

Optimal Partition Trees

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Abstract We revisit one of the most fundamental classes of data structure problems in computational geometry: range searching. Matoušek (Discrete Comput. Geom. 10:157–182, 1993) gave a partition tree method for d -dimensional simplex range searching achieving $O(n)$ space and $O(n^{1-1/d})$ query time. Although this method is generally believed to be optimal, it is complicated and requires $O(n^{1+\varepsilon})$ preprocessing time for any fixed $\varepsilon > 0$. An earlier method by Matoušek (Discrete Comput. Geom. 8:315–334, 1992) requires $O(n \log n)$ preprocessing time but $O(n^{1-1/d} \log^{O(1)} n)$ query time. We give a new method that achieves simultaneously $O(n \log n)$ preprocessing time, $O(n)$ space, and $O(n^{1-1/d})$ query time with high probability. Our method has several advantages:

- It is conceptually simpler than Matoušek's $O(n^{1-1/d})$ -time method. Our partition trees satisfy many ideal properties (e.g., constant degree, optimal crossing number at almost all layers, and disjointness of the children's cells at each node).
- It leads to more efficient multilevel partition trees, which are needed in many data structuring applications (each level adds at most one logarithmic factor to the space and query bounds, better than in all previous methods).
- A similar improvement applies to a shallow version of partition trees, yielding $O(n \log n)$ time, $O(n)$ space, and $O(n^{1-1/\lfloor d/2 \rfloor})$ query time for halfspace range emptiness in even dimensions $d \geq 4$.

Numerous consequences follow (e.g., improved results for computing spanning trees with low crossing number, ray shooting among line segments, intersection searching, exact nearest neighbor search, linear programming queries, finding extreme points, ...).

Keywords Simplex range searching · Halfspace range searching · Geometric data structures

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1 Introduction

Data structures for range searching [4, 5, 43, 47] are among the most important and most often used results within the computational geometry literature—countless papers applied such results and techniques to obtain the best theoretical computational bounds for a wide variety of geometric problems. However, to this day, some questions remain even concerning the most basic version of nonorthogonal range searching, simplex range searching. The purpose of this paper is to provide a (hopefully) final resolution to some of these lingering questions, by nailing down the best precise upper bounds.

History Formally, in *simplex range searching*, we want to preprocess n (weighted) points in \mathbb{R}^d so that we can quickly find (the sum of the weights of) all points inside a given query simplex. The dimension d is assumed to be a small constant. The problem not only is fundamental but has a remarkably rich history. We will confine the discussion of previous work to linear- or near-linear space data structures (otherwise, the number of results would multiply). The first published method was by Willard [53] in 1982, who gave a simple $O(n)$ -space *partition tree* achieving $O(n^{0.792})$ query time for $d = 2$. This prompted researchers to look for data structures with the best exponent in the query time. Many subsequent results appeared; for example, for $d = 2$, Willard himself improved the exponent to about 0.774, which was further reduced to 0.695 by Edelsbrunner and Welzl [33]; for $d = 3$, F. Yao [55] obtained 0.936, Dobkin and Edelsbrunner [31] 0.916, Edelsbrunner and Huber [32] 0.909, and Yao et al. [56] 0.899; in higher dimensions, Yao and Yao [54] obtained $1 - \lceil \lg(2^d - 1) \rceil / d$; and so on. A significant advance was made in Haussler and Welzl’s seminal paper [35], which introduced probabilistic techniques to computational geometry; they presented a partition tree with $O(n^{1-1/[d(d-1)+1]+\varepsilon})$ query time for any fixed $\varepsilon > 0$, greatly improving all results that came before, in all dimensions.

The right exponent turned out to be $1 - 1/d$. Four major papers described (near-)linear-space methods achieving the final near- $O(n^{1-1/d})$ query time:

1. In 1988,¹ Chazelle and Welzl [25, 52] devised the key technique of *iterative reweighting*² to compute *spanning trees with low crossing number*. They obtained an elegant $O(n)$ -space data structure where the answer to any query can be expressed using $O(n^{1-1/d} \log n)$ arithmetic (semigroup) operations on the weights (if subtractions on weights are allowed, the number of operations can be reduced to $O(n^{1-1/d} \alpha(n))$). Unfortunately, this combinatorial bound on the number of arithmetic operations does not bound the actual query time. Chazelle and Welzl obtained true “algorithmic” results only for $d = 2$, with $O(n)$ space and $O(\sqrt{n} \log n)$ query time, and for $d = 3$, with $O(n \log n)$ space and $O(n^{2/3} \log^2 n)$ query time. The preprocessing time of these data structures is also poor; for $d = 2$, one can

¹For better historical accuracy, we use dates of the earlier conference proceedings versions of papers.

²Also called the “multiplicative weights update method” [12] in some circles. In computational geometry, other favorite examples of the technique include one of Clarkson’s linear programming algorithm [28] and Brönnimann and Goodrich’s geometric set cover algorithm [15].

get $O(n^{3/2} \log^{1/2} n)$ preprocessing time by a randomized algorithm [52], but improvement to a near-linear bound requires further ideas from subsequent papers. Despite these issues, their approach was a remarkable breakthrough and set up the right direction for subsequent methods to follow.

2. Next, in 1990, Chazelle, Sharir, and Welzl [26] applied *cuttings* [23, 29] to get efficient data structures for simplex range searching. In the $O(n)$ -space case, their method has query time $O(n^{1-1/d+\varepsilon})$ with $O(n^{1+\varepsilon})$ preprocessing time for any fixed $\varepsilon > 0$. Although this brings us closer to the ideal bounds in any dimension, the method seems inherently suboptimal by at least a logarithmic factor (the data structure requires multiple separate cuttings, so as to guarantee that for every query, at least one of the cuttings is “good”).
3. In 1991, Matoušek [39] combined the iterative reweighting technique with the use of cuttings to prove a new *simplicial partition theorem*. Applying the theorem in a recursive fashion gives an $O(n)$ -space partition tree with $O(n^{1-1/d} \log^{O(1)} n)$ query time and $O(n \log n)$ preprocessing time.
The same paper also gives an alternative $O(n)$ -space structure with query time $O(n^{1-1/d} (\log \log n)^{O(1)})$ —or better still $O(n^{1-1/d} 2^{O(\log^* n)})$ if subtractions of weights are allowed—but the preprocessing time increases to $O(n^{1+\varepsilon})$. This result is subsumed by the next result, so will be ignored here.
4. Finally, in 1992, Matoušek [42] combined iterative reweighting with the *hierarchical cuttings* of Chazelle [21] to obtain an $O(n)$ -space partition tree with $O(n^{1-1/d})$ query time. The preprocessing time is $O(n^{1+\varepsilon})$. This method is considerably more complicated than Matoušek’s previous partition-theorem-based method.

Among data structures that are ordinary (unaugmented) partition trees, it is not difficult to show that the query cost must be $\Omega(n^{1-1/d})$ in the worst case [25]. More generally, Chazelle [20] proved a lower bound of $\Omega(\sqrt{n})$ in for $d = 2$ and $\Omega(n^{1-1/d} / \log n)$ in higher dimensions, for any simplex range searching data structure under the semigroup arithmetic model (see [13, 24] for more lower bound results); the $\log n$ factor in the denominator could likely be an artifact of the proof.

Matoušek’s final method thus seems to achieve asymptotically optimal query time. However, the story is not as neatly resolved as one would like. The main issue is preprocessing time: Matoušek’s final method is suboptimal by an n^ε factor. This explains why in actual algorithmic applications of range searching results, where we care about preprocessing and query times combined, researchers usually abandon his final method and resort to his earlier partition-theorem method. Unfortunately, this is the reason why in algorithmic applications, we frequently see appearances of extra $\log^{O(1)} n$ factors in time bounds, with unspecified numbers of log factors.

Even for $d = 2$, no ideal data structure with $O(n \log n)$ preprocessing time, $O(n)$ space, and $O(\sqrt{n})$ query time has been found.

New Results In this paper, we give a new method that rederives Matoušek’s final result of $O(n)$ space and $O(n^{1-1/d})$ query time for simplex range searching. Our new method is much simpler and easier to explain than Matoušek’s final method; it is a little more involved than the partition-theorem method, but not considerably so. The partition tree satisfies many desirable properties not achieved by Matoušek’s and,

for example, eliminates essentially all the disadvantages mentioned in the next-to-last paragraph of his paper [42] (we use a single tree and have complete control over the crossing number at almost all layers of the tree). Our tree also satisfies properties not achieved by the partition-theorem method [39] (we use a constant-degree tree and cells do not overlap). In the $d = 2$ case, we can even make our partition tree a BSP tree, like Willard’s original partition tree. These properties in themselves are already interesting, in the author’s opinion, but the approach has further major implications:

- Our method immediately yields improved results on *multilevel* versions of partition trees [5, 43], which are needed in applications that demand more sophisticated kinds of query. With our method, each level costs at most *one* more logarithmic factor in space and at most *one* more logarithmic factor in the query time—this behavior is exactly what we want and, for example, is analogous to what we see in the classical multilevel data structure, the range tree [5, 48], which can be viewed as a d -level 1-dimensional partition tree.

In contrast, Matoušek’s final method is not applicable at all, precisely because of the disadvantages mentioned. Matoušek’s paper [42] did explore multilevel results in detail but had to switch to a different method based on Chazelle, Sharir, and Welzl’s cutting approach [26], now with hierarchical cuttings. This method costs *two* logarithmic factors in space and *one* logarithmic factor in the query time per level; see [42, Theorem 5.2]. Furthermore, the preprocessing time is huge. An alternative version [42, Corollary 5.2] has $O(n^{1+\varepsilon})$ preprocessing time, but space and query time increase by an unspecified number of extra logarithmic factors. Matoušek’s partition-theorem method in the multilevel setting is worse, giving extra $O(n^\varepsilon)$ (or more precisely $2^{O(\sqrt{\log n})}$) factors in space and query time per level [39].

For concrete applications, consider the problems of preprocessing n simplices in \mathbb{R}^d so that we can (i) report all simplices containing a query point, or (ii) report all simplices contained inside a query simplex. These problems can be solved by a $(d + 1)$ -level partition tree. The previous method from [42] requires $O(n \log^{2d} n)$ space and $O(n^{1-1/d} \log^d n)$ query time. Our result reduces the space bound to $O(n \log^d n)$.

We get improved data structures for many other problems, even some traditional 2-dimensional problems, e.g., counting the number of intersections of line segments with a query line segment, and performing ray shooting among a collection of line segments.

Our new partition tree also leads to a new combinatorial theorem: any n -point set in \mathbb{R}^2 admits a Steiner triangulation of size $O(n)$ so that the maximum number of edges crossed by any line is $O(\sqrt{n})$. This crossing number bound is tight and improves previous upper bounds by a logarithmic factor [8].

- That our method is simpler allows us to examine the preprocessing time more carefully. We show that with some extra effort, a variant of our method indeed accomplishes what we set out to find: a simplex range searching structure with $O(n \log n)$ preprocessing time, $O(n)$ space, and $O(n^{1-1/d})$ query time with high probability (i.e., probability at least $1 - 1/n^{c_0}$ for an arbitrarily large constant c_0). The approach works in the multilevel setting as well, with one more logarithmic factor in the preprocessing time per level.

The improved preprocessing time should immediately lead to improved time bounds for a whole array of algorithmic applications.

- We can also obtain new bounds for the *halfspace range emptiness/reporting* problem. Here, we want to decide whether a query halfspace contains any data point, or report all points inside a query halfspace. Matoušek [40] described a *shallow* version of his partition theorem to obtain a data structure with $O(n \log n)$ preprocessing time, $O(n \log \log n)$ space, and $O(n^{1-1/\lfloor d/2 \rfloor} \log^{O(1)} n + k)$ query time for any $d \geq 4$, where k denotes the output size in the reporting case. The space was reduced to $O(n)$ by Ramos [49] for even d ; see also [1] for $d = 3$. The same paper by Matoušek also gave an alternative data structure with $O(n^{1+\varepsilon})$ preprocessing time, $O(n)$ space, and $O(n^{1-1/\lfloor d/2 \rfloor} 2^{O(\log^* n)})$ query time for halfspace range emptiness for any $d \geq 4$. Although it is conceivable that Matoušek's final method for simplex range searching with hierarchical cuttings could similarly be adapted to halfspace range emptiness for even d , to the best of the author's knowledge, no one has attempted to find out, perhaps daunted by the amount of technical details in the paper [42]. It is generally believed that the exponent $1 - 1/\lfloor d/2 \rfloor$ is tight, although a matching lower bound is currently not known.

With our simpler method, we can derive the following, likely optimal result for even dimensions without too much extra trouble: there is a data structure for halfspace range emptiness with $O(n \log n)$ preprocessing time, $O(n)$ space, $O(n^{1-1/\lfloor d/2 \rfloor})$ query time with high probability for any even $d \geq 4$. The same bounds hold for halfspace range reporting when the output size k is small ($o(n^{1-1/\lfloor d/2 \rfloor} / \log^{\omega(1)} n)$).

Halfspace range emptiness/reporting has its own long list of applications [17, 41, 45]. As a consequence, we get improved results for ray shooting and linear programming queries in intersections of halfspaces in even dimensions, exact nearest neighbor search in odd dimensions, and the computation of extreme points and convex hulls in even dimensions, to name just a few.

Organization The highlight of the paper is to be found in Sect. 3—in particular, the proof of Theorem 3.1—which describes the basic version of our new partition tree with $O(n)$ space and $O(n^{1-1/d})$ query time. This part is meant to be self-contained (except for the side remarks at the end), assuming only the background provided by Sect. 2. The rest of the paper—particularly Sect. 5 on getting $O(n \log n)$ preprocessing time, Sect. 6 on halfspace range searching, and Sect. 7 on applications—will get a bit more technical and incorporate additional known techniques, but mostly involves working out consequences of the basic new approach.

2 Background

Let P be the given set of n points in \mathbb{R}^d , which we assume to be in general position by standard perturbation techniques. A *partition tree* is a hierarchical data structure formed by recursively partitioning a point set into subsets. Formally, in this paper, we define a partition tree to be a tree T where each leaf stores at most a constant number of points, each point of P is stored in exactly one leaf, and each node v

stores a cell $\Delta(v)$ which encloses the subset $P(v)$ of all points stored at the leaves underneath v . Throughout the paper, we take “cell” to mean a polyhedron of $O(1)$ size, or to be more specific, a simplex. In general, the degree of a node is allowed to be nonconstant and the cells of the children of a node are allowed to overlap (although in our new basic method, we can guarantee constant degree and no overlaps of the cells). At each node v , we do not store any auxiliary information about $P(v)$, other than the sum of the weights in $P(v)$. Therefore, a partition tree is by our definition always an $O(n)$ -space data structure.

We can answer a query for a simplex q by proceeding recursively, starting at the root. When visiting a node, we only have to recurse in children whose cells cross the boundary ∂q , since each cell completely outside q can be ignored, and each cell completely inside q can be dealt with directly using the information stored at the node (or, in the case of the reporting problem, directly reporting all points underneath the node). We can identify the children cells crossing ∂q trivially in time linear in the number of children (the degree). The total query time is then bounded from above by the expression

$$\sum_{v \in T: \Delta(v) \cap \partial q \neq \emptyset} \text{deg}(v), \tag{1}$$

which we call the *query cost of ∂q* . (In the case of the reporting problem, we add an $O(k)$ term to the query time where k is the output size.) Since the boundary ∂q has $O(1)$ facets contained in $O(1)$ hyperplanes, it suffices to asymptotically bound the maximum query cost of an arbitrary hyperplane, which we refer to as the *query cost of the tree T* .

It is known [25] that there are point sets where the query cost is $\Omega(n^{1-1/d})$ for any partition tree under the above definitions. We want to devise a worst-case optimal method to construct partition trees matching this lower bound.

We begin by mentioning Matoušek’s simplicial partition theorem [39], to set the context. We will not use this theorem anywhere, but we will prove something stronger in Sect. 3.

Theorem 2.1 (Matoušek’s Partition Theorem) *Let P be a set of n points in \mathbb{R}^d . Then for any t , we can partition P into t subsets P_i and find t cells Δ_i , with $\Delta_i \supset P_i$, such that each subset contains $\Theta(n/t)$ points and each hyperplane crosses at most $O(t^{1-1/d})$ cells.*

A recursive application of the above theorem immediately gives a partition tree where the query cost satisfies the recurrence $Q(n) \leq O(t^{1-1/d})Q(n/t) + O(t)$. If we set t to an arbitrarily large constant, the recurrence has solution $Q(n) \leq O(n^{1-1/d+\varepsilon})$ for an arbitrarily small constant $\varepsilon > 0$. If instead we set $t = n^\delta$ for a sufficiently small constant $\delta > 0$, the solution becomes $Q(n) \leq O(n^{1-1/d} \log^{O(1)} n)$. To avoid the extra n^ε or $\log^{O(1)} n$ factors, we need a more refined method, since each time we recurse using Theorem 2.1, the hidden constant in the crossing number bound “blows up”.

Our new method is described in the next section, and is self-contained except for the use of two tools: a minor observation and an important lemma, both well-known and stated here without proofs.

First, we observe that in proving statements like Theorem 2.1, instead of considering an infinite number of possible hyperplanes, it suffices to work with a finite set of “test” hyperplanes. This type of observation was made in many of the previous papers [25, 26]; the particular version we need is a special case of one from [39]:

Observation 2.2 (Test Set) *Given a set P of n points in \mathbb{R}^d , we can construct a set H of $n^{O(1)}$ hyperplanes with the following property: For any collection of disjoint cells each containing at least one point of P , if the maximum number of cells crossed by a hyperplane in H is ℓ , then the maximum number of cells crossed by an arbitrary hyperplane is $O(\ell)$.*

For this version of the test-set observation, the proof is quite simple (briefly, we can just take H to be the $O(n^d)$ hyperplanes passing through d -tuples of points of P).

For the next lemma, we need a definition first. For a given set H of m hyperplanes, a $(1/r)$ -cutting [22, 44, 47] is a collection of disjoint cells such that each cell Δ is crossed by at most m/r hyperplanes. Let $\mathcal{A}(H)$ denote the arrangement of H . We let H_Δ denote the subset of all hyperplanes of H that cross Δ .

Lemma 2.3 (Cutting Lemma) *Let H be a set of m hyperplanes and Δ be any simplex in \mathbb{R}^d . For any r , we can find a $(1/r)$ -cutting of H into $O(r^d)$ disjoint cells whose union is Δ .*

If the number of vertices in $\mathcal{A}(H)$ inside Δ is X , then the number of cells can be reduced to $O(X(r/m)^d + r^{d-1})$.

The first part of the lemma is standard and was first shown by Chazelle and Friedman [23]. The X -sensitive variant is also known; e.g., see [21, 30]. The proof is by random sampling.³

To understand the intuitive significance of the cutting lemma, let H be a set of m test hyperplanes, $\Delta = \mathbb{R}^d$, and $r = t^{1/d}$. Then we get $O(t)$ cells each crossed by $O(m/t^{1/d})$ hyperplanes. The average number of points per cell is $O(n/t)$, and the average number of cells crossed by a test hyperplane is $O((t \cdot m/t^{1/d})/m) = O(t^{1-1/d})$. Thus, the cutting lemma “almost” implies the partition theorem. The challenge is to turn these average bounds into maximum bounds. For this purpose, Matoušek’s proof of his partition theorem adopts an *iterative reweighting* strategy by Welzl [25, 52] (roughly, in each iteration, we apply the cutting lemma to a multiset of hyperplanes in H to find one good cell, then increase the multiplicities of the hyperplanes crossing the cell, and repeat).

³Rough Proof Sketch: We draw a sample R of size r and take the cells of a *canonical triangulation* T of $\mathcal{A}(R)$ [27] restricted inside Δ . For a cell $\sigma \in T$ that is crossed by $a_\sigma m/r$ hyperplanes with $a_\sigma > 1$, we further subdivide the cell by taking a $(1/a_\sigma)$ -cutting of H_σ using any weaker method with, say, $a_\sigma^{O(1)}$ subcells (e.g., we can again use random sampling). To see why this procedure works, note that the size of the canonical triangulation is proportional to the size of $\mathcal{A}(R)$ restricted inside Δ , which is at most $O(X(r/m)^d + r^{d-1})$, since each vertex in $\mathcal{A}(H)$ shows up in $\mathcal{A}(R)$ with probability $O(r/m)^d$, and there are $O(r^{d-1})$ vertices in the intersection of $\mathcal{A}(R)$ with each $(d-1)$ -dimensional boundary facet of Δ . On the other hand, analysis by Clarkson and Shor [29] or Chazelle and Friedman [23] tells us that the parameter a_σ is $O(1)$ “on average” over all cells $\sigma \in T$.

3 The Basic Method

We now present our new method for simplex range searching with $O(n)$ space and $O(n^{1-1/d})$ query time. To keep the presentation simple, we defer discussion of pre-processing time to Sect. 5.

The key lies in the following theorem, where instead of constructing one partition of the point set from scratch as in Matoušek’s partition theorem, we consider the problem of *refining* a given partition. As we will see, the main result will follow just by repeated applications of our theorem in a straightforward manner.

Theorem 3.1 (Partition Refinement Theorem) *Let P be a set of at most n points and H be a set of m hyperplanes in \mathbb{R}^d . Suppose we are given t disjoint cells covering P , such that each cell contains at most $2n/t$ points of P and each hyperplane in H crosses at most ℓ cells. Then for any b , we can subdivide every cell into $O(b)$ disjoint subcells, for a total of at most bt subcells, such that each subcell contains at most $2n/(bt)$ points of P , and each hyperplane in H crosses at most the following total number of subcells:*

$$O((bt)^{1-1/d} + b^{1-1/(d-1)}\ell + b \log t \log m). \tag{2}$$

Note that the first term matches the bound from Matoušek’s partition theorem for a partition into $O(bt)$ parts. In the second term, it is crucial that the coefficient $b^{1-1/(d-1)}$ is asymptotically smaller than $b^{1-1/d}$; this will ensure that repeated applications of the theorem do not cause a constant-factor blow-up, if we pick b to be sufficiently large. The third term will not matter much in the end, and is purposely not written in the tightest manner. The proof of the theorem, like Matoušek’s, uses the cutting lemma repeatedly in conjunction with an iterated reweighting strategy as detailed below. The basic algorithm is simple:

Proof We maintain a multiset \widehat{H} , initially containing C copies of each hyperplane in H . The value of C is not important (for conceptual purposes, we can set C to be a sufficiently large power of b , which will ensure that future multiplicities are always integers). The size $|\widehat{H}|$ of a multiset H always refers to the sum of the multiplicities (the “weights”) of the elements. Let $X_\Delta(\widehat{H})$ denote the number of vertices inside Δ defined by the hyperplanes in \widehat{H} ; here, the multiplicity of a vertex is the product of the multiplicities of its defining hyperplanes.

The Algorithm Let $\Delta_t, \dots, \Delta_1$ be the given cells in a random order. For $i = t, \dots, 1$ do:

1. Subdivide Δ_i into disjoint subcells by building a $(1/r_i)$ -cutting of \widehat{H}_{Δ_i} inside Δ_i with

$$r_i := c \min \left\{ |\widehat{H}_{\Delta_i}| \left(\frac{b}{X_{\Delta_i}(\widehat{H})} \right)^{1/d}, b^{1/(d-1)} \right\}$$

for some constant c . By Lemma 2.3, the number of subcells inside Δ_i is $O(X_{\Delta_i}(\widehat{H})(r_i/|\widehat{H}_{\Delta_i}|)^d + r_i^{d-1})$, which can be made at most $b/4$ for c sufficiently small.

2. Further subdivide each subcell, e.g., by using vertical cuts, to ensure that each subcell contains at most $2n/(bt)$ points. The number of extra cuts required is $O(b)$ per cell Δ_i , and at most $bt/2 + o(bt)$ in total over all Δ_i . So, the number of subcells is $O(b)$ per cell, and at most $3bt/4 + o(bt) \ll bt$ in total.
3. For each hyperplane h , multiply the multiplicity of h in \widehat{H} by $(1 + 1/b)^{\lambda_i(h)}$ where $\lambda_i(h)$ denotes the number of subcells inside Δ_i crossed by h .

Analysis For a fixed hyperplane $h \in H$, let $\ell_i(h)$ be the number of cells in $\{\Delta_i, \dots, \Delta_1\}$ crossed by h . Since $\{\Delta_i, \dots, \Delta_1\}$ is a random subset of size i from a set of size t , we have $\ell_i(h) \leq O(\ell_i/t + \log(mt))$ w.h.p. $(mt)^4$ by a Chernoff bound (a version for sampling without replacement, as noted in the [Appendix](#)). So, defining $\ell_i := \max_{h \in H} \ell_i(h)$, we have $\ell_i \leq O(\ell_i/t + \log(mt))$ w.h.p. (t) .

In step 3, $\sum_{h \in \widehat{H}} \lambda_i(h) \leq O(b|\widehat{H}_{\Delta_i}|/r_i)$, since each of the $O(b)$ subcells is crossed by at most $|\widehat{H}_{\Delta_i}|/r_i$ hyperplanes of \widehat{H}_{Δ_i} . As $\lambda_i(h) \leq O(b)$, step 3 increases $|\widehat{H}|$ by

$$\sum_{h \in \widehat{H}} [(1 + 1/b)^{\lambda_i(h)} - 1] \leq O\left(\sum_{h \in \widehat{H}} \lambda_i(h)/b\right) \leq O(|\widehat{H}_{\Delta_i}|/r_i) = O(\alpha_i)|\widehat{H}|$$

$$\text{where } \alpha_i := \frac{|\widehat{H}_{\Delta_i}|}{r_i|\widehat{H}|} = O\left(\frac{1}{|\widehat{H}|} \cdot \left(\frac{X_{\Delta_i}(\widehat{H})}{b}\right)^{1/d} + \frac{|\widehat{H}_{\Delta_i}|}{b^{1/(d-1)}|\widehat{H}|}\right).$$

Note that $\sum_{j=1}^i |\widehat{H}_{\Delta_j}| \leq |\widehat{H}|\ell_i$ and $\sum_{j=1}^n X_{\Delta_j}(\widehat{H}) \leq |\widehat{H}|^d$ by disjointness of the cells. When conditioned to a fixed choice of $\Delta_t, \dots, \Delta_{i+1}$, the subset $\{\Delta_i, \dots, \Delta_1\}$ is fixed and Δ_i is a random element among this subset of size i , and so we have $E[|\widehat{H}_{\Delta_i}|] \leq |\widehat{H}|\ell_i/i$ and $E[X_{\Delta_i}(\widehat{H})] \leq |\widehat{H}|^d/i$; in particular, by Jensen’s inequality, $E[X_{\Delta_i}(\widehat{H})^{1/d}] \leq |\widehat{H}|/i^{1/d}$. Thus,

$$E[\alpha_i \mid \Delta_t, \dots, \Delta_{i+1}] \leq O\left(\frac{1}{(bi)^{1/d}} + \frac{\ell_i}{b^{1/(d-1)}i}\right).$$

Since $E[\ell_i] \leq O(\ell_i/t + \log(mt))$,

$$E[\alpha_i] \leq O\left(\frac{1}{(bi)^{1/d}} + \frac{\ell}{b^{1/(d-1)}t} + \frac{\log(mt)}{b^{1/(d-1)}i}\right).$$

At the end, $|\widehat{H}| \leq Cm \prod_{i=1}^t (1 + O(\alpha_i)) \leq Cm \exp(O(\sum_{i=1}^t \alpha_i))$, where

$$E\left[\sum_{i=1}^t \alpha_i\right] \leq O\left(\frac{t^{1-1/d}}{b^{1/d}} + \frac{\ell}{b^{1/(d-1)}} + \frac{\log(mt) \log t}{b^{1/(d-1)}}\right).$$

Let $\lambda(h)$ be the total number of subcells crossed by h . Since the final multiplicity of h in \widehat{H} is equal to $C(1 + 1/b)^{\lambda(h)} \leq |\widehat{H}|$, it follows that $\max_{h \in H} \lambda(h) \leq \log_{1+1/b}(|\widehat{H}|/C) \leq O(b \log(|\widehat{H}|/C))$, which has expected value at most $O(b \log m + (bt)^{1-1/d} + b^{1-1/(d-1)}\ell + b^{1-1/(d-1)} \log(mt) \log t)$, which is bounded from above by (2). □

⁴Throughout the paper, the notation “w.h.p. (n) ” (or “w.h.p.” for short, if n is understood) means “with probability at least $1 - 1/n^{c_0}$ ” for an arbitrarily large constant c_0 .

Theorem 3.2 *Given n points in \mathbb{R}^d , we can build a partition tree with query cost $O(n^{1-1/d})$.*

Proof Let H be the test set from Observation 2.2 of size $m \leq n^{O(1)}$. We construct a hierarchy of sets Π_t of cells for $t = 1, b, b^2, \dots$, where Π_1 consists of just a root cell containing all n points, and Π_{bt} is obtained by applying Theorem 3.1 to Π_t . The cells generated forms a partition tree of degree $O(b)$. Let $\ell(t)$ denote the maximum number of cells of Π_t crossed by an arbitrary hyperplane. By Theorem 3.1 and Observation 2.2,

$$\ell(bt) \leq O((bt)^{1-1/d} + b^{1-1/(d-1)}\ell(t) + b \log t \log n).$$

Fix an arbitrarily small constant $\delta > 0$. Set b to be a sufficiently large constant depending on δ . It is straightforward to verify, by induction over u , that

$$\ell(u) \leq c[u^{1-1/d} + u^{1-1/(d-1)+\delta} \log n] \tag{3}$$

for some constant c depending on δ and b . This follows since for a suitable absolute constant c_0 ,

$$\begin{aligned} & c_0[(bt)^{1-1/d} + cb^{1-1/(d-1)}[t^{1-1/d} + t^{1-1/(d-1)+\delta} \log n] + b \log t \log n] \\ & \leq c_0 \left[1 + \frac{c}{b^{1/(d-1)-1/d}} \right] (bt)^{1-1/d} + c_0 \left[\frac{c}{b^\delta} + b \right] (bt)^{1-1/(d-1)+\delta} \log n, \end{aligned}$$

so the induction carries through by choosing c sufficiently large to satisfy $c_0[1 + c/b^{1/(d-1)-1/d}] \leq c$ and $c_0[c/b^\delta + b] \leq c$.

Note that the first term of (3) dominates when u exceeds $\log^{c'} n$ for some constant c' , so we can write $\ell(u) \leq O(u^{1-1/d} + \log^{O(1)} n)$. We conclude that the query cost is $O(\sum_{t \leq n} b\ell(t)) \leq O(\sum_{t \leq n} [t^{1-1/d} + \log^{O(1)} n]) = O(n^{1-1/d})$, where the sums are over all t that are powers of b . □

Remarks To readers familiar with the proof of Matoušek’s partition theorem [39], we point out some differences from our proof of the partition refinement theorem. First, to ensure $\sum_j X_{\Delta_j}(\hat{H}) \leq |\hat{H}|^d$, we need disjointness of cells, which forces us to choose multiple subcells from a cutting instead of one subcell per iteration. This in turn forces us to use a different multiplier $(1 + 1/b)$ instead of doubling. Unlike Theorem 2.1, Theorem 3.1 guarantees an upper but not a lower bound on size of each subset (certain versions of the test set observation, e.g., Observation 5.1, require such a lower bound), though this is not an issue in our proof of Theorem 3.2.

To those familiar with Matoušek’s final method based on hierarchical cuttings [42], we note that our method share certain common ideas. Both apply iterative reweighting more “globally” to refine multiple cells simultaneously, as a way to avoid constant-factor blow-up in the partition-theorem method; and hierarchical cuttings originate from the same X -sensitive version of the cutting lemma. However, Matoušek’s method uses a far more complicated weight/multiplicity function, because it tries to deal with different layers of the tree all at once; in contrast, our partition refinement theorem works with one layer at a time and leads to a simpler, more modular design. Matoušek’s method also requires a special root of $O(n^{1/d} \log n)$ degree (whose children cells may overlap), and does not guarantee optimality for $\ell(t)$

except at the bottommost layer, whereas our method ensures optimality for almost all layers (except near the very top for very small t , which is not important). This will make a difference in the next section and in subsequent applications to multilevel data structures.

Random ordering is convenient but is not the most crucial part of the proof of Theorem 3.1: We can easily deterministically find a good cell Δ_i in each iteration if we do not care about the bound on ℓ_i . The naive bound $\ell_i \leq \ell$ can still lead to a weaker version of Theorem 3.1 with an extra logarithmic factor in the second term of (2), and Theorem 3.2 can still be derived but with some extra effort. (Alternatively, we can derandomize, by considering a potential function of the form $\sum_{h \in \widehat{H}} 2^{c\ell_i(h)t/i}$.)

In the $d = 2$ case, our partition tree can be made into a BSP tree, since the cuttings produced by taking canonical triangulations of the arrangements of samples are easily realizable as binary plane partitions (our algorithm computes such cuttings for $r \leq b^{O(1)}$ bounded by a constant).

4 Additional Properties

We point out some implications that follow from our method. Define the *order- γ query cost* of ∂q to be $\sum_{v \in T: \Delta(v) \cap \partial q \neq \emptyset} \sum_{w \text{ child of } v} |P(w)|^\gamma$. Define the order- γ query cost of T to be the maximum order- γ query cost of an arbitrary hyperplane. Our previous definition coincides with the case $\gamma = 0$. This extended definition is relevant in multilevel data structures where secondary data structures for $P(v)$ are stored at each node v ; see Sect. 7 for more details. We have the following new result:

Theorem 4.1 *Given n points in \mathbb{R}^d , we can build a partition tree with*

- (i) *Height $O(\log n)$ and order- $(1 - 1/d)$ query cost $O(n^{1-1/d} \log n)$.*
- (ii) *Height $O(\log \log n)$ and order- γ query cost $O(n^{1-1/d})$ for any fixed $\gamma < 1 - 1/d$.*

Proof

- (i) The same partition tree in the proof of Theorem 3.2 has order- $(1 - 1/d)$ query cost bounded by the following, where the sums are over all t that are powers of b :

$$\begin{aligned} & O\left(\sum_{t \leq n} b \left(\frac{n}{bt}\right)^{1-1/d} \ell(t)\right) \\ & \leq O\left(\sum_{t \leq n} (n/t)^{1-1/d} \cdot [t^{1-1/d} + t^{1-1/(d-1)+\delta} \log n]\right) \\ & = O\left(\sum_{t \leq n} \left[n^{1-1/d} + \frac{n^{1-1/d} \log n}{t^{1/(d-1)-1/d+\delta}}\right]\right) = O(n^{1-1/d} \log n). \end{aligned}$$

- (ii) To lower the height, we modify our partition tree T . Fix a sufficiently small constant $\varepsilon > 0$. We examine each node v of T in a top-down order, and whenever a child w of a node v has $|P(w)| > |P(v)|^{1-\varepsilon}$, we remove w by making w 's

children v 's children. In the new tree T' , every child w of a node v has $|P(w)| \leq |P(v)|^{1-\varepsilon}$, so the height is $O(\log \log n)$. For each child w of v in T' , w 's parent in T has size at least $|P(v)|^{1-\varepsilon}$, so the degree of v in T' is at most $b|P(v)|^\varepsilon \leq O((n/t)^\varepsilon)$ if $|P(v)| = O(n/t)$. Summing over all t that are powers of b , we obtain order- γ query cost

$$\begin{aligned} & O\left(\sum_{t \leq n} (n/t)^\varepsilon (n/t)^{1-\varepsilon} \ell(t)\right) \\ & \leq O\left(\sum_{t \leq n} (n/t)^{\gamma+\varepsilon} \cdot [t^{1-1/d} + \log^{O(1)} n]\right) \\ & = O\left(\sum_{t \leq n} \left[\frac{n^{1-1/d}}{(n/t)^{1-1/d-\gamma-\varepsilon}} + (n/t)^{\gamma+\varepsilon} \log^{O(1)} n\right]\right) = O(n^{1-1/d}). \quad \square \end{aligned}$$

We have omitted the case $\gamma > 1 - 1/d$, because a simple recursive application of Matoušek's partition theorem already gives optimal order- γ query cost $O(n^\gamma)$ with height $O(\log \log n)$ in this case.

Next, define a *B-partial partition tree* for a point set P to be the same as in our earlier definition of a partition tree, except that a leaf now may contain up to B points. Define the *query cost* in the same manner, as in (1). Note the query cost of a partial partition tree upper-bounds the number of internal and leaf cells crossed by a hyperplane, but does not account for the cost of any auxiliary data structures we plan to store at the leaves. The following theorem is useful in applications that need space/time tradeoffs, where we can switch to a larger-space data structure with small query time at the leaves; see Sect. 7 for more details. The theorem is also self-evidently useful in the external memory model (hence, the choice of name “B”): we get improved external-memory (even cache-oblivious) range searching data structures [10] immediately as a byproduct.

Theorem 4.2 *Given n points in \mathbb{R}^d and $B < n/\log^{\omega(1)} n$, we can build a B-partial partition tree with size $O(n/B)$ and query cost $O(n/B)^{1-1/d}$.*

Proof Stop the construction in the proof of Theorem 3.2 when t reaches n/B . Summing over all t that are powers of b , we obtain query cost $O(b \sum_{t \leq n/B} \ell(t)) = O(n/B)^{1-1/d}$. \square

5 Preprocessing Time

In this section, we examine the issue of preprocessing time. Obtaining polynomial time is not difficult with our method, and it might be possible to lower the bound to $O(n^{1+\varepsilon})$ by using recursion, as was done in Matoušek's final method [42]. Instead of recursion, we show how to directly speed up the algorithm in Sect. 3. We first aim for $O(n \log^{O(1)} n)$ preprocessing time, and later improve it to $O(n \log n)$.

First, we need a better version of Observation 2.2 with fewer test hyperplanes. The following observation was shown by Matoušek [39] (see also [47]).

Observation 5.1 (Test Set) *Given a set P of n points and any t , we can construct a set H_t of $O(t \log^{O(1)} N)$ test hyperplanes, in $O(t \log^{O(1)} N)$ time, satisfying the following property w.h.p.(N) for any $N \geq n$: For any collection of disjoint cells each containing at least n/t points of P , if the maximum number of cells crossed by a hyperplane in H_t is ℓ , then the maximum number of cells crossed by an arbitrary hyperplane is at most $O(\ell + t^{1-1/d})$.*

(Roughly interpreted, the proof involves drawing a random sample of $t^{1/d} \log N$ points and taking the $O(t \log^d N)$ hyperplanes through d -tuples of points in the sample; the time bound is clearly as stated. The $\log N$ factors can be removed with more work, using cuttings, but will not matter.)

Regarding Lemma 2.3, we can use known cutting algorithms [21, 23, 39] to get running time $O(mr^{O(1)})$ (randomized or deterministic).

To get a good time bound for Theorem 3.1, we need to modify the algorithm. The details become more involved, but there are two new main ideas:

- In step 1, we work with a sparser subset of \widehat{H} , by considering a random sample of \widehat{H} .
- More crucially, in step 3, we update the multiplicities less frequently, by considering a random sample of the subcells.

To prepare for the proof of the theorem below, we make two definitions, the first of which is standard: Given a (multi)set S , let a p -sample be a subset R generated by taking each (occurrence of an) element of S , and independently choosing to put the element in R with probability p . (Note that we can generate a p -sample in time linear in the output size rather than the size of S , by using a sequence of exponentially distributed random variables rather than 0-1 random variables.)

For a list $S = \langle s_1, \dots, s_K \rangle$ and a sublist R of S , where S itself (the elements s_i and the size K) may be random, we say that R is a *generalized p -sample of S* if the events $E_i = \{s_i \in R\}$ are all independent, each occurring with probability p . Note the subtle difference from the earlier definition (where S was thought of as fixed before R is chosen): it is acceptable for s_i to be dependent on E_1, \dots, E_{i-1} , as long as we decide to choose whether to put s_i to R independently of E_1, \dots, E_{i-1} and s_i .

Properties enjoyed by standard random samples may not necessarily hold for generalized p -samples. However, the Chernoff bound (see the Appendix) is still applicable to show that $|R \cap \{s_1, \dots, s_k\}| \leq O(pk + \log N)$ and $k \leq O(1/p)(|R \cap \{s_1, \dots, s_k\}| + \log N)$ for all $k \leq N$ w.h.p.(N). In particular, assuming $K \leq N$, we have $|R| \leq O(p|S| + \log N)$ and $|S| \leq O(1/p)(|R| + \log N)$ w.h.p.(N).

We now present a near-linear-time algorithm for the partition refinement theorem:

Theorem 5.2 *In Theorem 3.1, given the subset of points inside each cell, we can construct a subdivision into subcells, together with the subset of points inside each subcell, in time*

$$O(bn + b^{O(1)}(m + t) \log^{O(1)} N),$$

such that the stated properties are satisfied, except that the crossing number bound is now $O((bt)^{1-1/d} + b^{1-1/(d-1)}\ell + b \log t \log N)$ w.h.p.(N) for any $N \geq mt$.

Proof The Algorithm. Let $\Delta_t, \dots, \Delta_1$ be the given cells in a random order. For $i = t, \dots, 1$ do:

0. Let q be a power of 2 (with a negative integer exponent) closest to $\min\left\{\frac{(bi)^{1/d} \log N}{|\widehat{H}|}, \frac{b^{1/(d-1)} t \log N}{|\widehat{H}|^\ell}, \frac{b^{1/(d-1)} i}{|\widehat{H}|}\right\}$. If q is different from its value in the previous iteration, then create a new q -sample \widehat{R} of \widehat{H} .
1. Subdivide Δ_i into disjoint subcells by building a $(1/r_i)$ -cutting of \widehat{R}_{Δ_i} inside Δ_i with

$$r_i := c \min\left\{|\widehat{R}_{\Delta_i}| \left(\frac{b}{X_{\Delta_i}(\widehat{R})}\right)^{1/d}, b^{1/(d-1)}\right\}.$$

By Lemma 2.3, the number of subcells inside Δ_i is at most $b/4$ for a sufficiently small constant c .

2. Further subdivide each subcell by using extra cuts to ensure that each subcell contains at most $2n/(bt)$ points. The number of subcells remains $O(b)$ per cell Δ_i , and at most bt in total.
3. Take a p -sample ρ_i of the subcells of Δ_i with $p := \min\left\{\frac{b^{1/d} \log N}{t^{1-1/d}}, \frac{b^{1/(d-1)} \log N}{\ell}, \frac{b^{1/(d-1)}}{\log t}, 1\right\}$. If $\rho_i \neq \emptyset$, then
 - (a) For each $h \in H_{\Delta_i}$, add new copies of h to \widehat{H} so that the multiplicity of h in \widehat{H} gets multiplied by $(1 + 1/b)^{|\rho_i(h)|}$, where $\rho_i(h)$ denotes the set of subcells in ρ_i crossed by h .
 - (b) Insert a q -sample of the newly added hyperplanes of \widehat{H} to \widehat{R} .

During the algorithm, we also check for the validity of certain conditions, which will be delineated in the analysis below, and which are guaranteed to hold w.h.p. If one of these conditions fail, we declare failure and the algorithm may be re-run from scratch.

Analysis of the Crossing Number. Fix a value q and let $\widehat{R}^{(q)}$ denote the (dynamic) set \widehat{R} during the time interval when the power of 2 closest to $\min\left\{\frac{(bi)^{1/d} \log N}{|\widehat{H}|}, \frac{b^{1/(d-1)} t \log N}{|\widehat{H}|^\ell}, \frac{b^{1/(d-1)} i}{|\widehat{H}|}\right\}$ is equal to q . In step 1, $\widehat{R}^{(q)}$ starts off as a q -sample of \widehat{H} , and in step 3(b), we add q -samples of new hyperplanes in \widehat{H} to $\widehat{R}^{(q)}$. Thus, at any moment during its time interval, $\widehat{R}^{(q)}$ is a generalized q -sample of \widehat{H} , if we view the multiset \widehat{H} as a list where the hyperplanes are listed in the order they are added to \widehat{H} . Similarly, for any fixed simplex σ , $\widehat{R}_\sigma^{(q)}$ is a generalized q -sample of \widehat{H}_σ . Call two simplices σ_1 and σ_2 equivalent if $H_{\sigma_1} = H_{\sigma_2}$. The number of equivalence classes is polynomially bounded in N . Since $|\widehat{H}|$ is loosely bounded by $Cm(1 + 1/b)^{O(bt)}$ at all times, the total number of different q 's encountered is polynomially bounded in N . By the Chernoff bound, we can thus guarantee that the following holds at all times for all simplices σ and all q w.h.p.(N):

$$|\widehat{H}_\sigma| \leq O(1/q)(|\widehat{R}_\sigma^{(q)}| + \log N) \quad \text{and} \quad |\widehat{R}| \leq O(q|\widehat{H}| + \log N).$$

After step 1, w.h.p., for each subcell σ , we have $|\widehat{H}_\sigma| \leq O(1/q)(|\widehat{R}_\sigma| + \log N) \leq O(1/q)(|\widehat{R}_{\Delta_i}|/r_i + \log N)$, implying that $\sum_{h \in \widehat{H}} |\rho_i(h)| \leq |\rho_i| \cdot O(1/q)(|\widehat{R}_{\Delta_i}|/r_i + \log N)$. W.h.p., step 3(a) increases $|\widehat{H}|$ by the following amount:

$$\begin{aligned} & \sum_{h \in \widehat{H}} [(1 + 1/b)^{|\rho_i(h)|} - 1] \\ & \leq O\left(\sum_{h \in \widehat{H}} |\rho_i(h)|/b\right) \leq \frac{|\rho_i|}{bq} \cdot O(|\widehat{R}_{\Delta_i}|/r_i + \log N) \\ & \leq \frac{|\rho_i|}{bq} \cdot O(\alpha_i |\widehat{R}| + \log N) \\ & \text{where } \alpha_i := \frac{|\widehat{R}_{\Delta_i}|}{r_i |\widehat{R}|} = O\left(\frac{1}{|\widehat{R}|} \cdot \left(\frac{X_{\Delta_i}(\widehat{R})}{b}\right)^{1/d} + \frac{|\widehat{R}_{\Delta_i}|}{b^{1/(d-1)} |\widehat{R}|}\right) \\ & \leq \frac{|\rho_i|}{bq} \cdot O(\alpha_i q |\widehat{H}| + \log N) \leq O(\beta_i) |\widehat{H}| \quad \text{where } \beta_i := \frac{|\rho_i|}{b} \left(\alpha_i + \frac{\log N}{q |\widehat{H}|}\right). \end{aligned}$$

Let ℓ_i be as before. When conditioned to a fixed choice of $\Delta_r, \dots, \Delta_{i+1}$ and \widehat{R} , we have $E[|\widehat{R}_{\Delta_i}|] \leq |\widehat{R}| \ell_i / i$ and $E[X_{\Delta_i}(\widehat{R})] \leq |\widehat{R}|^d / i$, so $E[\alpha_i] \leq O(1/(bi)^{1/d} + \ell_i/(b^{1/(d-1)}i))$. Furthermore, $E[|\rho_i|] \leq O(bp)$ conditioned to any fixed value of α_i . Thus,

$$E[\beta_i \mid \Delta_r, \dots, \Delta_{i+1}, \widehat{R}] \leq p \cdot O(1/(bi)^{1/d} + \ell_i/(b^{1/(d-1)}i) + (\log N)/(q|\widehat{H}|)).$$

Since $E[\ell_i] \leq O(\ell_i/t + \log N)$,

$$E[\beta_i] \leq p \cdot O\left(\frac{1}{(bi)^{1/d}} + \frac{\ell}{b^{1/(d-1)}t} + \frac{\log N}{b^{1/(d-1)}i} + \frac{\log N}{q|\widehat{H}|}\right).$$

The last term disappears for our choice of q .

At the end, w.h.p., we can upper-bound $|\widehat{H}|$ by $Cm \prod_{i=1}^t (1 + O(\beta_i)) \leq Cm \exp(O(\sum_{i=1}^t \beta_i))$, where

$$E\left[\sum_{i=1}^t \beta_i\right] \leq p \cdot O\left(\frac{t^{1-1/d}}{b^{1/d}} + \frac{\ell}{b^{1/(d-1)}} + \frac{\log N \log t}{b^{1/(d-1)}}\right) \leq O(\log N)$$

for our choice of p . By Markov’s inequality, with probability $\Omega(1)$, we have $\sum_{i=1}^t \beta_i \leq O(\log N)$ and $|\widehat{H}| \leq CN^{O(1)}$. If not, we declare failure.

Let $\rho(h) = \bigcup_i \rho_i(h)$. Since the multiplicity of h in \widehat{H} is $C(1 + 1/b)^{|\rho(h)|} \leq |\widehat{H}|$, it follows that $\max_{h \in H} |\rho(h)| \leq \log_{1+1/b}(|\widehat{H}|/C) \leq O(b \log N)$ if algorithm does not fail. Observe that $\rho(h)$ is a generalized p -sample of the list $\lambda(h)$ of all subcells crossed by h , where the subcells are listed in the order they are created. Thus, w.h.p., we have $\max_{h \in H} |\lambda(h)| \leq O((1/p)(|\rho(h)| + \log N)) \leq O((b/p) \log N) \leq O((bt)^{1-1/d} + b^{1-1/(d-1)} \ell + b^{1-1/(d-1)} \log t \log N + b \log N)$ if the algorithm does not fail.

Analysis of Running Time. Conditioned to a fixed choice of $\Delta_r, \dots, \Delta_{i+1}$ and \widehat{R} , we have $E[|\widehat{R}_{\Delta_i}|] \leq |\widehat{R}| \ell_i / i$. Thus, $E[|\widehat{R}_{\Delta_i}|] \leq (q|\widehat{H}| + \log N) \cdot O(\ell_i/t + \log N)/i \leq (b \log N)^{O(1)}$ for our choice of q . Assuming that \widehat{R}_{Δ_i} is available, step 1 takes time $|\widehat{R}_{\Delta_i}| r_i^{O(1)}$, which has expected value $(b \log N)^{O(1)}$. We do not need to compute $X_{\Delta_i}(\widehat{R})$, but rather we try different values of r_i (powers of 2) until we find a cutting with the right number of subcells. Thus, the expected cost over all iterations is $O(t)(b \log N)^{O(1)}$. By Markov’s inequality, with probability $\Omega(1)$, this bound holds; if not, we declare failure.

In step 2, we can check the number of points per subcell by assigning points to subcells in $O(b)$ time per point, for a total cost of $O(bn)$. We can actually make the cost sublinear in n —namely, $O(t)(b \log N)^{O(1)}$ —by replacing P by a $\frac{bt \log N}{n}$ -sample Q of P . W.h.p., $|Q| = O(bt \log N)$, and for any simplex σ , $|Q \cap \sigma| \leq |Q|/(bt)$ implies $|P \cap \sigma| \leq O(n/(bt))$ by a Chernoff bound.

For step 3(a), we first need to compute the set H_{Δ_i} , i.e., report all hyperplanes of H intersecting a query cell Δ_i . In the dual, H becomes an m -point set and Δ_i becomes an $O(1)$ -size polyhedron, and this subproblem reduces to simplex range searching. We can use existing data structures, e.g., [39], to get $O(m \log m)$ preprocessing time and $O(m^{1-1/d} \log^{O(1)} m + |H_{\Delta_i}|)$ query time. Since $\sum_i |H_{\Delta_i}| \leq m\ell$ and the probability that $\rho_i \neq \emptyset$ is at most $O(bp)$, the total expected query time is $bp \cdot O(tm^{1-1/d} \log^{O(1)} m + m\ell) \leq O(t^{1/d} m^{1-1/d} + m)(b \log N)^{O(1)} \leq O(t + m)(b \log N)^{O(1)}$. Again, by Markov’s inequality, with probability $\Omega(1)$, this bound holds; if not, we declare failure. We can compute $|\rho_i(h)|$ for each $h \in H_{\Delta_i}$ naively in time $O(b)$, which can be absorbed in the $b^{O(1)}$ factor.

Finally, we account for the total cost of all operations done to $\widehat{R}^{(q)}$: initialization in step 0, insertions in step 3(b), and computing $\widehat{R}_{\Delta_i}^{(q)}$ in step 1. Fix q . We continue inserting to $\widehat{R}^{(q)}$ only if $q \leq 2 \frac{(bt)^{1/d} \log N}{|\widehat{H}|}$. Thus, w.h.p., at all times, $|\widehat{R}^{(q)}| \leq O(q|\widehat{H}| + \log N) \leq O(t^{1/d})(b \log N)^{O(1)}$. Computing $\widehat{R}_{\Delta_i}^{(q)}$ reduces to simplex range searching again. This time, we use an existing data structure that has larger space but supports faster querying, with preprocessing time and total insertion time $O(|\widehat{R}^{(q)}|^d \log^{O(1)} |\widehat{R}^{(q)}|) = O(t)(b \log N)^{O(1)}$ w.h.p., and query time $O(\log^{O(1)} |\widehat{R}^{(q)}| + |\widehat{R}_{\Delta_i}^{(q)}|)$. (Insertions are supported by standard amortization techniques [14].) Since $|\widehat{H}| \leq CN^{O(1)}$, the total number of different q ’s is $O(\log N)$, which can be absorbed in the $\log^{O(1)} N$ factor.

To summarize, our algorithm succeeds with probability $\Omega(1)$, and w.h.p. the stated bound holds if the algorithm does not fail. We can re-run the algorithm until success, with $O(\log N)$ number of trials w.h.p. The running time remains $O(m + t)(b \log N)^{O(1)}$. At the end, we can assign all the input points to subcells in $O(bn)$ additional time (this term can actually be reduced to $O(n \log b)$ with point location data structures). □

Theorem 5.3 *We can build data structures in $O(n \log n)$ time achieving the bounds stated in Theorems 3.2, 4.1, and 4.2 w.h.p.(n).*

Proof Let H be the union of the test sets H_t from Observation 5.1 over all t that are powers of 2; the total size is $m = O(n \log^{O(1)} N)$. From now on, summations involving the variable t are over all powers of 2 (instead of b).

Because Observation 5.1 requires a lower bound on the number of points per cell, we need to use a more careful, alternative construction in the proof of Theorem 3.2. Initially, our partition tree consists of just a root cell containing all n points. For $t = 2, 4, 8, \dots$ do:

Let Π_t be the set of all current leaf cells Δ such that the number of points inside Δ is between n/t and $2n/t$; the number of such cells is at most t . Apply

Theorem 3.1 to Π_t to get a set Π'_t of subcells where the number of points inside each subcell is at most $2n/(bt)$. Make the $O(b)$ subcells of each cell $\Delta \in \Pi_t$ the children of Δ in the partition tree (Δ is no longer a leaf but its children now are).

Let $\ell(t)$ denote the maximum number of cells of Π_t crossed by an arbitrary hyperplane. Let $\ell'(t)$ denote the maximum number of cells of Π'_t crossed by a hyperplane in H , which is bounded by (2). The maximum number of cells of Π_u crossed by a hyperplane in H is at most $\sum_{t \leq 2u/b} \ell'(t)$. By Observation 2.2, $\ell(u)$ is asymptotically bounded by the same expression plus $u^{1-1/d}$. We obtain the following recurrence:

$$\ell(u) \leq O\left(\sum_{t \leq 2u/b} [(bt)^{1-1/d} + b^{1-1/(d-1)}\ell(t) + b \log t \log n] + u^{1-1/d}\right). \tag{4}$$

For a sufficiently large constant b , we can again verify by induction that

$$\ell(u) \leq O(u^{1-1/d} + u^{1-1/(d-1)+\delta} \log n) \leq O(u^{1-1/d} + \log^{O(1)} n).$$

The query cost remains $O(\sum_{t \leq n} b\ell(t)) \leq O(n^{1-1/d})$.

Note that the above construction works just as well in the proofs of Theorems 4.1 and 4.2. In the proof of Theorem 4.2, the maximum number of internal node cells crossed by an arbitrary hyperplane is at most $O(\sum_{t \leq n/B} \ell(t)) = O(n/B)^{1-1/d}$. The maximum number of leaf cells crossed by an arbitrary hyperplane is at most b times this number, so the query cost remains $O(n/B)^{1-1/d}$.

By Theorem 5.3, we immediately get $O(n \log^{O(1)} n)$ preprocessing time w.h.p.(n). We now describe how to reduce the preprocessing time of Theorem 3.2 to $O(n \log n)$, by “bootstrapping”.

Build a B_0 -partial partition tree with $O(n/B_0)$ size and $O(n/B_0)^{1-1/d}$ query cost, by Theorem 4.2, with $B_0 = \log^c n$ for some constant c . Here, we can take H to be the union of H_t for all $t \leq n/B_0$, of total size $m = O((n/B_0) \log^{O(1)} n)$, and the time bound drops to $O(\sum_{t \leq n/B_0} [n + (n/B_0) \log^{O(1)} n]) = O(n \log n)$ for c sufficiently large. Using the preceding partition tree method with $P(B_0) = O(B_0 \log^{O(1)} B_0)$ and $Q(B_0) = O(B_0^{1-1/d})$ to handle the subproblems at the leaf cells, we then get a new partition tree with preprocessing time $O(n/B_0)P(B_0) + O(n \log n) = O(n \log n)$ and query time $O(n/B_0)^{1-1/d} Q(B_0) = O(n^{1-1/d})$.

The query bound is expected but can be made to hold w.h.p.(n): For a fixed query hyperplane, we are bounding a sum Z of independent random variables lying between 0 and $O(B_0)$ with $E[Z] \leq O(n^{1-1/d})$; by a Chernoff bound (see the Appendix), w.h.p.(n), we have $Z \leq O(n^{1-1/d} + B_0 \log n) = O(n^{1-1/d})$. Observe that every cell in our construction are defined by a constant number of input points, so the number of possible cells, and hence the number of combinatorially different query hyperplanes is polynomially bounded in n . Thus, the maximum query cost is indeed $O(n^{1-1/d})$ w.h.p.(n).

The same idea also reduces the preprocessing time of Theorems 4.1 and 4.2 to $O(n \log n)$. □

Note that the above result is Las Vegas, since the random choices made in the preprocessing algorithm only affect the query time, not the correctness of the query algorithm.

6 Shallow Version

In this section, we modify our method to solve the halfspace range reporting problem in an even dimension $d \geq 4$. It suffices to consider upper halfspaces, i.e., we want to report all points above a query hyperplane. We say that a hyperplane h is k -shallow if the number of points of P above h is at most k .

It can be checked that Observation 2.2 remains true for k -shallow hyperplanes, i.e., with the same test set H , for every k , if the maximum number of cells crossed by a k -shallow hyperplane in H is ℓ , then the maximum number of cells crossed by an arbitrary k -shallow hyperplane is at most $O(\ell)$.

Matoušek [40] gave a “shallow version” of the cutting lemma. Actually, a weaker version of the lemma suffices to derive a “shallow version” of the partition theorem e.g., as noted in [47] (the original shallow cutting lemma wants to cover all points in the $(\leq k)$ -level, whereas it suffices to cover a large number of points at low levels). We use this weaker cutting lemma, which has a simpler proof, because it makes an X -sensitive variant of the lemma easier to verify. Our version also has a nice new feature: all cells are *vertical*, i.e., they contain $(0, \dots, 0, -\infty)$.

Stating the right analog of “ X ” requires more care. Define $X_\Delta(H, p)$ to be the expected number of vertices of the lower envelope of a p -sample of H inside Δ . Define $\bar{X}_\Delta(H, p) = \sum_{p' \leq p} X_\Delta(H, p')$ where the sum is over powers p' of 2. Note that although $X_\Delta(\cdot, \cdot)$ may not be monotone increasing in p , $\bar{X}_\Delta(\cdot, \cdot)$ is, so it is more convenient to work with the latter quantity.

In order to state how many points ought to be covered, define $\mu_\Delta(H)$ (the “measure” of H) to be the total number of pairs (p, h) with $p \in P \cap \Delta$ and $h \in H$ such that p is above h .

Lemma 6.1 (Shallow Cutting Lemma, weak version) *Let H be a set of m hyperplanes, P be a set of points, and Δ be a vertical cell in \mathbb{R}^d . For any r , we can find an $O(1/r)$ -cutting of H into $O(r^{\lfloor d/2 \rfloor})$ disjoint vertical cells inside Δ that cover all but $O(\mu_\Delta(H)r/m)$ points of P .*

In terms of $\bar{X}(\cdot, \cdot)$, the number of cells can be reduced to $O(\bar{X}_\Delta(H, r/m) + r^{\lfloor (d-1)/2 \rfloor})$.

Proof Draw an (r/m) -sample R of H . Take the canonical triangulation T of the lower envelope of R restricted inside Δ . The expected number of cells is $O(\bar{X}_\Delta(H, r/m) + r^{\lfloor (d-1)/2 \rfloor})$, since there are $O(r^{\lfloor (d-1)/2 \rfloor})$ vertices in the lower envelope inside each $(d - 1)$ -dimensional boundary facet of Δ . For each cell $\sigma \in T$ with $|H_\sigma| = a_\sigma m/r$ ($a_\sigma > 1$), further subdivide σ as follows: pick a $(1/(2a_\sigma))$ -approximation A_σ of H_σ with $|A_\sigma| = O(a_\sigma^2 \log a_\sigma)$ (ε -approximations [35, 44, 47] can be found by random sampling); return any triangulation of the $(\leq |A_\sigma|/a_\sigma)$ -level L_σ of A_σ into $|A_\sigma|^{O(1)} \leq a_\sigma^{O(1)}$ vertical subcells intersected with σ . The total number of subcells is $\sum_{\sigma \in T} (r|H_\sigma|/m)^{O(1)}$, which has expected value $O(\bar{X}_\Delta(H, r/m) + r^{\lfloor (d-1)/2 \rfloor})$ by Clarkson and Shor’s or Chazelle and Friedman’s technique [23, 29].

Each vertex of L_σ has level at most $O(|H_\sigma|/a_\sigma) \leq O(m/r)$ in H by the approximation property of A_σ , so each subcell is indeed crossed by at most $O(m/r)$ hyperplanes (since a hyperplane crossing the subcell must lie below one of the d

vertices of the subcell). Inside a cell $\sigma \in T$, any uncovered point has level at least $|A_\sigma|/a_\sigma$ in A_σ and thus lies above at least $\Omega(|H_\sigma|/a_\sigma) \geq \Omega(m/r)$ hyperplanes in H again by the approximation property. So, the number of uncovered points in T is at most $O(\mu_\Delta(H)r/m)$. On the other hand, the number of uncovered points outside T is at most $\mu_\Delta(R)$, which has expected value $\mu_\Delta(H)r/m$. By Markov’s inequality, the bounds on the number of cells and uncovered points hold simultaneously to within constant factors with probability $\Omega(1)$. \square

Equipped with the shallow cutting lemma, we can adopt the same approach from Sect. 3 to solve the halfspace range reporting problem, with d replaced by $\lfloor d/2 \rfloor$ in the exponents. The assumption that d is even is critical, to ensure that $\lfloor (d - 1)/2 \rfloor$ is strictly less than $\lfloor d/2 \rfloor$.

Theorem 6.2 (Shallow Partition Refinement Theorem) *Let P be a set of at most n points and H be a set of m k -shallow hyperplanes in \mathbb{R}^d with $d > 2$. Suppose we are given t disjoint vertical cells covering P , such that each cell contains at most $2n/t$ points of P and each hyperplane in H crosses at most ℓ cells. Then for any b , we can find $O(b)$ disjoint vertical subcells inside every cell, for a total of at most bt cells, such that all but $O((bt)^{1/\lfloor d/2 \rfloor} k)$ points are covered by the subcells, each subcell contains at most $2n/(bt)$ points of P , and each hyperplane in H crosses at most the following total number of subcells:*

$$O((bt)^{1-1/\lfloor d/2 \rfloor} + b^{1-1/\lfloor (d-1)/2 \rfloor} \ell + b \log t \log m).$$

Proof We apply the same algorithm as in the proof of Theorem 3.1, except that in step 1 we apply Lemma 6.1 to \widehat{H}_{Δ_i} with

$$r_i \approx \min\{p^* |\widehat{H}_{\Delta_i}|, cb^{1/\lfloor (d-1)/2 \rfloor}\},$$

where p^* is the largest value at most $p_0 := (bi)^{1/\lfloor d/2 \rfloor} / \|\widehat{H}\|$ such that $\overline{X}_{\Delta_i}(\widehat{H}, r_i / |\widehat{H}_{\Delta_i}|) \leq \overline{X}_{\Delta_i}(\widehat{H}, p^*) \leq cb$ and thus the number of subcells in the cutting is $b/2$ for a sufficiently small constant c . (As a technicality, we should make $r_i / |\widehat{H}_{\Delta_i}|$ a power of 2.)

Note that $\sum_{j=1}^n \overline{X}_{\Delta_j}(\widehat{H}, p) \leq O(\sum_{p' \leq p} (p' |\widehat{H}|)^{\lfloor d/2 \rfloor}) = O((p |\widehat{H}|)^{\lfloor d/2 \rfloor})$, where the sum is over powers p' of 2. Conditioned to a fixed choice of $\Delta_1, \dots, \Delta_{i+1}$, we then have $E[\overline{X}_{\Delta_i}(\widehat{H}, p)] \leq O((p |\widehat{H}|)^{\lfloor d/2 \rfloor} / i)$. By Markov’s inequality, $\Pr[p^* < p] \leq \Pr[\overline{X}_{\Delta_i}(\widehat{H}, p) > cb] \leq O((p |\widehat{H}|)^{\lfloor d/2 \rfloor} / (bi))$ for any $p \leq p_0$. So,

$$E[1/p^*] \leq O\left(1/p_0 + \sum_{p \leq p_0} 1/p \cdot \frac{(p |\widehat{H}|)^{\lfloor d/2 \rfloor}}{bi}\right) = O\left(\frac{|\widehat{H}|}{(bi)^{1/\lfloor d/2 \rfloor}}\right),$$

where the sum is over powers p of 2. Now,

$$\alpha_i := \frac{|\widehat{H}_{\Delta_i}|}{r_i |\widehat{H}|} \leq O\left(\frac{1}{p^* |\widehat{H}|} + \frac{b^{1/\lfloor (d-1)/2 \rfloor} |\widehat{H}_{\Delta_i}|}{|\widehat{H}|}\right).$$

The first term has expected value $O(1/(bi)^{1/\lfloor d/2 \rfloor})$ by the above argument. The rest of the analysis for $E[\alpha_i]$ and the expected crossing number then carries through as before, with d and $d - 1$ replaced by $\lfloor d/2 \rfloor$ and $\lfloor (d - 1)/2 \rfloor$.

Lastly, we bound the number of points not covered by the subcells. Note that $\sum_{j=1}^n \mu_{\Delta_j}(\widehat{H}) \leq O(|\widehat{H}|k)$. Conditioned to a fixed choice of $\Delta_1, \dots, \Delta_{i+1}$, we have $E[\mu_{\Delta_i}(\widehat{H})] \leq O(|\widehat{H}|k/i)$, so the expected number of uncovered points in Δ_i is $O(|\widehat{H}|k/i \cdot r_i/|\widehat{H}_{\Delta_i}|) \leq O(p_0|\widehat{H}|k/i) \leq O(b^{1/\lfloor d/2 \rfloor} k/i^{1-1/\lfloor d/2 \rfloor})$. Summing over $i = 1, \dots, t$ gives the desired $O((bt)^{1/\lfloor d/2 \rfloor} k)$ bound. By Markov’s inequality, the bounds on crossing number and the number of uncovered points simultaneously hold to within constant factors with probability $\Omega(1)$. \square

Theorem 6.3 *Given n points in \mathbb{R}^d for an even $d \geq 4$, we can build a partition tree that has query cost $O(n^{1-1/\lfloor d/2 \rfloor} + k \log^{O(1)} n)$ for any k -shallow query hyperplane.*

Proof Let $k_t := n/(ct^{1/\lfloor d/2 \rfloor} \log n)$ for a constant c . We follow the partition tree construction from the proof of Theorem 5.3, where Π'_t is now obtained by applying Theorem 6.2 to Π_t and the k_{bt} -shallow hyperplanes in the test set H from Observation 2.2. We remove the points not covered. The total number of points removed is at most $O(\sum_t (bt)^{1/\lfloor d/2 \rfloor} k_{bt}) = O(n/c)$, which can be made at most $n/2$ for a sufficiently large c . We ignore the removed points for the time being.

In the analysis, we redefine $\ell(t)$ to be the maximum number of cells of Π_t crossed by an arbitrary k_t -shallow hyperplane. The same recurrence holds, with d and $d - 1$ replaced by $\lfloor d/2 \rfloor$ and $\lfloor (d - 1)/2 \rfloor$. Since d is even, $\lfloor (d - 1)/2 \rfloor = \lfloor d/2 \rfloor - 1$. We get $\ell(u) \leq O(u^{1-1/\lfloor d/2 \rfloor} + \log^{O(1)} n)$.

More generally, define $\ell(t, n, k)$ to be the maximum number of cells of Π_t crossed by a k -shallow hyperplane h . We now show that

$$\ell(u, n, k) \leq O(u^{1-1/\lfloor d/2 \rfloor} + (1 + ku/n) \log^{O(1)} n).$$

If $k < k_u$ or $k > k_1$, then the claim follows from the $\ell(u)$ bound or the trivial $O(u)$ bound. Otherwise, let $1 \leq t_* \leq u$ be such that $k \approx k_{t_*}$. There are $O(\sum_{t \leq t_*} \ell(t))$ cells crossed by h containing at least n/t_* points. For each cell Δ crossed by h which contains less than n/t_* points but whose parent contains at least n/t_* points, the cell Δ can contain at most $O((n/t_*)/(n/u)) = O(u/t_*)$ cells of Π_u . Thus, $\ell(u, n, k) \leq O(b \sum_{t \leq t_*} \ell(t) \cdot u/t_*) = O((t_*^{1-1/\lfloor d/2 \rfloor} + \log^{O(1)} n) \cdot (u/t_*)) = O((uk/n) \log^{O(1)} n)$.

The at most $n/2$ removed points can be handled recursively. We can join the $O(\log n)$ partition trees into one by creating a new root of degree $O(\log n)$. Define $\bar{\ell}(u, n, k)$ to be the maximum overall number of cells containing between n/u and $2n/u$ points that are crossed by a k -shallow hyperplane. Then $\bar{\ell}(u, n, k) \leq \ell(u, n, k) + \ell(u/2, n/2, k) + \dots \leq O(u^{1-1/\lfloor d/2 \rfloor} + (1 + ku/n) \log^{O(1)} n)$.

We conclude that the query cost is $O(\sum_{t \leq n} b\bar{\ell}(t, n, k)) \leq O(n^{1-1/\lfloor d/2 \rfloor} + k \log^{O(1)} n)$. \square

Theorem 6.4 *Given n points in \mathbb{R}^d for an even $d \geq 4$ and $B < n/\log^{\omega(1)} n$, we can build a B -partial partition tree with size $O(n/B)$ and query cost $O((n/B)^{1-1/\lfloor d/2 \rfloor} + (k/B) \log^{O(1)} n)$ for any k -shallow query hyperplane.*

Proof As in the proof of Theorems 4.2 and 5.3, the query cost is $O(\sum_{t \leq n/B} \bar{\ell}(t, n, k)) \leq O((n/B)^{1-1/\lfloor d/2 \rfloor} + (1 + k/B) \log^{O(1)} n)$. \square

Remarks Because of the recursive handling of removed points, we lose the property that the children cells are disjoint, but only at the root.

The above method is simpler than many of halfspace range reporting methods from Matoušek’s paper [40], in that we do not need auxiliary data structures. We bound the overall crossing number of a k -shallow hyperplane directly as a function of n and k .

We have purposely been sloppy about the second $O(k \log^{O(1)} n)$ term in Theorem 6.3, since our main interest is in the case when k is small. (For large $k > n^{1-1/\lfloor d/2 \rfloor} \log^{\omega(1)} n$, we already know how to get $O(k)$ time [40].) By bootstrapping as in the proof of Theorem 5.3 for j rounds, the polylogarithmic factor can be reduced to the j -th iterated logarithm $\log^{(j)} n$ for any constant j .

Preprocessing Time We now examine the preprocessing time of this method. First we need a more economical shallow version of the test set observation. This time, we cannot quote Matoušek’s paper [40] directly, since he did not state a sufficiently general version of the observation (he only considered (n/t) -shallow hyperplanes rather than near- $(n/t^{\lfloor d/2 \rfloor})$ -shallow hyperplanes, so the test set size became larger). We thus include a proof of the version we need (as it turns out, the dependence on t disappears in the case of vertical cells):

Observation 6.5 (Shallow Test Set) *Given a set P of n points in \mathbb{R}^d and any $k \geq \log N$, we can construct a set H_k of $O((n/k)^{\lfloor d/2 \rfloor} \log^{O(1)} N)$ $O(k)$ -shallow hyperplanes, in $O((n/k)^{\lfloor d/2 \rfloor} \log^{O(1)} N)$ time, satisfying the following property w.h.p.(N) for any $N \geq n$: For any collection of vertical cells, if the maximum number of cells crossed by a hyperplane in H is ℓ , then the maximum number of cells crossed by an arbitrary k -shallow hyperplane is $O(\ell)$.*

Proof In what follows, the dual of an object q is denoted by q^* . Draw a random sample R of P of size $(n/k) \log N$ and return the set H_k of all hyperplanes through d -tuples of points in R that are $(c \log N)$ -shallow with respect to R for a constant c . Note that in the dual, H_k^* corresponds to the vertices of the $(\leq c \log N)$ -level L of R^* and thus has size $O(|\hat{R}|^{\lfloor d/2 \rfloor} \log^{O(1)} N) = O((n/k)^{\lfloor d/2 \rfloor} \log^{O(1)} N)$ [29].

To prove correctness, first note that every hyperplane in H_k is $O(k)$ -shallow with respect to P w.h.p.(N) by a Chernoff bound (the number of combinatorially different hyperplanes is polynomially bounded). Let h be a k -shallow hyperplane with respect to P . Then by a Chernoff bound, w.h.p.(N), h is $(c \log N)$ -shallow with respect to R for a sufficiently large c , i.e., the point h^* is in L . Thus, h^* lies below a facet Δ^* of L , defined by d vertices $h_1^*, \dots, h_d^* \in H_k^*$. Suppose h crosses a vertical cell σ . Then some point $q \in \sigma$ lies above h , i.e., the hyperplane q^* lies below h^* . Then q^* must lie below one of the points h_1^*, \dots, h_d^* , i.e., q must lie above one of the hyperplanes $h_1, \dots, h_d \in H_k$. So, σ is crossed by one of h_1, \dots, h_d . \square

In the proof of Theorem 6.3, we can take H to be the union of the test sets H_{k_t} over all t ; the total size is $m = O(\sum_t (n/k_t)^{\lfloor d/2 \rfloor} \log^{O(1)} N) = O(\sum_t t \log^{O(1)} N) = O(n \log^{O(1)} N)$.

Lemma 6.1 requires $O(mr^{O(1)})$ time; we can repeat for $O(\log N)$ trials and return best result found to ensure correctness w.h.p.(N).

We can then proceed as in the proof of Theorem 5.2 to get a near-linear-time algorithm for Theorem 6.2. Since cells are vertical, the computation of H_{Δ_i} and R_{Δ_i} now reduces to halfspace range reporting in the dual (reporting hyperplanes crossing a vertical cell reduces to reporting hyperplanes lying below each of the d vertices of the cell).

It can be checked that the bootstrapping step in the proof of Theorem 5.3 carries through as well, yielding the final result:

Theorem 6.6 *We can build a partition tree in $O(n \log n)$ time achieving the bound stated in Theorems 6.3 and 6.4 w.h.p.(n).*

7 Some Applications

Although we dare not go through all papers in the literature that have used simplex or halfspace range searching data structures as subroutines, it is illustrative to at least briefly mention a few sample applications.

Spanning Trees (and Triangulations) with Low Crossing Number A series of papers [2, 7, 34, 38, 52] addressed the construction of spanning trees such that the maximum number of edges crossed by a hyperplane is small. This problem has direct applications to a number of geometric problems [3, 52]. We obtain the first $O(n \log n)$ -time algorithm that attains asymptotically optimal worst-case crossing number.

Corollary 7.1 *Given n points in \mathbb{R}^d , there is an $O(n \log n)$ -time Monte Carlo algorithm to compute a spanning tree with the property that any hyperplane crosses at most $O(n^{1-1/d})$ tree edges w.h.p.(n). For $d = 2$, we can ensure that the spanning tree is planar.*

Proof The result follows directly from Theorems 3.2 and 5.3: for each cell in the partition tree, we just select a constant ($O(b)$) number of edges to connect the subsets at the children.

In the $d = 2$ case, we can ensure that the spanning tree has no self-intersection, because our partition tree guarantees the disjointness of the children's cells: At each node v , we select connecting edges that lie outside the convex hulls of the children's subsets. For example, this can be done naively by triangulating the region between the convex hull of $P(v)$ and the convex hull of the children's subsets, and then picking out appropriate edges; the required time is linear in $|P(v)|$. The convex hulls themselves can be computed by merging bottom-up. The total time is $O(n \log n)$. \square

A related combinatorial problem is to find a Steiner triangulation with low crossing number for a 2-dimensional point set. The best upper bound known is $O(\sqrt{n} \log n)$ [8], which is obtained by combining spanning trees with low crossing number and a result of Hershberger and Suri [36] for simple polygons. We obtain the following new theorem:

Corollary 7.2 *Given a set P of n points in \mathbb{R}^2 , there exists a triangulation of $O(n)$ points which includes P , with the property that any line crosses at most $O(\sqrt{n})$ edges.*

Proof The result follows directly from Theorem 3.2, using again the property that our partition tree guarantees disjointness of the children’s cells: We introduce Steiner points at each vertex of every cell in the tree. At each node v , we shrink the children’s cells slightly and triangulate the “gap” between the cell at v and the children’s cells, which has constant $O(b)$ complexity. \square

Multilevel Data Structures Many query problems are solved using multilevel partition trees, where we build secondary data structures for each *canonical subset* and answer a query by identify relevant canonical subsets and querying their corresponding secondary structures. The corollary below encapsulates the main property of our partition tree for multilevel applications, and is a direct consequence of Theorems 4.1 and 5.3:

Corollary 7.3 *Given n points in \mathbb{R}^d ,*

- (i) *We can form $O(n)$ canonical subsets of total size $O(n \log n)$ in $O(n \log n)$ time, such that the subset of all points inside any query simplex can be reported as a union of disjoint canonical subsets C_i with $\sum_i |C_i|^{1-1/d} \leq O(n^{1-1/d} \log n)$ in time $O(n^{1-1/d} \log n)$ w.h.p.(n).*
- (ii) *For any fixed $\gamma < 1 - 1/d$, we can form $O(n)$ canonical subsets of total size $O(n \log \log n)$ in $O(n \log n)$ time, such that the subset of all points inside any query simplex can be reported as a union of disjoint canonical subsets C_i with $\sum_i |C_i|^\gamma \leq O(n^{1-1/d})$ in time $O(n^{1-1/d})$ w.h.p.(n).*

For example, we can get the following results:

Corollary 7.4 *Given n line segments in the plane, there is a data structure with*

- (a) *$O(n \log^2 n)$ preprocessing time and $O(n \log n)$ space such that we can report the k intersections with a query line in $O(\sqrt{n} \log n + k)$ expected time.⁵*
- (b) *$O(n \log^3 n)$ preprocessing time and $O(n \log^2 n)$ space such that we can report the k intersections with a query line segment in $O(\sqrt{n} \log^2 n + k)$ expected time.*
- (c) *$O(n \log^3 n)$ preprocessing time and $O(n \log^2 n)$ space such that we can find the first intersection by a query ray in $O(\sqrt{n} \log^2 n)$ expected time.*

Specifically, (a) follows by applying Corollary 7.3(i) twice; (b) follows by applying Corollary 7.3(i) once more; and (c) follows from (b) by a simple randomized reduction (e.g., [16]). For (c), the previous result by Agarwal [3] had $O(n\alpha(n) \log^4 n)$ space and $O(\sqrt{n\alpha(n)} \log^2 n)$ query time (for nonintersecting segments, his bounds of $O(n \log^3 n)$ space and $O(\sqrt{n} \log^2 n)$ query time are still worse than ours).

⁵For simplicity, we state expected time bounds rather than high-probability bounds in this section, although some can be made high-probability bounds with more care. In the expected bounds, we assume that the query objects are independent of the random choices made by the preprocessing algorithm.

For another example, we get the following result by applying Corollary 7.3(i) $d + 1$ times:

Corollary 7.5 *Given n simplices in \mathbb{R}^d , there is a data structure with $O(n \log^{d+1} n)$ preprocessing time and $O(n \log^d n)$ space, such that we can find all simplices containing a query point in $O(n^{1-1/d} \log^d n)$ expected time; and we can find all simplices contained inside a query simplex in $O(n^{1-1/d} \log^d n)$ expected time.*

In contrast, previous methods [39, 42] had $n^{O(1)}$ preprocessing time, $O(n \log^{2d} n)$ space, and $O(n^{1-1/d} \log^d n)$ query time; or $O(n^{1+\varepsilon})$ preprocessing time, $O(n \log^{O(1)} n)$ space, and $O(n^{1-1/d} \log^{O(1)} n)$ query time; or $O(n 2^{O(\sqrt{\log n})})$ preprocessing time and space, and $O(n^{1-1/d} 2^{O(\sqrt{\log n})})$ query time.

Other problems can be solved in a similar way, e.g., intersection searching among simplices in \mathbb{R}^d .

Applications of the Shallow Case Halfspace range emptiness dualizes to membership queries for an intersection \mathcal{P} of n halfspaces: decide whether a query point lies in \mathcal{P} . The problem is a special case of ray shooting in a convex polytope: find the intersection of a query ray with \mathcal{P} . In turn, ray shooting queries are special cases of j -dimensional linear programming queries: find a point in \mathcal{P} that is extreme along a query direction and lies inside a query j -flat. The author [16] (see also [50]) has given a randomized reduction of linear programming queries to halfspace range reporting queries in the case when the output size k is small ($O(\log n)$). This reduction does not increase the asymptotic time and space bounds w.h.p. if the query time bound grows faster than n^ε for some fixed $\varepsilon > 0$. Exact nearest neighbor search (finding the closest point in a given point set to a query point under the Euclidean metric) reduces to ray shooting in one higher dimension by the standard lifting map. Theorems 6.3 and 6.6 thus imply:

Corollary 7.6 *For $d \geq 4$ even, there are data structures for halfspace range emptiness queries for n points in \mathbb{R}^d , for ray shooting and linear programming queries inside the intersection of n halfspaces in \mathbb{R}^d , and exact nearest neighbor queries of n points in \mathbb{R}^{d-1} , with $O(n \log n)$ preprocessing time, $O(n)$ space, and $O(n^{1-1/\lfloor d/2 \rfloor})$ expected query time.*

Trade-offs Thus far, we have focused exclusively on linear-space data structures. By combining with a large-space data structure with logarithmic query time, one can usually obtain a continuous trade-off between space and query time. Specifically, such a trade-off for halfspace range counting, for example, can be obtained through the following corollary:

Corollary 7.7 *Suppose there is a d -dimensional halfspace range counting data structure for point sets of size at most B with $P(B)$ preprocessing time, $S(B)$ space, and $Q(B)$ (expected) query time. Then there is a d -dimensional halfspace range counting data structure for point sets of size at most n with*

- (i) $O(n/B)P(B) + O(n \log n)$ preprocessing time, $O(n/B)S(B) + O(n)$ space, and $O(n/B)^{1-1/d}Q(B) + O(n/B)^{1-1/d}$ expected query time, assuming $B < n/\log^{\omega(1)} n$.
- (ii) $O(n/B)^d P(B) + (n/B)^d B$ preprocessing time, $O(n/B)^d S(B) + O((n/B)^d B)$ space, and $Q(B) + O(\log(n/B))$ (expected) query time.

Proof Part (i) is an immediate consequence of Theorems 4.2 and 5.3. Part (ii) is known and follows from a direct application of hierarchical cuttings [21] (in particular, see [42, Theorem 5.1]). □

As space/query-time tradeoffs are already known [42], we look instead at preprocessing-time/query-time trade-offs, which are more important in algorithmic applications and where we get new results. We explain how such a trade-off can be obtained through the above corollary. Interesting, iterated logarithm arises:

Corollary 7.8 *There is a d -dimensional halfspace range counting data structure with $O(m2^{O(\log^* n)})$ preprocessing time and $O((n/m^{1/d})2^{O(\log^* n)})$ expected query time for any given $m \in [n \log n, n^d/\log^d n]$.*

Proof We first consider the extreme cases when m is $\Theta(n \log n)$ or $\Theta(n^d/\log^d n)$. Assume inductively that there is a data structure for problem size n_i with preprocessing and query time

$$P(n_i) \leq c_i n_i \log n_i, \quad Q(n_i) \leq c_i n_i^{1-1/d} / \log^{1/d} n_i.$$

Set n_{i+1} so that $\log n_{i+1} = n_i^{1-1/d} / \log^{1/d} n_i$. Apply Corollary 7.7(ii) to get a data structure for problem size n_{i+1} , with preprocessing and query time

$$P(n_{i+1}) \leq O((n_{i+1}/n_i)^d c_i n_i \log n_i) = O(c_i n_{i+1}^d / \log^d n_{i+1}),$$

$$Q(n_{i+1}) \leq c_i n_i^{1-1/d} / \log^{1/d} n_i + O(\log n_{i+1}) = O(\log n_{i+1}).$$

Next, set n_{i+2} so that $\log n_{i+2} = n_{i+1}^{d-1} / \log^d n_{i+1}$. Apply Corollary 7.7(i) to get a data structure for problem size n_{i+2} , with preprocessing and query time

$$P(n_{i+2}) \leq O((n_{i+2}/n_{i+1})c_i n_{i+1}^d / \log^d n_{i+1} + n_{i+2} \log n_{i+2}) = O(c_i n_{i+2} \log n_{i+2}),$$

$$Q(n_{i+2}) \leq O((n_{i+2}/n_{i+1})^{1-1/d} c_i \log n_{i+1}) = O(c_i n_{i+2}^{1-1/d} / \log^{1/d} n_{i+2}).$$

Thus, the induction carries through with $c_{i+2} \leq O(1)c_i$, implying $c_i \leq 2^{O(i)}$. Since $n_i \leq \log^{O(1)} n_{i+1}$, the number of iterations to reach problem size n is $O(\log^* n)$, so we conclude that there is a data structure with preprocessing time $O(2^{O(\log^* n)} n \log n)$ and query time $O(2^{O(\log^* n)} n^{1-1/d} / \log^{1/d} n)$.

To obtain the full trade-off, we use one final application of Corollary 7.7(ii), choosing B so that $B^{d-1} / \log B = n^d/m$, to get preprocessing time $O(2^{O(\log^* n)} (n/B)^d \times B \log B) = O(2^{O(\log^* n)} m)$ and query time $O(2^{O(\log^* n)} B^{1-1/d} / \log^{1/d} B + \log n) = O(2^{O(\log^* n)} n/m^{1/d})$. □

The above result is reminiscent of Matoušek’s $O(n^{4/3} 2^{O(\log^* n)})$ -time algorithm [42] for Hopcroft’s problem in 2-d: given n points and n lines in the plane,

find a point-line incidence (or similarly count the number of point-above-line pairs). Matoušek's solution required only cuttings and managed to avoid partition trees because the problem is off-line, i.e., the queries are known in advance. Corollary 7.8 can be viewed as an extension to on-line queries.

(Incidentally, we pick the example of halfspace range counting in the above, because for simplex range counting, the current best large-space data structure needed in Corollary 7.8(ii) has extra logarithmic factors [42], so our new method can eliminate some but not all logarithmic factors.)

We can obtain similar results for halfspace range emptiness:

Corollary 7.9 *Let $d \geq 4$ be even. Suppose there is a d -dimensional halfspace range emptiness data structure for point sets of size at most B with $P(B)$ (expected) preprocessing time, $S(B)$ (expected) space, and $Q(B)$ (expected) query time. Then there is a d -dimensional halfspace range emptiness data structure for point sets of at most size n with*

- (i) $O(n/B)P(B) + O(n \log n)$ (expected) preprocessing time, $O(n/B)S(B) + O(n)$ (expected) space, and $O(n/B)^{1-1/\lfloor d/2 \rfloor} Q(B) + O(n/B)^{1-1/\lfloor d/2 \rfloor}$ expected query time, assuming $B < n/\log^{\omega(1)} n$.
- (ii) $O(n/B)^d P(B) + O((n/B)^d B)$ expected preprocessing time, $O(n/B)^d S(B) + O((n/B)^d B)$ expected space, and $Q(B) + O(\log(n/B))$ expected query time.

Proof Part (i) is a consequence of Theorems 6.4 and 6.6. However, one technical issue arises: for emptiness queries, we could purport that the range is nonempty if the query cost exceeds the stated limit, but this would only yield a Monte Carlo algorithm. We describe a Las Vegas alternative. Store a random sample R of size r in any halfspace range emptiness data structure with $O(r \log r)$ preprocessing time, $O(r)$ space, and $O(r^{1-\delta})$ query time for some fixed $\delta > 0$. If the query halfspace h is not empty with respect to R , we are done. Otherwise, h is $O((n/r) \log n)$ -shallow w.h.p.(n) by a Chernoff bound (or the ε -net property of random samples [35, 44, 47]). Then w.h.p.(n), the query cost from Theorem 6.4 is $O((n/B)^{1-1/\lfloor d/2 \rfloor} + (n/(rB)) \log^{O(1)} n)$. Setting $r = (n/B)^{1/\lfloor d/2 \rfloor} \log^c n$ for a sufficiently large constant c yields the desired bound. (Note that for $d \geq 6$, the auxiliary data structure for R is unnecessary as $\delta = 0$ still works.)

Part (ii) follows from a direct application of hierarchical cuttings in the shallow context where we only need to cover a lower envelope of n halfspaces (although we are not aware of any references explicitly stating the result, see [11, 49] for the general idea, which involves just repeated applications of an X -sensitive shallow cutting lemma as in Lemma 6.1). \square

We can proceed as in the proof of Corollary 7.8 to get a preprocessing/query-time trade-off for halfspace range emptiness. The author [18] has given another randomized reduction from linear programming queries to halfspace range emptiness. This reduction does not increase the asymptotic expected query time if it grows faster than n^ε , and does not increase the asymptotic preprocessing time if it grows faster than $n^{1+\varepsilon}$. Therefore:

Corollary 7.10 *For $d \geq 4$ even, there is a d -dimensional halfspace range emptiness data structure with $O(m2^{O(\log^* n)})$ preprocessing time and $O((n/m^{1/\lfloor d/2 \rfloor})2^{O(\log^* n)})$ expected query time for any given $m \in [n \log n, n^{\lfloor d/2 \rfloor} / \log^{\lfloor d/2 \rfloor} n]$. The same result holds for ray shooting and linear programming queries inside the intersection of n halfspaces in \mathbb{R}^d , and exact nearest neighbor queries of n points in \mathbb{R}^{d-1} , assuming $m \in [n^{1+\delta}, n^{\lfloor d/2 \rfloor - \delta}]$ for some fixed $\delta > 0$.*

One Algorithmic Application Improved data structures often lead to improved algorithms for off-line problems. We close with just one such example: Given n points, we can find the extreme points (the vertices of the convex hull) by answering n linear programming queries. By choosing $m = n^{2-2/(\lfloor d/2 \rfloor + 1)}$, we get the following corollary, improving the best previous worst-case time bound of $O(n^{2-2/(\lfloor d/2 \rfloor + 1)} \log^{O(1)} n)$ [17, 41].

Corollary 7.11 *For $d \geq 4$ even, we can compute the extreme points of a given set of n points in \mathbb{R}^d in $O(n^{2-2/(\lfloor d/2 \rfloor + 1)} 2^{O(\log^* n)})$ expected time.*

By Seidel's algorithm [51], we can construct the convex hull itself in additional $O(f \log n)$ time where f denotes the output size. (For $f < n$, we could also slightly speed up the convex hull algorithm in [17] for $d \geq 6$ even, but an algorithm in [19, Theorem 6.3] is faster in this case.)

8 Conclusions

The new results of this paper completely subsume most of the results from Matoušek's previous paper [42], at least if randomized algorithms are acceptable (which are to most researchers nowadays). Although we have resolved some of the main open questions concerning upper bounds, there are still a few minor issues. For example, is it possible to eliminate the iterated logarithmic factor in the preprocessing-time/query-time tradeoffs (see Corollary 7.8)? For halfspace range reporting for even d , can one get $O(n)$ space and $O(n^{1-1/\lfloor d/2 \rfloor} + k)$ query time, without any extra iterated logarithmic factors in either term (see the remarks after Theorem 6.4)? More importantly, for halfspace emptiness for odd d , can one get $O(n)$ space and $O(n^{1-1/\lfloor d/2 \rfloor})$ query time?

In the other direction, for simplex range searching, can one prove a tight $\Omega(n^{1-1/d})$ lower bound on the query time for linear-space data structures in, say, the semigroup model [20], without any extra logarithmic factor? Can one prove lower bounds showing in some sense that logarithmic-factor increase is necessary for multi-level partition trees (see Corollary 7.3), or that extra logarithmic factors are necessary in the query time in the case of large near- $O(n^{\lfloor d/2 \rfloor})$ space [42]? More importantly, can one prove near optimal lower bounds for halfspace range emptiness?

It seems plausible that our approach could be adapted to other kinds of partition tree for semialgebraic sets [6, 9] in some cases (where the complexity of arrangements or lower envelopes in $d - 1$ dimensions is strictly less than in d dimensions). We do not know yet if our approach could lead to new results for dynamic partition trees that support insertions and deletions.

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Appendix: On Chernoff Bounds

We briefly mention the form of Chernoff bounds that we use. In our applications, we want high-probability bounds but do not need sharp concentrations (constant factor approximations suffice), so simplified versions of the bounds will do. One of the more commonly seen versions of the Chernoff bound (e.g., see [46, pp. 68–70]) appears as follows: If X is the sum of independent 0-1 variables, e.g., it is binomially distributed, with $E[X] = \mu$, then

$$\Pr[X > (1 + \delta)\mu] < \left[\frac{e^\delta}{(1 + \delta)^{1+\delta}} \right]^\mu \quad \text{for any } \delta > 0;$$

$$\Pr[X < (1 - \delta)\mu] < e^{-\mu\delta^2/2} \quad \text{for any } 0 < \delta \leq 1.$$

The first probability is at most $2^{-(1+\delta)\mu}$ if $\delta > 2e - 1$ (e.g., see [46, p. 72]). Thus, $\Pr[X > 2e\mu + c_0 \log N] < 1/N^{c_0}$, and we can write $X \leq O(\mu + \log N)$ w.h.p.(N).

The second probability for $\delta = 1/2$ is at most $e^{-\mu/8} < 1/N^{c_0}$ if $\mu > 8c_0 \log N$. Otherwise, $\mu/2 - 4c_0 \log N \leq 0$. In any case, $\Pr[X < \mu/2 - 4c_0 \log N] < 1/N^{c_0}$, and we can write $\mu \leq O(X + \log N)$ w.h.p.(N).

The first bound holds more generally when X is the sum of independent variables X_i where each X_i is bounded between 0 and 1. (If the variables are bounded by some other value, we can re-scale before applying.) The first bound also holds when X is hypergeometrically distributed, i.e., it is the number of elements of R in $\{1, \dots, \ell\}$ where R is a random subset of size i from $\{1, \dots, t\}$ chosen by sampling without replacement for a given i, ℓ, t (e.g., see [37]).

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