	6.2500	6.2500

TABLE II: Eigenvalues of the Laplacian of the optimal mirror graph for the second network in Example 1.

$\lambda_1(\hat{L})$	$\lambda_2(\hat{L})$	$\lambda_3(\hat{L})$	$\lambda_4(\hat{L})$	$\lambda_5(\hat{L})$
0.0000	4.0672	4.0672	4.0672	4.0672
$\lambda_6(\hat{L})$	$\lambda_7(\hat{L})$	$\lambda_8(\hat{L})$	$\lambda_9(\hat{L})$	
5.6734	5.6734	8.6922	8.6922	

Example 2: In this example, we design the cheapest (in the sense of communication cost) network for a network of 7 agents for different values of $\lambda_2^{\min}$ which is a user defined parameter that determines the minimum required convergence speed of the protocol. Optimization Program II is solved for all cases. Figure 2 shows the optimal graphs for four values of $\lambda_2^{\min}$, 2.5, 4, 4.75, 5. As can be seen in Figure 2, the communication cost of the optimal network is 12, 14, 18 and 21 for these four cases, respectively. The eigenvalue distribution of the Laplacian matrices of the optimal mirror graphs for different values of $\lambda_2^{\min}$ is shown in Table III. Figure 3 shows the communication cost of the optimal network for different values of $\lambda_2^{\min}$. This plot can be very useful in designing network topologies. It can be seen in Figure 3 that improving the network topology for a faster consensus is very costly in terms of communication cost when $\lambda_2^{\min} = 4.5$. In fact, changing $\lambda_2^{\min} = 4.5$ to $\lambda_2^{\min} = 5$ increases the communication cost by 50%. However, the network topology can be improved from $\lambda_2^{\min} = 3$ to $\lambda_2^{\min} = 4.5$ without increasing the communication cost of the network.

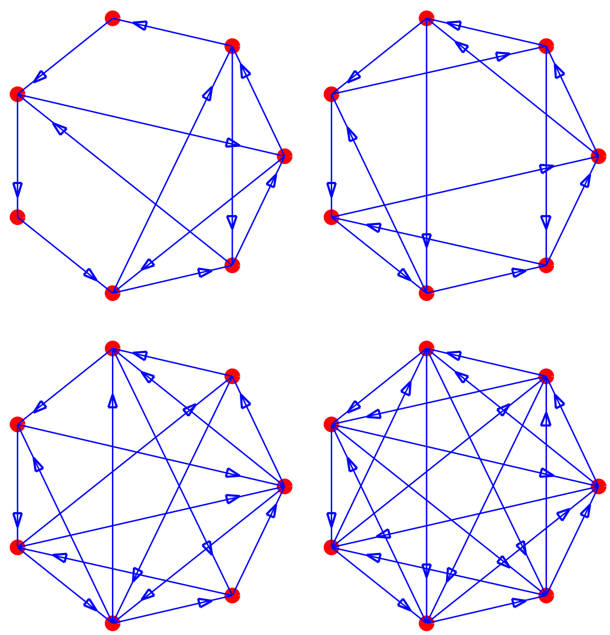

Fig. 2: $\lambda_2(\hat{L}) \geq 2.5$ (top, left), $\lambda_2(\hat{L}) \geq 4$ (top, right), $\lambda_2(\hat{L}) \geq 4.75$ (bottom, left), $\lambda_2(\hat{L}) \geq 5$ (bottom, right), corresponding to Example 2, obtained by solving Program.II.

VI. Conclusion

After generalizing some results proven for undirected graphs to the case of directed graphs, the problem of designing a network with optimal communication topology for efficient average-consensus protocol was considered. We posed the problem in two different ways. One approach was to find the topology which results in the fastest possible average-consensus while the communication cost, i.e. the number of communication links, is less than a given value and the network tolerates communications time delays of smaller than a given bound. In the second approach, under the same conditions, a minimum convergence speed is desired and the design problem is posed as finding the communication topology with the lowest communication cost that provides the desired speed performance. We formulated both design problems as a mixed integer semidefinite program.

The problem was solved for the case of directed graphs and the case of undirected graphs can be considered as a special class. Moreover, in applications where an undirected communication topology is desired, the convergence speed of the protocol can be improved by considering non-symmetric weights on the communication links, i.e. an undirected graph with a non-symmetric adjacency matrix.

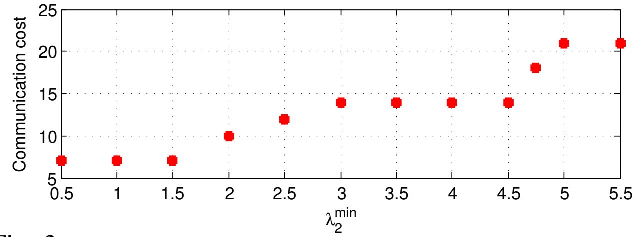

Fig. 3: Communication cost of the optimal graph for different values of $\lambda_2^{\min}$ for the network in Example 2.

TABLE III: Eigenvalues of the Laplacian of the optimal mirror graph for the network in Example 2.

$\lambda_2^{\min}$	$\lambda_2(\hat{L})$	$\lambda_3(\hat{L})$	$\lambda_4(\hat{L})$	$\lambda_5(\hat{L})$	$\lambda_6(\hat{L})$	$\lambda_7(\hat{L})$
1	1.1206	1.1206	3.6387	3.6387	5.6580	5.6580
2	2.0000	2.0000	4.9593	6.6657	7.2069	8.9154
3	3.4497	3.5003	4.6388	4.7704	6.5879	6.8518
4	4.0182	4.0299	4.6207	5.7061	7.6199	7.9180
5	5.4412	5.4473	5.4533	5.4579	5.4616	5.4690
5.7	5.7650	5.7658	5.7674	5.7679	5.7684	5.7703

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