Optimal Orthogonal Boundary Filter Banks

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Abstract - The use of a paraunitary filter bank for image processing requires a special treatment at image boundaries to ensure perfect reconstruction and orthogonality of these regions. Using time-varying boundary filter banks, we will discuss a procedure that explores all degrees of freedom of the border filters in a method essentially independent of signal extensions, allowing us to design optimal boundary filter banks, while maintaining fast implementation algorithms.

I INTRODUCTION

The applications of multirate filter banks [1] in image processing are receiving increasing attention and so are the problems resulting from the processing of finite length signals. This work is a continuation of the one in [2], and here we explore the concept of time-varying orthogonal filter banks so that filter banks near the borders are changed to force overall orthogonality [3]–[8].

We use a PR critically-decimated paraunitary uniform filter bank [1] of M FIR filters. The filters are assumed to have a maximum length $L = NM$, where N is also called the overlap factor, and the analysis and synthesis filters have impulse reponses $f_k(n)$ and $g_k(n)$ (k=0,1,...,M-1, n=0,1,...,L-1), respectively. We will refer to such system shortly as a paraunitary filter bank (PUFB). The input signal, $x(n)$ is, thus, transformed by the analysis filter bank into the subband signals $y_k(m)$. In a PUFB, we can define a lapped transform matrix $\mathbf{P}$ with elements p_{ij} as

$$p_{ij} = f_i(L-1-j) = g_i(j), \qquad (1)$$

and $\mathbf{P}$ can be segmented into N square matrices as

$$\mathbf{P} = [\mathbf{P}_0 \; \mathbf{P}_1 \; \cdots \; \mathbf{P}_{N-1}], \qquad (2)$$

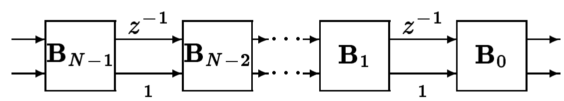

Figure 1: Flow graph for tha analysis section of a paraunitary FIR filter bank where $\mathbf{E}(z)$ can be factorized using symmetric delays and N stages.

Let the input signal $x(n)$ be expressed in its polyphase components $x_i(m) = x(mM+i)$ and define a signal $y(n)$ whose polyphase components are the subbands, such that $y_i(m) = y(mM+i)$. Thus, the two sets of M signals are related by a polyphase transfer matrix (PTM) $\mathbf{E}(z)$ [1] which is paraunitary, i.e. $\mathbf{E}^{-1}(z) = \mathbf{E}^T(z^{-1})$. We will consider the PUFBs which can be parameterized using the symmetric delay factorization (SDF). Let

$$\Lambda(z) = \begin{bmatrix} z^{-1}\mathbf{I}_{M/2} & 0 \\ 0 & \mathbf{I}_{M/2} \end{bmatrix}. \qquad (3)$$

The SDF of the PTM is given by

$$\mathbf{E}(z) = \mathbf{B}_0 \prod_{i=1}^{N-1} (\Lambda(z)\mathbf{B}_i) \qquad (4)$$

where all stages $\mathbf{B}_i$ are allowed to be arbitrary $M \times M$ orthogonal matrices. The flow graph for implementing a PUFB which can be parameterized using SDF is shown in Fig. 1. The use of SDF is not very restrictive in practice, as, for M even, most PUFBs with any practical advantage can be expressed in this way. Examples of such filter banks are linear-phase PUFBs [10], cosine modulated filter banks [1, 9], and, of course, all 2-channel PUFB. Its great advantage is that it spares us the task of developing different algorithms for each border of the signal.

If the vectors $\mathbf{x}$ and $\mathbf{y}$ contain the signals $x(n)$ and $y(n)$, respectively, then the analysis and synthesis sections can be represented in matrix notation as

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$$\mathbf{y} = \mathbf{T}_\infty \mathbf{x} \qquad \mathbf{x} = \mathbf{T}_\infty^T \mathbf{y} \tag{5}$$

where

$$\mathbf{T}_\infty = \begin{bmatrix} \ddots & \ddots & & \ddots & & & 0 \\ & \mathbf{P}_0 & \mathbf{P}_1 & \cdots & \mathbf{P}_{N-1} & & \\ & & \mathbf{P}_0 & \mathbf{P}_1 & \cdots & \mathbf{P}_{N-1} & \\ 0 & & & \ddots & \ddots & & \ddots \end{bmatrix}. \tag{6}$$

It is easy to show that the paraunitariness constraint implies that $\mathbf{T}_\infty$ is orthogonal.

Suppose the signal $x(n)$ has only N_x samples and assume $N_x = N_B M$, where N_B is an integer representing the number of blocks, with M samples per block. To avoid the expansion of the number of samples, we require $y(n)$ to have N_x samples, so that each subband would have N_B samples. It is clear that there is a size-limited linear transform $\mathbf{T}$ leading $\mathbf{x}$ into $\mathbf{y}$ so that

$$\mathbf{y} = \mathbf{T}\mathbf{x} \qquad \mathbf{x} = \mathbf{T}^{-1}\mathbf{y}. \tag{7}$$

II Boundary filters

Let $\mathbf{T}$ be composed of time-varying filter banks as

$$\mathbf{T} = \begin{bmatrix} \mathbf{P}(0) & & & \\ & \mathbf{P}(1) & & \\ & & \ddots & \\ & & & \mathbf{P}(N_B - 1) \end{bmatrix}. \tag{8}$$

Let K be the greatest integer smaller than $N/2$ (the same as integer division as $K = N/2$). Hence, there are K filter banks, at each border, which have their basis functions crossing the signal boundaries. We call this the minimal complete design (MCD) when only K filter banks at each border are changed in order to achieve orthogonality of $\mathbf{T}$. We could change all N_B filter banks but only $2K$ of them have any influence to the borders, so that we will often assume MCD. Then, we have

$$\mathbf{P}(m) = \mathbf{P} \quad \text{for } K \le m \le N_B - K - 1 \tag{9}$$

and the remaining filter banks are redesigned, but remaining instantaneously paraunitary [4, 12], and obeying PR rules for time-varying filter banks.

For $\mathbf{T}$ orthogonal, the entries of the boundary filters are such that there is no overlap of the filters across the signal borders [12]. For an infinite-length signal, we can draw the flow-graph relating the input and output of the analysis section, which accounts for permutations and orthogonal matrices [4, 5, 12], and represents an orthogonal system as

$$\mathbf{T}_\infty = \tilde{\mathbf{B}}_0 \prod_{i=1}^{N-1} \mathbf{W}\tilde{\mathbf{B}}_i \tag{10}$$

where $\tilde{\mathbf{B}}_i = diag\{\cdots, \mathbf{B}_i, \mathbf{B}_i, \mathbf{B}_i, \cdots\}$ and $\mathbf{W}$ is a permutation matrix that can be derived from the factorization of $\mathbf{E}(z)$ [4, 5, 12]. Hence, the synthesis process, defined by $\mathbf{T}_\infty^T$, would be represented by the same flow-graph reversing the direction to follow the paths and substituting the orthogonal matrices by their transposes. Since the transform cannot allow overlap across the signal borders, the two adjacent size-limited transforms have to be completely independent. So, the algorithm described by Fig. 1 is applied to a hypothetical unlimited-length signal and the stages $\mathbf{B}_i$ are modified (however, orthogonality is maintained) along the time-index so that transitions among SDF PUFBs are achieved. The input and output signals are segmented into blocks of M samples, and blocks are labelled 0 through $N_B - 1$ for the actual support region of $x(n)$ and $y(n)$. A simple way to find the complete SDF relevant for the signal is: (i) Construct the flow-graph for the hypothetical infinite-length signal; (ii) Eliminate unnecessary paths and boxes, used for the signal outside the bounds. (iii) From the remaining boxes, those which are connected to output blocks numbered K through $N_B - K - 1$ are the same as in the time-invariant SDF and are not changed for an MCD, while the remaining can be any orthogonal matrix (maintaining their sizes) and are responsible for the degrees of freedom in the transitory boundary filter banks.

A straightforward algorithm to perform steps (ii) and (iii) is now presented. Let the i-th stage be the one with all matrices $\mathbf{B}_i$. Note that each box labelled $\mathbf{B}_i$ has two input or output branches (each carrying $M/2$ samples). To prune unnecessary branches and boxes, start by disconnecting the input samples outside signal bounds from the flow-graph. For i varying from $i = N - 1$ through $i = 0$, check all boxes in stage $N - 1$, then proceed with stage $N - 2$ towards stage 0. For each box in each stage, check its input branches. If both of its input branches are disconnected, erase this box and its output branches. If only one input branch is disconnected, erase one output branch and make the box in question a $M/2 \times M/2$ orthogonal matrix. If both input branches are connected, leave the box as a $M \times M$ orthogonal matrix. For example, for $N = 4$ and $N_B = 6$ the resulting flow-graph is shown in Fig. 2, where the generic orthogonal matrices of sizes $M/2 \times M/2$ or $M \times M$ are marked by $\circ$ and $\times$ respectively.

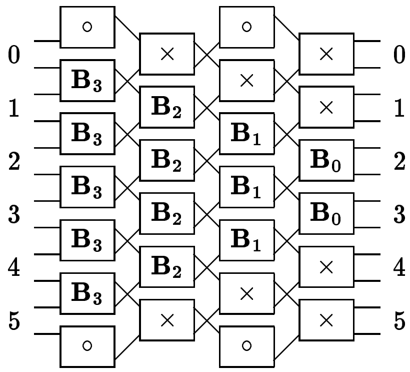

Figure 2: Flow-graph for a size-limited orthogonal implementation of a PUFB for $N=4$ and $N_B=6$. Each branch carries $M/2$ samples.

When the prunning process is complete, and the boxes belonging to the transitory boundary filter banks are selected, we will have some orthogonal matrices as degrees of freedom. An $n\times n$ orthogonal matrix has $n(n-1)/2$ degrees of freedom corresponding to its plane rotation angles. The reader can check that, for each border, the number of generic orthogonal boundary matrices is

stage $2i$: $K-i$ matrices of size $M\times M$

stage $2i+1$: $K-i-1$ matrices of size $M\times M$ and one of size $\frac{M}{2}\times\frac{M}{2}$

The number of degrees of freedom for each border is

$$\begin{aligned}\nu &= \left(\sum_{i=1}^{K} i + \sum_{i=1}^{K-1} i\right)\frac{M(M-1)}{2} + K\frac{\frac{M}{2}(\frac{M}{2}-1)}{2}\\ &= [(4K+1)(M-1)-1]\frac{KM}{8} \qquad (11)\end{aligned}$$

In the design of the boundary filter banks, for an optimal orthogonal solution we shall span all degrees of freedom in a search for the minimum of a specific cost function. As the relation among the plane rotations and cost functions is generally non-linear, an optimization algorithm would generally have slow convergence and lead to a local minimum. So, a large number of variables to optimize can be burdensome. Note that ν can be a very big number.

III Design examples

In a simple example, for $M=2$, $N=4$ and $N_B=6$ (see Fig. 2) we have 4 degrees of freedom at each border. ($M/2=1$ and the 1×1 "orthogonal" matrices are set to 1.) We started with a 2-channel 8-tap PR PUFB shown in Fig. 3(a) and used an unconstrained non-linear optimization routine to optimize the border matrices (one plane rotation angle per matrix), where the function maximized was an average of the stopband atenuation of the boundary filters. The 12 resulting bases for the 12-sample signal are shown in Fig. 3(b), where the relation of the basis functions and $\mathbf{P}(m)$ ($m=0,\ldots,5$) is indicated. Note that $\mathbf{P}(2)=\mathbf{P}(3)=\mathbf{P}$ for MCD, and the 4 bases in the middle of Fig. 3(b) are the same as those in Fig. 3(a).

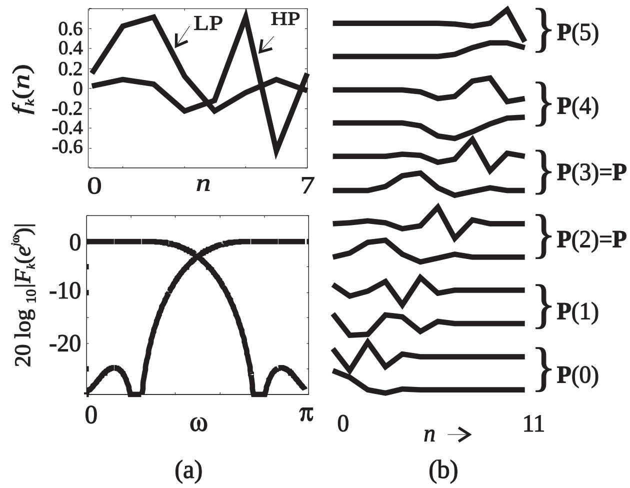

Figure 3: Design example of orthogonal boundary filter banks based on a 2-channel PUFB.

As a second example, more tuned to image coding applications, we used the modulated lapped transform (MLT) [9] with $N=2$ and $M=8$. We have 34 degrees of freedom at each border since just one filter bank needs to be optimized because $K=1$. Malvar [9] provided a standard boundary solution for the MLT which is orthogonal and, therefore, it is a special case among all solutions wherein the 34 degrees of freedom per border would span. Here, we maximized the transform coding gain G_{TC} [11] for the boundary filter bank. Assuming $x(n)$ has autocorrelation $r_x(n)=0.95^{|n|}$ and an autocorrelation matrix $\mathbf{R}_{xx}$ with entries $R_{ij}=0.95^{|i-j|}$ $(0\leq(i,j)\leq L-1)$, and denoting the diagonal elements of $\mathbf{P}\mathbf{R}_{xx}\mathbf{P}^T$ as σ_0^2 through σ_{M-1}^2, then

$$G_{TC}=10\log_{10}\left(\frac{1}{M}\sum_{i=0}^{M-1}\sigma_i^2\right)\left(\prod_{i=0}^{M-1}\sigma_i^2\right)^{-1/M}. \quad (12)$$

In one border, the G_{TC} for the optimal boundary filter bank is 9.19dB, compared to 5.66dB from that of Malvar. As a reference, the MLT has G_{TC} ranging from 8.25dB through 9.22dB (it depends upon a design parameter [9]) and the popular discrete cosine transform (DCT) has $G_{TC}=8.83$dB.

We carried image coding tests, using the MLT, a 48×48-pels image and adopting $M=8$. For a two-dimensional separable implementation, there are

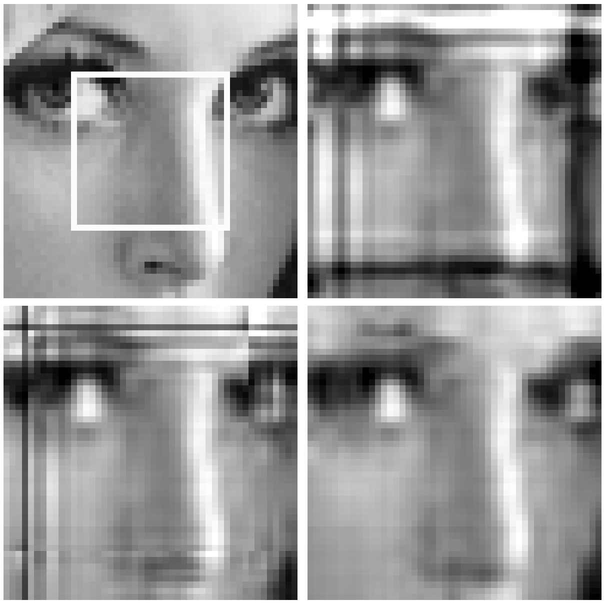

Figure 4: High-compression simulation with the $M = 8$ MLT. From top to bottom and left to right: original image; periodic extension; standard boundary MLT; boundary filter bank optimized for maximum G_{TC}.

$8^2 = 64$ subbands, and 36 coefficients in each subband, where 20 of them result from boundary filter banks. So, as each basis function (filter) has 16 elements ($N = 2$), boundary PUFBs will affect a 12-pixels-deep region of the reconstructed image starting at each border. We carried a comparison, using periodic extension (circular convolution), the standard boundary orthogonal solution proposed by Malvar [9] and our optimal solution (maximum G_{TC}). We quantized only 8 out of 64 subbands and the results are shown in Fig. 4, where it can be seen that the optimal boundary solution is free of border distortion at high-compression rates, while the other methods are not.

IV Conclusions

We have developed a technique to construct orthogonal boundary filter banks for PUFBs. The restrictions imposed are minimal, and the results can be changed to accomodate an odd number of channels M or a non-symmetric factorization, in which case the basic ideas would not change, but the presentation would be greatly complicated. Simplification, in fact, is the reason behind the choice for restricting the length of the signal N_x to be a multiple of M. The motivation for studying finite-length signals is for the application of PUFBs in image processing/coding. In this case, the image dimensions are often chosen as a multiple of M, otherwise, the image is artificially extended, as do most of the image coders. Methods to construct orthogonal boundary filter banks have been reported earlier [4]–[8], however we have presented a general solution (as long as the PUFB obeys the SDF), and have explicitely pointed the degrees of freedom of such transitions. This allowed us to easily design optimal boundary filter banks. The absence of border distortion is also clear from our image coding tests using optimized boundary filter bank, providing a great improvement in relation to existing orthogonalization methods for the MLT.

A more complete discussion of the work in this paper and of [2] can be found in [5, 12].

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