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Josef Dalík

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OPTIMAL-ORDER QUADRATIC INTERPOLATION IN VERTICES OF UNSTRUCTURED TRIANGULATIONS*

Josef Dalík, Brno

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Abstract. We study the problem of Lagrange interpolation of functions of two variables by quadratic polynomials under the condition that nodes of interpolation are vertices of a triangulation. For an extensive class of triangulations we prove that every inner vertex belongs to a local six-tuple of vertices which, used as nodes of interpolation, have the following property: For every smooth function there exists a unique quadratic Lagrange interpolation polynomial and the related local interpolation error is of optimal order. The existence of such six-tuples of vertices is a precondition for a successful application of certain post-processing procedures to the finite-element approximations of the solutions of differential problems.

Keywords: interpolation of functions of two variables, strongly regular classes of triangulations, poised sets of vertices

MSC 2010: 41A05, 41A10, 65D05

1. Introduction

Lagrange interpolation of functions in several variables belongs to the classical topics of numerical analysis. See for example Beresin, Shidkow [3], Prenter [14] or the basic recent results in Liang, Lü, Feng [12], Sauer, Xu [16] and Gasca, Sauer [8].

We denote by (x_1, x_2) the cartesian coordinates of a point $x \in \mathbb{R}^2$ and put

$$D(a,b,c) = \frac{1}{2}\begin{vmatrix} a_1 - c_1 & a_2 - c_2 \\ b_1 - c_1 & b_2 - c_2 \end{vmatrix}$$

for arbitrary points $a, b, c \in \mathbb{R}^2$. It is known that $D(a,b,c) > 0$ if and only if the ordered triple (a,b,c) is oriented positively and $A(\overline{abc}) = |D(a,b,c)|$ is the area of the triangle $\overline{abc}$.

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We denote by $\mathcal{P}^2$ the space of (real) polynomials of total degree less than or equal to two of the (real) variables x_1, x_2. As for every $P \in \mathcal{P}^2$ there exist $\alpha_1, \ldots, \alpha_6$ in $\mathbb{R}$ such that

$$P(x) = \alpha_1 + \alpha_2 x_1 + \alpha_3 x_2 + \alpha_4 (x_1)^2 + \alpha_5 x_1 x_2 + \alpha_6 (x_2)^2, \tag{1}$$

one would expect that interpolants from $\mathcal{P}^2$ are determined by their values in six nodes of interpolation. This is not the case in general.

According to Sauer, Xu [16], we call points $b^1, \ldots, b^6$ *poised* whenever for arbitrary given $p_1, \ldots, p_6 \in \mathbb{R}$ there exists a unique $P \in \mathcal{P}^2$ such that

$$P(b^i) = p_i \quad \text{for } i = 1, \ldots, 6. \tag{2}$$

If we write $P(b^i)$ in the form (1), conditions (2) assume the form

$$M\alpha = p \tag{3}$$

with

$$M = \begin{bmatrix} 1 & b_1^1 & b_2^1 & (b_1^1)^2 & b_1^1 b_2^1 & (b_2^1)^2 \\ 1 & b_1^2 & b_2^2 & (b_1^2)^2 & b_1^2 b_2^2 & (b_2^2)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & b_1^6 & b_2^6 & (b_1^6)^2 & b_1^6 b_2^6 & (b_2^6)^2 \end{bmatrix}, \quad \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_6 \end{bmatrix}, \quad p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_6 \end{bmatrix}.$$

We can see from (3) that the points $b^1, \ldots, b^6$ are poised if and only if the matrix M is non-singular and this is equivalent to the fact that only the trivial linear combination of the columns of M is a zero vector. This means exactly that the points $b^1, \ldots, b^6$ cannot be located on any quadratic curve.

In Section 2 we present a simple construction of a quadratic polynomial $l_1(x)$ for given points $b^1, \ldots, b^6$ such that $l_1(b^i) = 0$ for $i = 2, \ldots, 6$ and formulate the statement $16\, l_1(b^1) = |M|$. From this essential identity we derive basic properties of l_1 and of the related polynomials $l_2, \ldots, l_6$. In Section 3 we denote by $\mathbf{F}$ a strongly regular family of triangulations of a fixed bounded domain $\Omega \subset \mathbb{R}^2$ whose triangles have no obtuse inner angles. For every triangulation $\mathcal{T}_h \in \mathbf{F}$ we describe a simple procedure which selects a five-tuple $b^1, \ldots, b^5$ from the set of neighbours of any given inner vertex $a = b^6$ of $\mathcal{T}_h$ and prove that the set $b^1, \ldots, b^6$ is poised and stable in a certain sense. Analogous result has been proved for the so-called rings of vertices $b^1, \ldots, b^6$ around triangles from $\mathcal{T}_h$ in Dalík [5]. In Section 4 we prove for all the above-mentioned poised sets $\{b^1, \ldots, b^6\}$ that for every function $u \in \mathbf{C}^3(\overline{\Omega})$ the quadratic interpolation polynomial L of u in $b^1, \ldots, b^6$ satisfies the estimates

$|\partial(u-L)^{|m|}/\partial x^m| < C\,h^{3-|m|}$ for all multiindices m with $|m| \leqslant 2$ in a convex local set containing $b^1, \ldots, b^6$. The parameter C depends on the function u only.

According to these error-estimates, the gradient ∇L is an approximation of ∇u with a local error of size $O(h^2)$. As is outlined in Křížek [9], this gives rise to a recovery operator in the sense of Křížek, Neittaanmäki [10], investigated in Durán, Muschietti, Rodríguez [6], Durán, Rodríguez [7], Ainsworth, Craig [1] and in a large amount of recent papers and books. See Ainsworth, Oden [2], Ovall [15] and the references therein.

2. Poised six-tuples of points

We derive the polynomial l_1 in a natural way and present a "geometric characterization" of the determinant $|M|$ in Lemma 1. By this statement, Lemma 2 and Corollaries 1, 2 follow immediately. Let us put

$$Q_0(x) = D(x, b^5, b^6)D(x, b^2, b^3), \quad Q_1(x) = D(x, b^3, b^5)D(x, b^6, b^2)$$

and

$$Q(x) = \alpha Q_0(x) + \beta Q_1(x)$$

for arbitrary points $b^2, \ldots, b^6$ and real numbers α, β. It is easy to see that

$$Q_0(x) = Q_1(x) = Q(x) = 0 \quad \text{for} \quad x = b^2, b^3, b^5, b^6.$$

Setting $\alpha = D(b^4, b^5, b^3)D(b^4, b^6, b^2)$ and $\beta = D(b^4, b^5, b^6)D(b^4, b^2, b^3)$, we get $Q(x) = 0$ for $x = b^4$, too. In this case, we write l_1 instead of Q.

Definition 1. For arbitrary points $b^1, \ldots, b^6 \in \mathbb{R}^2$, we put

$$\begin{aligned} l_1(x) = {} & D(x, b^5, b^6)D(x, b^2, b^3)D(b^4, b^5, b^3)D(b^4, b^6, b^2) \\ & + D(x, b^3, b^5)D(x, b^6, b^2)D(b^4, b^5, b^6)D(b^4, b^2, b^3) \end{aligned}$$

and

$$l(b^1, \ldots, b^6) = l_1(b^1).$$

Properties of the expression $l(b^1, \ldots, b^6)$, formulated in Lemma 2 and in Corollaries 1, 2, can be easily derived from the following basic statement.

Lemma 1. *For arbitrary points $b^1, \ldots, b^6 \in \mathbb{R}^2$ we have*

$$|M| = 16\, l(b^1, \ldots, b^6).$$

P r o o f. This statement has been proved by a symbolic computation using the symbolic algebra system MAPLE. □

We denote by $\operatorname{tr}(i_1, \ldots, i_6)$ the number of transpositions transforming the permutation $(1, \ldots, 6)$ to the permutation $(i_1, \ldots, i_6)$.

Lemma 2. *For arbitrary points $b^1, \ldots, b^6 \in \mathbb{R}^2$ and for every permutation $(i_1, \ldots, i_6)$ of indices $1, \ldots, 6$ we have*

$$l(b^{i_1}, \ldots, b^{i_6}) = (-1)^{\operatorname{tr}(i_1, \ldots, i_6)} l(b^1, \ldots, b^6). \tag{4}$$

We adopt the following convention.

C o n v e n t i o n. For arbitrary points $x^1, \ldots, x^k \in \mathbb{R}^2$, operations $+$ and $-$ on the set $\{1, \ldots, k\}$ of indices mean addition and subtraction modulo k.

Definition 2. For arbitrary points $b^1, \ldots, b^6 \in \mathbb{R}^2$ and for $i = 1, \ldots, 6$ we put

$$l_i(x) = l(x, b^{i+1}, \ldots, b^{i+5}).$$

Corollary 1. *The following statements* a)–c) *are valid for arbitrary points $b^1, \ldots, b^6 \in \mathbb{R}^2$ and index $i \in \{1, \ldots, 6\}$:*

a) $l_i \in \mathcal{P}^2$,

b) $l_i(b^j) = 0$ *for all* $j \neq i$,

c) $l_i(b^i) = (-1)^{i-1} l_1(b^1)$.

Corollary 2. *The following statements* a)–c) *are equivalent for arbitrary points $b^1, \ldots, b^6 \in \mathbb{R}^2$:*

a) $b^1, \ldots, b^6$ *are poised,*

b) $l_i(b^i) \neq 0$ *for some* $i \in \{1, \ldots, 6\}$,

c) $l_i(b^i) \neq 0$ *for all* $i \in \{1, \ldots, 6\}$.

3. Poised six-tuples of vertices

In this section we define our class $\mathbf{F}$ of strongly regular triangulations and discuss the notion of a ring of vertices around a triangle. Then we describe the process of reduction of the set of neighbours of any inner vertex of a triangulation from $\mathbf{F}$ and prove Theorem 2 saying that the result of this process is a poised set satisfying a uniform stability condition.

Definition 3. We denote by $\mathcal{T}_h$ a non-empty finite set of triangles such that the *meshsize* h is the longest length of their sides, by $\mathcal{V}_h$ the set of vertices of triangles from $\mathcal{T}_h$ and put

$$\Omega_h = \bigcup_{T \in \mathcal{T}_h} T.$$

We call $\mathcal{T}_h$ a *triangulation* of Ω whenever Ω is a bounded domain in $\mathbb{R}^2$ and the following conditions a)–c) are satisfied:

a) The intersection of any two different triangles T_1, T_2 from $\mathcal{T}_h$ is either a common side of T_1, T_2 or a common vertex of T_1, T_2 or an empty set.

b) $\mathcal{V}_h \subseteq \overline{\Omega}$ and $\mathcal{V}_h \cap \partial\Omega = \mathcal{V}_h \cap \partial\Omega_h$.

c) The interior of Ω_h is connected.

Definition 4. Let $\mathcal{T}_h$ be a triangulation of Ω and $a \in \mathcal{V}_h$. We say that

$$\mathcal{N}_h(a) = \{b \in \mathcal{V}_h \colon \overline{ab} \text{ is an edge of } \mathcal{T}_h\}$$

is the *set of neighbours* of a and call a an *inner vertex* of $\mathcal{T}_h$ whenever $a \notin \partial\Omega$.

Definition 5. A family $(\mathcal{T}_h)_{h\in I}$ of triangulations of a fixed Ω is called *strongly regular* whenever I is a set of positive meshsizes such that 0 belongs to the closure $\overline{I}$ and there exists a $\nu_0 > 0$ with the property

$$A(T) > \nu_0 h^2 \tag{5}$$

for all $T \in \mathcal{T}_h$ and $h \in I$.

It is easy to see that each triangle from a triangulation belonging to a strongly regular family has all sides longer than $2\nu_0 h$ and all inner angles greater than $\arcsin(2\nu_0)$.

Notation.

1. We denote by $\mathbf{F}$ a strongly regular family of triangulations $\mathcal{T}_h$ with at least six vertices and without obtuse inner angles of triangles.
2. We reserve the symbols $C, \overline{C}, C_0, \overline{C}_0, \dots$ for generic constants independent of the meshsize h.

Definition 6. Let $\mathcal{T}_h \in \mathbf{F}$, $T_1 \in \mathcal{T}_h$ and $b^1, \ldots, b^6 \in \mathcal{V}_h$. We call $b^1, \ldots, b^6$ a *ring* around T_1 if $T_1 = \overline{b^1 b^3 b^5}$ and the triangles

$$T_2 = \overline{b^1 b^2 b^3}, \quad T_3 = \overline{b^3 b^4 b^5}, \quad T_4 = \overline{b^1 b^5 b^6}$$

belong to $\mathcal{T}_h - \{T_1\}$.

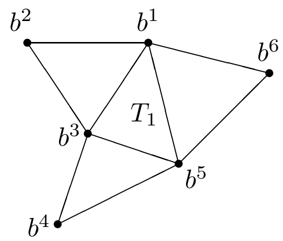

Figure 1.

In Fig. 1, a ring around the triangle T_1 is illustrated. The following theorem and condition (5) say that rings around triangles are poised.

Theorem 1. *There exists a constant $C > 0$ such that for any ring $b^1, \ldots, b^6$ around a triangle $T_1 \in \mathcal{T}_h \in \mathbf{F}$ we can find $k \in \{1, \ldots, 4\}$ satisfying*

$$|l_1(b^1)| > CA(T_k)A(T_2)A(T_3)A(T_4).$$

P r o o f. It is the content of Dalík [5]. $\square$

We prove an analogous statement for rings around inner vertices of triangulations from $\mathbf{F}$.

Definition 7. Let a be an inner vertex of a triangulation $\mathcal{T}_h \in \mathbf{F}$.

a) We call $b^1, \ldots, b^k$ an *orientation* of a set $B \subseteq \mathcal{N}_h(a)$ if $\{b^1, \ldots, b^k\} = B$, $D(a, b^{i-1}, b^i) > 0$ for $i = 1, \ldots, k$ and $\alpha_1 + \ldots + \alpha_k = 2\pi$ for $\alpha_i = \angle b^{i-1} a b^i$. In this case we say that the set B is *oriented* and put $\beta_i = \angle b^i b^{i-1} a$, $\gamma_i = \angle a b^i b^{i-1}$ for $i = 1, \ldots, k$. See Fig. 2.

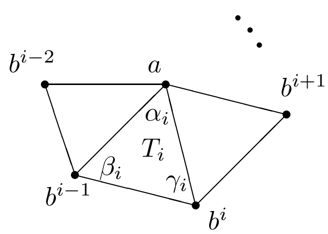

Figure 2.

b) We call $b^1, \dots, b^n, a$ a *ring* (around a in $\mathcal{T}_h$) whenever $b^1, \dots, b^n$ is an orientation of the set $\mathcal{N}_h(a)$. In this case we put $T_1 = \overline{ab^nb^1}$, $T_2 = \overline{ab^1b^2}$, ..., $T_n = \overline{ab^{n-1}b^n}$.

It is easy to see that $T_1, \dots, T_n$ are just the triangles from $\mathcal{T}_h$ with vertex a and, as $\alpha_i \leqslant \pi/2$ for $i = 1, \dots, n$, $n \geqslant 4$.

Definition 8. Let a be an inner vertex of a triangulation $\mathcal{T}_h \in \mathbf{F}$ with n neighbours.

a) In the case $n \geqslant 5$ we say that $b^1, \dots, b^5, a$ is a *reduced ring* (around a in $\mathcal{T}_h$) if $B_5 = \{b^1, \dots, b^5\}$ is an oriented subset of $B_n = \mathcal{N}_h(a)$ such that $B_5 = \mathcal{N}_h(a)$ in the case $n = 5$ and B_5 is a result of the following process of *reduction* in the case $n > 5$: Successively for $k = n, n-1, \dots, 6$, we put $B_{k-1} = B_k - \{b^i\}$ for a vertex $b^i \in \{b^1, \dots, b^k\}$ whenever $b^1, \dots, b^k$ is an orientation of B_k and

$$\alpha_i + \alpha_{i+1} = \min\{\alpha_j + \alpha_{j+1} \colon j = 1, \dots, k\}.$$

In Fig. 3, the process of reduction is illustrated.

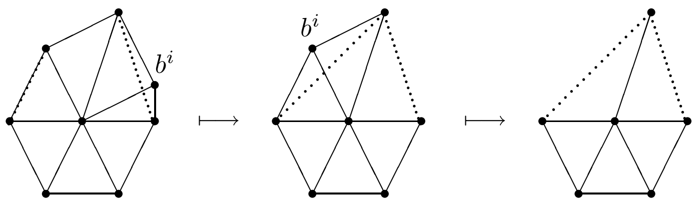

Figure 3.

b) In the case $n = 4$, let $b^1, \dots, b^4, a$ be a ring around a. Because $|\mathcal{V}_h| \geqslant 6$ and the interior of Ω_h is connected, there exists a triangle T_5 in $\mathcal{T}_h$ different from $T_1, \dots, T_4$ whose one side is the segment $\overline{b^1b^2}$, $\overline{b^2b^3}$, $\overline{b^3b^4}$ or $\overline{b^4b^1}$. We choose an orientation $b^1, \dots, b^4$, so that $T_5 = \overline{b^1b^4b^5}$. See Fig. 4. Then we say that $b^1, \dots, b^5, a$ is a *reduced ring* (around a in $\mathcal{T}_h$).

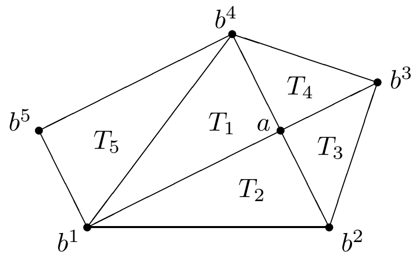

Figure 4.

Lemma 3. *Let a be an inner vertex of a triangulation $\mathcal{T}_h \in \mathbf{F}$ with $n \geqslant 5$ neighbours and let $b^1, \ldots, b^5, a$ be a reduced ring. If*

$$\alpha_{\min} = \min\{\alpha_1, \ldots, \alpha_5\} \quad \textit{and} \quad \alpha_{\max} = \max\{\alpha_1, \ldots, \alpha_5\}$$

then the following statements a)–d) *are valid:*

a) $\max\{\alpha_{\max}, \frac{1}{2}\pi\} \leqslant \alpha_i + \alpha_{i+1}$ *for* $i = 1, \ldots, 5$,

b) $\arcsin(2\nu_0) \leqslant \alpha_{\min}$, $\alpha_{\max} \leqslant \frac{2}{3}\pi$,

c) $\pi < \alpha_i + \alpha_{i+1}$ *for at most one index* i,

d) $\beta_i \leqslant \frac{1}{2}\pi$, $\gamma_i \leqslant \frac{1}{2}\pi$ *for* $i = 1, \ldots, 5$.

Proof. 1. Assume that $n = 5$. As $\alpha_{\max} \leqslant \frac{1}{2}\pi$ and $\alpha_1 + \ldots + \alpha_5 = 2\pi$, we have $\frac{1}{2}\pi \leqslant \alpha_i + \alpha_{i+1} \leqslant \pi$ for $i = 1, \ldots, 5$ and a)–d) follow immediately.

2. In the case $n > 5$, we first prove the following statements i), ii).

i) $\alpha_i + \alpha_{i+1} < \alpha_j \Longrightarrow \alpha_j \leqslant \frac{1}{2}\pi$ for $i, j = 1, \ldots, 5$: If $\alpha_i + \alpha_{i+1} < \alpha_j$, then the angle α_j is not a sum of smaller angles constructed during the process of reduction because the construction of $\alpha_i + \alpha_{i+1}$ would precede the construction of α_j. But then α_j is an inner angle of a triangle from $\mathcal{T}_h$ and we have $\alpha_j \leqslant \frac{1}{2}\pi$.

ii) $\frac{1}{2}\pi \leqslant \alpha_i + \alpha_{i+1}$ for $i = 1, \ldots, 5$: If $\alpha_i + \alpha_{i+1} < \frac{1}{2}\pi$ and $j \notin \{i, i+1\}$ then $\alpha_j \leqslant \alpha_i + \alpha_{i+1} \Longrightarrow \alpha_j < \frac{1}{2}\pi$ obviously and $\alpha_i + \alpha_{i+1} < \alpha_j \Longrightarrow \alpha_j \leqslant \frac{1}{2}\pi$ by i). But then $\alpha_1 + \ldots + \alpha_5 < 2\pi$, a contradiction.

Statement a) follows by i), ii) immediately.

Proof of b): We already know that $\arcsin(2\nu_0) \leqslant \alpha_{\min}$. Let $\alpha_{\max} = \alpha_1$ for unicity. Then $\alpha_1 \leqslant \alpha_2 + \alpha_3$, $\alpha_1 \leqslant \alpha_4 + \alpha_5$ by a) and, as $\alpha_1 + \ldots + \alpha_5 = 2\pi$, we conclude $\alpha_1 \leqslant \frac{2}{3}\pi$.

Proof of c): Assume that $\pi < \alpha_1 + \alpha_2$. Then $\alpha_3 + \alpha_4 < \pi$ and $\alpha_4 + \alpha_5 < \pi$ because $\alpha_3 + \alpha_4 + \alpha_5 = 2\pi - \alpha_1 - \alpha_2 < \pi$. The implications

$$\alpha_5 + \alpha_1 \geqslant \pi \Longrightarrow \alpha_2 + \alpha_3 + \alpha_4 \leqslant \pi < \alpha_1 + \alpha_2 \Longrightarrow \alpha_3 + \alpha_4 < \alpha_1$$

and statement a) lead to $\alpha_5 + \alpha_1 < \pi$. The relation $\alpha_2 + \alpha_3 < \pi$ can be proved analogously.

Proof of d): Let $b^1, \ldots, b^n$ be an orientation of $\mathcal{N}_h(a)$. Then $\beta_i \leqslant \frac{1}{2}\pi$, $\gamma_i \leqslant \frac{1}{2}\pi$ for $i = 1, \ldots, n$, so that the n-gon $\overline{b^1 b^2 \ldots b^n}$ is convex. A successive removal of vertices during reduction preserves convexity and the angles β_i, γ_i do not increase. □

Theorem 2. *There exists a constant $C > 0$ such that*

$$l(a, b^1, \ldots, b^5) > Ch^8$$

for all reduced rings $b^1, \ldots, b^5, a$ in triangulations $\mathcal{T}_h \in \mathbf{F}$.

P r o o f. For brevity, we write $D(abc)$ instead of $D(a,b,c)$ in this proof. Let $b^1,\dots,b^5$, a be a reduced ring around an inner vertex a in a triangulation $\mathcal{T}_h \in \mathbf{F}$ with n neighbours. We first assume that $n \geqslant 5$. The value of

$$l = l(a,b^1,\dots,b^5) = D(ab^4b^5)D(ab^1b^2)D(b^3b^4b^2)D(b^3b^5b^1) + D(ab^2b^4)D(ab^5b^1)D(b^3b^4b^5)D(b^3b^1b^2)$$

does not depend on the choice of orientation $b^1,\dots,b^5$ due to (4). According to Lemma 3 c), we can choose such an orientation that

$$\alpha_3+\alpha_4 \leqslant \pi, \quad \alpha_5+\alpha_1 \leqslant \pi \quad \text{and} \quad \alpha_1+\alpha_2 \leqslant \pi.$$

The inequalities $\alpha_3+\alpha_4 \leqslant \pi$ and $\beta_4+\gamma_3 \leqslant \pi$, see Lemma 3 d), imply

$$D(ab^2b^4) + D(b^3b^4b^2) = D(ab^2b^3) + D(ab^3b^4) \tag{6}$$

and

$$D(ab^2b^4) \geqslant 0, \quad D(b^3b^4b^2) \geqslant 0. \tag{7}$$

As $D(ab^5b^1) > D(b^3b^5b^1)$ implies $\alpha_2+\alpha_3 < \gamma_1$ or $\alpha_4+\alpha_5 < \beta_1$ and these conclusions are in contradiction to Lemma 3 a), d), we have

$$D(ab^5b^1) \leqslant D(b^3b^5b^1). \tag{8}$$

The convexity of the pentagon $\overline{b^1b^2b^3b^4b^5}$ and (8) give us

$$D(b^3b^4b^5) \geqslant 0, \quad D(b^3b^1b^2) \geqslant 0, \quad \text{and} \quad D(b^3b^5b^1) \geqslant 0. \tag{9}$$

As the segments $\overline{ab^1},\dots,\overline{ab^5}$ are sides of triangles from $\mathcal{T}_h$, $|ab^i| \geqslant 2\nu_0 h$ for $i = 1,\dots,5$. These inequalities and Lemma 3 b) say that

$$D(ab^{i-1}b^i) > 4\nu_0^3h^2 \quad \text{for } i = 1,\dots,5. \tag{10}$$

If $D(ab^2b^4) \leqslant D(b^3b^4b^2)$ then the second term of l is non-negative due to (10), (7), (9) and, after omitting it, we obtain

$$l \geqslant D(ab^4b^5)D(ab^1b^2)\frac{1}{2}[D(ab^2b^3)+D(ab^3b^4)]D(ab^5b^1) \geqslant C\,h^8$$

by (6), (8) and (10).

In the case $D(b^3b^4b^2) < D(ab^2b^4)$, the valid inequalities $\alpha_1 + \alpha_2 \leqslant \pi$, $\alpha_5 + \alpha_1 \leqslant \pi$ lead to

$$D(b^3b^4b^2) < D(ab^3b^4) \Longrightarrow D(ab^3b^4) < D(b^5b^3b^4), \tag{11}$$

$$D(b^3b^4b^2) < D(ab^2b^3) \Longrightarrow D(ab^2b^3) < D(b^1b^2b^3). \tag{12}$$

If either $D(b^3b^4b^2) \geqslant D(ab^3b^4)$ or $D(b^3b^4b^2) \geqslant D(ab^2b^3)$ then the second term of l is non-negative due to (9), (10). After its omission, we obtain

$$l \geqslant D(ab^4b^5)D(ab^1b^2)\min\{D(ab^3b^4), D(ab^2b^3)\}D(ab^5b^1) > Ch^8$$

by (8) and (10). If both the assumptions in (11), (12) are valid then we omit the first summand from l (it is non-negative due to (10), (7), (9)) and obtain

$$l \geqslant \frac{1}{2}[D(ab^2b^3) + D(ab^3b^4)]D(ab^5b^1)D(ab^3b^4)D(ab^2b^3) > Ch^8$$

according to (6), (11) and (12).

If $n = 4$ and $b^1, \ldots, b^5, a$ is a reduced ring from Definition 8 b) then, as is illustrated in Fig. 4, vertices $b^1, b^2, a, b^3, b^4, b^5$ create a ring around the triangle $T_1 \in \mathcal{T}_h$. The statement follows by Theorem 1 and by (5). □

4. Quadratic interpolation in poised six-tuples of vertices

We prove local uniform optimal-order error-estimates of interpolation of functions from $\mathbf{C}^3(\overline{\Omega})$ by quadratic polynomials in the poised sets from Theorems 1 and 2. These are generalizations of the estimates from Dalík [4].

Definition 9. Let $\mathcal{T}_h \in \mathbf{F}$ and let $b^1, \ldots, b^6$ be either

a) a ring around a triangle from $\mathcal{T}_h$ or

b) a reduced ring around an inner vertex $a = b^6$ in $\mathcal{T}_h$.

Then we call $\{b^1, \ldots, b^6\}$ a *local poised set*. We put $B = \{b^1, \ldots, b^6\}$ in the case a), $B = \{a\} \cup \mathcal{N}_h(a)$ in the case b) and call the set

$$\mathcal{E}(b^1, \ldots, b^6) = \{x \in \mathcal{V}_h \colon \overline{xyz} \in \mathcal{T}_h \text{ for some } y, z \in B\}$$

an *extension* of $\{b^1, \ldots, b^6\}$. For every nonempty set $E \subseteq \mathcal{V}_h$ we denote by $\operatorname{conv}(E)$ the convex closure of E. Instead of $\operatorname{conv}(\mathcal{E}(b^1, \ldots, b^6))$ we briefly write $\operatorname{conv}(\mathcal{E})$.

For any local poised set $\{b^1, \ldots, b^6\}$ we approximate functions $u \in \mathbf{C}^3(\overline{\Omega})$ by quadratic interpolation polynomials in the nodes $b^1, \ldots, b^6$ and estimate the local interpolation error on the set $\operatorname{conv}(\mathcal{E})$. Fig. 5 illustrates the fact that $\operatorname{conv}(\mathcal{E}) \not\subseteq \overline{\Omega}$ may occur. In this case we take an open ball Ω_e such that $\overline{\Omega} \subset \Omega_e$. Obviously, $\operatorname{conv}(\mathcal{E}) \subseteq \Omega_e$ for all local poised sets. Due to the Whitney Theorem, see Theorem 1.8.10 in Kufner, John, Fučík [11], each function $u \in \mathbf{C}^3(\overline{\Omega})$ has an extension $U \in \mathbf{C}^3(\overline{\Omega}_e)$ and we identify u with its extension U on $\overline{\Omega}_e$. In this sense we guarantee that functions $u \in C^3(\overline{\Omega})$ belong to $C^3(\operatorname{conv}(\mathcal{E}))$ for all poised sets.

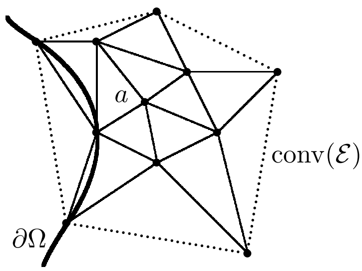

Figure 5.

Definition 10. We relate the *Lagrange basis functions*

$$L_i(x) = \frac{l_i(x)}{l_i(b^i)} \quad \text{for } i = 1, \ldots, 6$$

to each local poised set $\{b^1, \ldots, b^6\}$ in $\mathcal{T}_h \in \mathbf{F}$. Then

$$L(x) = \sum_{i=1}^{6} u(b^i) L_i(x)$$

is the *Lagrange interpolation polynomial* of a function $u \in \mathbf{C}(\overline{\Omega})$ at the points $b^1, \ldots, b^6$.

If $\{b^1, \ldots, b^6\}$ is a local poised set in $\mathcal{T}_h \in \mathbf{F}$ then $|l_i(b^i)| \geqslant Ch^8$ for $i = 1, \ldots, 6$ by Theorems 1, 2 and assumption (5). Moreover, $|l_i(x)| \leqslant \overline{C}_1 h^8$ and $|\partial l_i / \partial x_\iota(x)| \leqslant \overline{C}_2 h^7$ for all $x \in \operatorname{conv}(\mathcal{E})$ and $\iota = 1, 2$ are obvious. Hence the following estimates are valid.

Lemma 4. *There exists a constant $\nu_1 > 0$ such that*

$$|L_i(x)| \leqslant \nu_1, \quad \Big|\frac{\partial L_i}{\partial x_\iota}(x)\Big| \leqslant \nu_1 h^{-1} \tag{13}$$

for all triangulations $\mathcal{T}_h \in \mathbf{F}$, all local poised sets $\{b^1, \ldots, b^6\}$ in $\mathcal{T}_h$, all $x \in \operatorname{conv}(\mathcal{E})$, $i = 1, \ldots, 6$ and $\iota = 1, 2$.

Lemma 5. *Assume that $\{b^1, \ldots, b^6\}$ is a local poised set in $\mathcal{T}_h \in \mathbf{F}$ and $P \in \mathcal{P}^2$ satisfies*

$$|P(b^i)| \leqslant ch^3 \quad \text{for } i = 1, \ldots, 6$$

for some $c \geqslant 0$. Then

$$|P(x)| \leqslant 6\nu_1 ch^3 \quad \forall\, x \in \operatorname{conv}(\mathcal{E}).$$

Proof. $P(x) = \sum_{i=1}^{6} P(b^i)L_i(x)$ and (13) yield the statement. □

Lemma 6. *For every function $u \in \mathbf{C}^3(\overline{\Omega})$ and every $C_1 > 0$ there exists $C_2 > 0$ such that*

$$\Big|\frac{\partial^{|m|}(u-P)}{\partial x^m}(x)\Big| \leqslant C_2 h^{3-|m|} \quad \forall\, x \in \operatorname{conv}(\mathcal{E})$$

for all multiindices m, $|m| \leqslant 2$, all poised sets $\{b^1, \ldots, b^6\}$ in $\mathcal{T}_h \in \mathbf{F}$ and all $P \in \mathcal{P}^2$ satisfying $|(u-P)(x)| < C_1 h^3$ in $\operatorname{conv}(\mathcal{E})$.

Proof. Let us consider $u \in \mathbf{C}^3(\overline{\Omega})$, a local poised set $\{b^1, \ldots, b^6\}$ in $\mathcal{T}_h \in \mathbf{F}$ and $P \in \mathcal{P}^2$ satisfying $|(u-P)(x)| < C_1 h^3$ in $\operatorname{conv}(\mathcal{E})$. Let T be the second degree Taylor polynomial of u at a point $y \in \operatorname{conv}(\mathcal{E})$. Then for every multiindex m with $|m| \leqslant 2$, $\partial^{|m|}T/\partial x^m$ is a Taylor polynomial of $\partial^{|m|}u/\partial x^m$ at point y of degree $2 - |m|$ and

$$\Big|\frac{\partial^{|m|}(u-T)}{\partial x^m}(x)\Big| < \overline{C}_2 h^{3-|m|} \quad \forall\, x \in \operatorname{conv}(\mathcal{E}) \tag{14}$$

for $\overline{C}_2$ depending on u only. This result for $|m| = 0$ and our assumption give $|(T-P)(x)| < (C_1 + \overline{C}_2)h^3$ for all $x \in \operatorname{conv}(\mathcal{E})$. As $T - P \in \mathcal{P}^2$, $\operatorname{conv}(\mathcal{E})$ is a convex compact domain in $\mathbb{R}^2$ whose width corresponds to h, we obtain

$$\Big|\frac{\partial^{|m|}(T-P)}{\partial x^m}(x)\Big| \leqslant \overline{C}_3 h^{3-|m|} \quad \forall\, x \in \operatorname{conv}(\mathcal{E}) \tag{15}$$

for all m, $|m| \leqslant 2$ by the generalization from Wilhelmsen [17] of Markov's inequality published in Markov [13] originally. Our conclusions (14), (15) yield the statement for $C_2 = \overline{C}_2 + \overline{C}_3$. □

Theorem 3. *For every function* $u \in \mathbf{C}^3(\overline{\Omega})$ *there exists a constant* $C > 0$ *such that*

$$\left|\frac{\partial^{|m|}(u-L)}{\partial x^m}(x)\right| \leqslant Ch^{3-|m|} \quad \forall\, x \in \operatorname{conv}(\mathcal{E}) \tag{16}$$

is valid for all multiindices m *with* $|m| \leqslant 2$, *all* $\mathcal{T}_h \in \mathbf{F}$, *all local poised sets* $\{b^1, \ldots, b^6\}$ *in* $\mathcal{T}_h$ *and for the Lagrange interpolation polynomial* L *of* u *at the points* $b^1, \ldots, b^6$.

Proof. Let us consider a triangulation $\mathcal{T}_h \in \mathbf{F}$ and a local poised set $\{b^1, \ldots, b^6\}$ in $\mathcal{T}_h$. The interpolant $L \in \mathcal{P}^2$ exists and is unique by the poisedness of $b^1, \ldots, b^6$. If T is a second-degree Taylor polynomial of u at a point $y \in \operatorname{conv}(\mathcal{E})$ then $|(u-T)(x)| < \overline{C}_1 h^3$ in $\operatorname{conv}(\mathcal{E})$ by the Taylor theorem. Then $|(T-L)(b^i)| < \overline{C}_1 h^3$ for $i = 1, \ldots, 6$ and $|(T-L)(x)| < \overline{C}_2 h^3$ in $\operatorname{conv}(\mathcal{E})$ by Lemma 5. But then $|(u-L)(x)| < C_1 h^3$ for $C_1 = \overline{C}_1 + \overline{C}_2$ and the statement follows by Lemma 6. □

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Author's address: *J. Dalík*, Department of Mathematics, Brno University of Technology, Žižkova 17, 602 00 Brno, Czech Republic, e-mail: dalik.j@fce.vutbr.cz.