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# Optimal partial regularity for very weak solutions to a class of nonlinear elliptic systems

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**Abstract**

We consider optimal partial regularity for very weak solutions to a class of nonlinear elliptic systems and obtain the general criterion for a very weak solution to be regular in the neighborhood of a given point. First, by Hodge decomposition and the technique of filling holes, we establish the relation between the very weak solution and the classical weak solution. Furthermore, combining the technique of *p*-harmonic approximation with the method of Hodge decomposition, we obtain the partial regularity result. In particular, the partial regularity we obtained is optimal.

**Keywords:** Optimal partial regularity; Very weak solution; Hodge decomposition; *p*-harmonic approximation technique; Nonlinear elliptic system

## 1 Introduction

In this paper, we are concerned with optimal partial regularity for very weak solutions of nonlinear elliptic systems of the following type:

$$-\operatorname{div} A(x,u,\nabla u)=f(x)+\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big),\quad x\in\Omega, \tag{1.1}$$

where $\Omega\subset R^n$ is a bounded domain, $n\geq 2$, $N>1$, $1<p<+\infty$, the Caratheodony function $A(x,u,h):\Omega\times R^n\times R^{nN}\longrightarrow R^n$ satisfies the following conditions:

(H1) There exists a constant $\alpha>0$ such that

$$A(x,u,h)h\geq\alpha|h|^p,\quad \forall h\in R^{nN},x\in\Omega,u\in R^n;$$

(H2) There exists a constant $\beta>0$ such that

$$\big\langle A(x,u_1,h_1)-A(x,u_2,h_2),(h_1-h_2)\big\rangle\geq\beta\big(|h_1|+|h_2|\big)^{p-2}|h_1-h_2|^2;$$

(H3) There exists a constant $\beta\leq\gamma<+\infty$ such that

$$\big|A(x,u,h)\big|\leq\gamma\big(|h|^{p-1}+|u|^{(p-1)\alpha}+\varphi(x)\big),\quad x\in\Omega,u\in R^n,h\in R^{nN},$$

where $0<\alpha<\frac{n}{n-(p-1)}$, $\varphi(x)\in L^{\frac{p}{p-1}}(\Omega)$.

Now, we can definite the very weak solutions of a nonlinear elliptic system.

**Definition 1.1** We call a function $u \in W^{1,r}(\Omega)$ ($\max\{1, p-1\} \le r < p$) a very weak solution to the nonlinear elliptic system (1.1), if the integral equality

$$\int_\Omega A(x,u,\nabla u)\cdot\nabla\phi\,dx = \int_\Omega f(x)\cdot\phi\,dx - \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\phi\,dx$$

holds for all functions $\phi(x) \in C_0^\infty(\Omega)$.

The definition of a "very weak solution" was put forward by Iwaniec [14]. In 1994, Iwaniec observed that: in the integral sense, the integrable index of a weak solution should be no less than the natural index minus 1. Then, he defined a "very weak solution", and established the relation between the very weak solution and the classical weak solution for the homogeneous A-harmonic equation. The conclusion has been extended to the case of an inhomogeneous A-harmonic system by Zhao and Chen [25].

Greco and Luigi et al. [12, 19] generalized this result to a $p$-Laplace-type system with the form

$$\operatorname{div}\Big[\big[G(x)\nabla u,\nabla u\big]^{\frac{p-2}{2}}G(x)\nabla u\Big] = 0.$$

The same result of inhomogeneous $p$-Laplace-type systems with the form

$$\operatorname{div}\Big[\big[A(x)\nabla u,\nabla u\big]^{\frac{p-2}{2}}A(x)\nabla u\Big] = \operatorname{div}\big(\sqrt{A(x)}F(x)\big),$$

had been found by Stroffolini [20].

Soon afterwards, similar results were extended to the elliptic system, parabolic system, $p$-Laplace system, etc., under all kinds of conditions. Of course, they obtained fruitful results [1–8, 13, 15–18, 22–24]. Note that the A-operator in the equations considered above is independent of the very weak solution $u$. Furthermore, the results obtained in the above works only prove that the very weak solution is in fact the classical weak solution.

Motivated by the above works, we study the partial regularity theory of very weak solutions to the nonlinear elliptic system (1.1) in this paper. The differences from the previous works are the following three cases:

(i) The A-harmonic operator $A(x,u,\nabla u)$ in the system (1.1) is not only dependent on the very weak solution $u$, but also on the gradient of the very weak solution $\nabla u$;

(ii) The inhomogeneous term $f(x) + \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ has both a general function term $f(x)$, and a divergence term with $p$-Laplace-type $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$.

(iii) Here, we not only consider the relation between the very weak solution $u$ and the classical weak solution, but also establish the optimal partial regularity for the very weak solution, i.e., $u \in C^{1,1}(\Omega\Omega_0)$.

All of this means that we not only need to solve the problems caused by the very weak solution $u$ in the A-harmonic operator, but also should overcome the difficulties from the inhomogeneous term $f(x) + \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. The elemental but most important item is the inhomogeneous term composing two completely different form functions, which allow us to find some new appropriate methods.

In order to overcome these difficulties and obtain the desired conclusion, we use the method of Hodge decomposition to reveal the relation between the very weak solution and the weak solution. Then, by the technique of $p$-harmonic approximation, we establish the optimal partial regularity. Now, let us show them one by one.

First, in order to handle the problems from the very weak solution $u$ of the A-harmonic operator, we should construct an appropriate type of Hodge decomposition. Then, combining the Sobolev embedding theorem, Young's inequality, and the estimations of Hodge decomposition, we resolve the problem.

For the inhomogeneous term $f(x) + \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, since it is composed of two terms: the general function $f(x)$ and the $p$-Laplace-type divergence function $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$, if we select one of the Hodge decomposition terms $\phi$ as the test function in the definition of the very weak solution, we can obtain the required estimation for the general function term $f(x)$ in the proof of the Caccioppoli second inequality. On the other hand, if the function $\eta^2 u$ is selected as the test function, the divergence term $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ cannot be processed.

To solve these problems, we combine the method of the A-harmonic approximation technique with Hodge decomposition $|\nabla(\eta v)|^{-\varepsilon}\nabla(\eta v) = \nabla\phi + H$. Select $\phi$ of Hodge decomposition as the test function, and then match the estimator of each item, in particular the estimator of $H$ with minimal coefficient $\varepsilon$, and then combine all kinds of inequalities. The key thing is making full use of the minimal coefficient $\varepsilon$ in the $H$ estimator again and again. Finally, we obtain the suitable Coccioppoli second inequality.

However, due to the divergence term $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ in the inhomogeneous term [9, 20], we cannot obtain the conditions of the A-harmonic approximation lemma [1, 21]. Fortunately, the conditions for the $p$-harmonic approximation lemma [10] can be derived. Thus, in this paper, we choose the $p$-harmonic approximation method to establish the decay estimation.

Finally, by the standard iterative method, the optimal partial regularity for the very weak solution $u$ of the system (1.1) is obtained. That is,

**Theorem 1.1** *Assume that* $f \in L_{\text{loc}}^{\frac{nq}{n(p-1)+q}}(\Omega)$, $q > p > 1 + \frac{1}{n}$, $u \in W^{1,r}(\Omega)$, $(\max\{1, p-1\} \le r < p)$ *is a very weak solution of the system* (1.1) *under the conditions* (H1)–(H3). *Then,* $u \in W^{1,p}(\Omega)$ *and there exists an open set* $\Omega_0 \subset \Omega$, *such that* $u \in C^{1,1}(\Omega \setminus \Omega_0)$, *where*

$$\Omega\backslash\Omega_0 = \Sigma_1 \cup \Sigma_2,$$

*with*

$$\Sigma_1 = \left\{x_0 \in \Omega : \liminf_{\rho\to 0^+} \fint_{B_\rho(x_0)} |\nabla u - (\nabla u)_{x_{0,\rho}}|^p \, dx > 0\right\},$$

*and*

$$\Sigma_2 = \left\{x_0 \in \Omega : \limsup_{\rho\to 0^+} \big(|u_{x_{0,\rho}}| + |(\nabla u)_{x_{0,\rho}}|\big) = \infty\right\}.$$

*In particular,*

$$\operatorname{meas}(\Omega\backslash\Omega_0) = 0.$$

## 2 Preliminaries

In this section, we introduce the $p$-harmonic approximation lemma and some basic results, which we will use in the proof of the main theorem. The first one we recall is the $p$-harmonic approximation lemma [10].

**Lemma 2.1** ([8]) *For any $\epsilon > 0$, there exists a positive constant $\delta \in (0,1]$, depending only on $n$, $N$, $p$ and $\epsilon$, such that: Whenever $u \in W^{1,p}(B_\rho, R^N)$ with $\rho^{p-n}\int_{B_\rho}|Du|^p\,dx \leq 1$ is approximately p-harmonic in the sense that:*

$$\left|\rho^{p-n}\int_{B_\rho}|Du|^{p-2}Du\cdot D\varphi\,dx\right| \leq \delta\rho\sup_{B_\rho}|D\varphi| \tag{2.1}$$

*holds for all $\varphi \in C_0^1(B_\rho, R^N)$. Then, there exists a p-harmonic function $h \in W^{1,p}(B_\rho, R^N)$ such that:*

$$\rho^{p-n}\int_{B_\rho}|Dh|^p\,dx \leq 1, \tag{2.2}$$

*and*

$$\rho^{-n}\int_{B_\rho}|h-u|^p\,dx \leq \epsilon^p. \tag{2.3}$$

The next feature we study is a standard estimate for second-order homogeneous elliptic systems with a constant coefficient coming from Campanato [6]. The result is established by the Caccioppoli inequality for $h$ and its derivative of any order.

**Lemma 2.2** *For $A$, $\alpha$, $\beta$ and $\gamma$ given in the conditions ($H1$)-($H3$), there exists a constant $C_0$ (without loss of generality, we assume that $C_0 \geq 1$) depending only on $n$, $N$, $\alpha$, $\beta$ and $\gamma$, such that for arbitrarily p-harmonic function $h$ on $B_\rho(x_0)$, the following inequality holds:*

$$\rho^p\sup_{B_{\frac{\rho}{2}}(x_0)}|\nabla h|^p + \rho^{2p}\sup_{B_{\frac{\rho}{2}}(x_0)}\left|\nabla^2 h\right|^p \leq C_0\rho^{p-n}\int_{B_\rho(x_0)}|\nabla h|^p\,dx.$$

The following lemma in this section is a result of the property for a very weak solution. That is,

**Lemma 2.3** ([26]) *Assume that $f \in L^{\frac{nq}{n(p-1)+q}}(\Omega)$, $q > p$, then there exists an integral exponent $1 < r_1 = r_1(n,p,\alpha,\beta) < p < r_2 = r_2(n,p,\alpha,\beta) < +\infty$, such that for every very weak solution $u \in W^{1,r_1}(\Omega)$, there is $u \in W^{1,r_2}(\Omega)$. This means that the very weak solution $u$ in fact is a classical weak solution.*

Hodge decomposition is a critical tool to obtain the desired regularity result.

**Lemma 2.4** ([12]) *Let $\Omega \subset R^n$ be a regular domain, $\omega \in W_0^{1,r}(\Omega, R^N)$, $r > 1$, and $-1 < \varepsilon < r-1$. Then there exist $\phi \in W_0^{1,\frac{r}{1+\varepsilon}}(\Omega, R^N)$ and a divergence free matrix-field $H \in L^{\frac{r}{1+\varepsilon}}(\Omega, R^{nN})$ such that*

$$|\nabla\omega|^\varepsilon\nabla\omega = \nabla\phi + H. \tag{2.4}$$

Moreover,

$$\|H\|_{\frac{r}{1+\varepsilon}} \leq C_r(\Omega, m)|\varepsilon|\|\nabla\omega\|_r^{1+\varepsilon}. \tag{2.5}$$

Here, the most important case is where $\varepsilon$ is negative. For $u \in W^{1,r}_{\mathrm{loc}}(\Omega, R^N)$, one can apply (2.4) with $\omega = u - u_0$ and $\varepsilon = r - p$. Note that $\nabla\phi \in L^{\frac{r}{r-p+1}}(\Omega, R^{nN})$, thus $\phi$ can be illustrated as a test function in the very weak solution definition.

The following idea has been found in the context of quasiregular mappings [11].

**Lemma 2.5** ([11]) *Let $u(x) \in L^p(B_R)$, $B_R \subset \Omega$, $f \in L^t(B_R)$, $t > p$, and the integral inequality*

$$\fint_{B_{\frac{R}{2}}} |u|^p\,dx \leq K\left(\fint_{B_R} |u|^s\,dx\right)^{\frac{p}{s}} + \theta \fint_{B_R} |u|^p\,dx + \fint_{B_R} |f|^p\,dx$$

*holds for $1 \leq s < p$, $0 \leq \theta \leq 1$; Then, there exists an integral coefficient $p' = p'(K, n, p, \theta)$, $(t \geq p' > p)$, such that $u \in L^{p'}_{\mathrm{loc}}(\Omega)$, and for some constant $C' = C'(n, p, K, \theta)$, we have*

$$\left(\fint_{B_{\frac{R}{2}}} |u|^{p'}\,dx\right)^{\frac{1}{p'}} \leq C'\left(\fint_{B_R} |u|^p\,dx\right)^{\frac{1}{p}} + C'\left(\fint_{B_R} |f|^{p'}\,dx\right)^{\frac{1}{p'}}.$$

The last result we would introduce in this section is an elemental but necessary inequality.

**Lemma 2.6** ([14]) *Suppose $X$ and $Y$ are vectors of an inner product space. Then,*

$$\left||X|^\varepsilon X - |Y|^\varepsilon Y\right| \leq \frac{1-\varepsilon}{1+\varepsilon} 2^{-\varepsilon}|X - Y|^{1+\varepsilon}$$

*for $-1 < \varepsilon \leq 0$, and*

$$\left||X|^\varepsilon X - |Y|^\varepsilon Y\right| \leq 1 + \varepsilon\big(|Y| + |X - Y|\big)^\varepsilon |X - Y|$$

*for $\varepsilon \geq 0$.*

## 3 Caccioppoli second inequality

To establish optimal partial regularity for the very weak solution to the inhomogeneous A-harmonic system (1.1), we should establish a suitable Caccioppoli-type inequality.

**Theorem 3.1** (Caccioppoli second inequality) *Assume that $u \in W^{1,r_1}_{\mathrm{loc}}(\Omega, R^N)$ with $1 + \frac{1}{n} < r_1 = r_1(n, p, \alpha, \beta) < p < r_2 = r_2(n, p, \alpha, \beta) < +\infty$ is a very weak solution to the inhomogeneous A-harmonic system* (1.1) *under the conditions* (H1)-(H3), *$f \in L^{\frac{nq}{n(p-1)+q}}(\Omega)$, $q > p$. Then for every $x_0 \in \Omega$, $u_0 \in R^N$, $p_0 \in R^{nN}$ and arbitrarily $\rho, R : 0 < \rho < R < \min(1, \mathrm{dist}(x_0, \partial\Omega))$, we have $u \in W^{1,p}_{\mathrm{loc}}(\Omega, R^N)$ and*

$$\begin{aligned}\int_{B_{R/2}(x_0)} |\nabla v|^p\,dx &\leq \tilde{C}_1 \int_{B_R(x_0)} |v\nabla\eta|^p\,dx + \tilde{C}_2 \int_{B_R(x_0)} R^p\,dx \\ &\quad + \tilde{C}_3\left(-\int_{B_R(x_0)} |\nabla v|^{p-\varepsilon}\,dx\right)^{\frac{p}{p-\varepsilon}} \big(\alpha_n R^n\big)^{1+\frac{p}{n}},\end{aligned}$$

*where constants $\tilde{C}_1$, $\tilde{C}_2$ and $\tilde{C}_3$ depend only on $\beta$, $\alpha$, $\gamma$, and $C(n, p)$.*

*Proof* Consider a cut-off function $\eta \in C_0^\infty(B_R(x_0))$, satisfying $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_{R/2}(x_0)$ and $|\nabla \eta| < \frac{C}{R}$. Assume that $u \in W^{1,p-\varepsilon}_{\mathrm{loc}}(\Omega)$ $(0 < \varepsilon < \frac{1}{2})$ is a very weak solution to the system (1.1), then for fixed constant $u_0 \in R^N$, $p_0 \in R^{nN}$, $x_0 \in \Omega$, we can find that $v = u - u_0 - p_0(x - x_0) \in W^{1,p-\varepsilon}_{\mathrm{loc}}(\Omega)$ $(0 < \varepsilon < \frac{1}{2})$.

Consider Hodge decomposition (Lemma 2.4) of the following type:

$$\left|\nabla(\eta v)\right|^{-\varepsilon} \nabla(\eta v) = \nabla \phi + H, \quad \text{with } \phi \in W^{1,\frac{p-\varepsilon}{1-\varepsilon}}_{\mathrm{loc}}(\Omega), \tag{3.1}$$

here, $H \in L^{\frac{p-\varepsilon}{1-\varepsilon}}(\Omega)$ is a vector field with zero divergence, and satisfies

$$\|\nabla \phi\|_{\frac{p-\varepsilon}{1-\varepsilon}} \le C(n,p) \left\|\nabla(\eta v)\right\|^{1-\varepsilon}_{p-\varepsilon}, \tag{3.2}$$

$$\|H\|_{\frac{p-\varepsilon}{1-\varepsilon}} \le C(n,p)\varepsilon \left\|\nabla(\eta v)\right\|^{1-\varepsilon}_{p-\varepsilon}. \tag{3.3}$$

Let

$$E(\eta, v) = \left|\nabla(\eta v)\right|^{-\varepsilon} \nabla(\eta v) - |\eta \nabla v|^{-\varepsilon} \eta \nabla v. \tag{3.4}$$

Using the element inequality:

$$\left||X|^{-\varepsilon} X - |Y|^{-\varepsilon} Y\right| \le 2^\varepsilon \frac{1+\varepsilon}{1-\varepsilon} |X - Y|^{1-\varepsilon}, \quad 0 < \varepsilon < 1, X, Y \in R^n,$$

which means that

$$\left|E(\eta, v)\right| \le 2^\varepsilon \frac{1+\varepsilon}{1-\varepsilon} |v \nabla \eta|^{1-\varepsilon}.$$

Now, from (3.1) and (3.4), we can find that

$$\nabla \phi = E(\eta, v) + |\eta \nabla v|^{-\varepsilon} \eta \nabla v - H. \tag{3.5}$$

Take the function $\phi$ in the Hodge decomposition (3.1) as a test function, by the definition of a very weak solution, we have

$$\begin{aligned}
&\int_\Omega \left[A(x,u,\nabla u) - A(x,u,p_0)\right] \cdot \nabla \phi \, dx \\
&\quad = \int_\Omega f(x) \cdot \phi \, dx - \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx - \int_\Omega A(x,u,p_0) \cdot \nabla \phi \, dx.
\end{aligned}$$

Substitute the expression of $\nabla \phi$ into the above equation to obtain

$$\begin{aligned}
&\int_{B_R(x_0)} \left[A(x,u,\nabla u) - A(x,u,p_0)\right] \cdot |\eta \nabla v|^{-\varepsilon} \eta \nabla v \, dx \\
&\quad = \int_{B_R(x_0)} \left[A(x,u,\nabla u) + |\nabla u|^{p-2} \nabla u\right] \cdot H \, dx \\
&\qquad - \int_{B_R(x_0)} \left[A(x,u,\nabla u) + |\nabla u|^{p-2} \nabla u\right] \cdot E(\eta, v) \, dx \\
&\qquad - \int_{B_R(x_0)} |\nabla u|^{p-2} \nabla u \cdot |\eta \nabla v|^{-\varepsilon} \eta \nabla v \, dx
\end{aligned}$$

$$
\begin{aligned}
&- \int_{B_R(x_0)} A(x,u,p_0) \cdot |\eta \nabla v|^{-\varepsilon} \eta \nabla v \, dx \\
&+ \int_{B_R(x_0)} f(x) \cdot \phi \, dx. \qquad (3.6)
\end{aligned}
$$

It is given by condition (H2) that

$$
\begin{aligned}
&\beta \int_{B_R(x_0)} |\nabla v|^{p-1} |\eta \nabla v|^{-\varepsilon} \eta \nabla v \, dx \\
&\quad \le \beta \int_{B_R(x_0)} \big(|\nabla u| + |p_0|\big)^{p-2} |\nabla u - p_0| \cdot |\eta \nabla v|^{-\varepsilon} \eta \nabla v \, dx \\
&\quad \le \int_{B_R(x_0)} \big[A(x,u,\nabla u) - A(x,u,p_0)\big] \cdot |\eta \nabla v|^{-\varepsilon} \eta \nabla v \, dx. \qquad (3.7)
\end{aligned}
$$

Using the monotonicity of $p$-Laplace operators:

$$
\big(|\nabla u|^{p-2} \nabla u - |p_0|^{p-2} p_0\big)(\nabla u - p_0) \ge 0.
$$

We have

$$
\begin{aligned}
&\beta \int_{B_R(x_0)} |\nabla v|^{p-1} |\eta \nabla v|^{-\varepsilon} \eta \nabla v \, dx \\
&\quad \le \int_{B_R(x_0)} \big[A(x,u,\nabla u) + |\nabla u|^{p-2} \nabla u\big] \cdot H \, dx \\
&\qquad - \int_{B_R(x_0)} \big[A(x,u,\nabla u) + |\nabla u|^{p-2} \nabla u\big] \cdot E(\eta, v) \, dx \\
&\qquad - \int_{B_R(x_0)} \big[|\nabla u|^{p-2} \nabla u - |p_0|^{p-2} p_0\big](\nabla u - p_0) \cdot |\eta \nabla v|^{-\varepsilon} \eta \, dx \\
&\qquad - \int_{B_R(x_0)} \big[A(x,u,p_0) + |p_0|^{p-2} p_0\big] \cdot |\eta \nabla v|^{-\varepsilon} \eta \nabla v \, dx \\
&\qquad + \int_{B_R(x_0)} f(x) \cdot \phi \, dx \\
&\quad \le \int_{B_R(x_0)} \big[A(x,u,\nabla u) + |\nabla u|^{p-2} \nabla u\big] \cdot H \, dx \\
&\qquad - \int_{B_R(x_0)} \big[A(x,u,\nabla u) + |\nabla u|^{p-2} \nabla u\big] \cdot E(\eta, v) \, dx \\
&\qquad - \int_{B_R(x_0)} \big[A(x,u,p_0) + |p_0|^{p-2} p_0\big] \cdot |\eta \nabla v|^{-\varepsilon} \eta \nabla v \, dx \\
&\qquad + \int_{B_R(x_0)} f(x) \cdot \phi \, dx. \\
&\quad \le I_1 + I_2 + I_3 + I_4. \qquad (3.8)
\end{aligned}
$$

By the boundedness (H3) of the operator $A$, we can obtain

$$
I_1 = \int_{B_R(x_0)} \big[\big|A(x,u,\nabla u)\big| + |\nabla u|^{p-1}\big] \cdot H \, dx
$$

$$
\begin{aligned}
&\le (\gamma+1)\int_{B_R(x_0)} |\nabla u|^{p-1}\cdot H\,dx \\
&\quad + \gamma\int_{B_R(x_0)} |u|^{(p-1)\alpha}\cdot H\,dx + \gamma\int_{B_R(x_0)} \varphi(x)\cdot H\,dx \\
&\le 2^{p-1}(\gamma+1)\int_{B_R(x_0)} |\nabla v|^{p-1}\cdot H\,dx + 2^{(p-1)\alpha}\gamma\int_{B_R(x_0)} |v|^{(p-1)\alpha}\cdot H\,dx \\
&\quad + \int_{B_R(x_0)} \bigl[2^{p-1}(\gamma+1)|p_0|^{p-1} + 2^{(p-1)\alpha}\gamma\bigl|u_0+p_0(x-x_0)\bigr|^{(p-1)\alpha} + \gamma\varphi(x)\bigr]\cdot H\,dx \\
&= J_1+J_2+J_3.
\end{aligned}
\tag{3.9}
$$

From Holder's inequality, the estimate of $H$, and Young's inequality, we can find that

$$
\begin{aligned}
J_1 &= 2^{p-1}(\gamma+1)\int_{B_R(x_0)} |\nabla v|^{p-1}\cdot H\,dx \\
&\le 2^{p-1}(\gamma+1)\biggl(\int_{B_R(x_0)} |\nabla v|^{p-\varepsilon}\,dx\biggr)^{\frac{p-1}{p-\varepsilon}}\cdot\biggl(\int_{B_R(x_0)} H^{\frac{p-\varepsilon}{1-\varepsilon}}\biggr)^{\frac{1-\varepsilon}{p-\varepsilon}} \\
&\le 2^{p-1}(\gamma+1)C(n,p)\varepsilon\biggl(\int_{B_R(x_0)} |\nabla v|^{p-\varepsilon}\,dx\biggr)^{\frac{p-1}{p-\varepsilon}}\cdot\biggl(\int_{B_R(x_0)} \bigl|\nabla(\eta v)\bigr|^{p-\varepsilon}\biggr)^{\frac{1-\varepsilon}{p-\varepsilon}} \\
&\le 2^{p-\varepsilon}(\gamma+1)C(n,p)\varepsilon\int_{B_R(x_0)} \eta^{1-\varepsilon}|\nabla v|^{p-\varepsilon}\,dx \\
&\quad + 2^{p-\varepsilon}(\gamma+1)C(n,p)\varepsilon\biggl(\int_{B_R(x_0)} |\nabla v|^{p-\varepsilon}\,dx\biggr)^{\frac{p-1}{p-\varepsilon}}\cdot\biggl(\int_{B_R(x_0)} |v\nabla\eta|^{p-\varepsilon}\,dx\biggr)^{\frac{1-\varepsilon}{p-\varepsilon}} \\
&\le 2^{p-\varepsilon}(\gamma+1)C(n,p)\varepsilon\int_{B_R(x_0)} \bigl(\eta^{1-\varepsilon}+\eta\bigr)|\nabla v|^{p-\varepsilon}\,dx \\
&\quad + 2^{p-\varepsilon}(\gamma+1)C(\eta)C(n,p)\varepsilon\int_{B_R(x_0)} |v\nabla\eta|^{p-\varepsilon}\,dx.
\end{aligned}
\tag{3.10}
$$

By Holder's inequality and the inequality (3.3), we can find that

$$
\begin{aligned}
J_2 &= 2^{(p-1)\alpha}\gamma\int_{B_R(x_0)} |v|^{(p-1)\alpha}\cdot H\,dx \\
&\le 2^{(p-1)\alpha}\gamma\biggl(\int_{B_R(x_0)} |v|^{(p-\varepsilon)\alpha}\,dx\biggr)^{\frac{p-1}{p-\varepsilon}}\cdot\biggl(\int_{B_R(x_0)} |H|^{\frac{p-\varepsilon}{1-\varepsilon}}\,dx\biggr)^{\frac{1-\varepsilon}{p-\varepsilon}} \\
&\le 2^{(p-1)\alpha}\gamma C(n,p)\varepsilon\biggl(\int_{B_R(x_0)} |v|^{(p-\varepsilon)\alpha}\,dx\biggr)^{\frac{p-1}{p-\varepsilon}}\cdot\biggl(\int_{B_R(x_0)} \bigl|\nabla(\eta v)\bigr|^{p-\varepsilon}\,dx\biggr)^{\frac{1-\varepsilon}{p-\varepsilon}}.
\end{aligned}
$$

Noting that $0<\alpha<\frac{n}{n-(p-1)}$, and letting

$$
p' = \frac{n(p-\varepsilon)}{n+1-\varepsilon} < p-\varepsilon, \qquad p'' = \frac{np'}{n-p'} = \frac{n(p-\varepsilon)}{n-p+1} > p-\varepsilon.
$$

Then, when $1 \le \alpha < \frac{n}{n-(p-1)}$, by Holder's inequality, Sobolev's inequality, and Young's inequality, we have

$$\begin{aligned}
J_2 &\le 2^{(p-1)\alpha}\gamma C(n,p)\varepsilon\left(\int_{B_R(x_0)}|v|^{\frac{np'}{n-p'}}\,dx\right)^{\frac{n-p'}{np'}(p-1)\alpha} \\
&\quad\cdot\left(\int_{B_R(x_0)}dx\right)^{1-\frac{n-p'}{np'}(p-1)\alpha}\left(\int_{B_R(x_0)}\left|\nabla(\eta v)\right|^{p-\varepsilon}\,dx\right)^{\frac{1-\varepsilon}{p-\varepsilon}} \\
&\le 2^{(p-1)\alpha}\gamma C(n,p)\varepsilon\left(\int_{B_R(x_0)}|\nabla v|^{p'}\,dx\right)^{\frac{1}{p'}(p-1)\alpha} \\
&\quad\cdot\left(\int_{B_R(x_0)}dx\right)^{1-\frac{n-p'}{np'}(p-1)\alpha}\left(\int_{B_R(x_0)}|v\nabla\eta+\eta\nabla v|^{p-\varepsilon}\,dx\right)^{\frac{1-\varepsilon}{p-\varepsilon}} \\
&\le 2^{(p-1)\alpha}\gamma C(n,p)\varepsilon\left(\alpha_n R^n\right)^{[1-\frac{(n-p')(p-1)\alpha}{np'}+(1-\frac{p'}{p-\varepsilon})\cdot\frac{1}{p'}(p-1)\alpha]\cdot\frac{p-\varepsilon}{p-1}} \\
&\quad\cdot\left(\int_{B_R(x_0)}|\eta\nabla v|^{p-\varepsilon}\,dx\right)^{\alpha} \\
&\quad+2^{(p-1)\alpha}\gamma C(n,p)C(\eta)\varepsilon\left(\int_{B_R(x_0)}|v\nabla\eta|^{p-\varepsilon}+|\eta\nabla v|^{p-\varepsilon}\,dx\right) \\
&\le 2^{(p-1)\alpha}\gamma C(n,p)\varepsilon\left(\alpha_n R^n\right)^{[\frac{(p-\varepsilon)\alpha}{n}+\frac{p-\varepsilon}{p-1}-\alpha]}\cdot\left(\int_{B_R(x_0)}|\eta\nabla v|^{p-\varepsilon}\,dx\right)^{\alpha} \\
&\quad+2^{(p-1)\alpha+(1-\varepsilon)}\gamma C(\eta)C(n,p)\varepsilon\left[\int_{B_R(x_0)}|v\nabla\eta|^{p-\varepsilon}\,dx+\int_{B_R(x_0)}|\eta\nabla v|^{p-\varepsilon}\,dx\right] \\
&\le C_1\varepsilon\int_{B_R(x_0)}|\eta\nabla v|^{p-\varepsilon}\,dx+C_2\varepsilon\int_{B_R(x_0)}|v\nabla\eta|^{p-\varepsilon}\,dx,
\end{aligned}\tag{3.11}$$

where

$$\begin{aligned}
C_1 &= 2^{(p-1)\alpha}\gamma C(n,p)\|v\|_{W^{1,p-\varepsilon}}^{(\alpha-1)(p-\varepsilon)}|B_R|^{[\frac{(p-\varepsilon)\alpha}{n}+\frac{p-\varepsilon}{p-1}-\alpha]}+2^{(p-1)\alpha+(1-\varepsilon)}\gamma C(\eta)C(n,p), \\
C_2 &= 2^{(p-1)\alpha+(1-\varepsilon)}\gamma C(\eta)C(n,p).
\end{aligned}$$

If $0 < \alpha < 1$, using Holder's inequality twice and then combining Young's inequality and Sobolev's inequality, yields

$$\begin{aligned}
J_2 &= 2^{(p-1)\alpha}\gamma C(n,p)\varepsilon\left(\int_{B_R(x_0)}|v|^{(p-\varepsilon)\alpha}\,dx\right)^{\frac{p-1}{p-\varepsilon}}\cdot\left(\int_{B_R(x_0)}\left|\nabla(\eta v)\right|^{p-\varepsilon}\,dx\right)^{\frac{1-\varepsilon}{p-\varepsilon}} \\
&\le 2^{(p-1)\alpha}\gamma C(n,p)\varepsilon\left(\int_{B_R(x_0)}\left(1+|v|^{(p-\varepsilon)}\right)dx\right)^{\frac{p-1}{p-\varepsilon}} \\
&\quad\cdot\left(\int_{B_R(x_0)}\left|\nabla(\eta v)\right|^{p-\varepsilon}\,dx\right)^{\frac{1-\varepsilon}{p-\varepsilon}} \\
&\le 2^{(p-1)\alpha}\gamma C(n,p)\varepsilon\left[\int_{B_R(x_0)}dx+\int_{B_R(x_0)}|v|^{(p-\varepsilon)}\,dx\right] \\
&\quad+2^{(p-1)\alpha}\gamma C(n,p)\varepsilon\int_{B_R(x_0)}\left|\nabla(\eta v)\right|^{p-\varepsilon}\,dx
\end{aligned}$$

$$
\begin{aligned}
&\le 2^{(p-1)\alpha}\gamma C(n,p)\varepsilon \int_{B_R(x_0)} dx + 2^{(p-1)\alpha}\gamma C(n,p)\varepsilon \int_{B_R(x_0)} \left|\nabla(\eta v)\right|^{p-\varepsilon} dx \\
&\quad + 2^{(p-1)\alpha}\gamma C(n,p)\varepsilon \int_{B_R(x_0)} |v\nabla\eta|^{(p-\varepsilon)} dx \\
&\le 2^{(p-1)\alpha}\gamma C(n,p)\varepsilon \left[2^{p-\varepsilon} \int_{B_R(x_0)} \eta^{p-\varepsilon}|\nabla v|^{p-\varepsilon} dx + \left(2^{p-\varepsilon}+1\right) \int_{B_R(x_0)} |v\nabla\eta|^{p-\varepsilon} dx\right] \\
&\quad + 2^{(p-1)\alpha}\gamma C(n,p)\varepsilon \int_{B_R(x_0)} dx. \qquad (3.12)
\end{aligned}
$$

Combining the inequalities (3.11) with (3.12), we can find that

$$
J_2 \le \varepsilon\left[C_3 \int_{B_R(x_0)} \eta^{p-\varepsilon}|\nabla v|^{p-\varepsilon} dx + C_4 \int_{B_R(x_0)} |v\nabla\eta|^{p-\varepsilon} dx + C_5 \int_{B_R(x_0)} dx\right], \qquad (3.13)
$$

where

$$
\begin{aligned}
&C_3 = \max\left\{2^{(p-1)\alpha+(p-\varepsilon)}\gamma C(n,p), C_1\right\}, \\
&C_4 = \max\left\{2^{(p-1)\alpha}\gamma C(n,p)\cdot\left(2^{p-\varepsilon}+1\right), C_2\right\}, \\
&C_5 = 2^{(p-1)\alpha}\gamma C(n,p).
\end{aligned}
$$

Using Young's inequality and Holder's inequality, we can find that

$$
\begin{aligned}
J_3 &= \int_{B_R(x_0)} \left[2^{p-1}(\gamma+1)|p_0|^{p-1} + 2^{(p-1)\alpha}\gamma\left|u_0 + p_0(x-x_0)\right|^{(p-1)\alpha} + \gamma\varphi(x)\right]\cdot H\, dx \\
&\le \left[\int_{B_R(x_0)} \left[2^{p-1}(\gamma+1)|p_0|^{p-1} + 2^{(p-1)\alpha}\gamma|u_0 + p_0R|^{(p-1)\alpha} + \gamma\left|\varphi(x)\right|\right]^{\frac{p-\varepsilon}{p-1}} dx\right]^{\frac{p-1}{p-\varepsilon}} \\
&\quad \cdot\left(\int_{B_R(x_0)} H^{\frac{p-\varepsilon}{1-\varepsilon}} dx\right)^{\frac{1-\varepsilon}{p-\varepsilon}} \\
&\le C(n,p)\varepsilon\left[\int_{B_R(x_0)} \left[2^{p-1}(\gamma+1)|p_0|^{p-1} + 2^{(p-1)\alpha}\gamma|u_0 + p_0R|^{(p-1)\alpha} + \gamma\left|\varphi(x)\right|\right]^{\frac{p-\varepsilon}{p-1}} dx\right]^{\frac{p-1}{p-\varepsilon}} \\
&\quad \cdot\left(\int_{B_R(x_0)} \left|\nabla(\eta v)\right|^{p-\varepsilon} dx\right)^{\frac{1-\varepsilon}{p-\varepsilon}} \\
&\le C(n,p)\varepsilon \int_{B_R(x_0)} \left[2^{p-1}(\gamma+1)|p_0|^{p-1} + 2^{(p-1)\alpha}\gamma|u_0 + p_0R|^{(p-1)\alpha} + \gamma\left|\varphi(x)\right|\right]^{\frac{p-\varepsilon}{p-1}} dx \\
&\quad + C(n,p)\varepsilon \int_{B_R(x_0)} |\eta\nabla v|^{p-\varepsilon} dx + C(n,p)\varepsilon \int_{B_R(x_0)} |v\nabla\eta|^{p-\varepsilon} dx.
\end{aligned}
$$

By the estimates of $J_1$, $J_2$, and $J_3$, one can derive that

$$
\begin{aligned}
I_1 &\le 2^{p-\varepsilon}(\gamma+1)C(n,p)\varepsilon \int_{B_R(x_0)} \left(\eta^{1-\varepsilon}+\eta\right)|\nabla v|^{p-\varepsilon} dx \\
&\quad + 2^{p-\varepsilon}(\gamma+1)C(\eta)C(n,p)\varepsilon \int_{B_R(x_0)} |v\nabla\eta|^{p-\varepsilon} dx \\
&\quad + \varepsilon\left[C_3 \int_{B_R(x_0)} \eta^{p-\varepsilon}|\nabla v|^{p-\varepsilon} dx + C_4 \int_{B_R(x_0)} |v\nabla\eta|^{p-\varepsilon} dx + C_5 \int_{B_R(x_0)} dx\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ C(n,p)\varepsilon \int_{B_R(x_0)} \left[2^{p-1}(\gamma+1)|p_0|^{p-1} + 2^{(p-1)\alpha}\gamma|u_0 + p_0 R|^{(p-1)\alpha} + \gamma\left|\varphi(x)\right|\right]^{\frac{p-\varepsilon}{p-1}} dx \\
&+ C(n,p)\varepsilon \int_{B_R(x_0)} |\eta\nabla v|^{p-\varepsilon} dx + C(n,p)\varepsilon \int_{B_R(x_0)} |v\nabla\eta|^{p-\varepsilon} dx \\
&\leq \varepsilon\left[C_6 \int_{B_R(x_0)} \eta^{1-\varepsilon}|\nabla v|^{p-\varepsilon} dx + C_7 \int_{B_R(x_0)} |v\nabla\eta|^{p-\varepsilon} dx + C_8 \int_{B_R(x_0)} dx\right],
\end{aligned}
$$

where

$$
\begin{aligned}
&C_6 = 2^{p+1-\varepsilon}(\gamma+1)C(n,p) + C_3 + C(n,p), \\
&C_7 = 2^{p-\varepsilon}(\gamma+1)C(\eta)C(n,p) + C_4 + C(n,p), \\
&C_8 = C(n,p)\left[2^{p-1}(\gamma+1)|p_0|^{p-1} + 2^{(p-1)\alpha}\gamma|u_0 + p_0|^{(p-1)\alpha} + \gamma\left\|\varphi(x)\right\|_{L^{\frac{p-\varepsilon}{p-1}}}\right]^{\frac{p-\varepsilon}{p-1}} + C_5.
\end{aligned}
$$

Now, we can obtain the following inequality by the boundedness (H3) of the operator $A(x,u,\nabla u)$,

$$
\begin{aligned}
I_2 &\leq (\gamma+1)\int_{B_R(x_0)} |\nabla u|^{p-1}\left|E(\eta,v)\right| dx + \gamma\int_{B_R(x_0)} |u|^{(p-1)\alpha}\left|E(\eta,v)\right| dx \\
&\quad + \gamma\int_{B_R(x_0)} \left|\varphi(x)\right|\left|E(\eta,v)\right| dx \\
&\leq 2^{p-1}(\gamma+1)\int_{B_R(x_0)} |\nabla v|^{p-1}\left|E(\eta,v)\right| dx + 2^{(p-1)\alpha}\gamma\int_{B_R(x_0)} |v|^{(p-1)\alpha}\left|E(\eta,v)\right| dx \\
&\quad + \int_{B_R(x_0)} \left(2^{p-1}(\gamma+1)|p_0|^{p-1} + 2^{(p-1)\alpha}\gamma\left(|u_0| + |p_0|\right)^{(p-1)\alpha} + \gamma\left|\varphi(x)\right|\right)\left|E(\eta,v)\right| dx \\
&= K_1 + K_2 + K_3.
\end{aligned}
$$

From Young's inequality and the estimate of $|E(\eta,v)|$, we have

$$
\begin{aligned}
K_1 &= 2^{p-1}(\gamma+1)\int_{B_R(x_0)} |\nabla v|^{p-1}\left|E(\eta,v)\right| dx \\
&\leq 2^{p-1+\varepsilon}(\gamma+1)\frac{1+\varepsilon}{1-\varepsilon}\int_{B_R(x_0)} |\nabla v|^{p-1}|v\nabla\eta|^{1-\varepsilon} dx \\
&\leq \varepsilon\int_{B_R(x_0)} |\eta\nabla v|^{p-\varepsilon} dx + C\left(\varepsilon\eta^{p-\varepsilon}\right)\left(2^{p-1+\varepsilon}(\gamma+1)\frac{1+\varepsilon}{1-\varepsilon}\right)^{\frac{p-\varepsilon}{1-\varepsilon}} \\
&\quad \times\int_{B_R(x_0)} |v\nabla\eta|^{p-\varepsilon} dx. \qquad (3.14)
\end{aligned}
$$

For $0 \leq \alpha < \frac{n}{n-(p-1)}$ with $p\prime = \frac{n(p-\varepsilon)}{n+1-\varepsilon} < p-\varepsilon$, $p\prime\prime = \frac{np\prime}{n-p\prime} = \frac{n(p-\varepsilon)}{n-p+1} > p-\varepsilon$, combining Holder's inequality, Young's inequality, and the Sobolev inequality, we have

$$
\begin{aligned}
K_2 &= 2^{(p-1)\alpha}\gamma\int_{B_R(x_0)} |v|^{(p-1)\alpha}\left|E(\eta,v)\right| dx \\
&\leq 2^{(p-1)\alpha+\varepsilon}\gamma\frac{1+\varepsilon}{1-\varepsilon}\int_{B_R(x_0)} |v|^{(p-1)\alpha}|v\nabla\eta|^{1-\varepsilon} dx
\end{aligned}
$$

$$\begin{aligned}
&\le 2^{(p-1)\alpha+\varepsilon}\gamma\frac{1+\varepsilon}{1-\varepsilon}\left(\int_{B_R(x_0)}|v|^{(p-\varepsilon)\alpha}\,dx\right)^{\frac{p-1}{p-\varepsilon}}\left(\int_{B_R(x_0)}|v\nabla\eta|^{p-\varepsilon}\,dx\right)^{\frac{1-\varepsilon}{p-\varepsilon}}\\
&\le 2^{(p-1)\alpha+\varepsilon}\gamma\frac{1+\varepsilon}{1-\varepsilon}\left(\int_{B_R(x_0)}|v|^{\frac{np'}{n-p'}}\,dx\right)^{\frac{n-p'}{np'}(p-1)\alpha}\left(\alpha_nR^n\right)^{\frac{n-(n-p+1)\alpha}{n}\cdot\frac{p-1}{p-\varepsilon}}\\
&\quad\cdot\left(\int_{B_R(x_0)}|v\nabla\eta|^{p-\varepsilon}\,dx\right)^{\frac{1-\varepsilon}{p-\varepsilon}}\\
&\le 2^{(p-1)\alpha+\varepsilon}\gamma\frac{1+\varepsilon}{1-\varepsilon}\left(\int_{B_R(x_0)}|\nabla v|^{p'}\,dx\right)^{\frac{1}{p'}(p-1)\alpha}\left(\alpha_nR^n\right)^{\frac{n-(n-p+1)\alpha}{n}\cdot\frac{p-1}{p-\varepsilon}}\\
&\quad\cdot\left(\int_{B_R(x_0)}|v\nabla\eta|^{p-\varepsilon}\,dx\right)^{\frac{1-\varepsilon}{p-\varepsilon}}.
\end{aligned}$$

For $1\le\alpha<\frac{n}{n-(p-1)}$, by Young's inequality and then Holder's inequality, we have

$$\begin{aligned}
K_2&\le 2^{(p-1)\alpha+\varepsilon}\gamma\frac{1+\varepsilon}{1-\varepsilon}\left(\alpha_nR^n\right)^{\frac{n-(n-p+1)\alpha}{n}+\frac{(1-\varepsilon)\alpha}{n}}\varepsilon\eta\left(\int_{B_R(x_0)}|\nabla v|^{p-\varepsilon}\,dx\right)^{\alpha}\\
&\quad+2^{(p-1)\alpha+\varepsilon}\gamma\frac{1+\varepsilon}{1-\varepsilon}C(\varepsilon\eta)\int_{B_R(x_0)}|v\nabla\eta|^{p-\varepsilon}\,dx\\
&\le 2^{(p-1)\alpha+\varepsilon}\gamma\frac{1+\varepsilon}{1-\varepsilon}\left(\alpha_nR^n\right)^{\frac{n-(n-p+\varepsilon)\alpha}{n}}C\left(\|v\|_{W^{1,p-\varepsilon}}\right)\cdot\varepsilon\int_{B_R(x_0)}\eta|\nabla v|^{p-\varepsilon}\,dx\\
&\quad+2^{(p-1)\alpha+\varepsilon}\gamma\frac{1+\varepsilon}{1-\varepsilon}C(\varepsilon\eta)\int_{B_R(x_0)}|v\nabla\eta|^{p-\varepsilon}\,dx.
\end{aligned}\tag{3.15}$$

If $0<\alpha<1$, then using Young's inequality, Holder's inequality, and Young's inequality in turn, we can find that

$$\begin{aligned}
K_2&\le 2^{(p-1)\alpha+\varepsilon}\gamma\frac{1+\varepsilon}{1-\varepsilon}\varepsilon\eta\left(\int_{B_R(x_0)}|\nabla v|^{p'}\,dx\right)^{\frac{1}{p'}(p-\varepsilon)\alpha}\left(\alpha_nR^n\right)^{\frac{n-(n-p+1)\alpha}{n}}\\
&\quad+2^{(p-1)\alpha+\varepsilon}\gamma\frac{1+\varepsilon}{1-\varepsilon}C(\varepsilon\eta)\int_{B_R(x_0)}|v\nabla\eta|^{p-\varepsilon}\,dx\\
&\le 2^{(p-1)\alpha+\varepsilon}\gamma\frac{1+\varepsilon}{1-\varepsilon}\varepsilon\eta\left(\int_{B_R(x_0)}|\nabla v|^{p-\varepsilon}\,dx\right)^{\alpha}\left(\alpha_nR^n\right)^{\frac{n-(n-p+\varepsilon)\alpha}{n}}\\
&\quad+2^{(p-1)\alpha+\varepsilon}\gamma\frac{1+\varepsilon}{1-\varepsilon}C(\varepsilon\eta)\int_{B_R(x_0)}|v\nabla\eta|^{p-\varepsilon}\,dx\\
&\le 2^{(p-1)\alpha+\varepsilon}\gamma\frac{1+\varepsilon}{1-\varepsilon}\varepsilon\int_{B_R(x_0)}\eta|\nabla v|^{p-\varepsilon}\,dx+2^{(p-1)\alpha+\varepsilon}\gamma\frac{1+\varepsilon}{1-\varepsilon}\varepsilon\eta\left(\alpha_nR^n\right)^{1+\frac{(p-\varepsilon)\alpha}{n(1-\alpha)}}\\
&\quad+2^{(p-1)\alpha+\varepsilon}\gamma\frac{1+\varepsilon}{1-\varepsilon}C(\varepsilon\eta)\int_{B_R(x_0)}|v\nabla\eta|^{p-\varepsilon}\,dx.
\end{aligned}\tag{3.16}$$

From the estimates of (3.15) and (3.16), we have

$$K_2\le 2^{(p-1)\alpha+\varepsilon}\gamma\frac{1+\varepsilon}{1-\varepsilon}\max\left\{\left(\alpha_nR^n\right)^{\frac{n-(n-p+\varepsilon)\alpha}{n}}C\left(\|v\|_{W^{1,p-\varepsilon}}\right),1\right\}\varepsilon\int_{B_R(x_0)}\eta|\nabla v|^{p-\varepsilon}\,dx$$

$$+ 2^{(p-1)\alpha+\varepsilon}\gamma\frac{1+\varepsilon}{1-\varepsilon}C(\varepsilon\eta)\int_{B_R(x_0)}|v\nabla\eta|^{p-\varepsilon}\,dx$$
$$+ 2^{(p-1)\alpha+\varepsilon}\gamma\frac{1+\varepsilon}{1-\varepsilon}\varepsilon\eta\big(\alpha_n R^n\big)^{1+\frac{(p-\varepsilon)\alpha}{n(1-\alpha)}}.$$

Proceed to estimate $K_3$ by Young's inequality, the estimate of $E(\eta,v)$ and the definition of $\varphi(x)$, we can find that

$$\begin{aligned}
K_3 &= \int_{B_R(x_0)}\big(2^{p-1}(\gamma+1)|p_0|^{p-1} + 2^{(p-1)\alpha}\gamma\big(|u_0|+|p_0|\big)^{(p-1)\alpha} + \gamma\big|\varphi(x)\big|\big)\big|E(\eta,v)\big|\,dx \\
&\le 2^{\varepsilon}\frac{1+\varepsilon}{1-\varepsilon} \\
&\quad\times\int_{B_R(x_0)}\big(2^{p-1}(\gamma+1)|p_0|^{p-1} + 2^{(p-1)\alpha}\gamma\big(|u_0|+|p_0|\big)^{(p-1)\alpha} + \gamma\big|\varphi(x)\big|\big)|v\nabla\eta|^{1-\varepsilon}\,dx \\
&\le 2^{\varepsilon}\frac{1+\varepsilon}{1-\varepsilon}\varepsilon\int_{B_R(x_0)}\big(2^{p-1}(\gamma+1)|p_0|^{p-1} + 2^{(p-1)\alpha}\gamma\big(|u_0|+|p_0|\big)^{(p-1)\alpha} + \gamma\big|\varphi(x)\big|\big)^{\frac{p-\varepsilon}{p-1}}\,dx \\
&\quad+ 2^{\varepsilon}\frac{1+\varepsilon}{1-\varepsilon}C(\varepsilon)\int_{B_R(x_0)}|v\nabla\eta|^{p-\varepsilon}\,dx.
\end{aligned}$$

From the estimates of $K_1$, $K_2$ and $K_3$ we find that

$$I_2 \le C_9\varepsilon\int_{B_R(x_0)}|\eta\nabla v|^{p-\varepsilon}\,dx + C_{10}\int_{B_R(x_0)}|v\nabla\eta|^{p-\varepsilon}\,dx + C_{11}\varepsilon\int_{B_R(x_0)}dx,$$

where

$$\begin{aligned}
C_9 &= 1 + 2^{(p-1)\alpha+\varepsilon}\gamma\frac{1+\varepsilon}{1-\varepsilon}\max\big\{\big(\alpha_n R^n\big)^{\frac{n-(n-p+\varepsilon)\alpha}{n}}C\big(\|v\|_{W^{1,p-\varepsilon}}\big),1\big\}, \\
C_{10} &= C\big(\varepsilon\eta^{p-\varepsilon}\big)\bigg(2^{p-1+\varepsilon}(\gamma+1)\frac{1+\varepsilon}{1-\varepsilon}\bigg)^{\frac{p-\varepsilon}{1-\varepsilon}} + 2^{(p-1)\alpha+\varepsilon}\gamma\frac{1+\varepsilon}{1-\varepsilon}C(\varepsilon\eta) + 2^{\varepsilon}\frac{1+\varepsilon}{1-\varepsilon}C(\varepsilon), \\
C_{11} &= 2^{\varepsilon}\frac{1+\varepsilon}{1-\varepsilon}\Big[\big(2^{p-1}(\gamma+1)|p_0|^{p-1} + 2^{(p-1)\alpha}\gamma\big(|u_0|+|p_0|\big)^{(p-1)\alpha} + \gamma\big\|\varphi(x)\big\|_{L^{\frac{p-\varepsilon}{p-1}}}\big)^{\frac{p-\varepsilon}{p-1}} \\
&\quad+ 2^{(p-1)\alpha}\eta\big(\alpha_n R^n\big)^{\frac{(p-\varepsilon)\alpha}{n(1-\alpha)}}\Big].
\end{aligned}$$

By the condition (H3) and the estimate of $K_2$, we can obtain that

$$\begin{aligned}
I_3 &\le \int_{B_R(x_0)}\big((\gamma+1)|p_0|^{p-1} + \gamma|u|^{(p-1)\alpha} + \gamma\varphi(x)\big)|\eta\nabla v|^{1-\varepsilon}\,dx \\
&\le 2^{(p-1)\alpha}\gamma\int_{B_R(x_0)}|v|^{(p-1)\alpha}|\eta\nabla v|^{1-\varepsilon}\,dx \\
&\quad+ \int_{B_R(x_0)}\big((\gamma+1)|p_0|^{p-1} + 2^{(p-1)\alpha}\gamma\big(|u_0|+|p_0|\big)^{(p-1)\alpha} + \gamma\varphi(x)\big)|\eta\nabla v|^{1-\varepsilon}\,dx \\
&\le I_{31} + I_{32}.
\end{aligned}$$

Combing Holder's and Young's inequalities with Sobolev's theorem,

$$I_{31} = 2^{(p-1)\alpha}\gamma\int_{B_R(x_0)}|v|^{(p-1)\alpha}|\eta\nabla v|^{1-\varepsilon}\,dx$$

$$
\begin{aligned}
&\le 2^{(p-1)\alpha}\gamma\left(\int_{B_R(x_0)}|\eta\nabla v|^{p-\varepsilon}\,dx\right)^{\frac{1-\varepsilon}{p-\varepsilon}}\left(\int_{B_R(x_0)}|v|^{(p-\varepsilon)\alpha}\,dx\right)^{\frac{p-1}{p-\varepsilon}}\\
&\le 2^{(p-1)\alpha}\gamma\varepsilon\int_{B_R(x_0)}|\eta\nabla v|^{p-\varepsilon}\,dx+2^{(p-1)\alpha}\gamma C(\varepsilon)\int_{B_R(x_0)}|v|^{(p-\varepsilon)\alpha}\,dx\\
&\le 2^{(p-1)\alpha}\gamma\varepsilon\int_{B_R(x_0)}|\eta\nabla v|^{p-\varepsilon}\,dx\\
&\quad+2^{(p-1)\alpha}\gamma C(\varepsilon)\left(\int_{B_R(x_0)}|v|^{\frac{np'}{n-p'}}\,dx\right)^{\frac{n-p'}{np'}\cdot(p-\varepsilon)\alpha}\left(\alpha_nR^n\right)^{1-\frac{n-p'}{np'}\cdot(p-\varepsilon)\alpha}\\
&\le 2^{(p-1)\alpha}\gamma\varepsilon\int_{B_R(x_0)}|\eta\nabla v|^{p-\varepsilon}\,dx\\
&\quad+2^{(p-1)\alpha}\gamma C(\varepsilon)\left(\int_{B_R(x_0)}|\nabla v|^{p'}\,dx\right)^{\frac{1}{p'}\cdot(p-\varepsilon)\alpha}\left(\alpha_nR^n\right)^{1-\frac{n-p+1}{n}\cdot\alpha}.
\end{aligned}
$$

In the case of $1\le\alpha\le\frac{n}{n-(p-1)}$, noting that $v\in W^{1,p-\varepsilon}(\Omega)$ and $p'<p-\varepsilon$, we can find that

$$
\begin{aligned}
I_{31}&\le 2^{(p-1)\alpha}\gamma\varepsilon\int_{B_R(x_0)}|\eta\nabla v|^{p-\varepsilon}\,dx\\
&\quad+2^{(p-1)\alpha}\gamma C\left(\|v\|_{W^{1,p'}},\varepsilon\right)\left(-\!\!\!\!\!\!\int_{B_R(x_0)}|\nabla v|^{p'}\,dx\right)^{\frac{1}{p'}\cdot(p-\varepsilon)}\left(\alpha_nR^n\right)^{1+\frac{p-\varepsilon}{n}}.
\end{aligned}
$$

If $0<\alpha<1$, then by Young's inequality, we can obtain that

$$
\begin{aligned}
I_{31}&\le 2^{(p-1)\alpha}\gamma\varepsilon\int_{B_R(x_0)}|\eta\nabla v|^{p-\varepsilon}\,dx\\
&\quad+2^{(p-1)\alpha}\gamma C(\varepsilon)\left(\int_{B_R(x_0)}|\nabla v|^{p'}\,dx\right)^{\frac{1}{p'}\cdot(p-\varepsilon)\alpha}\left(\alpha_nR^n\right)^{\frac{(p-1)\alpha}{n}}\cdot\left(\alpha_nR^n\right)^{\frac{n(1-\alpha)}{n}}\\
&\le 2^{(p-1)\alpha}\gamma\varepsilon\int_{B_R(x_0)}|\eta\nabla v|^{p-\varepsilon}\,dx+2^{(p-1)\alpha}\gamma\varepsilon\alpha_nR^n\\
&\quad+2^{(p-1)\alpha}\gamma C(\varepsilon)\left(-\!\!\!\!\!\!\int_{B_R(x_0)}|\nabla v|^{p'}\,dx\right)^{\frac{1}{p'}\cdot(p-\varepsilon)}\left(\alpha_nR^n\right)^{1+\frac{(p-\varepsilon)}{n}}.
\end{aligned}
$$

This means that

$$
\begin{aligned}
I_{31}&\le 2^{(p-1)\alpha}\gamma\varepsilon\int_{B_R(x_0)}|\eta\nabla v|^{p-\varepsilon}\,dx+2^{(p-1)\alpha}\gamma\varepsilon\alpha_nR^n\\
&\quad+2^{(p-1)\alpha}\gamma\max\left\{C(\varepsilon),C\left(\|v\|_{W^{1,p'}},\varepsilon\right)\right\}\left(-\!\!\!\!\!\!\int_{B_R(x_0)}|\nabla v|^{p'}\,dx\right)^{\frac{1}{p'}\cdot(p-\varepsilon)}\left(\alpha_nR^n\right)^{1+\frac{(p-\varepsilon)}{n}}.
\end{aligned}
$$

Using Holder's inequality and then Young's inequality, yields that

$$
\begin{aligned}
I_{32}&=\int_{B_R(x_0)}\left((\gamma+1)|p_0|^{p-1}+2^{(p-1)\alpha}\gamma\left(|u_0|+|p_0|\right)^{(p-1)\alpha}+\gamma\varphi(x)\right)|\eta\nabla v|^{1-\varepsilon}\,dx\\
&\le\left(\int_{B_R(x_0)}|\eta\nabla v|^{p'}\,dx\right)^{\frac{1-\varepsilon}{p'}}\left(\alpha_nR^n\right)^{\frac{1-\varepsilon}{n}}\left(\alpha_nR^n\right)^{1-\frac{1-\varepsilon}{p'}-\frac{1-\varepsilon}{n}}
\end{aligned}
$$

$$
\begin{aligned}
&\quad \cdot \Big[(\gamma+1)|p_0|^{p-1} + 2^{(p-1)\alpha}\gamma\big(|u_0|+|p_0|\big)^{(p-1)\alpha} + \gamma\,\big\|\varphi(x)\big\|_{L^{\frac{p'}{p'-(1-\varepsilon)}}}\Big] \\
&\le C(\varepsilon)\bigg(-\!\!\!\!\!\!\int_{B_R(x_0)} |\eta\nabla v|^{p'}\,dx\bigg)^{\frac{p-\varepsilon}{p'}}\big(\alpha_n R^n\big)^{1+\frac{p-\varepsilon}{n}} + \varepsilon\big(\alpha_n R^n\big)^{\frac{n+1-\varepsilon}{n}} \\
&\quad \cdot \Big[(\gamma+1)|p_0|^{p-1} + 2^{(p-1)\alpha}\gamma\big(|u_0|+|p_0|\big)^{(p-1)\alpha} + \gamma\,\big\|\varphi(x)\big\|_{L^{\frac{p'}{p'-(1-\varepsilon)}}}\Big]^{\frac{p-\varepsilon}{p-1}}.
\end{aligned}
$$

Now, from the estimates of $I_{31}$ and $I_{32}$, we can find that

$$
\begin{aligned}
I_3 &\le 2^{(p-1)\alpha}\gamma\varepsilon\int_{B_R(x_0)} |\eta\nabla v|^{p-\varepsilon}\,dx \\
&\quad + C_{12}\bigg(-\!\!\!\!\!\!\int_{B_R(x_0)} |\nabla v|^{p'}\bigg)^{\frac{1}{p'}\cdot(p-\varepsilon)}\big(\alpha_n R^n\big)^{1+\frac{p-\varepsilon}{n}} + C_{13}\varepsilon\int_{B_R(x_0)} dx,
\end{aligned}
$$

with

$$
\begin{aligned}
C_{12} &= 2^{(p-1)\alpha}\gamma\max\big\{C(\varepsilon), C\big(\|v\|_{W^{1,p'}},\varepsilon\big)\big\} + C(\varepsilon), \\
C_{13} &= 2^{(p-1)\alpha}\gamma \\
&\quad + \big(\alpha_n R^n\big)^{\frac{1-\varepsilon}{n}}\cdot\Big[(\gamma+1)|p_0|^{p-1} + 2^{(p-1)\alpha}\gamma\big(|u_0|+|p_0|\big)^{(p-1)\alpha} + \gamma\,\big\|\varphi(x)\big\|_{L^{\frac{p'}{p'-(1-\varepsilon)}}}\Big]^{\frac{p-\varepsilon}{p-1}}.
\end{aligned}
$$

Finally, noting that $1+\frac{1}{n} < p < n$ and $p' = \frac{n(p-\varepsilon)}{n+1-\varepsilon} < p-\varepsilon$, then using Holder's inequality, Sobolev's inequality and Hodge decomposition in turn, we have

$$
\begin{aligned}
I_4 &= \int_{B_R(x_0)} |f||\phi|\,dx \\
&\le \bigg(\int_{B_R(x_0)} |f|^{\frac{np'/(1-\varepsilon)}{(n+1)p'/(1-\varepsilon)-n}}\,dx\bigg)^{\frac{(n+1)p'/(1-\varepsilon)-n}{np'/(1-\varepsilon)}}\bigg(\int_{B_R(x_0)} |\phi|^{\frac{np'/(1-\varepsilon)}{n-p'/(1-\varepsilon)}}\,dx\bigg)^{\frac{n-p'/(1-\varepsilon)}{np'/(1-\varepsilon)}} \\
&\le \bigg(\int_{B_R(x_0)} |f|^{\frac{np'/(1-\varepsilon)}{(n+1)p'/(1-\varepsilon)-n}}\,dx\bigg)^{\frac{(n+1)p'/(1-\varepsilon)-n}{np'/(1-\varepsilon)}}\bigg(\int_{B_R(x_0)} |\nabla\phi|^{p'/(1-\varepsilon)}\,dx\bigg)^{(1-\varepsilon)/p'} \\
&\le \bigg(\int_{B_R(x_0)} |f|^{\frac{np'/(1-\varepsilon)}{(n+1)p'/(1-\varepsilon)-n}}\,dx\bigg)^{\frac{(n+1)p'/(1-\varepsilon)-n}{np'/(1-\varepsilon)}} \\
&\quad \cdot\bigg(\int_{B_R(x_0)} \big|E(\eta,v) + |\eta\nabla v|^{-\varepsilon}\eta\nabla v - H\big|^{p'/(1-\varepsilon)}\,dx\bigg)^{(1-\varepsilon)/p'} \\
&\le \bigg(\int_{B_R(x_0)} |f|^{\frac{np'/(1-\varepsilon)}{(n+1)p'/(1-\varepsilon)-n}}\,dx\bigg)^{\frac{(n+1)p'/(1-\varepsilon)-n}{np'/(1-\varepsilon)}}\cdot 2^{1+\varepsilon}\frac{1+\varepsilon}{1-\varepsilon}\bigg(\int_{B_R(x_0)} |v\nabla\eta|^{p'}\,dx\bigg)^{(1-\varepsilon)/p'} \\
&\quad + 2\bigg(\int_{B_R(x_0)} |f|^{\frac{np'/(1-\varepsilon)}{(n+1)p'/(1-\varepsilon)-n}}\,dx\bigg)^{\frac{(n+1)p'/(1-\varepsilon)-n}{np'/(1-\varepsilon)}}\bigg(\int_{B_R(x_0)} |\eta\nabla v|^{p'}\,dx\bigg)^{(1-\varepsilon)/p'} \\
&\quad + 2\bigg(\int_{B_R(x_0)} |f|^{\frac{np'/(1-\varepsilon)}{(n+1)p'/(1-\varepsilon)-n}}\,dx\bigg)^{\frac{(n+1)p'/(1-\varepsilon)-n}{np'/(1-\varepsilon)}}\bigg(\int_{B_R(x_0)} |H|^{p'/(1-\varepsilon)}\,dx\bigg)^{(1-\varepsilon)/p'} \\
&= L_1 + L_2 + L_3.
\end{aligned}
$$

Noting that $p\prime < p - \varepsilon$, using Holder's inequality and then Young's inequality, we can find that

$$\begin{aligned}
L_1 &= \left(\int_{B_R(x_0)} |f|^{\frac{np\prime/(1-\varepsilon)}{(n+1)p\prime/(1-\varepsilon)-n}}\,dx\right)^{\frac{(n+1)p\prime/(1-\varepsilon)-n}{np\prime/(1-\varepsilon)}} \cdot 2^{1+\varepsilon}\frac{1+\varepsilon}{1-\varepsilon}\left(\int_{B_R(x_0)} |v\nabla\eta|^{p'}\,dx\right)^{(1-\varepsilon)/p\prime} \\
&\le 2^{1+\varepsilon}\frac{1+\varepsilon}{1-\varepsilon}\left(\int_{B_R(x_0)} |f|^{\frac{np\prime/(1-\varepsilon)}{(n+1)p\prime/(1-\varepsilon)-n}}\,dx\right)^{\frac{(n+1)p\prime/(1-\varepsilon)-n}{np\prime/(1-\varepsilon)}} \\
&\quad \cdot \left(\alpha_n R^n\right)^{(1-\frac{p\prime}{p-\varepsilon})(\frac{1-\varepsilon}{p\prime})}\left(\int_{B_R(x_0)} |v\nabla\eta|^{p-\varepsilon}\,dx\right)^{\frac{(1-\varepsilon)}{p-\varepsilon}} \\
&\le 2^{1+\varepsilon}\frac{1+\varepsilon}{1-\varepsilon}\varepsilon\|f\|^{\frac{p-\varepsilon}{p-1}}_{L^{\frac{np\prime/(1-\varepsilon)}{(n+1)p\prime/(1-\varepsilon)-n}}}\left(\alpha_n R^n\right)^{1+\frac{p-\varepsilon}{n(p-1)}} \\
&\quad + 2^{1+\varepsilon}\frac{1+\varepsilon}{1-\varepsilon}C(\varepsilon)\int_{B_R(x_0)} |v\nabla\eta|^{p-\varepsilon}\,dx.
\end{aligned}$$

By Young's inequality, we have

$$\begin{aligned}
L_2 &= 2\left(\int_{B_R(x_0)} |f|^{\frac{np\prime/(1-\varepsilon)}{(n+1)p\prime/(1-\varepsilon)-n}}\,dx\right)^{\frac{(n+1)p\prime/(1-\varepsilon)-n}{np\prime/(1-\varepsilon)}}\left(\int_{B_R(x_0)} |\eta\nabla v|^{p'}\,dx\right)^{(1-\varepsilon)/p\prime} \\
&\le 2\|f\|_{L^{\frac{np\prime/(1-\varepsilon)}{(n+1)p\prime/(1-\varepsilon)-n}}}\left(\alpha_n R^n\right)^{1+\frac{1}{n}-\frac{(n+p-\varepsilon)(1-\varepsilon)}{n(p-\varepsilon)}} \\
&\quad \times \left(\fint_{B_R(x_0)} |\eta\nabla v|^{p'}\,dx\right)^{(1-\varepsilon)/p\prime}\left(\alpha_n R^n\right)^{\frac{(n+1-\varepsilon)(1-\varepsilon)}{n(p-\varepsilon)}+\frac{(p-1)(1-\varepsilon)}{n(p-\varepsilon)}} \\
&\le \varepsilon\|f\|^{\frac{p-\varepsilon}{p-1}}_{L^{\frac{np\prime/(1-\varepsilon)}{(n+1)p\prime/(1-\varepsilon)-n}}}\left(\alpha_n R^n\right)^{1+\frac{\varepsilon(p-\varepsilon)}{n(p-1)}} \\
&\quad + C(\varepsilon)\left(\fint_{B_R(x_0)} |\eta\nabla v|^{p'}\,dx\right)^{\frac{(p-\varepsilon)}{p\prime}}\left(\alpha_n R^n\right)^{1+\frac{p-\varepsilon}{n}}.
\end{aligned}$$

Using the estimate of $H$, Holder's and Young's inequalities, in turn, we have

$$\begin{aligned}
L_3 &= 2\left(\int_{B_R(x_0)} |f|^{\frac{np\prime/(1-\varepsilon)}{(n+1)p\prime/(1-\varepsilon)-n}}\,dx\right)^{\frac{(n+1)p\prime/(1-\varepsilon)-n}{np\prime/(1-\varepsilon)}}\left(\int_{B_R(x_0)} |H|^{p\prime/(1-\varepsilon)}\,dx\right)^{(1-\varepsilon)/p\prime} \\
&\le 2\left(\int_{B_R(x_0)} |f|^{\frac{np\prime/(1-\varepsilon)}{(n+1)p\prime/(1-\varepsilon)-n}}\,dx\right)^{\frac{(n+1)p\prime/(1-\varepsilon)-n}{np\prime/(1-\varepsilon)}} \\
&\quad \cdot \left(\alpha_n R^n\right)^{(1-\frac{p\prime}{p-\varepsilon})\frac{1-\varepsilon}{p\prime}}\left(\int_{B_R(x_0)} |H|^{\frac{p-\varepsilon}{1-\varepsilon}}\,dx\right)^{\frac{1-\varepsilon}{p-\varepsilon}} \\
&\le 2\|f\|_{L^{\frac{np\prime/(1-\varepsilon)}{(n+1)p\prime/(1-\varepsilon)-n}}}\left(\alpha_n R^n\right)^{1+\frac{1}{n}-\frac{(n+1-\varepsilon)(1-\varepsilon)}{n(p-\varepsilon)}} \\
&\quad \cdot \left(\alpha_n R^n\right)^{\frac{(1-\varepsilon)^2}{n(p-\varepsilon)}}C(n,p)\varepsilon\left(\int_{B_R(x_0)} \left|\nabla(\eta v)\right|^{p-\varepsilon}\,dx\right)^{\frac{1-\varepsilon}{p-\varepsilon}} \\
&\le 2^{2-\varepsilon}\|f\|_{L^{\frac{np\prime/(1-\varepsilon)}{(n+1)p\prime/(1-\varepsilon)-n}}}\left(\alpha_n R^n\right)^{1+\frac{1}{n}-\frac{n(1-\varepsilon)}{n(p-\varepsilon)}}
\end{aligned}$$

$$
\begin{aligned}
&\cdot C(n,p)\varepsilon\left[\left(\int_{B_R(x_0)}|\eta\nabla v|^{p-\varepsilon}\,dx\right)^{\frac{1-\varepsilon}{p-\varepsilon}}+\left(\int_{B_R(x_0)}|v\nabla\eta|^{p-\varepsilon}\,dx\right)^{\frac{1-\varepsilon}{p-\varepsilon}}\right]\\
&\le 2^{3-\varepsilon}C(n,p)\varepsilon\|f\|^{\frac{p-\varepsilon}{p-1}}_{L^{\frac{np'/(1-\varepsilon)}{(n+1)p'/(1-\varepsilon)-n}}}\left(\alpha_nR^n\right)^{1+\frac{p-\varepsilon}{n(p-1)}}\\
&\quad+2^{2-\varepsilon}C(n,p)\varepsilon\left[\int_{B_R(x_0)}|\eta\nabla v|^{p-\varepsilon}\,dx+\int_{B_R(x_0)}|v\nabla\eta|^{p-\varepsilon}\,dx\right].
\end{aligned}
$$

From $L_1$, $L_2$, and $L_3$, we can find that

$$
\begin{aligned}
I_4&\le C_{14}\int_{B_R(x_0)}|v\nabla\eta|^{p-\varepsilon}\,dx+2^{2-\varepsilon}C(n,p)\varepsilon\int_{B_R(x_0)}|\eta\nabla v|^{p-\varepsilon}\,dx\\
&\quad+C(\varepsilon)\left(\fint_{B_R(x_0)}|\eta\nabla v|^{p'}\,dx\right)^{\frac{(p-\varepsilon)}{p'}}\left(\alpha_nR^n\right)^{1+\frac{p-\varepsilon}{n}}\\
&\quad+C_{15}\varepsilon\|f\|^{\frac{p-\varepsilon}{p-1}}_{L^{\frac{np'/(1-\varepsilon)}{(n+1)p'/(1-\varepsilon)-n}}}\left(\alpha_nR^n\right),
\end{aligned}
$$

with

$$
\begin{aligned}
C_{14}&=2^{1+\varepsilon}\frac{1+\varepsilon}{1-\varepsilon}C(\varepsilon)+2^{2-\varepsilon}C(n,p)\varepsilon,\\
C_{15}&=2^{1+\varepsilon}\frac{1+\varepsilon}{1-\varepsilon}\left(\alpha_nR^n\right)^{\frac{p-\varepsilon}{n(p-1)}}+\left(\alpha_nR^n\right)^{\frac{\varepsilon(p-\varepsilon)}{n(p-1)}}+2^{3-\varepsilon}C(n,p)\left(\alpha_nR^n\right)^{\frac{p-\varepsilon}{n(p-1)}}.
\end{aligned}
$$

From the estimates of $I_1$, $I_2$, $I_3$ and $I_4$, we have

$$
\begin{aligned}
&\beta\int_{B_R(x_0)}\eta^{1-\varepsilon}|\nabla v|^{p-\varepsilon}\,dx\\
&\quad\le\varepsilon\left[C_6+C_9+2^{(p-1)\alpha}\gamma+2^{2-\varepsilon}C(n,p)\right]\int_{B_R(x_0)}\eta^{1-\varepsilon}|\nabla v|^{p-\varepsilon}\,dx\\
&\qquad+[C_7\varepsilon+C_{10}+C_{14}]\int_{B_R(x_0)}|v\nabla\eta|^{p-\varepsilon}\,dx\\
&\qquad+\left[C_{12}+C(\varepsilon)\right]\left(\fint_{B_R(x_0)}|\nabla v|^{p'}\,dx\right)^{\frac{1}{p'}(p-\varepsilon)}\left(\alpha_nR^n\right)^{1+\frac{p-\varepsilon}{n}}\\
&\qquad+\varepsilon\left[C_8+C_{11}+C_{13}+C_{15}\|f\|^{\frac{p-\varepsilon}{p-1}}_{L^{\frac{np'/(1-\varepsilon)}{(n+1)p'/(1-\varepsilon)-n}}}\right]\left(\alpha_nR^n\right).
\end{aligned}
$$

Choosing $\varepsilon$ small enough such that it satisfies

$$
\beta-\varepsilon\left[C_6+C_9+2^{(p-1)\alpha}\gamma+2^{2-\varepsilon}C(n,p)\right]>0\quad\text{and}\quad 0<\varepsilon<R^p,
$$

and further letting the integral region of the left side in the former estimate formula is $B_{R/2}(x_0)$, then by Lemma 2.5, we can find that there exists an integral coefficient $r>p-\varepsilon$, such that $u\in W^{1,r}(\Omega)$ and

$$
\begin{aligned}
\int_{B_{R/2}(x_0)}|\nabla v|^r\,dx&\le C_1'\int_{B_R(x_0)}|v\nabla\eta|^r\,dx+C_2'\int_{B_R(x_0)}R^p\,dx\\
&\quad+C_3'\left(-\int_{B_R(x_0)}|\nabla v|^{p-\varepsilon}\,dx\right)^{\frac{r}{p-\varepsilon}}\left(\alpha_nR^n\right)^{1+\frac{r}{n}}.
\end{aligned}
$$

Applying Lemma 2.5 to repeat the above derivation process over and over again, we can finally complete the proof of Theorem 3.1. □

## 4 Decay estimate

In this section, our primary purpose is to establish the decay estimate. This is a critical step for proving the regularization of very weak solutions to system (1.1). Here, we use the $p$-harmonic approximation technique to establish the regularity result. Therefore, we should verify the conditions of the $p$-harmonic approximation lemma first. That is,

**Lemma 4.1** *Suppose that $u \in W^{1,p-\varepsilon}(\Omega)$ is a very weak solution to system (1.1) under the conditions (H1)-(H4), then for every $x_0 \in \Omega$, $u_0 \in R^N$, $p_0 \in R^{nN}$, $0 < \rho \leq R \leq 1$, and arbitrary $\phi \in C_0^1(B_\rho(x_0), R^N)$ with $\sup_{B_\rho(x_0)} |\nabla\phi| \leq 1$, the integral inequality*

$$\left|\rho^{p-n}\int_{B_\rho(x_0)} |\nabla v|^{p-2}\nabla v\cdot\nabla\phi\,dx\right| \leq \alpha_n\rho^p\left[C_{19}\Phi\big(x_0,\rho,(\nabla u)_{x_o,\rho}\big)+C_{18}\right]\cdot\sup_{B_\rho(x_0)}|\nabla\phi|,$$

*holds for $v = u - u_0 - p_0(x - x_0)$ and*

$$\Phi\big(x_0,\rho,(\nabla u)_{x_o,\rho}\big) = \fint_{B_\rho(x_0)} |\nabla v|^p\,dx.$$

*Proof* By Theorem 3.1, we can find that $u \in W^{1,p}(\Omega)$.

Let $B_\rho(x_0) \subset B_R(x_0)$ be an arbitrary ball, by the definition of very weak solutions to the system (1.1), we can find that

$$\begin{aligned}&\int_{B_\rho(x_0)}\big[A(x,u,\nabla u)-A(x,u,p_0)\big]\cdot\nabla\phi\,dx\\&\quad=\int_{B_\rho(x_0)} f(x)\cdot\phi\,dx-\int_{B_\rho(x_0)}|\nabla u|^{p-2}\nabla u\cdot\nabla\phi\,dx-\int_{B_\rho(x_0)}A(x,u,p_0)\cdot\nabla\phi\,dx.\end{aligned}$$

By the condition (H2), we can deduce that

$$\begin{aligned}&\beta\int_{B_\rho(x_0)}|\nabla u-p_0|^{p-2}(\nabla u-p_0)\cdot\nabla\phi\,dx\\&\quad\leq\beta\int_{B_\rho(x_0)}\big(|\nabla u|+|p_0|\big)^{p-2}(\nabla u-p_0)\cdot\nabla\phi\,dx\\&\quad\leq\int_{B_\rho(x_0)}\big[A(x,u,\nabla u)-A(x,u,p_0)\big]\cdot\nabla\phi\,dx.\end{aligned}$$

Combining the above two equations, we can obtain that,

$$\beta\int_{B_\rho(x_0)}|\nabla u-p_0|^{p-2}(\nabla u-p_0)\cdot\nabla\phi\,dx\leq L_{41}+L_{42}+L_{43},$$

where

$$L_{41}=\int_{B_\rho(x_0)} f(x)\cdot\phi\,dx,$$

$$L_{42} = -\int_{B_\rho(x_0)} |\nabla u|^{p-2}\nabla u \cdot \nabla\phi \, dx,$$

$$L_{43} = -\int_{B_\rho(x_0)} A(x,u,p_0) \cdot \nabla\phi \, dx.$$

For $\phi \in C_0^1(B_\rho(x_0), R^N)$, by Holder's inequality and then Sobolev's inequality, we can estimate that

$$\begin{aligned} L_{41} &\le \left(\int_{B_\rho(x_0)} \left|f(x)\right|^{\frac{n(p-\varepsilon)}{n(p-1)+(p-\varepsilon)}} dx\right)^{\frac{n(p-1)+(p-\varepsilon)}{n(p-\varepsilon)}} \left(\int_{B_\rho(x_0)} |\phi|^{\frac{n(p-\varepsilon)}{n(1-\varepsilon)-(p-\varepsilon)}}\right)^{\frac{n(1-\varepsilon)-(p-\varepsilon)}{n(p-\varepsilon)}} \\ &\le \left(\int_{B_\rho(x_0)} \left|f(x)\right|^{\frac{n(p-\varepsilon)}{n(p-1)+(p-\varepsilon)}} dx\right)^{\frac{n(p-1)+(p-\varepsilon)}{n(p-\varepsilon)}} \left(\int_{B_\rho(x_0)} |\nabla\phi|^{\frac{p-\varepsilon}{1-\varepsilon}} dx\right)^{\frac{1-\varepsilon}{p-\varepsilon}} \\ &\le \sup_{B_\rho(x_0)} |\nabla\phi| \cdot \|f\|_{L^{\frac{n(p-\varepsilon)}{n(p-1)+(p-\varepsilon)}}} \left(\alpha_n\rho^n\right)^{\frac{n+1}{n}}. \end{aligned}$$

Using Holder's inequality and Young's inequality again, yields that

$$\begin{aligned} L_{42} &\le \sup_{B_\rho(x_0)} |\nabla\phi| \int_{B_\rho(x_0)} |\nabla u|^{p-1} dx \\ &\le \sup_{B_\rho(x_0)} |\nabla\phi| \cdot 2^{p-1}\left[\int_{B_\rho(x_0)} |\nabla u - p_0|^{p-1} dx + \int_{B_\rho(x_0)} |p_0|^{p-1} dx\right] \\ &\le \sup_{B_\rho(x_0)} |\nabla\phi| \cdot 2^{p-1}\left[\left(\int_{B_\rho(x_0)} |\nabla u - p_0|^p dx\right)^{\frac{p-1}{p}} \left(\int_{B_\rho(x_0)} dx\right)^{\frac{1}{p}} + |p_0|^{p-1}\alpha_n\rho^n\right] \\ &\le \sup_{B_\rho(x_0)} |\nabla\phi| \cdot 2^{p-1}\left[\left(\fint_{B_\rho(x_0)} |\nabla u - p_0|^p dx\right) + \left(|p_0|^{p-1}+1\right)\right]\alpha_n\rho^n. \end{aligned}$$

Finally, by the condition (H3), we find that

$$\begin{aligned} L_{43} &\le \sup_{B_\rho(x_0)} |\nabla\phi| \int_{B_\rho(x_0)} \left|A(x,u,p_0)\right| dx \\ &\le \sup_{B_\rho(x_0)} |\nabla\phi| \int_{B_\rho(x_0)} \gamma\left[|p_0|^{p-1} + |u|^{(p-1)\alpha} + \varphi(x)\right] dx \\ &\le \gamma \sup_{B_\rho(x_0)} |\nabla\phi| \Big[\left(|p_0|^{p-1} + 2^{(p-1)\alpha}\left(|u_0|^{(p-1)\alpha} + |p_0|^{(p-1)\alpha}\right) + \left\|\varphi(x)\right\|_{L^1}\right)\alpha_n\rho^n \\ &\quad + \int_{B_\rho(x_0)} \left|u - u_0 - p_0(x-x_0)\right|^{(p-1)\alpha} dx\Big]. \end{aligned}$$

If $0 < \alpha \le 1$, by Holder's inequality, Young's inequality, and Poincare's inequality, in turn, we have

$$\begin{aligned} &\int_{B_\rho(x_0)} \left|u - u_0 - p_0(x-x_0)\right|^{(p-1)\alpha} dx \\ &\quad \le \left(\int_{B_\rho(x_0)} \left|u - u_0 - p_0(x-x_0)\right|^p dx\right)^{\frac{(p-1)\alpha}{p}} \left(\int_{B_\rho(x_0)} dx\right)^{1-\frac{(p-1)\alpha}{p}} \end{aligned}$$

$$
\begin{aligned}
&\le \int_{B_\rho(x_0)} \left|u-u_0-p_0(x-x_0)\right|^p dx + \int_{B_\rho(x_0)} dx \\
&\le \int_{B_\rho(x_0)} |\nabla u - p_0|^p dx + \int_{B_\rho(x_0)} dx = \left[ ⨍_{B_\rho(x_0)} |\nabla u - p_0|^p dx + 1 \right] \alpha_n \rho^n.
\end{aligned}
$$

Furthermore, in the case of $1 < \alpha < \frac{n}{n-(p-1)}$, using Holder's inequality, Young's inequality, and then Sobolev's inequality, we can derive that

$$
\begin{aligned}
&\int_{B_\rho(x_0)} \left|u-u_0-p_0(x-x_0)\right|^{(p-1)\alpha} dx \\
&\le \left( \int_{B_\rho(x_0)} \left|u-u_0-p_0(x-x_0)\right|^{\frac{n(p-1)}{n-(p-1)}} dx \right)^{\frac{n-(p-1)\alpha}{n}} \left( \int_{B_\rho(x_0)} dx \right)^{1-\frac{n-(p-1)\alpha}{n}} \\
&\le \int_{B_\rho(x_0)} \left|u-u_0-p_0(x-x_0)\right|^{\frac{n(p-1)}{n-(p-1)}} dx + \int_{B_\rho(x_0)} dx \\
&\le \left( \int_{B_\rho(x_0)} |\nabla u - p_0|^{p-1} dx \right)^{\frac{n}{n-(p-1)}} + \int_{B_\rho(x_0)} dx \\
&\le \left( \int_{B_\rho(x_0)} |\nabla u - p_0|^{p} dx \right)^{\frac{n(p-1)}{p[n-(p-1)]}} \left( \int_{B_\rho(x_0)} dx \right)^{\frac{n}{p[n-(p-1)]}} + \int_{B_\rho(x_0)} dx \\
&\le \left( \int_{B_\rho(x_0)} |\nabla u - p_0|^{p} dx \right)^{\frac{n}{n-(p-1)}} + \left( \int_{B_\rho(x_0)} dx \right)^{\frac{n}{n-(p-1)}} + \int_{B_\rho(x_0)} dx \\
&\le \alpha_n \rho^n \left[ C\left(\|v\|_{W^{1,p}}\right) ⨍_{B_\rho(x_0)} |\nabla u - p_0|^p dx + \left( \left(\alpha_n \rho^n\right)^{\frac{p-1}{n-(p-1)}} + 1 \right) \right].
\end{aligned}
$$

Combining the estimates of both the case of $0 < \alpha \le 1$ and the case of $1 < \alpha < \frac{n}{n-(p-1)}$ for $\int_{B_\rho(x_0)} |u-u_0-p_0(x-x_0)|^{(p-1)\alpha} dx$, we can find that

$$
\begin{aligned}
L_{43} &\le \gamma \alpha_n \rho^n \sup_{B_\rho(x_0)} |\nabla \phi| \left[ |p_0|^{p-1} + 2^{(p-1)\alpha} \left( |u_0|^{(p-1)\alpha} + |p_0|^{(p-1)\alpha} \right) + \left\| \varphi(x) \right\|_{L^1} \right. \\
&\quad \left. + \max\left\{ C\left(\|v\|_{W^{1,p}}\right), 1 \right\} ⨍_{B_\rho(x_0)} |\nabla u - p_0|^p dx + \left(\alpha_n \rho^n\right)^{\frac{p-1}{n-(p-1)}} + 1 \right] \\
&\le \alpha_n \rho^n \sup_{B_\rho(x_0)} |\nabla \phi| \left[ C_{16} + C_{17} ⨍_{B_\rho(x_0)} |\nabla u - p_0|^p dx \right],
\end{aligned}
$$

where

$$
\begin{aligned}
&C_{16} = \gamma \left[ |p_0|^{p-1} + 2^{(p-1)\alpha} \left( |u_0|^{(p-1)\alpha} + |p_0|^{(p-1)\alpha} \right) + \left\| \varphi(x) \right\|_{L^1} + \left(\alpha_n \rho^n\right)^{\frac{p-1}{n-(p-1)}} + 1 \right], \\
&C_{17} = \gamma \max\left\{ C\left(\|v\|_{W^{1,p}}\right), 1 \right\}.
\end{aligned}
$$

Now, from the estimates of $L_{41}$, $L_{42}$ and $L_{43}$ we have that

$$
\begin{aligned}
&\int_{B_\rho(x_0)} |\nabla u - p_0|^{p-2} (\nabla u - p_0) \cdot \nabla \phi \, dx \\
&\le \alpha_n \rho^n \sup_{B_\rho(x_0)} |\nabla \phi| \cdot \frac{1}{\beta} \left[ \left(\alpha_n \rho^n\right)^{\frac{1}{n}} \|f\|_{L^{\frac{n(p-\varepsilon)}{n(p-1)+(p-\varepsilon)}}} \right.
\end{aligned}
$$

$$+ \left(2^{p-1} + C_{17}\right) \fint_{B_\rho(x_0)} |\nabla u - p_0|^p \, dx + 2^{p-1}\left(|p_0|^{p-1} + 1\right) + C_{16}\Big]$$
$$\le \alpha_n \rho^n \sup_{B_\rho(x_0)} |\nabla \phi| \left[ C_{18} + C_{19} \fint_{B_\rho(x_0)} |\nabla u - p_0|^p \, dx \right],$$

with

$$C_{18} = \frac{1}{\beta}\left[\left(\alpha_n \rho^n\right)^{\frac{1}{n}} \|f\|_{L^{\frac{n(p-\varepsilon)}{n(p-1)+(p-\varepsilon)}}} + 2^{p-1}\left(|p_0|^{p-1} + 1\right) + C_{16}\right],$$
$$C_{19} = \frac{1}{\beta}\left[2^{p-1} + C_{17}\right].$$

Let

$$\Phi\left(x_0, \rho, (\nabla u)_{x_0,\rho}\right) = \fint_{B_\rho(x_0)} |\nabla u - p_0|^p \, dx = \fint_{B_\rho(x_0)} |\nabla v|^p \, dx,$$

then,

$$\int_{B_\rho(x_0)} |\nabla u - p_0|^{p-2}(\nabla u - p_0) \cdot \nabla \phi \, dx \le \alpha_n \rho^n \left[C_{18} + C_{19} \Phi\left(x_0, \rho, (\nabla u)_{x_0,\rho}\right)\right] \sup_{B_\rho(x_0)} |\nabla \phi|.$$

The proof of Lemma 4.1 is complete. □

**Lemma 4.2** *Assume that $u \in W^{1,p-\varepsilon}(\Omega)$ is a very weak solution to the system* (1.1) *under the conditions* (H1)-(H3), *then there exists a constant $\delta = \delta(n, N, p) > 0$ such that for $0 < \rho < R \le 1$, the smallness condition*

$$\Phi\left(x_0, \rho, (\nabla u)_{x_0,\rho}\right) \le \left(\frac{\delta}{2}\right)^p$$

*holds, then for $0 < \theta < \frac{1}{4}$, the decay estimate*

$$\Phi\left(x_0, \theta\rho, (\nabla u)_{x_0,\theta\rho}\right) \le \theta^p\left[C_{20} \Phi\left(x_0, \rho, (\nabla u)_{x_0,\rho}\right) + C_{21}\rho^p\right],$$

*holds.*

*Proof* To prove the decay estimate, we should use the Caccioppoli second inequality.

Now, taking $u_0 = u_{x_0,2\theta\rho}$ and $p_0 = p_0 + \mu \nabla h(x_0)$ in Theorem 3.1, we can obtain that

$$\begin{aligned} &\int_{B_{\theta\rho}(x_0)} |\nabla u - (p_0 + \mu \nabla h(x_0)|^p \, dx \\ &\le \tilde{C}_1 \int_{B_{2\theta\rho}(x_0)} \frac{|u - u_{x_0,2\theta\rho} - (p_0 + \mu \nabla h(x_0))(x - x_0)|^p}{(2\theta\rho)^p} \, dx + \tilde{C}_2 \int_{B_{2\theta\rho}(x_0)} (2\theta\rho)^p \, dx \\ &\quad + \tilde{C}_3 (\fint_{B_{\theta\rho}(x_0)} |\nabla u - \left(p_0 + \mu \nabla h(x_0)\right)|^{p-\varepsilon} \, dx)^{\frac{p}{p-\varepsilon}} \left(\alpha_n (2\theta\rho)^n\right)^{1+\frac{p}{n}}, \end{aligned} \tag{4.1}$$

where

$$\mu = \left[\rho^p \alpha_n^{\frac{p}{p-1}} \left(C_{19}^{\frac{p}{p-1}} \Phi\left(x_0, \rho, (\nabla u)_{x_0,\rho}\right) + \left(\frac{2}{\delta} C_{18}\right)^{\frac{p}{p-1}}\right)\right]^{\frac{1}{p}}.$$

For establishing a decay estimate, the main aim now is to control the first term in the right-hand side of (4.1).

Thus, for the corresponding constant $\delta \in (0,1]$ in the $p$-harmonic mapping Lemma 2.1, we suppose that

$$\begin{aligned} w(x) &= \frac{u - u_{x_0,\rho} - p_0(x-x_0)}{\mu} \\ &= \frac{u - u_{x_0,\rho} - p_0(x-x_0)}{[\rho^p \alpha_n^{\frac{p}{p-1}} (C_{19}^{\frac{p}{p-1}} \Phi(x_0,\rho,(\nabla u)_{x_0,\rho}) + (\frac{2}{\delta}C_{18})^{\frac{p}{p-1}})]^{\frac{1}{p}}}. \end{aligned}$$

By Lemma 2.1, we can find that for arbitrary $\phi \in C_0^1(B_\rho(x_0), R^N)$, that

$$\begin{aligned} &\left| \rho^{p-n} \int_{B_\rho(x_0)} |\nabla w|^{p-2} \nabla w \cdot \nabla \phi \, dx \right| \\ &\quad \leq \frac{\rho^{p-n} \int_{B_\rho(x_0)} |\nabla u - p_0|^{p-2} (\nabla u - p_0) \cdot \nabla \phi \, dx}{[\rho^p \alpha_n^{\frac{p}{p-1}} (C_{19}^{\frac{p}{p-1}} \Phi(x_0,\rho,(\nabla u)_{x_0,\rho}) + (\frac{2}{\delta}C_{18})^{\frac{p}{p-1}})]^{\frac{p-1}{p}}} \\ &\quad \leq \frac{\rho^p [C_{18} + C_{19} \Phi(x_0,\rho,(\nabla u)_{x_0,\rho})]}{[\rho^p (C_{19}^{\frac{p}{p-1}} \Phi(x_0,\rho,(\nabla u)_{x_0,\rho}) + (\frac{2}{\delta}C_{18})^{\frac{p}{p-1}})]^{\frac{p-1}{p}}} \cdot \sup_{B_\rho(x_0)} |\nabla \phi| \\ &\quad \leq \rho \left[ \frac{\delta}{2} + \Phi^{\frac{1}{p}} \left( x_0,\rho,(\nabla u)_{x_0,\rho} \right) \right] \cdot \sup_{B_\rho(x_0)} |\nabla \phi|. \end{aligned}$$

Assume that the smallness condition

$$\Phi^{\frac{1}{p}}(x_0,\rho,p_0) \leq \frac{\delta}{2}$$

holds, then we can compute that

$$\left| \rho^{p-n} \int_{B_\rho(x_0)} |\nabla w|^{p-2} \nabla w \cdot \nabla \phi \, dx \right| \leq \rho\delta \cdot \sup_{B_\rho(x_0)} |\nabla \phi|$$

and

$$\begin{aligned} \rho^{p-n} \int_{B_\rho(x_0)} |\nabla w|^p \, dx &\leq \frac{\rho^{p-n} \int_{B_\rho(x_0)} |\nabla u - p_0|^p \, dx}{\rho^p \alpha_n^{\frac{p}{p-1}} [C_{19}^{\frac{p}{p-1}} \Phi(x_0,\rho,(\nabla u)_{x_0,\rho}) + (\frac{2}{\delta}C_{18})^{\frac{p}{p-1}}]} \\ &\leq \frac{\rho^p \alpha_n \Phi(x_0,\rho,(\nabla u)_{x_0,\rho})}{\rho^p C_{19}^{\frac{p}{p-1}} \alpha_n^{\frac{p}{p-1}} \Phi(x_0,\rho,(\nabla u)_{x_0,\rho})} \\ &\leq \frac{1}{C_{19}^{\frac{p}{p-1}} \alpha_n^{\frac{1}{p-1}}} < 1. \end{aligned}$$

Note that

$$\left| \rho^{p-n} \int_{B_\rho(x_0)} |\nabla w|^{p-2} \nabla w \cdot \nabla \phi \, dx \right| \leq \rho\delta \cdot \sup_{B_\rho(x_0)} |\nabla \phi|$$

and

$$\rho^{p-n} \int_{B_\rho(x_0)} |\nabla w|^p \, dx < 1$$

are exactly the two conditions required in the $p$-harmonic approximation lemma. Therefore, we can apply Lemma 2.1 to find that: there exists a $p$-harmonic approximation function $h \in W^{1,p}(B_\rho(x_0), R^N)$ such that the following smallness conditions hold,

$$\rho^{p-n} \int_{B_\rho(x_0)} |\nabla h|^p \, dx \le 1$$

and

$$\rho^{-n} \int_{B_\rho(x_0)} |w - h|^p \, dx \le \epsilon^p,$$

where $\delta = \delta(n, N, p, \epsilon) \in (0,1)$ is the corresponding function required in Lemma 2.1.

With these two smallness conditions, we can estimate the first term on the right-hand side of the equation (4.1). First, note that the mean of $u(x) - (p_0 + \mu \nabla h(x_0))(x - x_0)$ over $B_{2\theta\rho}(x_0)$ is $u_{x_0,2\theta\rho}$, and by the minimality theorem of the mean value, we can deduce that

$$\begin{aligned}
&(2\theta\rho)^{-n-p} \int_{B_{2\theta\rho}(x_0)} \left| u - u_{x_0,2\theta\rho} - \big(p_0 + \mu \nabla h(x_0)\big)(x - x_0) \right|^p dx \\
&\quad \le (2\theta\rho)^{-n-p} \int_{B_{2\theta\rho}(x_0)} \left| u - u_{x_0,\rho} - p_0(x - x_0) - \mu\big[h(x_0) + \nabla h(x_0)(x - x_0)\big] \right|^p dx \\
&\quad = (2\theta\rho)^{-n-p} \mu^p \int_{B_{2\theta\rho}(x_0)} \left| w(x) - h(x_0) - \nabla h(x_0)(x - x_0) \right|^p dx \\
&\quad \le 2^p (2\theta\rho)^{-n-p} \mu^p \int_{B_{2\theta\rho}(x_0)} \Big[ \left| w(x) - h(x) \right|^p + \left| h(x) - h(x_0) - \nabla h(x_0)(x - x_0) \right|^p \Big] dx \\
&\quad \le 2^p (2\theta\rho)^{-p} \mu^p \Bigg[ (2\theta\rho)^{-n} \int_{B_{2\theta\rho}(x_0)} \left| w(x) - h(x) \right|^p dx \\
&\qquad + \sup_{B_{2\theta\rho}(x_0)} \left| h(x) - h(x_0) - \nabla h(x_0)(x - x_0) \right|^p \Bigg].
\end{aligned} \tag{4.2}$$

According to the properties of the function $w(x)$ and the $p$-harmonic approximation function $h(x)$, it can be obtained that

$$(2\theta\rho)^{-n} \int_{B_{2\theta\rho}(x_0)} \left| w(x) - h(x) \right|^p dx \le \epsilon^p$$

and

$$\begin{aligned}
&\sup_{B_{2\theta\rho}(x_0)} \left| h(x) - h(x_0) - \nabla h(x_0)(x - x_0) \right|^p \\
&\quad = \sup_{B_{2\theta\rho}(x_0)} \left| \frac{h(x) - h(x_0)}{x - x_0}(x - x_0) - \nabla h(x_0)(x - x_0) \right|^p
\end{aligned}$$

$$
\begin{aligned}
&\le (2\theta\rho)^p \sup_{B_{2\theta\rho}(x_0)} \left|\nabla h(x) - \nabla h(x_0)\right|^p \\
&\le (2\theta\rho)^{2p} \sup_{B_{2\theta\rho}(x_0)} \left|\nabla^2 h(x)\right|^p \\
&\le (2\theta\rho)^{2p} \sup_{B_{\frac{\rho}{2}}(x_0)} \left|\nabla^2 h(x)\right|^p \\
&= (2\theta)^{2p}\rho^{2p} \sup_{B_{\frac{\rho}{2}}(x_0)} \left|\nabla^2 h(x)\right|^p \\
&\le (2\theta)^{2p} C_0 \rho^{p-n} \int_{B_\rho(x_0)} |\nabla h|^p \, dx \\
&\le C_0(2\theta)^{2p}, \qquad (4.3)
\end{aligned}
$$

here we have used Lemma 2.2.

Substituting the above two estimates into the inequality (4.2), we can obtain

$$
\begin{aligned}
&(2\theta\rho)^{-n-p} \int_{B_{2\theta\rho}(x_0)} \left|u - u_{x_0,2\theta\rho} - \big(p_0 + \mu\nabla h(x_0)\big)(x-x_0)\right|^p dx \\
&\le 2^p(2\theta\rho)^{-p}\mu^p\left[\epsilon^p + C_0(2\theta)^{2p}\right] \\
&\le 2^p(2\theta\rho)^{-p}\left[\rho^p\alpha_n^{\frac{p}{p-1}}\left(C_{19}^{\frac{p}{p-1}}\Phi\big(x_0,\rho,(\nabla u)_{x_0,\rho}\big) + \left(\frac{2}{\delta}C_{18}\right)^{\frac{p}{p-1}}\right)\right]\left[\epsilon^p + C_0(2\theta)^{2p}\right] \\
&= \alpha_n^{\frac{p}{p-1}}\left[C_{19}^{\frac{p}{p-1}}\Phi\big(x_0,\rho,(\nabla u)_{x_0,\rho}\big) + \left(\frac{2}{\delta}C_{18}\right)^{\frac{p}{p-1}}\right]\left[\theta^{-p}\epsilon^p + C_0 2^{2p}\theta^p\right]. \qquad (4.4)
\end{aligned}
$$

Taking the estimate (4.4) into Theorem 3.1 with $u_0 = u_{x_0,2\theta\rho}$ and $p_0 = p_0 + \mu\nabla h(x_0)$, we can find that

$$
\begin{aligned}
&(2\theta\rho)^{-n} \int_{B_{\theta\rho}(x_0)} |\nabla u - (p_0 + \mu\nabla h(x_0)|^p \, dx \\
&\le \tilde{C}_1\alpha_n^{\frac{p}{p-1}}\left[C_{19}^{\frac{p}{p-1}}\Phi\big(x_0,\rho,(\nabla u)_{x_0,\rho}\big) + \left(\frac{2}{\delta}C_{18}\right)^{\frac{p}{p-1}}\right]\left[\theta^{-p}\epsilon^p + C_0 2^{2p}\theta^p\right] \\
&\quad + \tilde{C}_2(2\theta\rho)^{-n}\int_{B_{2\theta\rho}(x_0)} (2\theta\rho)^p \, dx \\
&\quad + \tilde{C}_3\Big(\fint_{B_{\theta\rho}(x_0)} |\nabla u - \big(p_0 + \mu\nabla h(x_0)|^{p-\varepsilon} \, dx\Big)^{\frac{p}{p-\varepsilon}} (\alpha_n)^{1+\frac{p}{n}}(2\theta\rho)^p. \qquad (4.5)
\end{aligned}
$$

Then, we use the minimality principle of the mean value, and letting $p_0 = (\nabla u)_{x_0,\theta\rho}$, together with $\epsilon = \theta^{2p}$, yields

$$
\begin{aligned}
&\Phi\big(x_0,\theta\rho,(\nabla u)_{x_0,\theta\rho}\big) \\
&= \alpha_n^{-1}(\theta\rho)^{-n}\int_{B_{\theta\rho}(x_0)} \left|\nabla u - (\nabla u)_{x_0,\theta\rho}\right|^p dx \\
&\le \alpha_n^{-1}(\theta\rho)^{-n}\int_{B_{\theta\rho}(x_0)} |\nabla u - (p_0 + \mu\nabla h(x_0)|^p \, dx
\end{aligned}
$$

$$\begin{aligned}
&\le \tilde{C}_1 2^n \alpha_n^{\frac{1}{p-1}} \left[ C_{19}^{\frac{p}{p-1}} \Phi\big(x_0, \rho, (\nabla u)_{x_0,\rho}\big) + \left(\frac{2}{\delta} C_{18}\right)^{\frac{p}{p-1}} \right] \big[1 + C_0 2^{2p}\big] \theta^p \\
&\quad + \tilde{C}_2 2^{n+p} (\theta\rho)^p \\
&\quad + \tilde{C}_3 2^{n+p} \Big( \fint_{B_{\theta\rho}(x_0)} \big|\nabla u - \big(p_0 + \mu \nabla h(x_0)\big|^{p-\varepsilon}\, dx\Big)^{\frac{p}{p-\varepsilon}} (\alpha_n)^{\frac{p}{n}} (\theta\rho)^p \\
&\le \theta^p \big[ C_{20} \Phi\big(x_0, \rho, (\nabla u)_{x_0,\rho}\big) + C_{21} \rho^p \big],
\end{aligned} \tag{4.6}$$

with

$$\begin{aligned}
C_{20} &= 2^n \tilde{C}_1 C_{19}^{\frac{p}{p-1}} \alpha_n^{\frac{p}{p-1}} \big[1 + C_0 2^{2p}\big], \\
C_{21} &= 2^n \tilde{C}_1 \alpha_n^{\frac{p}{p-1}} \left(\frac{2}{\delta} C_{18}\right)^{\frac{p}{p-1}} \big[1 + C_0 2^{2p}\big] + \tilde{C}_2 2^{n+p} + \tilde{C}_3 2^{n+p} \|u\|^p_{W^{1,p-\varepsilon}} (\alpha_n)^{\frac{p}{n}}.
\end{aligned}$$

Noting that $\theta \in (0, \frac{1}{4})$, and letting $p_0 = (\nabla u)_{x_0,\theta\rho}$, we have

$$\Phi\big(x_0, \theta\rho, (\nabla u)_{x_0,\theta\rho}\big) \le \theta^p \big[ C_{20} \Phi\big(x_0, \rho, (\nabla u)_{x_0,\rho}\big) + C_{21} \rho^p \big].$$

The proof of Lemma 4.2 completed. □

## 5 Proof of the main results

The main purpose of this section is to establish the desired partial regularity, by the method of standard iteration.

*Proof of Theorem* 1.1 To obtain the result of Theorem 1.1, we have proved Caccioppolli's second inequality and the decay estimate already. Finally, we should iterate the decay estimate. That is, we should show that for $\forall j \in N$, there holds

$$\Phi\big(x_0, \theta^j\rho, (\nabla u)_{x_0,\theta^j\rho}\big) \le \theta^{jp} \big[ C_{20} \Phi\big(x_0, \rho, (\nabla u)_{x_0,\rho}\big) + C_{21} \rho^p \big].$$

Now, we take a constant $t_0 = t_0(n, N, p, \epsilon) > 0$ such that

$$t_0 \le \left(\frac{\delta}{2}\right)^p, \tag{5.1}$$

and choose $\rho_0 > 0$ small enough, such that

$$C_{21} \frac{1 - C_{20}^{j+1}}{1 - C_{20}} \rho_0^p < \frac{1}{2} t_0. \tag{5.2}$$

Then, for $\rho \in (0, \rho_0]$, from Lemma 4.2, we have

$$\Phi\big(x_0, \theta\rho, (\nabla u)_{x_0,\theta\rho}\big) \le \theta^p \big[ C_{20} \Phi\big(x_0, \rho, (\nabla u)_{x_0,\rho}\big) + C_{21} \rho^p \big].$$

In fact, if we can ensure that for $j \in N$, the inequality

$$\Phi\big(x_0, \theta^j\rho, (\nabla u)_{x_0,\theta^j\rho}\big) \le t_0$$

holds. Then, the conclusion

$$\Phi\big(x_0,\theta^{j+1}\rho,(\nabla u)_{x_0,\theta^{j+1}\rho}\big) \le \theta^p\big[C_{20}\Phi\big(x_0,\theta^j\rho,(\nabla u)_{x_0,\theta^j\rho}\big) + C_{21}\big(\theta^j\rho\big)^p\big]$$

is satisfied.

Let us iterate this procedure now. Assuming that the iterative formula is valid for $j = 0,1,2,\ldots,j-1$, then

$$\begin{aligned}
&\Phi\big(x_0,\theta^j\rho,(\nabla u)_{x_0,\theta^j\rho}\big)\\
&\quad\le \theta^p\big[C_{20}\Phi\big(x_0,\theta^{j-1}\rho,(\nabla u)_{x_0,\theta^{j-1}\rho}\big) + C_{21}\big(\theta^{j-1}\rho\big)^p\big]\\
&\quad\le C_{20}\theta^p\big[\theta^p\big(C_{20}\Phi\big(x_0,\theta^{j-2}\rho,(\nabla u)_{x_0,\theta^{j-2}\rho}\big) + C_{21}\big(\theta^{j-2}\rho\big)^p\big)\big] + C_{21}\big(\theta^j\rho\big)^p\\
&\quad= C_{20}^2\theta^{2p}\Phi\big(x_0,\theta^{j-2}\rho,(\nabla u)_{x_0,\theta^{j-2}\rho}\big) + C_{21}\big(\theta^j\rho\big)^p[1 + C_{20}]\\
&\quad\le C_{20}^3\theta^{3p}\Phi\big(x_0,\theta^{j-3}\rho,(\nabla u)_{x_0,\theta^{j-3}\rho}\big) + C_{21}\big(\theta^j\rho\big)^p\big[1 + C_{20} + C_{20}^2\big]\\
&\quad\le \cdots\\
&\quad\le C_{20}^j\theta^{jp}\Phi\big(x_0,\rho,(\nabla u)_{x_0,\rho}\big) + C_{21}\big(\theta^j\rho\big)^p\big[1 + C_{20} + C_{20}^2 + \cdots + C_{20}^j\big]\\
&\quad= \theta^{jp}\left[C_{20}^j\Phi\big(x_0,\rho,(\nabla u)_{x_0,\rho}\big) + C_{21}\frac{1-C_{20}^{j+1}}{1-C_{20}}\rho\right] \le \theta^{jp}t_0.
\end{aligned}$$

Now, we complete the whole iterative process.

Finally, according to the Holder continuous theorem, we can find that, if $\Phi(x_0,\rho,(\nabla u)_{x_0,\rho}) \le (\frac{\delta}{2})^p$ and $C_{21}\frac{1-C_{20}^{j+1}}{1-C_{20}}\rho^p \le \frac{t_0}{2}$, then, the very weak solution $u \in W^{1,p-\varepsilon}(\Omega)$ of the system (1.1) satisfies the result of Theorem 1.1. That is, $u \in C^{1,1}(\Omega\Omega_0)$.

The proof of Theorem 1.1 completed. □

## 6 Conclusion

In this paper, we mainly consider the partial regularity result of a class of the nonlinear elliptic system (1.1). The inhomogeneous term is made up of two terms, the general function $f(x)$ and the $p$-Laplace-type divergence function $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$. First, using Hodge decomposition, we find the relation between the very weak solution and the classical weak solution. That is, the very weak solution of the system (1.1) in fact is a classical weak solution of the system (1.1). Then, by the $p$-harmonic approximation technique, we obtain the partial regularity theory of the very weak solution of the system (1.1). In particular, the partial regularity result we obtained is optimal.

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**Availability of data and materials**
All data and materials in this paper are real and available.

## Declarations

**Competing interests**
The authors declare that they have no competing interests.

**Author contribution**
SHC participated in the design of the study and drafted the manuscript. ZT participated in conceiving the study and amending the paper. All authors read and approved the final manuscript.

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