Open access • Journal Article • DOI:10.1007/S00199-010-0577-3

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Published on: 01 Aug 2012 - Economic Theory (Springer-Verlag)

Topics: AK model, Smoothing, Consumption smoothing and Endogenous growth theory

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Discussion Paper 2010-29

Institut de Recherches Économiques et Sociales
de l'Université catholique de Louvain

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M. Bambi*, G. Fabbri† and F. Gozzi‡

August 29, 2010

Abstract

In this paper the dynamic programming approach is exploited in order to identify the closed loop policy function, and the consumption smoothing mechanism in an endogenous growth model with time to build, linear technology and irreversibility constraint in investment. Moreover the link among the time to build parameter, the real interest rate, and the magnitude of the smoothing effect is deeply investigated and compared with what happens in a vintage capital model characterized by the same technology and utility function. Finally we have analyzed the effect of time to build on the speed of convergence of the main aggregate variables.

Key words: Time-to-build, AK model, Dynamic programming, optimal strategies, closed loop policy.
Journal of Economic Literature: E22, E32, O40.
AMS classification (2000): 49L20, 49K25, 34K35.

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1 Introduction

Since the seminal contribution of Kalecki [14] very few authors have investigated the implications of time-to-build in continuous time growth models. To the best of our knowledge, El Hodiri et al. [12] were the first to introduce gestation lags in production in an optimal control framework. In a similar setting, Rustichini [18] provided some key theoretical results on the rising of deterministic (Hopf) cycles while Asea and Zak [1] and Bambi [3] applied these results in an exogenous and endogenous growth model, respectively. The main reason for these few contributions in growth theory is that the dimensionality of the problem switches from finite to infinite as soon as capital takes time to become productive; then unusual techniques as complex analysis, functional analysis, and nonstandard optimal control theory, become necessary to handle this kind of models.[1] The methodological approach used in the previously cited contributions consists in applying a modified version of the Maximum Principle (see Kolmanovsky and Mishkis [15]) and then an *open loop control* to determine the optimal trajectory for the aggregate economic variables and the possibility of

[1] A completely different picture for discrete time models. There, the dimensionality of the problem remains always finite independently by the presence of time to build, Bambi and Gori [4]. This is probably the reason why the RBC and neo-keynesian literature is richer of contributions and time-to-build is often used to increase the explanatory power of the models (see for example Kydland and Prescott [16], or more recently Edge [11]).

(Hopf) cycles. However the impossibility to identify explicitly the *closed loop policy (CLP) function*, is the main limitation of this approach since it prevents a deep understanding of the economic implications of these models.

In this paper we want to move further and investigate not only the balanced growth path properties and the transitional dynamics (Asea and Zak [1], and Bambi [3]) but also the consumption smoothing mechanism and the relation among delays in production, the real interest rate, and the magnitude of the smoothing effect, characterizing an endogenous growth model with time to build and linear technology. Dealing with these "new" questions means to find the explicit formula of the CLP function between consumption and capital which cannot be anymore a linear function of the present value of capital as in the standard AK model (Barro Sala-i-Martin [2], page 208) because the presence of damping oscillations in capital, induced by the delay in production, would trigger the same dynamics on consumption.

The most natural way to identify this function is through the method of Dynamic Programming as soon as its associated Hamilton-Jacobi-Bellman equation (HJB) can be solved explicitly. The counterpart of this method is that, in the case of time-to-build, the HJB equation is a Partial Differential Equation in infinite dimension which does not admit explicit solutions unless specific assumptions on the production and utility function are introduced.

Luckily the specific structure of our problem (linear production function and homogeneity of the utility function) let us to develop an ad hoc approach in order to calculate explicitly the HJB equation and then the CLP function which, as explained before, will be the key element in unfolding the consumption smoothing mechanism at work in a time to build model. Once identified, the CLP function will unveil the following smoothing effect: the perfect foresight agents know that a share of their past investments are installed but not yet productive machines which will become fully operative as soon as the time to build period is expired. Then these machines enter in the consumers' total wealth but with a discounted value as shown in Section 5. For this reason, the rational agents anticipate today part of their future consumption, smoothing in this way the oscillations transmitted by present capital to present consumption.

Moreover a comparison with a vintage capital model characterized by the same linear technology and utility function, is also proposed.[2] The CLP function for this case was identified for the first time by Fabbri and Gozzi [13], using a DP approach which presents several nontrivial differences with respect to that one proposed here as clearly discussed at the beginning of Section 3. What will emerge from this comparison is a completely different nature of the consumption smoothing mechanism in the two frameworks. In fact, there is no anticipation of future consumption in a vintage capital setup but the smoothing effect is entirely due to the replacement activity of the old machines which prevents the economy (and then consumption) to shrink over time. It is worth noting that the mechanism reflects again a forward looking behaviour since the consumers'

[2]Following the seminal contribution of Benhabib and Rustichini [5], Boucekkine at al. [8] were the first to deal with an AK vintage capital model through the Maximum Principle.

total wealth internalizes the expected future obsolescence cost of the machines.

Finally, several considerations are also proposed on the speed of convergence of the optimal path and on the efficiency of the DP approach and the Maximum Principle concerning the balanced growth path and the transitional dynamics parameters restrictions.

The paper is organized as follows. In Section 2, the model setup is introduced and its main features presented. Section 3 explains how the problem can be rewritten in infinite dimension and how to handle it with the Hamilton-Jacobi-Bellman equation in order to find a solution of the problem. The closed loop policy function and the properties of the optimal paths are derived and described in Section 4. The next section, 5, explains in details the economic implications of the results developed with a particular attention to the consumption smoothing effects. A comparison with vintage capital models and some considerations on the speed of convergence are also investigated in this section. Finally Section 6 concludes. The Appendix contains all the proofs.

2 The model and its main features

2.1 Basic setup

We model time-to-build in the simplest possible way by assuming, as suggested by Kalecki [14], that capital goods produced at time t become operative at time $t+d$, the time-to-build delay d being strictly positive.[3] This assumption is appended to an AK endogenous growth model with an irreversibility constraint on investment. The social planner problem can be considered since no distortions are present:

$$\max \int_0^\infty \frac{c(t)^{1-\sigma}-1}{1-\sigma} e^{-\rho t} dt$$

subject to

$$
\begin{aligned}
\dot{k}(t) &= \tilde{A}k(t-d) - c(t) \qquad && \forall t \geq 0 && (1)\\
\dot{k}(t) &\geq -\delta k(t-d), \qquad \forall t \geq 0 && && (2)\\
k(t) &\geq 0 \qquad \forall t \geq 0 && && (3)\\
c(t) &\geq 0, \quad \forall t \geq 0 && && (4)\\
k(t) &= k_0(t), \qquad k_0(t) \geq 0,\ k_0(t) \not\equiv 0 \quad \forall t \in [-d, 0] && && (5)
\end{aligned}
$$

All the variables are per capita. The parameter $\tilde{A} = (A-\delta) > 0$ depends on the productivity level A, and the usual capital depreciation rate $\delta \geq 0$.[4] As usual $\rho > 0$ indicates the intertemporal preference discount factor, while $\sigma > 0$ with $\sigma \neq 1$ is the inverse of the elasticity of substitution. The inequality (2) is

[3]Kalecki refers to the parameter d as *"gestation period"* of any investment. This period starts with the investment orders and ends with the deliveries of finished industrial equipments.

[4]Differently from Bambi [3], the dynamic programming approach proposed here let us to completely characterize the dynamics of the economy without any further assumption on capital depreciation.

the irreversible investment constraint. Irreversibility means that once installed, capital has no value unless used in production. It is worth noting that the problem can be analyzed through the dynamic programming approach with or without the irreversibility constraint (2). As in the standard AK model, the set of initial conditions which lead to a corner solution is smaller when irreversibility is not introduced. In what follows, we focus on the interior solutions, and we will find that the optimal strategies and trajectories coincide with those of the same problem without the irreversibility constraint even if the latter is characterized by a wider set of initial conditions. Finally, relation (5) is the relevant history of capital in the interval $[-d, 0]$.

2.2 The associated optimal control problem

In this subsection we rephrase the model presented above as an optimal control problem of a differential delay equation. Given any initial datum $k_0(\cdot) \in C([-d,0];\mathbb{R}^+)$ and any control strategy $c(\cdot) \in L^1_{loc}([0,+\infty);\mathbb{R})$, where $L^1_{loc}([0,+\infty);\mathbb{R})$ is the set of all functions from $[0,+\infty)$ to $\mathbb{R}$ that are Lebesgue measurable and integrable on all bounded intervals, we call $k_{k_0(\cdot),c(\cdot)}(\cdot)$ the unique related capital trajectory, that is the unique (see [7] Theorem 4.1 page 222) absolutely continuous solution of (1). Moreover, given any initial datum $k_0(\cdot) \in C([-d,0];\mathbb{R}^+)$, $c(\cdot)$ is an admissible consumption strategy for such initial datum if

$$c(\cdot) \in \mathcal{A}(k_0(\cdot)) := \Big\{ c \in L^1_{loc}([0,+\infty);\mathbb{R}) \ : \\ : \ c(t) \geq 0 \text{ and } Ak_{k_0(\cdot),c(\cdot)}(t-d) - c(t) \geq 0 \text{ for all } t \geq 0 \Big\}. \tag{6}$$

The functional to maximize is (dropping the constant $-(1-\sigma)^{-1}$ which does not change the optimal strategies)

$$J(k_0(\cdot), c(\cdot)) := \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt. \tag{7}$$

The value function of the problem is defined as

$$V(k_0(\cdot)) := \sup_{c(\cdot)\in\mathcal{A}(k_0(\cdot))} J(k_0(\cdot), c(\cdot)) \tag{8}$$

with the agreement that $V(k_0(\cdot)) = -\infty$ if $\mathcal{A}(k_0(\cdot)) = \emptyset$ or if J is always $-\infty$.

2.3 The equation for the maximal growth of capital

When we set consumption equal to 0 we obtain the equation describing the maximal growth path of capital, $k^M(\cdot)$, which is indeed described by the homogeneous part of the capital accumulation equation (1):

$$\begin{cases} \dot{k^M}(t) = \tilde{A}k^M(t-d) \\ k^M(s) = k_0^M(s) \qquad \text{for all } s \in [-d, 0]. \end{cases} \tag{9}$$

In this subsection we study the properties of this equation, which will be crucial to fully characterize the solution of our problem. Observe first that this equation has a unique continuous solution. The characteristic equation of (9) is the transcendental equation

$$z = \tilde{A} e^{-zd}. \tag{10}$$

whose spectrum of roots is described in the next proposition.

Proposition 2.1. *Concerning the roots of the characteristic equation (10) we have the following.*

(a) There is only one real root ξ of (10). This root is simple and satisfies[5]

$$0 < \xi_0 := \tilde{A} \frac{e^{-\tilde{A}d}(\tilde{A}d + 1)}{1 + \tilde{A}de^{-\tilde{A}d}} < \xi < \tilde{A}. \tag{11}$$

(b) The characteristic equation (10) has only simple roots.

(c) There are two real sequences $\{\mu_k,\ k = 1, 2, ...\}$ and $\{\nu_k,\ k = 1, 2, ...\}$ such that all the complex and nonreal roots of (10) are given by $\{\lambda_k^+ = \mu_k + i\nu_k,\ k = 1, 2, ...\}$ and $\{\lambda_k^- = \mu_k - i\nu_k,\ k = 1, 2, ...\}$.

(d) For each k we have $d \cdot \nu_k \in ((2k-1)\pi, 2k\pi)$.

(e) The real sequence $\{\mu_k,\ k = 1, 2, ...\}$, is strictly decreasing to $-\infty$. We have $\mu_1 = 0$ iff $\nu_1 = \tilde{A}d = \frac{3\pi}{2}$. Finally

$$\mu_1 < 0 \Longleftrightarrow \tilde{A}d < \frac{3\pi}{2}, \tag{12}$$

$$\nu_1 < \frac{3\pi}{2} \Longleftrightarrow \tilde{A}d < \frac{3\pi}{2}. \tag{13}$$

Note that in the paper [3] the main results on the optimal equilibrium path and its characteristics are based on the assumption $\tilde{A}d < \frac{3\pi}{2}$. Here we extend the results without imposing such constraint on the delay parameter. See the proof of Proposition 4.6.

In the next proposition, we also prove how the first two characteristic roots of (10) depend on the main parameters of the economy. This information will be useful later when the global speed of convergence will be studied.

Proposition 2.2. *The roots ξ and $\mu_1 + i\nu_1$ of (10) satisfy the following.*

(a) $\frac{\partial \xi}{\partial \tilde{A}} = \frac{1}{\tilde{A}d} \cdot \frac{\xi d}{1+\xi d} > 0, \qquad \frac{\partial \xi}{\partial d} = -\frac{1}{d^2} \cdot \frac{(\xi d)^2}{1+\xi d} < 0,$

(b) $\frac{\partial \mu_1}{\partial \tilde{A}} = \frac{1}{\tilde{A}d} \cdot \frac{\mu_1 d + (\mu_1 d)^2 + (\nu_1 d)^2}{(1+\mu_1 d)^2 + (\nu_1 d)^2} > 0, \qquad \frac{\partial \mu_1}{\partial d} = \frac{1}{d^2}\left[-\mu_1 d + \frac{\mu_1 d + (\mu_1 d)^2 + (\nu_1 d)^2}{(1+\mu_1 d)^2 + (\nu_1 d)^2}\right],$

$\frac{\partial \nu_1}{\partial \tilde{A}} = \frac{1}{\tilde{A}d} \cdot \frac{\nu_1 d}{(1+\mu_1 d)^2 + (\nu_1 d)^2} > 0, \qquad \frac{\partial \nu_1}{\partial d} = \frac{1}{d^2}\left[-\nu_1 d + \frac{\nu_1 d}{(1+\mu_1 d)^2 + (\nu_1 d)^2}\right] < 0,$

[5] In the degenerate case $d = 0$ we have $\xi = \tilde{A}$ which is the real interest rate in the standard model.

Now we use the above Proposition 2.1 to derive a condition on the parameters that guarantees the finiteness of the value function.

Proposition 2.3. *We have the following facts:*

(i) For all $c(\cdot) \in L^1_{loc}([0,+\infty);\mathbb{R})$ *with* $c(\cdot) \geq 0$ *we have that* $k_{k_0(\cdot),c(\cdot)}(t) \leq k^M(t)$ *for all* $t \geq 0$.

(ii) For all $\varepsilon > 0$ *we have that*

$$\lim_{t\to+\infty} \frac{k^M(t)}{e^{t(\xi+\varepsilon)}} = 0$$

Proposition 2.4. *Suppose that*

$$\rho > \xi(1-\sigma). \tag{14}$$

then $-\infty < V(k_0(\cdot)) < +\infty$ *for all* $k_0(\cdot) \in C([-d,0];\mathbb{R}^+)$.

Before proceeding, it is worth noting that the highest real root ξ of (10) is indeed the (constant) real interest rate of the economy. This can be seen looking at the firm's behaviour and more precisely at its intertemporal investment decisions to maximize the present value of current and future dividends[6]

$$\max_{\{k(t),i(t)\}_{t=0}^{+\infty}} \int_0^\infty (Ak(t-d) - i(t))e^{-rt}dt$$

$$\begin{array}{rcl} \dot{k}(t) &=& i(t) - \delta k(t-d) \\ k(t) &=& k_0(t) \qquad t \in [-d,0] \end{array}$$

The Hamiltonian is $H := [Ak(t-d) - i(t)]\, e^{-rt} + q(t)[i(t) - \delta k(t-d)]$ and its first order conditions:

$$\begin{array}{rcl} q(t) &=& e^{-rt} \\ \dot{q}(t) &=& -Ae^{-r(t+d)} + \delta q(t+d) \end{array}$$

lead to the relation $r = \tilde{A}e^{-rd}$ and then to the transcendental equation (10) studied before. Taking into account this fact, relation (14) is the standard condition in endogenous growth theory that the discount factor ρ has to be large enough to the objective be bounded.

A similar reasoning can be extended to the case of a vintage capital model with linear technology. The problem becomes

$$\max_{\{k(t),i(t)\}_{t=0}^{+\infty}} \int_0^\infty (Ak(t) - i(t))e^{-rt}dt$$

[6]The discounted dividends a firm pays out are equal to earnings $Ak(t-d)$ less investment expenditure. Observe also that the real interest rate used to discount is assumed constant because our guessed real interest rate ξ was proved to be time invariant due to the linear technology assumption.

$$\begin{aligned} \dot{k}(t) &= i(t) - i(t-T) \\ k(0) &= \int_{-T}^{0} i_0(z)dz \\ i(t) &= i_0(t) \ \ with \ \ t \in [-T, 0) \end{aligned}$$

The first order conditions of this problem are:

$$\begin{aligned} q(t) - q(t+T) &= -e^{-rt} \\ \dot{q}(t) &= Ae^{-rt} \end{aligned}$$

which lead to the relation $r = A(1 - e^{-rT})$ which is exactly the same equation found by Fabbri and Gozzi [13], equation 14, page 340, for the maximal rate of reproduction of capital ξ.

2.4 A useful change of variables

Here we introduce a suitable change of variables that will allow us to treat more efficiently the problem. Before proceeding we need to ask a bit more on the initial datum $k_0(\cdot)$, namely we assume that $k_0(\cdot) \in H^1([-d, 0]; \mathbb{R}^+)$[7]. We also assume that $c(\cdot) \in L^2_{loc}([0, +\infty); \mathbb{R})$, this is not a strong assumption since such set contains the optimal strategies of our problem[8]. Chosen $k_0 \in H^1([-d, 0]; \mathbb{R}^+)$ and $c(\cdot) \in L^2_{loc}([0, +\infty); \mathbb{R})$, the equation (1) admits a unique continuous solution and such a solution belongs to $H^1_{loc}([-d, +\infty); \mathbb{R})$ as proved in [7] page 287[9].

As usual we denote by $y(t)$, $i(t)$, $j(t) = \dot{k}(t)$ respectively the output, the gross investment, the net investment at time t. We now rewrite the optimal control problem in term of output, $y(t) = Ak(t-d)$ (for $t \geq 0$) and adjusted net investment, $u(t) = (A/\tilde{A})\dot{k}(t)$ (for $t \geq -d$) since this is convenient from a mathematical point of view. To do this we first observe that, multiplying both sides of the capital accumulation equation (1) by $(A/\tilde{A})$ and using the definition of adjusted net investment ($u(\cdot)$), we get

$$u(t) = y(t) - \frac{A}{\tilde{A}} c(t).$$

Moreover taking into account the resource constraint of the economy $y(t) = i(t) + c(t)$ it follows immediately that $u(t) \in [j(t), i(t)]$ or, in term of $y(t)$,

$$u(t) \in \left[\left(1 - \frac{A}{\tilde{A}}\right) y(t), y(t) \right] \tag{15}$$

[7] $H^1([-d, 0]; \mathbb{R}^+)$ is the set of the absolutely continuous functions $f\colon [-d, 0] \to \mathbb{R}^+$ such that $\int_{-d}^{0} |f'(r)|^2 \, \mathrm{d}r < +\infty$.

[8] $L^2_{\mathrm{loc}}([0, +\infty); \mathbb{R})$ is the set of all functions from $[0, +\infty)$ to $\mathbb{R}$ that are Lebesgue measurable and square integrable on all bounded intervals.

[9] The space $H^1_{loc}([-d, +\infty); \mathbb{R})$ is the set of all functions f from $[-d, +\infty)$ to $\mathbb{R}$ that are absolutely continuous and such that, for every $T > -d$

$$\int_{-d}^{T} |f'(s)| \, \mathrm{d}s < +\infty.$$

Then, maximizing the functional (7) is equivalent to maximize

$$\overline{J}(k_0(\cdot), c(\cdot)) := \int_0^\infty e^{-\rho t} \frac{\left(\frac{A}{\tilde{A}} c(t)\right)^{1-\sigma}}{1-\sigma} \, \mathrm{d}s = \int_0^\infty e^{-\rho t} \frac{(y(t) - u(t))^{1-\sigma}}{1-\sigma} \, \mathrm{d}s \quad (16)$$

subject to the state equation

$$\begin{cases} \dot{y}(t) = \tilde{A} u(t-d) & t \geq 0 \\ u(s) = u_0(s) \ \left(= \frac{A}{\tilde{A}} \dot{k}(s)\right) & s \in [-d, 0) \\ y(0) = y_0 \ \ (= A k(-d)) \end{cases} \quad (17)$$

and the constraints (15). Observe that the state equation (17) is obtained by time differentiating the production function and applying the definition of adjusted net investment. Observe also that in (17) the initial datum is now a couple (y_0, u_0) where $y_0 \in \mathbb{R}$ (indeed in $\mathbb{R}^+$ as $k(-d) \geq 0$) and $u_0 \in L^2([-d, 0); \mathbb{R})$ while the control strategy is the function $u(\cdot) \in L^2_{loc}([0, +\infty); \mathbb{R})$.

Given any initial data $y_0 \in \mathbb{R}$ and $u_0 \in L^2([-d, 0); \mathbb{R})$, and any control strategy $u(\cdot) \in L^2_{loc}([0, +\infty); \mathbb{R})$ we call $y_{(y_0, u_0(\cdot)), u(\cdot)}(\cdot)$ the unique related output trajectory, that is the unique (see [7] Theorem 4.1 page 222) absolutely continuous solution of (17).

Remark 2.5. *To apply the above change of variables we need to assume that k_0 belongs $H^1([-d, 0]; \mathbb{R}^+)$. Indeed with a limiting procedure we could study also the case when k_0 is only continuous and positive. Since this would not add useful information from the economic point of view we will always assume that $k_0 \in H^1([-d, 0]; \mathbb{R}^+)$.*

3 Solution through the infinite dimensional approach

In this section we rewrite the optimal control problem (16)-(17)-(15) in a suitable infinite dimensional form and then we solve it with the Dynamic Programming approach. The study of the associated infinite dimensional problem is done following the basic steps of the Dynamic Programming approach as in [13]. We recall that our problem has three important differences with respect to the one of [13]

- the presence of delay in the state and not in the control (exactly the opposite of what happens in [13]);
- the presence of a state-control constraint with a delay (while in [13] there was no delay in the state-control constraint);
- the initial condition which is given as the historic path of capital (while in [13] it is the historic path of investments that also determines the present capital).

These three facts complicates the problem with respect to [13], especially for the key point: finding the closed loop policy function (also called *optimal feedback*). This means that the infinite dimensional study made in [13] cannot be repeated here.

We sketch the "road map" to solve the problem mentioning the points where the technical difficulties arise and where we cannot use the arguments of [13].

- (Section 3.1) First rewrite the problem in a suitable infinite dimensional space. The main point here is the choice of the state variable of the system (the so called *structural state*) in Definition 3.1 which is different from [13] and makes the associated infinite dimensional problem solvable.

- (Section 3.2) Write the associated HJB equation computing exactly the Hamiltonians, define the right concept of solution of it and find an explicit solution. To guess this explicit solution we proceed as in [13] taking the power $1-\sigma$ of a suitable linear function of the structural state. However the spaces where the function is defined are different from the case treated in [13] due to the different constraints of our problem.

- (Section 3.3) Prove that the explicit solution of the HJB found in Section 3.2 is indeed the value function and find the Closed Loop Policy (CLP) function in infinite dimension. The form of the candidate CLP is obvious from the form of the explicit solution. What is absolutely nontrivial is to prove that this candidate CLP gives optimal strategies. This task is much harder than in [13] and requires a different set of assumptions, see the discussion before Proposition 3.11.

Once this is done we only have to translate the results into the "finite dimensional" language. This will be done in Section 4.

3.1 The problem rewritten in infinite dimension

There are various ways to write an infinite dimensional problem associated to (16)-(17)-(15): as in [13] we choose the approach depicted in [19] as it is the one that fits better into our problem. We have first to define a new state variable (the structural state) that lives in a suitable infinite dimensional space. Then we will write the state equation for the this new state variable and finally rewrite the objective functional.

The infinite dimensional space where we rewrite the problem is the Hilbert space $M^2 := \mathbb{R} \times L^2([-d,0];\mathbb{R})$[10]. The inner product on M^2 is defined as:

$$\langle (x^0, x^1), (z^0, z^1) \rangle_{M^2} := x^0 z^0 + \langle x^1, z^1 \rangle_{L^2} = x^0 z^0 + \left(\int_{-d}^0 x^1(s) z^1(s) \, \mathrm{d}s \right)$$

[10] We recall that for L^2 spaces the extrema of the interval are not important so $L^2([-d,0];\mathbb{R}) = L^2([-d,0);\mathbb{R})$. Here we use the closed interval as it is more convenient to define the second element of the state on it.

for every $(x^0, x^1), (z^0, z^1) \in M^2$. We will avoid the subscript $_{M^2}$ when it is not ambiguous.

We now introduce the new state variable (the structural state).

Definition 3.1. *Given the initial data $y_0 \in \mathbb{R}$ and $u_0 \in L^2([-d, 0]; \mathbb{R})$, and the control strategy $u(\cdot) \in L^2_{loc}([0, +\infty); \mathbb{R})$ we define the* structural state *of the system at time $t \geq 0$ the couple*[11]

$$x_{(y_0,u_0(\cdot)),u(\cdot)}(t) = (x^0_{(y_0,u_0(\cdot)),u(\cdot)}(t), x^1_{(y_0,u_0(\cdot)),u(\cdot)}(t))$$

$$:= (y_{(y_0,u_0(\cdot)),u(\cdot)}(t), \gamma(t)[\cdot]) \in M^2,$$

where $\gamma(t)[\cdot]$ is the element of $L^2([-d, 0]; \mathbb{R})$ defined as:

$$\begin{cases} \gamma(t)[\cdot] \colon [-d, 0] \to \mathbb{R} \\ \gamma(t)[s] := \tilde{A}u(t - d - s) \end{cases} \tag{18}$$

In the following we will often avoid to write the dependence of $x(\cdot)$, $y(\cdot)$ on $y_0(\cdot)$, $u_0(\cdot)$ and $u(\cdot)$ to obtain a more compact notation.

Now we are going to rewrite the state equation. We need first to introduce some operators. We start defining the unbounded operator G on M^2

$$\begin{cases} D(G) := \{(\psi^0, \psi^1) \in M^2 : \psi^1 \in W^{1,2}([-d, 0]; \mathbb{R}),\ \psi^0 = \psi^1[0]\} \\ G \colon D(G) \to M^2 \\ G(\psi^0, \psi^1) := (0, \frac{d}{ds}\psi^1). \end{cases}$$

The operator G^* is (see [7] Section 4.6 page 242) the generator of a C_0 semigroup on M^2.

Now we want to define the Dirac's delta δ_{-d}, (i.e. the evaluation of a function at the point $-d$) on the elements of $D(G)$. To do this we first recall that, given a function $f[\cdot] \in C([-d, 0]; \mathbb{R})$ the Dirac's delta at the point $-d$ (denoted by δ_{-d}) is simply $f[-d]$. With this definition δ_{-d} is a linear continuous functional from $C([-d, 0]; \mathbb{R})$ to $\mathbb{R}$. Since (by the Sobolev embedding Theorem) $W^{1,2}([-d, 0]; \mathbb{R}) \subseteq C([-d, 0]; \mathbb{R})$, it is possible to calculate $\delta_{-d} f = f[-d]$ for all $f[\cdot] \in W^{1,2}([-d, 0]; \mathbb{R})$. This means that, for $\psi = (\psi^0, \psi^1) \in D(G)$, we can calculate $\delta_{-d}\psi^1 = \psi^1[-d]$. From now on, with an abuse of notation, we will agree that, for every $\psi = (\psi^0, \psi^1) \in D(G)$,

$$\delta_{-d}(\psi^0, \psi^1) = \delta_{-d}\psi^1 = \psi^1[-d] \in \mathbb{R}. \tag{19}$$

We are now ready to rewrite the state equation of our starting problem as an ODE in M^2. We have the following theorem whose proof can be found in ([7] Theorem **5.1** page 258).

[11]Note that, for a fixed $t \geq 0$, $\gamma(t)$ is a function that belongs to $L^2([-d, 0]; \mathbb{R})$. We use from now on the notation $\gamma(t)[s]$ to mean its evaluation in the point $s \in [-d, 0)$. We will use the same notation to denote the evaluation of a function, defined on $[-d, 0]$, at a point $s \in [-d, 0]$.

Theorem 3.2. *Given any initial data $y_0 \in \mathbb{R}$, $u_0 \in L^2([-d,0];\mathbb{R})$, any control strategy $u(\cdot) \in L^2_{loc}([0,+\infty);\mathbb{R})$, the structural state $x_{(y_0,u_0(\cdot)),u(\cdot)}(\cdot)$, introduced in Definition 3.1, is the unique solution of the equation*

$$\begin{cases} \frac{d}{dt}x(t) = G^* x(t) + u(t)\tilde{A}\delta_{-d}, & t \geq 0 \\ x(0) = p = (y(0), \gamma(0)[\cdot]) \end{cases} \tag{20}$$

($\gamma(0)[\cdot]$ is defined as function of $u_0(\cdot)$ as in (18)) in the space

$$\Pi := \left\{ f \in C(0,+\infty; M^2) \ : \ \frac{d}{\mathrm{d}t} f \in L^2_{loc}(0,+\infty, D(G)') \right\}$$

in the following weak sense: for every $\psi \in D(G)$

$$\begin{cases} \frac{d}{dt}\langle \psi, x(t)\rangle = \langle G\psi, x(t)\rangle + \tilde{A}\psi^1[-d]u(t), & t \geq 0 \\ \langle \psi, x(0)\rangle = \psi^0 x^0 + \left\langle \psi^1, x^1(0)\right\rangle_{L^2} = \psi^0 y(0) + \int_{-d}^0 \psi^1[s]u(-s-d)\,\mathrm{d}s \end{cases} \tag{21}$$

Note (see [7] page 258) that (20) has a unique solution for every initial datum $p \in M^2$ and control strategy $u(\cdot) \in L^2_{loc}([0,+\infty);\mathbb{R})$, we call such a solution $x_{p,u(\cdot)}(\cdot)$. We will give here some definitions that work for a generic $p \in M^2$. The constraints in the new language become

$$u(t) \in \left[\left(1 - \frac{A}{\tilde{A}}\right) x^0(t), x^0(t) \right], \qquad t \geq 0,$$

so the set of admissible control strategies for a given initial datum $p \in M^2$ is given by

$$\mathcal{A}_0(p) := \Big\{ u \in L^2_{loc}([0,+\infty);\mathbb{R}^+) \ : \\ : \ u(t) \in \left[\left(1 - \frac{A}{\tilde{A}}\right) x^0_{p,u(\cdot)}(t), x^0_{p,u(\cdot)}(t) \right] \text{ for all } t \geq 0 \Big\}. \tag{22}$$

Note that if $x^0_{p,u(\cdot)}(t) < 0$ then $\left[\left(1 - \frac{A}{\tilde{A}}\right) x^0_{p,u(\cdot)}(t), x^0_{p,u(\cdot)}(t) \right] = \emptyset$, so the condition for the admissibility imply $x^0_{p,u(\cdot)}(t) \geq 0$ for all $t \geq 0$. The functional to be maximized becomes

$$J_0(p, u(\cdot)) := \int_0^\infty e^{-\rho s} \frac{(x^0_{p,u(\cdot)}(t) - u(t))^{1-\sigma}}{(1-\sigma)} \,\mathrm{d}s. \tag{23}$$

The only difference with (16) is the dependence on $p \in M^2$. The value function is:

$$V_0(p) := \sup_{u(\cdot) \in \mathcal{A}_0(p)} J_0(p, u(\cdot))$$

where we mean $V_0(p) = -\infty$ if $\mathcal{A}_0(p)$ is empty or if J_0 is always $-\infty$.

3.2 The HJB equation and its explicit solution

First we introduce the *current value* Hamiltonian: it will be defined on a subset of $M^2 \times M^2 \times \mathbb{R}$ called E:

$$E := \left\{ ((x^0, x^1), P, u) \in M^2 \times D(G) \times \mathbb{R} \ : x^0 \geq 0, \ u \in \left[\left(1 - \frac{A}{\tilde{A}} \right) x^0, x^0 \right] \right\}$$

The current value Hamiltonian $\mathcal{H}_{CV}$ is then defined as:

$$\begin{cases} \mathcal{H}_{CV} \colon E \to \mathbb{R} \\ \mathcal{H}_{CV}((x^0, x^1), P, u) := \langle (x^0, x^1), GP \rangle_{M^2} + \left\langle u\tilde{A}\delta_{-d}, P \right\rangle_{M^2} + \dfrac{(x^0 - u)^{1-\sigma}}{1-\sigma} \\ \qquad\qquad = \left\langle x^1, \frac{\mathrm{d}}{\mathrm{d}s} P^1 \right\rangle_{L^2} + u\tilde{A}P^1[-d] + \dfrac{(x^0 - u)^{1-\sigma}}{1-\sigma} \end{cases}$$

in the points where $u < x^0$ or $\sigma < 1$. When $u = x^0$ and $\sigma > 1$ we define $\mathcal{H}_{CV} = -\infty$.

The (maximum value) Hamiltonian of the system is defined as follows: we call S the subset of $M^2 \times M^2$ given by:

$$S := \{ ((x^0, x^1), P) \in M^2 \times M^2 \ : \ x^0 \geq 0, \ P \in D(G) \};$$

the Hamiltonian becomes then:

$$\begin{cases} \mathcal{H} \colon S \to \overline{\mathbb{R}} \\ \mathcal{H} \colon ((x^0, x^1), P) \mapsto \sup_{u \in \left[\left(1 - \frac{A}{\tilde{A}}\right) x^0, x^0 \right]} \mathcal{H}_{CV}((x^0, x^1), P, u). \end{cases}$$

The *HJB* equation of the problem is then:

$$\rho V(x^0, x^1) \ - \mathcal{H}((x^0, x^1), DV(x^0, x^1)) = 0 \tag{24}$$

We now give the definition of "regular" solution of the HJB equation (24) that takes into account the fact that the domain where we want to define the solution is not open.

Definition 3.3. *Let Ω be an open set of M^2 and $\Omega_1 \subseteq \Omega$ be a closed subset. An application $g \in C^1(\Omega; \mathbb{R})$ is a solution of the HJB equation (24) on Ω_1 if for all (p^0, p^1) in Ω_1 we have*

$$\begin{cases} ((p^0, p^1), (Dg(p^0, p^1))) \in S, \\ \rho g(p^0, p^1) - \mathcal{H}\big((p^0, p^1), Dg(p^0, p^1)\big) = 0. \end{cases}$$

Remark 3.4. *If $P \in D(G)$ and $(\tilde{A}P^1[-d])^{-1/\sigma} \in (0, +\infty)$ the function*

$$\mathcal{H}_{CV}(x, P, \cdot) \colon \left[\left(1 - \frac{A}{\tilde{A}} \right) x^0, x^0 \right] \to \mathbb{R} \tag{25}$$

admits a unique maximum point at

$$u^{MAX} = \begin{cases} x^0 - (\tilde{A}P^1[-d])^{-1/\sigma}, & \text{if } (\tilde{A}P^1[-d])^{-1/\sigma} \in \left(0, \frac{A}{\tilde{A}}x^0\right], \\ x^0, & \text{otherwise,} \end{cases}$$

and we can write the Hamiltonian as

$$\mathcal{H}((x^0,x^1),P) = \begin{cases} \langle (x^0,x^1), GP\rangle_{M^2} + x^0\tilde{A}P^1[-d] + \frac{\sigma}{1-\sigma}(\tilde{A}P^1[-d])^{\frac{\sigma-1}{\sigma}}, \\ \qquad\qquad\qquad\qquad \text{if } (\tilde{A}P^1[-d])^{-1/\sigma} \in \left(0, \frac{A}{\tilde{A}}x^0\right], \\ \langle (x^0,x^1), GP\rangle_{M^2} + \frac{1}{1-\sigma}(x^0)^{1-\sigma}, \qquad \text{otherwise.} \end{cases} \tag{26}$$

The interesting case ("no bad corner solutions") is when $(\tilde{A}P^1[-d])^{-1/\sigma} \in \left(0, \frac{A}{\tilde{A}}x^0\right]$*, so the unique maximum point* u^{MAX} *belongs to* $\left[\left(1-\frac{A}{\tilde{A}}\right)x^0, x^0\right)$*. The expression for* u^{MAX} *will be crucial to write the solution of the original problem in closed-loop form so to find the Closed Loop Policy function.*

Remark 3.5. *If we consider the problem without the irreversibility constraint we can use the simplified form of the Hamiltonian in a wider range of points. In this case we let* u *vary on the whole interval* $(-\infty, x^0)$*, so, for all* $P \in D(G)$ *with* $(\tilde{A}P^1[-d])^{-1/\sigma} > 0$*, the function*

$$\mathcal{H}_{CV}(x,P,\cdot)\colon \left(-\infty, x^0\right] \to \mathbb{R}$$

admits a unique maximum point at

$$u^{MAX} = x^0 - (\tilde{A}P^1[-d])^{-1/\sigma} \ \in \left(-\infty, x^0\right)$$

and the Hamiltonian has the simplified form:

$$\mathcal{H}((x^0,x^1),P) = \langle (x^0,x^1), GP\rangle_{M^2} + x^0\tilde{A}P^1[-d] + \frac{\sigma}{1-\sigma}(\tilde{A}P^1[-d])^{\frac{\sigma-1}{\sigma}}. \tag{27}$$

Now we want to find an explicit solution of the (24). Since (24) is analogous to the one-dimensional HJB equation related to the linear problem with CRRA utility functional we guess that a possible form of the solution can be $v(x) = \nu(\Gamma(x))^{1-\sigma}$ where ν is a constant and $\Gamma(\cdot)$ is a linear function on M^2. This is indeed the case. However, differently from the standard one dimensional AK model it is difficult to find the form of $\Gamma(\cdot)$ and to identify the spaces Ω and Ω_1 where the solution lives. We first define the function $\Gamma(\cdot) : M^2 \to \mathbb{R}$ as

$$\Gamma(x^0,x^1) = x^0 + \int_{-d}^{0} e^{\xi s} x^1[s]\,\mathrm{d}s.$$

If we consider the function

$$\theta(\cdot)\colon [-d,0] \to \mathbb{R}, \qquad \theta(s) = e^{\xi s}$$

and we define $\psi \in M^2$ as

$$\psi = (\psi^0, \psi^1) := (1, \theta) \tag{28}$$

we can express $\Gamma(\cdot)$ as

$$\Gamma(x) = \langle x, \psi \rangle_{M^2}.$$

Note that

$$\psi \in D(G). \tag{29}$$

Using $\Gamma(\cdot)$ we can define

$$X := \Big\{ x \in M^2 \ : \ \Gamma(x) > 0 \Big\}.$$

Moreover we call

$$\alpha = \frac{\rho - \xi(1-\sigma)}{\sigma\xi} \tag{30}$$

and

$$Y := \left\{ x = (x^0, x^1) \in X \ : \ \Gamma(x) \leq x^0 \left(\frac{1}{\alpha} \frac{A}{\bar{A}} \right) \right\}. \tag{31}$$

It is easy to see that X is an open set of M^2 and Y a closed subset of X.

We have the following:

Proposition 3.6. *Under the assumption* (14) *the function* $v \colon X \to \mathbb{R}$ *given by*

$$v(x) := \nu \Gamma(x)^{1-\sigma} \tag{32}$$

with

$$\nu = \alpha^{-\sigma} \frac{1}{(1-\sigma)\xi}$$

is differentiable in all $x = (x^0, x^1) \in X$ *and is a solution of the HJB equation (24) in* Y *in the sense of Definition 3.3.*

Remark 3.7. *If we consider the problem without the irreversibility constraint, as we have seen in Remark 3.5, we can use the simplified form of the Hamiltonian and, arguing exactly as in Proposition 3.6 we obtain that*

$$v(x) = \nu \Gamma(x)^{1-\sigma}$$

is a solution of the HJB equation (24) on the whole set X.

3.3 Closed Loop Policy in infinite dimensions

We call $C(M^2)$ the set of the continuous functions from M^2 to $\mathbb{R}$. We give first some definitions concerning feedback strategies (or closed loop policies).

Definition 3.8. *Given $p \in M^2$ we call $\varphi \in C(M^2)$ a* feedback strategy related to p *if the equation.*

$$\begin{cases} \frac{d}{dt}x(t) = G^* x(t) + \tilde{A}\delta_{-d}(\varphi(x(t))), & t > 0 \\ x(0) = p \end{cases} \tag{33}$$

has a unique solution $x_\varphi(t)$ in Π (in the sense of (21)). We denote by FS_p the set of feedback strategies related to p.

Definition 3.9. *Given $p \in M^2$ and $\varphi \in FS_p$ we say that φ is an* admissible feedback strategy related to p *if the unique solution $x_\varphi(t)$ of the equation (33) satisfies: $\varphi(x_\varphi(\cdot)) \in \mathcal{A}_0(p)$. We call AFS_p the set of admissible feedback strategies related to p.*

Definition 3.10. *Given $p \in M^2$ and $\varphi \in AFS_p$ we say that φ is an* optimal feedback strategy related to p *if*

$$V_0(p) = \int_0^{+\infty} e^{-\rho t} \frac{\left(x_\varphi(t) - \varphi(x_\varphi(t))\right)^{1-\sigma}}{(1-\sigma)} \, dt$$

We denote by OFS_p the set of optimal feedback strategies related to p.

While it is easy to write the candidate optimal feedback, it is difficult to prove that it is really optimal. and the procedure and the assumptions are different from [13] and more difficult. The main reason for this difficulty is the nature of initial datum of the problem. Indeed such datum is done by two component: the present (belonging to $\mathbb{R}$) and the past (belonging to L^2). In [13] the present (the initial capital) is always determined by the past (the history of investments). Here this is not true: the present (the initial output) is not determined by the past (the history of the adjusted net investments). So in our problem we have one more degree of freedom in the datum. So the set of admissible initial data (which is the domain of the candidate optimal feedback) become more complex to study.

We start proving that our candidate feedback is in FS_p.

Proposition 3.11. *For every $p \in M^2$ the map*

$$\begin{cases} \phi \colon M^2 \to \mathbb{R} \\ \phi(x) := x^0 - \alpha\Gamma(x) \end{cases} \tag{34}$$

is in FS_p.

Now we prove the following crucial invariance properties.

Theorem 3.12. *Along the trajectories driven by the feedback ϕ defined in (34) we have that*

$$\Gamma(x_\phi(t)) = \Gamma(x_\phi(0))e^{gt}$$

where

$$g := (\xi(1-\alpha)) = \left(\xi - \frac{\rho - \xi(1-\sigma)}{\sigma}\right) = \frac{\xi - \rho}{\sigma} \tag{35}$$

so in particular, if $p \in X$ then the evolution of (62) remains in X. Moreover, if $\alpha < 1$ (which is equivalent to $\rho < \xi$) the sets

$$I_c := \left\{(x^0, x^1) \in M^2 \ : \ x^0 > 0 \text{ and } x^1[s] \in \left[0, cx^0\right] \text{ for almost all } s \in [-d, 0]\right\} \tag{36}$$

are invariant for the flow of the the autonomous ODE:

$$\frac{d}{\mathrm{d}t} x_\phi(t) = G^* x_\phi(t) + \tilde{A}\delta_{-d}(\phi(x_\phi(t))). \tag{37}$$

when

$$c < \bar{c} := \left(\frac{1}{\alpha} - 1\right)\left(\frac{\xi \tilde{A}}{\tilde{A} - \xi}\right)$$

Corollary 3.13. *The set*

$$I := \bigcup_{c < \bar{c}} I_c \tag{38}$$

is invariant for the flow of (37).

From now on we assume the following.

Hypothesis 3.14. $\alpha < 1$ *i.e.* $\rho < \xi$.

Observe that this assumption has a clear economic interpretation: it guarantees endogenous growth. Indeed the growth rate of the optimal strategy will be exactly $g = (\xi - \rho)\sigma^{-1}$.

In the standard AK model, endogenous growth is guaranteed only when the real interest rate is higher than the intertemporal preference discount rate ρ; exactly the same relation holds here since we have shown that the maximal growth rate of capital ξ is also the real interest rate of the economy once the time to build assumption is introduced. Moreover, from (9) we have that

$$\tilde{A}\frac{(\tilde{A}d + 1)e^{-\tilde{A}d}}{1 + \tilde{A}de^{-\tilde{A}d}} := \xi_0 < \xi < \tilde{A}$$

and, for $d \to 0^+$ we have $\xi_0 \to \tilde{A}^-$ so also $\xi \to \tilde{A}^-$ and then the return to capital ξ converges to $A - \delta$ as soon as $d \to 0^+$.

Theorem 3.15. *Assume (14) and Hypothesis 3.14. Then*

1. *The set I defined in (38) is a subset of Y and then for every $p \in I$ the map ϕ defined in (34) is in AFS_p.*

2. *For every $p \in I$ the map ϕ defined in (34) is also in OFS_p.*

4 Explicit form of the value function, of the closed loop policy and properties of the optimal paths

We now use the results of the previous subsection to write the solution of the original optimal control problem in the delay differential equation setting. From Proposition 3.6 we have the following.

Proposition 4.1. *Assume (14) and Hypothesis 3.14. Given an initial datum* $(y_0, u_0(\cdot)) \in I$ *the value function* V *related to the problem is*

$$V(y_0, u_0(\cdot)) = \nu\Big(\int_{-d}^{0} e^{\xi s}\tilde{A}u_0(-d-s)\,\mathrm{d}s + y_0\Big)^{1-\sigma}$$

where

$$\nu = \alpha^{-\sigma}\frac{1}{(1-\sigma)\xi}$$

Moreover, from Theorem 3.15 we can give a solution in closed form of the problem

Proposition 4.2. *Let assume to have (14). Given an initial datum* $(y_0, u_0(\cdot)) \in I$ *the optimal control* $u^*(\cdot)$ *and the related state trajectory* $y^*(\cdot)$ *satisfy for all* $t \geq 0$*:*

$$u^*(t) = y^*(t) - \alpha\Big(y^*(t) + \int_{-d}^{0} \tilde{A}e^{\xi s}u^*(t-s-d)\,\mathrm{d}s\Big) \tag{39}$$

Corollary 4.3. *Assume (14) and Hypothesis 3.14. Given an initial datum* $(y_0, u_0(\cdot)) \in I$ *the optimal control* $u^*(\cdot)$ *is the only absolutely continuous solution on* $[0, +\infty)$ *of the delay differential equation.*

$$\begin{cases} \dot{u}^*(t) = \tilde{A}u^*(t-d)\,(1-\alpha) - \\ \quad -\alpha\left(\xi\tilde{A}e^{\xi t}\int_{-d-t}^{-t} e^{\xi s}u^*(-d-s)\,\mathrm{d}s + \tilde{A}(-u^*(t-d) + e^{-d\xi}u^*(t))\right) \\ u^*(s) = u_0(s) \qquad \text{for } s \in [-d, 0) \\ u^*(0) = (1-\alpha)\,y_0 - \alpha\int_{-d}^{0} e^{\xi s}u_0(-d-s)(s)\,\mathrm{d}s \end{cases} \tag{40}$$

Now we observe that $y^*(\cdot) - u^*(\cdot)$ (and so the optimal consumption path) has constant growth rate.

Lemma 4.4. *Assume (14) and Hypothesis 3.14. Given any initial datum* $(y_0, u_0(\cdot)) \in I$ *there exists a* Λ *such that along the optimal trajectory the optimal control* $u^*(\cdot)$ *and the related state trajectory* $y^*(\cdot)$ *satisfy for all* $t \geq 0$*:*

$$y^*(t) - u^*(t) = \Lambda e^{gt} \tag{41}$$

where $g = \frac{\xi-\rho}{\sigma}$*. Moreover we can compute explicitly the value of* Λ*; it is given by*

$$\Lambda = \alpha\Big(\int_{-d}^{0} \tilde{A}e^{\xi s}u_0(-s-d)\,\mathrm{d}s + y_0\Big)$$

and

An immediate consequence of the above result is the following.

Corollary 4.5. *Assume (14) and Hypothesis 3.14. Given any initial datum $(y_0, u_0(\cdot)) \in I$, define the detrended state and control variables as:*

$$\bar{y}(t) := e^{-gt} y^*(t)$$

$$\bar{u}(t) := e^{-gt} u^*(t),$$

we have that $\bar{c}(t) \doteq \frac{\tilde{A}}{A}\big(\bar{y}(t) - \bar{u}(t)\big)$ *is constant on optimal trajectories, and its value is* $\frac{\tilde{A}}{A}\Lambda$.

Proposition 4.6. *Assume (14) and Hypothesis 3.14. Given any initial datum $(y_0, u_0(\cdot)) \in I$, let $\bar{u}(\cdot)$ and $\bar{y}(\cdot)$ be the detrended variables defined as in Corollary 4.5. Then*

$$\lim_{t\to\infty} \bar{y}(t) = y_L \qquad \textit{and} \qquad \lim_{t\to\infty} \bar{u}(t) = u_L$$

where

$$y_L = \Lambda \left(1 - \frac{1-\alpha}{1 + \frac{1-e^{-(\xi-g)d}}{\xi-g}\alpha \tilde{A} e^{-gd}} \right)^{-1} \tag{42}$$

and

$$u_L = \Lambda \left[\left(1 - \frac{1-\alpha}{1 + \frac{1-e^{-(\xi-g)d}}{\xi-g}\alpha \tilde{A} e^{-gd}} \right)^{-1} - 1 \right]. \tag{43}$$

In Subsection 2.4 we rephrased the control problem with the variables $y(\cdot)$ (state) and $u(\cdot)$ (control). Now we express the obtained results using the original variables: $k(\cdot)$ (state) and $c(\cdot)$ (control). In particular we assume to have, as initial datum, the history of k in the interval $[-d, 0]$ (the same that in (1)). More precisely we assume to know the history of $k_0(\cdot) \in H^1(-d, 0)$. Recalling (17) we have

$$u_0(s) = \frac{A}{\tilde{A}} \dot{k}_0(s) \qquad \text{for } s \in (-d, 0) \tag{44}$$

and

$$y_0 = A k_0(-d). \tag{45}$$

We can also rewrite the set I in terms of k_0, obtaining that $(y_0, u_0(\cdot)) \in I$ if and only if $k_0 \in \mathcal{K}$ where:

$$\mathcal{K} := \Big\{ k_0 \in H^1(-d,0) \ : \ k_0(-d) \geq 0 \text{ and } \dot{k}_0(s) \in [0, \bar{c} k_0(-d)] \Big\}.$$

Using the previous results of this section we have the following theorem.

Theorem 4.7. *Let us consider the optimal control problem with state equation (1), target functional (7) and set of controls (6). Let assume to have (14), if $k_0 \in \mathcal{K}$ we have the following facts:*

1. *The optimal consumption $c^*(t)$ is given by:*
$$c^*(t) = \tilde{A}\Lambda_0 e^{gt} \tag{46}$$
where $g = \frac{\xi-\rho}{\sigma}$ and
$$\Lambda_0 = \left(\frac{\rho - \xi(1-\sigma)}{\sigma\xi}\right)\left(\int_{-d}^{0} e^{\xi s}\dot{k}_0(-s-d)\,\mathrm{d}s + k_0(-d)\right).$$

2. *The trajectory of the capital along the optimal path is the unique solution of the following DDE:*
$$\begin{cases} \dot{k}^*(t) = \tilde{A}k^*(t-d) - \tilde{A}\Lambda_0 e^{gt} \\ k^*(s) = k_0(s) \qquad \text{for all } s \in [-d, 0) \\ k^*(0) = k_0(0) \end{cases} \tag{47}$$
where g and Λ_0 are defined above.

3. *The explicit expression for the value function, defined in (8), is*
$$V(k_0(\cdot)) = \tilde{A}^{1-\sigma}\nu\left(\int_{-d}^{0} e^{\xi s}\dot{k}_0(-d-s)\,\mathrm{d}s + k_0(-d)\right)^{1-\sigma}$$
where
$$\nu = \left(\frac{\rho - \xi(1-\sigma)}{\sigma\xi}\right)^{-\sigma}\frac{1}{(1-\sigma)\xi}.$$

4. *The detrended trajectory of the capital along the optimal path admits a limit for $t \to +\infty$. More precisely if we define $\bar{k}(t) := e^{-gt}k^*(t)$ we have*
$$\lim_{t\to+\infty}\bar{k}(t) = \Lambda_0\left(1 - \frac{1-\alpha}{1 + \frac{1-e^{-(\xi-g)d}}{\xi-g}\alpha\tilde{A}e^{-gd}}\right)^{-1} =: k_L$$
where Λ_0 is defined above.

5. *The optimal capital trajectory can be written as:*
$$k^*(t) = k_L e^{gt} + \sum_{j=1}^{+\infty} e^{\mu_j t}\left[k_j^1\cos(\nu_j t) + k_j^2\sin(\nu_j t)\right].$$
where $\{\mu_j\}$ and $\{\nu_j\}$ are defined in Proposition 2.1-(c), k_L is known from the point 4 above while k_j^1, k_j^2 can be calculated from k_0 and the other parameters of the model.

5 Economic implications of the model

5.1 Disentangling the consumption smoothing effect

It is well known that in the standard AK model, optimal consumption is a constant rate of total wealth (capital) since the interest rate of the economy is time invariant. For the same argument the economy jumps immediately on its balanced growth path. However when the time to build assumption is introduced, transition to the balanced growth path is no more instantaneous; it has been shown indeed in Theorem 4.7 that the agents' optimal decision is characterized by smooth consumption (namely detrended consumption is constant) but fluctuations in all the other aggregate variables. Similar results in a time to build context were found by Collard et al. [9], where a Ramsey model is solved numerically, and by Bambi and Gori [4] in a model with indivisible labor supply.

These contributions justify the consumption smoothing behavior by pointing out to the advanced nature of the Euler-type equation but no further effort in explaining the mechanisms which links the time to build structure of capital to this specific consumption dynamics has been done yet. In the following, we fill this gap by showing how the closed loop policy function for $c^*(t)$ together with a rational expectation argument can be used to explain consumption smoothing in a time to build context.

First of all we take the closed loop policy function developed in Proposition 4.2 and Corollary 4.3 and we rewrite it in terms of optimal consumption and optimal investment:[12]

$$c^*(t) = \alpha A \left(\int_{-\infty}^{t-d} i^*(s)ds + \int_{t-d}^{t} i^*(s)e^{\xi(t-s-d)}ds \right) \tag{48}$$

The representative agent chooses a consumption path at time t which is a constant share of total wealth. Differently from the standard AK model, the total wealth, namely the term in parenthesis in (48), is characterized by the sum of two components. The first component corresponds to $k(t-d)$, and it remains the only one determining the optimal consumption path as soon as the delay parameter, d, goes to zero. Under this circumstance, the parameter αA converges to $\frac{1}{\sigma}[\rho-(A-\delta)(1-\sigma)]$ and then the CLP function becomes exactly that one in the standard AK model (see for example Barro, and Sala-i-Martin [2], page 208).

Since a strictly positive choice of the delay parameter leads to oscillations in capital (the first term in parenthesis in (48)) as proved in Corollary 4.3, and Lemma 4.4 then the total wealth's second component has to play a key role in offsetting the fluctuations transmitted through capital to consumption. Broadly speaking the smoothness of the optimal consumption path proved in Corollary 4.5 is achieved through a smoothing effect induced by the last element

[12] In the proposed discussion the depreciation rate δ is assumed equal to zero and then investment, $i(t)$, is the key variable in the optimal feedback policy. However all the results still hold when $\delta > 0$ and the key variable is the adjusted net investment $u(t)$.

in parenthesis of (48). This component represents the value of capital produced between $t-d$ and t, which is not yet operative; investments are discounted, using the interest rate ξ, for the period still remaining until the machines become operative for the first time. Observe also that these investments will lead to new productive machines from $t+d$ on, whose arrival (and discounted value) is already known at time t by the perfect foresight agent. Then part of his future consumption is moved backward, $CS^{-}(t+d)$, conditioning on his rational expectations on future production:

$$c^{*}(t) = \alpha A \left(k^{*}(t) + CS^{-}(t+d)\right) \tag{49}$$

This mechanism can be exploited even more when the CLP function is written in terms of the optimal level of consumption at time $t+d$ as a function of the optimal level of consumption at time t:

$$c^{*}(t+d) = c^{*}(t) + \alpha A \left(\int_{t-d}^{t} i^{*}(s)ds - \int_{t-d}^{t} i^{*}(s)e^{\xi(t-s-d)}ds + \int_{t}^{t+d} i^{*}(s)e^{\xi(t-s)}ds \right) \tag{50}$$

which, once rewritten in terms of the optimal capital variation between period t and $t+d$ and the backward movements in consumption, CS^{-}, becomes:

$$c^{*}(t+d) = c^{*}(t) + \alpha A \left(\Delta_{d} k^{*}(t+d) - CS^{-}(t+d) + CS^{-}(t+2d)\right) \tag{51}$$

It is now evident how part of consumption at time $t+d$ is moved backward in order to smooth consumption at time t while part of the consumption at time $t+2d$ is moved backward in order to offset the fluctuations at time $t+d$ rising from the output variation and the smoothing mechanism between $(t, t+d)$. Summing up, two conditions are required to achieve consumption smoothing in the economy. Firstly, investment has to fluctuate to fully compensate for output fluctuations. Secondly, investment fluctuations have to be consistent with a smooth path for total wealth, since consumption is a constant rate of it.

It is also possible to compare and underline the analogies and differences with a vintage capital model with linear technology. In this case, the CLP function is given by the following relation ([13], page 23):

$$c^{*}(t) = \alpha A \left(\int_{t-T}^{t} i^{*}(s)ds - \int_{t-T}^{t} i^{*}(s)e^{\xi(t-T-s)}ds \right) \tag{52}$$

Optimal consumption is again determined by a share of total wealth, namely the object in parenthesis, which depends on two components. The first term is a share of the output as before but with a technology induced by the vintage capital structure where T indicates the machine life span. The second term represents the obsolescence costs associated to scrapping, and it is forward-looking, since it subtracts the expected future obsolescence cost from the value of total wealth. Finally the main difference in the consumption smoothing mechanism between vintage capital and time to build lies on a different definition of total wealth.

5.2 Speed of convergence to the balanced growth path

Once time to build (or vintage capital) is embedded in the AK model, the economy displays transitional dynamics in the main aggregate variables. Moreover, it has been proved in Corollary 4.5 and in Theorem 4.7 that the detrended path $\bar{x}(t)$ of the aggregate variable $x(t)$, where $\bar{x}(t) = x(t)e^{-gt}$, converges to a constant value, x_L. Then, it becomes interesting to analyze the speed of convergence of $\bar{y}(t)$, $\bar{k}(t)$, and $\bar{u}(t)$ to y_L, k_L, and u_L respectively, in order to understand how much emphasis has to be placed on the transition or on the long run behavior.[13] More precisely, a low speed of convergence indicates a relevant role of the transitional dynamics in ascertaining the predictive power of the model even in an endogenous growth model.

It is also worth noting that in our framework with linear technology we are able to derive analytically the *global* speed of convergence while in previous contributions the main focus was on its *local* version (see for example Ortigueira and Santos [17]).[14] Then it is possible to identify the parameters in the economy which may affect the global dynamics and then the speed of convergence of the stationary solutions. Of course, the main role is played by the delay parameter which avoids the immediate adjustment of all the aggregate variables to their balanced growth path switching their speed of convergence from infinite to a finite value. In particular, the speed of convergence is measured by $\hat{\lambda} = |Re(\lambda_{\max}) - g|$, with $\lambda_{\max}$ the complex (and non real) root of the characteristic equation (10) having the highest real part; changes in the speed of convergence due to different choices of the time to build parameter are reported in Figure 1 after having calibrated the economy yearly.[15] In the same graph, we have also reported a green line showing the speed of convergence to the steady state of a neoclassical growth model with Cobb Douglas technology and no time to build.[16]. For a yearly calibration, the Ramsey model's rate of convergence is around 7 per cent. On the other hand, the red line, at around 2 per cent, points out the empirical estimated value of the speed of convergence as documented in the literature (for a survey on econometric contributions refer to [17]).

This analysis indicates how time to build has to be considered a new different channel through which reducing the speed of convergence of growth models. Moreover a high level of the time to build parameter, useful to meet an empirical plausible speed of convergence, induces large oscillations in the aggregate variables (see for several numerical examples Bambi [3], Figure 7) and amplifies in this way the magnitude of the smoothing mechanism necessary to keep

[13]Consumption is kept aside from this analysis since $\bar{c}(t)$ jumps immediately to the constant c_L.

[14]In this sense, our measure of the global speed of convergence is more accurate since we avoid computational errors induced by calculating numerically the stable manifold.

[15]More precisely we have set $\delta = 0.1$, and $\sigma = 1.5$; the level of technology A and the intertemporal preference rate ρ are let to vary in order to pin down the real interest rate to five per cent a year.

[16]The parameters δ, and σ are the same as in the AK case while the real interest rate is again set to five percent by adjusting accordingly the level of technology A and the intertemporal preference rate ρ once the share of capital is set to 0.3.

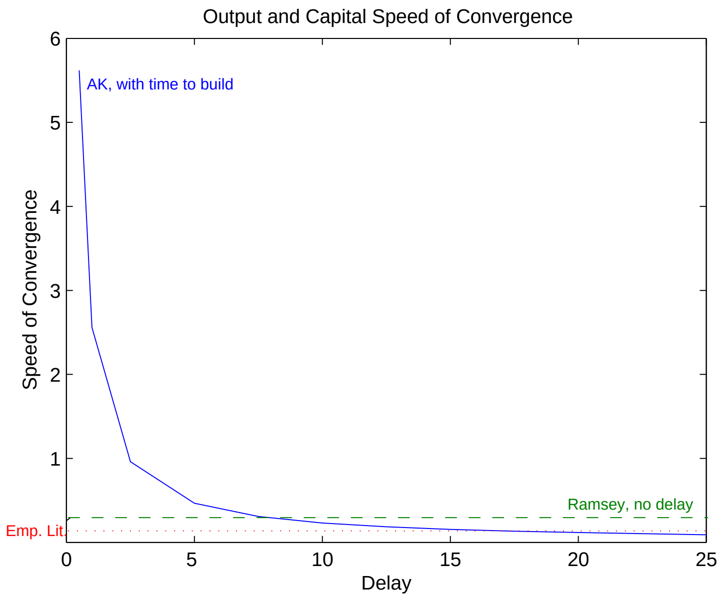

Figure 1: Speed of Convergence for different choices of d.

detrended consumption a constant share of total wealth. At $d = 1$, $d = 5$, and $d = 10$ the smoothing mechanism offsets variations in detrended consumption from its steady state level of a maximum magnitude of 0.08%, 3%, and 36% respectively. Then a negative trade off between the speed of convergence and the magnitude of the smoothing effect emerges. Finally, the presence of time to build triggers also in an AK model, the usual relations between the level of technology, the rate of intertemporal preference and the depreciation rate on the speed of convergence as pointed out in Proposition 2.2.

6 Conclusion

In this paper, we have shown how the close form policy function of an AK model with time to build can be found by using a not-standard Dynamic Programming approach, and how this result let us to fully explain the consumption smoothing effects induced by gestation lags in production. The differences and similarities with a vintage capital model having linear technology are also exploited by comparing the closed loop policy function in the two different frameworks and enlightening the different role of the equivalent capital. Finally several considerations on how delay in production may affect the global speed of convergence are proposed.

Acknowledgments. We are grateful to an anonymous referee for several comments on a previous version of the paper. We also thank the participants at the "Agents interactions, market interdependencies, and aggregate instabilities" conference in Paris and at the 2009 SAET conference in Ischia, for their useful feedbacks.

Appendix: Proofs

Proof of Proposition 2.1. First of all we prove (a). Let us define the function

$$\begin{cases} f(\cdot)\colon \mathbb{R} \to \mathbb{R} \\ f(\cdot)\colon z \mapsto z - \tilde{A}e^{-zd}. \end{cases}$$

It can be easily seen that

$$\lim_{z\to-\infty} f(z) = -\infty \qquad \text{and} \qquad \lim_{z\to\infty} f(z) = \infty. \tag{53}$$

Moreover the derivative of $f(\cdot)$ is

$$f'(z) = 1 + \tilde{A}de^{-zd} > 0$$

so f is strictly increasing and by (53) it has a unique zero ξ and this prove the first statement. Since $f(0) = -\tilde{A} < 0$ we have that $\xi > 0$. Moreover

$$0 < \tilde{A}(1 - e^{\tilde{A}d}) = f(\tilde{A})$$

so, since $f(\cdot)$ is strictly growing, $\xi < \tilde{A}$. This prove the second inequality of the (11). The first can be proved observing first that $f(\cdot)$ is concave, indeed it second derivative is given by

$$f''(z) = -\tilde{A}d^2e^{dz} < 0.$$

So, in particular, for all real $z \neq \tilde{A}$ we have

$$f(z) < f(\tilde{A}) + f'(\tilde{A})(z - \tilde{A}) = \tilde{A}(1 - e^{-\tilde{A}d}) + (z - \tilde{A})(1 + \tilde{A}de^{-\tilde{A}d}), \tag{54}$$

and if we consider the unique zero

$$\xi_0 = \tilde{A}\frac{e^{-\tilde{A}d}(\tilde{A}d + 1)}{1 + \tilde{A}de^{-\tilde{A}d}} \neq \tilde{A}$$

of the right hand side of (54) (it is just a straight line varying z in $\mathbb{R}$) we have $f(\xi_0) < 0$ and since f is growing and ξ is its unique zero the first inequality of (11) follows.

To prove the other parts observe first that z is a root of (10) if and only if $w = zd$ is a root of

$$w = \tilde{A}de^{-w}. \tag{55}$$

Now it is enough to apply Theorem 3.1 p. 312 of [10] to get (b), (c), (d).

The first statement of (e) follows from Theorem 3.12 p.315 of [10]. Indeed there it is stated that the sequence μ_k is strictly decreasing. The fact that $\mu_k \to -\infty$ as $k \to +\infty$ follows since, rewriting (10) we have

$$d\mu_k = \tilde{A}de^{-d\mu_k}\cos(d\nu_k), \qquad d\nu_k = -\tilde{A}de^{-d\mu_k}\sin(d\nu_k),$$

So from the second equation and the fact (coming from (d)) that $\nu_k \to +\infty$ as $k \to +\infty$, the claim follows.

The second statement of (e) follows from Lemma 3.3 p. 312 of [10]. The final statement follows from the second statement and from the fact that (see Exercise 3.11, p.315 of [10]) μ_1 and ν_1 are strictly increasing functions of $\tilde{A}d$. $\square$

Proof of Proposition 2.2. It is a simple application of the implicit function theorem. For the root ξ one considers the function $F(\tilde{A}, d, \xi) = \xi - \tilde{A}e^{-\xi d}$ and observe that

$$\frac{\partial \xi}{\partial \tilde{A}} = -\frac{\frac{\partial F}{\partial \tilde{A}}}{\frac{\partial F}{\partial \xi}} \qquad \frac{\partial \xi}{\partial d} = -\frac{\frac{\partial F}{\partial d}}{\frac{\partial F}{\partial \xi}},$$

and make the straightforward computations.

For the root $\mu_1 + i\nu_1$ to simplify computations we use the fact that $z = \mu + i\nu$ is a root of (10) if and only if $w = zd =: \bar{\mu} + i\bar{\nu}$ is a root of

$$w = \beta e^{-w} \Longleftrightarrow \begin{cases} \bar{\mu} = \beta e^{-\bar{\mu}} \cos \bar{\nu} \\ \bar{\nu} = -\beta e^{-\bar{\mu}} \sin \bar{\nu} \end{cases} . \tag{56}$$

where $\beta = \tilde{A}d$. Then we use the implicit function theorem to find $\frac{d\bar{\mu}}{d\beta}$, $\frac{d\bar{\nu}}{d\beta}$ and then we use the fact that $\bar{\mu} = d\mu$, $\bar{\nu} = d\nu$ and that $\beta = \tilde{A}d$ so

$$\frac{\partial \mu}{\partial \tilde{A}} = \frac{1}{d} \cdot \frac{\partial \bar{\mu}}{\partial \beta} \cdot \frac{\partial \beta}{\partial \tilde{A}} = \frac{\partial \bar{\mu}}{\partial \beta}$$

$$\frac{\partial \mu}{\partial d} = -\frac{1}{d^2}\bar{\mu} + \frac{1}{d}\frac{\partial \bar{\mu}}{\partial \beta} \cdot \frac{\partial \beta}{\partial d} = -\frac{\mu}{d^2} + \frac{\tilde{A}}{d} \cdot \frac{\partial \bar{\mu}}{\partial \beta}$$

and then the claim follows by straightforward computations. □

Proof of Proposition 2.3. The first part follows easily from the definition of $k^M(\cdot)$ and the positivity of $c(\cdot)$. As proved in [3] ξ is the solution of (10) with highest real part, so the claim follows from [10] page 34. □

Proof of Proposition 2.4. For $\sigma > 1$ it is obvious since $J(k_0(\cdot); c(\cdot)) < 0$ always. For $\sigma \in (0, 1)$ we observe that for every $c(\cdot) \in L^1_{\text{loc}}([0, +\infty); \mathbb{R}^+)$,

$$J(k_0(\cdot); c(\cdot)) \leq \frac{1}{1-\sigma} \int_0^{+\infty} e^{-\rho t} (Ak_{k_0,c}(t))^{1-\sigma} \, \mathrm{d}t \leq \\ \leq \frac{1}{1-\sigma} \int_0^{+\infty} e^{-\rho t} (Ak^M(t))^{1-\sigma} \, \mathrm{d}t < +\infty. \tag{57}$$

where the last inequality follows from part (2) of Proposition 2.3. □

Proof of Theorem 3.2. The proof (in a more general case) can be found in [7] Theorem **5.1** page. 258. □

Proof of Proposition 3.6. v is of course continuous and differentiable in every point of X and its differential in x is

$$Dv(x) = (\nu(1-\sigma)\Gamma(x)^{-\sigma}, (1-\sigma)\nu\Gamma(x)^{-\sigma}\psi^1\}) = \nu\Gamma(x)^{-\sigma}\psi$$

So $Dv(x) \in D(G)$ everywhere in X.
We can also calculate explicitly GDv and $\tilde{A}\delta_{-d}Dv$, we have (using that ξ satisfies the characteristic equation (10) and then $\tilde{A}\delta_{-d}(\psi^1) = \xi$):

$$GDv(x) = (0, (1-\sigma)\nu\Gamma^{-\sigma}\xi\psi^1\}) \tag{58}$$
$$\tilde{A}\delta_{-d}Dv(x) = (1-\sigma)\nu\Gamma^{-\sigma}\xi > 0 \tag{59}$$

so

$$(\tilde{A}\delta_{-d}Dv(x))^{-1/\sigma} = \alpha\Gamma(x) \tag{60}$$

For the definition of X $(\tilde{A}\delta_{-d}Dv)^{-1/\sigma} > 0$.
If $x = (x^0, x^1) \in Y$ then

$$\Gamma(x) \leq \frac{1}{\alpha}\frac{A}{\tilde{A}}x^0 \tag{61}$$

and then $(\tilde{A}\delta_{-d}Dv)^{-1/\sigma} \leq \frac{A}{\tilde{A}}x^0$. So we can use Remark 3.4 and use the Hamiltonian in the form of equation (26).
Now it is sufficient substitute (58) and (59) in (26) and verify, by easy calculations, the relation:

$$\rho v(x^0, x^1) - \langle (x^0, x^1), GDv(x^0, x^1)\rangle_{M^2} -$$
$$-x^0\tilde{A}\delta_{-d}Dv((x^0, x^1) - \frac{\sigma}{1-\sigma}(\tilde{A}\delta_{-d}Dv((x^0, x^1))^{\frac{\sigma-1}{\sigma}} = 0$$

□

Proof of Proposition 3.11. Clearly $\phi \in C(M^2)$. Given $p \in M^2$ we have to prove that

$$\begin{cases} \frac{d}{dt}x_\phi(t) = G^* x_\phi(t) + \tilde{A}\delta_{-d}(\phi(x_\phi(t))), & t > 0 \\ x_\phi(0) = p \end{cases} \tag{62}$$

has a unique solution in Π. Unfortunately this cannot be done using known theorems available in the literature so we do it directly.
Informal description of the approach
We begin with an informal description of our approach: along the trajectories driven by the (candidate) feedback ϕ we have (using the DDE notation, with u and y):

$$u(t) = y(t) - \alpha\left(y(t) + \int_{-d}^{0} e^{\xi s}\tilde{A}u(t-d-s)\,\mathrm{d}s + \right) =$$
$$y(t)\alpha - \alpha e^{\xi t}\int_{t}^{t+d} e^{-\xi r}\tilde{A}u(r-d)\,\mathrm{d}r. \tag{63}$$

If we take the derivative of such an expression and impose $\dot{y}(t) = \tilde{A}u(t-d)$ we find

$$
\begin{aligned}
&\dot{u}(t) = \tilde{A}u(t-d)(1-\alpha)- \\
&-\alpha\left(\xi\tilde{A}e^{\xi t}\int_t^{t+d} e^{-\xi s}u(s-d)\,\mathrm{d}s + \tilde{A}(-u(t-d)+e^{-d\xi}u(t))\right).
\end{aligned}
\tag{64}
$$

and $u(0) = y(0)(1-\alpha) - \alpha\int_{-d}^0 e^{\xi s}u(-d-s = \,\mathrm{d}s$. In the (rigorous) proof we will consider (64), together with the equations $\dot{y}(t) = \tilde{A}u(t-d)$ and the initial conditions, as a starting point. We will prove the existence and uniqueness of the solution of such a DDE and, eventually, tranforming such DDE in the infinite dimensional setting, the existence and the uniqueness of the solution for (62).

End of the informal description of the approach

We consider the following DDE in $\tilde{u}$ and $\tilde{y}$:

$$
\begin{cases}
\dot{\tilde{u}}(t) = \tilde{A}\tilde{u}(t-d)\,(1-\alpha) - \\
\quad -\alpha\left(\xi\tilde{A}e^{\xi t}\int_t^{t+d} e^{-\xi s}\tilde{u}(-d+s)\,\mathrm{d}s + \tilde{A}(-\tilde{u}(t-d)+e^{-d\xi}\tilde{u}(t))\right) \quad t \geq 0 & \text{(65a)} \\
\dot{\tilde{y}}(t) = \tilde{A}\tilde{u}(t-d) \qquad t \geq 0 & \text{(65b)} \\
\tilde{y}(0) = y(0) & \text{(65c)} \\
\tilde{u}(s) = u(s) \qquad \text{for } s \in [-d, 0) & \text{(65d)} \\
\tilde{u}(0) = (1-\alpha)\,y(0) - \alpha\int_{-d}^0 e^{\xi s}\tilde{A}u(-d-s)\,\mathrm{d}s & \text{(65e)}
\end{cases}
$$

that has an absolute continuous solution $(\tilde{u}, \tilde{y})$ on $[0, +\infty)$ (see for example [7] page 287 for a proof). Setting $\tilde{x} := (\tilde{y}, \tilde{\gamma}(t))$ where

$$
\tilde{\gamma}(t)[s] = \tilde{A}\tilde{u}(t-d-s) \qquad \text{for } s \in [-d, s),
$$

thanks to Theorem 3.2, $\tilde{x}(\cdot)$ satisfies, by (65b), (65c) and (65d),

$$
\begin{cases}
\frac{d}{dt}\tilde{x} = G^*\tilde{x}(t) + \tilde{A}\delta_{-d}(\tilde{u}(t)), \quad t > 0 \\
\tilde{x}(0) = (y(0), \gamma(0))
\end{cases}
$$

Moreover, integrating (65a),

$$
\begin{aligned}
\tilde{u}(t) = \tilde{u}(0) + &\int_0^t \tilde{A}\tilde{u}(s-d)\,(1-\alpha)\,\mathrm{d}s- \\
-\alpha &\int_0^t \left[\xi\tilde{A}e^{\xi s}\int_s^{s+d} e^{-\xi r}\tilde{u}(-d+r)\,\mathrm{d}r + \tilde{A}(-\tilde{u}(s-d)+e^{-d\xi}\tilde{u}(s))\right]\mathrm{d}s =
\end{aligned}
\tag{65}
$$

(integrating by part in the double-integral term)

$$
\begin{aligned}
= \tilde{u}(0) + &\int_0^t \tilde{A}\tilde{u}(s-d)\,(1-\alpha)\,\mathrm{d}s- \\
&-\alpha\left(\int_{-d}^0 e^{\xi r}\tilde{u}(t-d-r)\,\mathrm{d}r\right) + \alpha\tilde{A}\int_0^d e^{-\xi r}\tilde{u}(-d+r)\,\mathrm{d}r =
\end{aligned}
\tag{66}
$$

(using (65e))

$$= (1-\alpha)\,\tilde{y}(0) + \int_0^t \dot{\tilde{y}}(s)\,(1-\alpha)\,\mathrm{d}s - \alpha\left(\int_{-d}^0 e^{\xi r}\tilde{u}(t-d-r)\,\mathrm{d}r\right) =$$

$$= \tilde{y}(t)\,(1-\alpha) - \alpha\left(\int_{-d}^0 e^{\xi r}\tilde{u}(t-d-r)\,\mathrm{d}r\right) =$$

$$= \tilde{x}^0(t)\,(1-\alpha) - \alpha\left(\int_{-d}^0 e^{\xi r}\tilde{x}^1(t)[r]\,\mathrm{d}r\right) = \phi(\tilde{x}(t)) \quad (67)$$

and so

$$\begin{cases} \frac{d}{dt}\tilde{x}(t) = G^*\tilde{x}(t) + \tilde{A}\delta_{-d}(\phi(\tilde{x}(t))), \quad t > 0 \\ \tilde{x}(0) = (y(0), \gamma(0)) \end{cases}$$

and then $\tilde{x}(t)$ is a solution of (62). The uniqueness follows from the linearity of ϕ so. This prove that $\phi \in FS_p$. □

Proof of Theorem 3.12. To prove the first statement we take the derivative of the expression $\Gamma(x_phi(t)) = \langle \psi, x_\phi(t)\rangle$. Note that, since ϕ is a feedback strategy (Proposition 3.11) and $\phi \in D(G)$ (as observed in (29)) such derivative exists and (from (21)) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma(x_\phi(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\langle \psi, x_\phi(t)\rangle = \langle G\psi, x_\phi(t)\rangle + \tilde{A}\delta_{-d}\psi\phi(x_\phi(t)) =$$

(thanks to the definition of ψ given in (28)

$$= \xi\left\langle \psi^1, x_\phi(t)\right\rangle + \tilde{A}e^{-\xi d}\Big((x_\phi^0(t) - \alpha\Gamma(x_\phi(t)))\Big) = \qquad \Big(\text{since } \xi = \tilde{A}e^{-\xi d}\Big)$$

$$= \left[\xi\left\langle \psi^1, x_\phi(t)\right\rangle + \xi(x_\phi^0(t)\right] - \xi\alpha\Gamma(x_\phi(t))) = \xi(1-\alpha)\Gamma(x_\phi(t))). \quad (68)$$

This conclude the proof of the first statement.

To prove the invariance of I_c let us take a $c < \bar{c}$ and a $p = (p^0, p^1) \in I_c$. For $t \geq 0$ we have that (we call x_ϕ simply x)

$$u(t) = \phi(x(t)) := x^0(t) - \alpha\bigg(\int_{-d}^0 e^{\xi s}x^1(t)[s]\,\mathrm{d}s + x(t)^0\bigg) \quad (69)$$

where $(x^0(t), x^1(t))$ is the trajectory starting from p. Since, thanks to Theorem 3.11, $\phi \in FS_p$ then the trajectory $(x^0(\cdot), x^1(\cdot))$ is continuous and then $u(\cdot)$ is continuous on $[0, +\infty)$. Let $\bar{t} \in [0, +\infty)$ be, by contradiction, the first time such that $u(\bar{t}) \leq 0$ or $u(\bar{t}) \geq x^0(\bar{t})$. We have

$$u(\bar{t}) = x^0(\bar{t}) - \alpha\bigg(\int_{-d}^0 e^{\xi s}x^1(\bar{t})[s]\,\mathrm{d}s + x(\bar{t})^0\bigg) \quad (70)$$

Since $p^1 \geq 0$ and $u(t) > 0$ for all $t \in [0, \bar{t})$ then $x^0(t)$ is always growing[17] on $[0, \bar{t}]$. Now for $t \geq 0$ and $s \in [-d, 0]$ we have:

$$x^1(t)[s] = \begin{cases} p^1[s-t] & \text{if } s-t > -d \\ \tilde{A}u(t-d-s) & \text{if } s-t < -d \end{cases} \tag{71}$$

Then, since $p \in I$, we have, for almost every $s \in (-d, 0)$,

$$0 \leq x^1(\bar{t})[s] \leq c x^0(\bar{t})$$

and so $\int_{-d}^0 e^{\xi s} x^1(\bar{t})[s] \, \mathrm{d}s \leq \frac{c}{\xi}\left(1 - e^{-\xi d}\right) x(\bar{t})^0$, then

$$0 \ < \ \alpha\left(\int_{-d}^0 e^{\xi s} x^1(\bar{t})[s] \, \mathrm{d}s \ + \ x(\bar{t})^0\right) \ \leq \ \alpha\left(\frac{c}{\xi}\left(1 - e^{-\xi d}\right) + 1\right) x(\bar{t})^0 \tag{72}$$

where the first inequality follows from the fact that $x^0(\bar{t}) \geq x^0(0) > 0$. So, from the first inequality of the (72) and from (70), we have immediately that $u(\bar{t}) < x^0(\bar{t})$. Moreover from (70) and the second inequality of (72) we have

$$u(\bar{t}) \geq x^0(\bar{t}) \left[1 - \alpha\left(\frac{c}{\xi}\left(1 - e^{-\xi d}\right) + 1\right)\right]$$

and then, thanks to the fact that $c < \bar{c}$ we have

$$0 < u(\bar{t}).$$

Summarizing $u(\bar{t}) > 0$ and $u(\bar{t}) < x^0(\bar{t})$ and this is a contradiction with the definition of $\bar{t}$. So, for $t \geq 0$, $u(t) \in (0, x^0(t))$. This also implies that $x^0(t)$ is always growing and then (since $x^0(0) > 0$) anways strictly positive. Thanks to the relation (71) I_c is an invariant set and we have the claim. □

Proof of Corollary 3.13. It follows easily by the fact that by Theorem 3.12, every I_c is invariant. □

Proof of Theorem 3.15. 1. To prove that $I \subseteq Y$ we have only to verify that for every I_c (with $c < \bar{c}$) $I_c \subseteq X$ and the inequality appearing in the (31) is satisfied. The first fact follows by the strict positivity of x^0 and by the positivity of $x^1(\cdot)$ of the element of I_c. To prove the inequality appearing in (31) we have only to observe that, on I

$$\left(\int_{-d}^0 e^{\xi s} x^1[s] \, \mathrm{d}s + x^0\right) \leq \left(\frac{c}{\xi}\left(1 - e^{-\xi d}\right) + 1\right) x^0 < \frac{1}{\alpha} x^0 \leq \frac{A}{\tilde{A}} \frac{1}{\alpha} x^0$$

[17] Since $x^0(t)$ solves the DDE:

$$x^0(t) = p^0(0) + \int_0^{t \wedge d} \frac{p^1[-s]}{\tilde{A}} \, \mathrm{d}s + \int_0^{(t-d) \wedge d} u(s) \, \mathrm{d}s.$$

This fact easily follows by the fact that $x^0(t) = y(t)$ where $y(t)$ follows the DDE in (65).

where the first inequality follows from the definition of I_c (as in (72)) and the second by Hypothesis 3.14 and by the definition of $\bar{c}$. So we have that $I \subseteq Y$. We take now $p \in I$, in particular $p \in I_c$ for some I_c with $c < \bar{c}$. Considering the evolution of the system starting from p and driven by the feedback ϕ is the same that considering the evolution of equation (37) starting from p. But from Theorem 3.12 we know that I_c is invariant for the flow of (37) and then the trajectory starting from $p \in I_c$ remains in I_c and then, since $I_c \subseteq Y$, remains in Y and then, thanks to the definition of Y and the fact that along the paths of (37) we have (69) we have that $u(\cdot) \in \mathcal{A}_0(p)$ and so $\phi \in AFS_p$.

2. Now we prove that $\phi \in OFS_p$. We consider v as defined in Proposition 3.6. From what we have just said on the admissibility of $u(t)$ follows that $x(\cdot)$ remains in Y as defined in (31) and so the Hamiltonian can be expressed in the simplified form (26) recalled in Remark 3.4. Moreover, thanks to Theorem 3.6 v is a solution of HJB on the points of the trajectory.
We introduce:

$$\begin{cases} \tilde{v}(t,x) \colon \mathbb{R} \times X \to \mathbb{R} \\ \tilde{v}(t,x) := e^{-\rho t} v(x) \qquad (v \text{ is defined in (32)}). \end{cases} \tag{73}$$

Using that $(Dv(x(t))) \in D(G)$ and that the function $x \mapsto Dv(x)$ is continuous with respect the norm of $D(G)$ (see the proof of Proposition 3.6 for the explicit form of $Dv(x)$), we find:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\tilde{v}(t,x) &= -\rho\tilde{v}(t,x(t)) + \langle D_x\tilde{v}(t,x(t)), G^* x(t) + (\tilde{A}\delta_{-d})^* u(t)\rangle_{D(G)\times D(G)'} \\ &- \rho e^{-\rho t} v(x(t)) + e^{-\rho t}\Big(\langle GDv(x(t)), x(t)\rangle_{M^2} + (\tilde{A}\delta_{-d})Dv(x(t))u(t)\Big) \end{aligned} \tag{74}$$

By definition (recalling that $u(\cdot) = \phi(x)(\cdot)$):

$$v(p) - J_0(p, u(\cdot)) = v(x(0)) - \int_0^\infty e^{-\rho t}\frac{(x^0(t) - \phi(x)(t))^{1-\sigma}}{(1-\sigma)}\,\mathrm{d}t =$$

Then, using (74) (using Proposition 2.3 to guarantee that the integral is finite and that the "boundary term at ∞" vanishes), we obtain

$$= \int_0^\infty e^{-\rho t}\Big(\rho v(x(t)) - \langle GDv(x(t)), x(t)\rangle_{M^2} - \langle(\tilde{A}\delta_{-d})Dv(x(t)), u(t)\rangle_{\mathbb{R}}\Big)\,\mathrm{d}t -$$

$$- \int_0^\infty e^{-\rho t}\left(\frac{(x^0(t) - u(t))^{1-\sigma}}{(1-\sigma)}\right)\mathrm{d}t =$$

$$= \int_0^\infty e^{-\rho t}\Big(\rho v(x(t)) - \langle GDv(x(t)), x(t)\rangle_{M^2}$$

$$-\langle(\tilde{A}\delta_{-d})Dv(x(t)), u(t)\rangle_{\mathbb{R}} - \frac{(x^0(t) - u(t))^{1-\sigma}}{(1-\sigma)}\Big)\,\mathrm{d}t =$$

using Theorem 3.6

$$= \int_0^{\infty} e^{-\rho t} \Big(\mathcal{H}(x(t), Dv(x(t))) - \mathcal{H}_{CV}(x(t), Dv(x(t)), u(t)) \Big) \, \mathrm{d}t \tag{75}$$

The conclusion follows by three observations:

1. Noting that $\mathcal{H}(x(t), Dv(x(t))) \geq \mathcal{H}_{CV}(x(t), Dv(x(t)), u(t))$ the (75) implies that, for every admissible control $\lambda(\cdot)$, $v(p) - J_0(p, \lambda(\cdot)) \geq 0$ and then $v(p) \geq V_0(p)$.
2. The original maximization problem is equivalent to the problem of find a control $\lambda(\cdot)$ that minimize $v(p) - J_0(p, \lambda(\cdot))$
3. The feedback strategy ϕ achieves $v(p) - J_0(p, u(\cdot)) = 0$ that is the minimum in view of point 1. Moreover this implies that $v(p) \geq V_0(p)$.

□

Proof of Lemma 4.4. The first statement follows by Theorem 3.12. In view of Proposition 4.2 along optimal trajectory we have:

$$\Lambda e^{gt} = y^*(t) - u^*(t) = \alpha \bigg(\int_{-d}^{0} \tilde{A} e^{\xi s} u^*(t - s - d) \, \mathrm{d}s + y^*(t) \bigg)$$

so to compute the explicit value of Λ we only have to compute the value of the right side at time 0 and we find

$$\Lambda = \alpha \bigg(\int_{-d}^{0} e^{\xi s} \tilde{A} u(-s - d) \, \mathrm{d}s + y(0) \bigg).$$

This conclude the proof. □

Proof of Proposition 4.6. The existence of the limit y_L for $\bar{y}(t)$ is proved in [3] (in Proposition 2 page 1027 the author proves the existence of the limit for $\bar{k}(t) = \frac{1}{A} y(t + d)$). This implies, thanks to Corollary 4.5 the existence of the limit u_L. We can here compute explicitly the value of such limits using the explicit form of the optimal feedback (39). Namely we have only to impose, from (39)

$$u_L = y_L - \alpha \left(y_L + \tilde{A} \int_{-d}^{0} e^{\xi s} u_L e^{-gs} e^{-gd} \, \mathrm{d}s \right) =$$

$$= y_l (1 - \alpha) - \frac{1 - e^{-(\xi - g)d}}{\xi - g} u_L \alpha \tilde{A} e^{-gd} \tag{76}$$

and then

$$u_L = y_L \frac{1 - \alpha}{1 + \frac{1 - e^{-(\xi - g)d}}{\xi - g} \alpha \tilde{A} e^{-gd}}. \tag{77}$$

Moreover from Corollary 4.5 we have that

$$u_L = y_L - \Lambda. \tag{78}$$

Using (77) and (78) we find:

$$y_L = \Lambda \left(1 - \frac{1-\alpha}{1 + \frac{1-e^{-(\xi-g)d}}{\xi-g} \alpha \tilde{A} e^{-gd}} \right)^{-1}$$

and

$$u_L = \Lambda \left[\left(1 - \frac{1-\alpha}{1 + \frac{1-e^{-(\xi-g)d}}{\xi-g} \alpha \tilde{A} e^{-gd}} \right)^{-1} - 1 \right]$$

and so we have the claim. □

Proof of Theorem 4.7. All the statements are corollaries of the results of Section 4. More precisely:

1. Follows from Lemma 4.4 and by relations (44)-(45).
2. Follows from the previous point and (1).
3. Follows from Proposition 4.1 and by relations (44)-(45).
4. Follows from Proposition 4.6 and by relations (44)-(45) and by (17).
5. Follows from the point 4 above and [6].

□

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ISSN 1379-244X D/2010/3082/029