Optimal Portfolios in Wishart Models and Effects of Discrete Rebalancing on Portfolio Distribution and Strategy Selection

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Preface

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Chapter 1

Introduction and outline

As illustrated by the title, this dissertation is mainly devoted to the research of two problems – the continuous-time portfolio optimization in different Wishart models and the effects of discrete rebalancing on portfolio wealth distribution and optimal portfolio strategy. The first objective is to study the continuous-time portfolio optimization problems in different Wishart models. In continuous-time models, agents can make investment decisions at any time during the investment period in order to maximize their expected utility from terminal wealth with respect to some utility function. This subject has been extensively studied since the development of stochastic analysis around the 1960's and the contribution of Merton in 1971 [40]. While Merton considered an asset price model with non-stochastic volatility, a lot of recent research is done in models incorporating their own stochastic volatility processes, since the non-stochastic models are not flexible enough to model some economic phenomena, such as the "smile effect" and the "leverage effect" etc. However, the most research done so far is based on an one-factor stochastic volatility model, for example, Kraft and Zariphopoulou investigated in [35, 47] extensively the optimization problems involving stochastic volatility in the setting of a Heston model, which is an one-dimensional case of the Wishart model. Because of the necessity of modeling a complete portfolio of assets, it is of interest for us to study optimal portfolios in the Wishart stochastic volatility model.

The Wishart model is a multivariate extension of the Heston model. In the Wishart model, the multidimensional asset price process evolves as some diffusion dynamic, where the covariance matrix process follows a Wishart process. The Wishart processes possess a desirable property, i.e. the affine property. Roughly speaking, an affine process is a process whose logarithm of its Laplace transform is affine dependent on the initial state of the process (see [18]). As we will see later, this property makes the portfolio optimization regarding Wishart processes tractable in many cases. In light of the computational tractability and flexibility in capturing many of the empirical features of financial dynamics, the Wishart model has been widely studied and used in recent years. More information on the Wishart models and their financial applications may be found in [10, 11, 13, 14, 20, 23].

In the Wishart model for our continuous-time portfolio optimization problem, we denote by $(S_t)_{t\geq 0} = (S_{t,1}, \ldots, S_{t,d})_{t\geq 0}$ the vector process of the risky assets price. The joint dynamics of the risky assets price process $(S_t)_{t\geq 0}$ and its stochastic covariance matrix process $(\Sigma_t)_{t\geq 0}$

are given by the following stochastic differential system:

$$dS_t = diag\left(S_t\right) B\left(\Sigma_t\right) dt + diag\left(S_t\right) \Sigma_t^{1/2} dW_t^S,$$
$$d\Sigma_t = \left(\Omega\Omega^T + M\Sigma_t + \Sigma_t M^T\right) dt + \Sigma_t^{1/2} dW_t^\sigma Q + Q^T \left(dW_t^\sigma\right)^T \Sigma_t^{1/2},$$

where $\left(W_t^S\right)_{t\geq 0}$, $\left(W_t^\sigma\right)_{t\geq 0}$ are a d dimensional Brownian motions vector and a $d \times d$ Brownian motions matrix respectively on the probability space $(\Omega, \mathcal{F}, P)$. The parameter $B(\Sigma) : S_d^+(\mathbb{R}) \to \mathbb{R}^d$ is measurable, whereas Ω, M, Q are $d \times d$ matrices with Ω invertible. Furthermore, we assume that besides the d risky assets, there is still one bond with constant risk-free rate r in the financial market. Given an investor maximizing utility from terminal wealth with respect to a utility function U, the portfolio optimization problem can be formulated as

$$\Phi\left(t, x\right) = \max_\pi E^{t,x}\left[U\left(X_T\right)\right]$$

with X_T being the portfolio wealth at T and $x = X_t$. We denote by $\pi = (\pi_t)$ the vector process of the investment proportion in the d risky assets.

We will consider two cases of U – logarithmic utility and power utility. The optimal portfolio strategy with logarithmic utility in the Wishart models can be derived by replacing X_T by its explicit solution, whereas the portfolio optimization problems with power utility, i.e. extensions of the *Merton problem*, are generally handled either by methods from stochastic control theory, which lead to Hamilton-Jacobi-Bellmann equations (see [35, 47]) or by martingale methods (see [32]). Previous to a detailed description of the methods used in this thesis, let us first introduce two cases of Wishart models, namely the uncorrelated and the correlated Wishart models. A *uncorrelated Wishart volatility model* is a Wishart model with $\left(W_t^S\right)_{t\geq 0}$ and $\left(W_t^\sigma\right)_{t\geq 0}$ being uncorrelated. A *correlated Wishart volatility model* is defined likewise as a model with correlated $\left(W_t^S\right)_{t\geq 0}$ and $\left(W_t^\sigma\right)_{t\geq 0}$. In the uncorrelated Wishart model, we can derive the optimal portfolio strategy and represent the value function as the expectation of a stochastic exponential by the application of the Girsanov theorem. This is possible due to the assumption of no correlations. We especially consider the case of $\Sigma_t \mathbf{v} = B(\Sigma_t) - r$ with $\mathbf{v} \in \mathbb{R}^d$ and show that the value function can in this case be expressed as an exponential function with all coefficients given in closed-form by Proposition 4.2.7 In the correlated Wishart model we can not apply the Girsanov theorem to solve the optimization problem, since the Girsanov density is not automatically a martingale and furthermore, we can not interchange the maximizing and expectation operations as in the uncorrelated case (see Remark 3.1.2 for more details). Instead of the Girsanov theorem, we use the Hamilton-Jacobi-Bellman (HJB) principle to derive the optimal portfolios and the value functions, which extend the results of Kraft and Zariphopoulou in [35, 47]. We again consider two special models. One model has a general asset process but a special Q matrix and a special correlation between $\left(W_t^S\right)$ and $\left(W_t^\sigma\right)$, whereas the other model possesses a special assets drift with $\Sigma_t \mathbf{v} = B(\Sigma_t) - r$, but less restrictions on Q and the correlations. We first derive the Feynman-Kac representations of the candidate value functions for both models. The most challenging part in this section is to identify when the HJB equation owns a finite solution. An example illustrating the blow-up of the HJB solution is presented by Korn & Kraft [34]. Note that in the setting of a one-dimensional Heston model, the sufficient conditions for a finite value function has been derived by Kraft [35]. For the second model, we can show by Theorem 4.2.5 and Corollary 4.2.7 that there exists a finite explicit value function under some conditions and a verification result to the HJB equation is also presented at the end of this section.

For the sake of completeness, we also consider the portfolio selection problems with no risk-free asset in the market. These problems could be regarded as the original optimization problems subject to the additional constraint $\pi^T\mathbf{1} = 1$. The optimal portfolio strategies can be derived by applying the constraint directly, namely replacing the d-th asset weight π_d by $1 - \sum_{i=1}^{d-1} \pi_i$.

The second major part of this thesis is devoted to the analysis of the effects of discrete rebalancing on the portfolio wealth distribution and optimal portfolio strategy. As pointed out by Bertsimas, Kogan and Andrew in [6], continuous-time stochastic processes are only approximations of physically realizable phenomena and continuous trading within a portfolio is only theoretically feasible. Hence, it is necessary to study portfolios with constraints on rebalancing frequency. While Bertsimas [6] focused on the asymptotic distribution of the tracking error, which results from the implementation of the continuous delta-hedging in discrete-time, we pay attention to the value evolution of a discretely rebalanced portfolio and the modification of the optimal portfolio strategy under rebalancing frequency constraint.

The discretely rebalanced portfolio's profit and loss distribution is critical for the risk measurement of less liquid portfolios and there is already some earlier work done in this field. Motivated by the proposal of a new market risk measure – the incremental risk charge (IRC), which, loosely speaking, equals a 99.9% VaR of the terminal wealth distribution of a less liquid portfolio over a one-year horizon, Glasserman approximated in [21] the loss distribution for the discretely rebalanced portfolio relative to the continuously rebalanced portfolio. The approximation is derived from a limiting result for the difference between the discretely and continuously rebalanced portfolios as the rebalancing frequency increases. The IRC has been introduced by the Basel Committee on Banking Supervision in 2007. It is able to capture the risk in long-term fluctuations of less liquid securities compared with the traditional ten-day 99% VaR used in the banking industry. For further introduction to IRC please refer to [3, 4, 21].

Contrasting the model of the assets dynamics in Glasserman's paper, we introduce in this thesis a new and more sophisticated model with assets dynamics satisfying the *regular conditions* listed in Section 5.1. We show that the limiting result for the difference between the discretely and continuously rebalanced portfolios can be extended to our new model and the impact of discrete rebalancing on the portfolio wealth distribution can be corrected by the *volatility adjustment* (an adjustment that corrects the volatility for discrete rebalancing) and the *conditional mean adjustment* (an adjustment that adjusts the tail of the discrete portfolio's loss distribution conditional on a large loss in the continuous portfolio). Besides being used as an approximation of the discretely rebalanced portfolio, the limiting result can also be used to measure the relative error between the continuously and discretely rebalanced portfolios. Such a measure is usually called the *temporal granularity* and is defined as the standard deviation of the limit distribution in the limit theorem (Theorem 5.2.3). As we shall show, the temporal granularity helps us to find some rebalancing frequency to keep the relative error within some sufficiently small value.

The other main result in this part is concerned with the optimal portfolio strategy of a Δt-periodic rebalanced portfolio with asset price processes following geometric Brownian motions. Since the price process of a discretely rebalanced portfolio does not have a "simple" explicit solution like the continuously rebalanced portfolio, we can not deal with the optimization problem in the usual way. This makes it rather difficult to find an explicit analytical portfolio strategy. Thus, we focus on finding a better portfolio strategy that provides a larger utility compared with π^* – the optimal strategy for the continuously rebalanced portfolios for Δt sufficiently small. The key approach we apply in this part is one iteration of Newton's method with π^* being the initial value.

This thesis is organized as follows: In Chapter 2 we provide the readers with some necessary backgrounds of Wishart processes and some new properties regarding the affine property of the Wishart processes as well as the corresponding proofs. In the last section of Chapter 2, a multivariate Wishart stochastic volatility model is introduced.

Chapter 3 is devoted to determining the optimal portfolio strategies with respect to logarithmic and power utility, in the uncorrelated Wishart volatility model. The optimal portfolio strategies of our optimization problems are given explicitly for two cases – with and without risk-free asset in the financial market.

In Chapter 4 we proceed with the determination of the optimal portfolio strategies in the correlated Wishart volatility model. Again, two utility functions are taken into consideration, namely the logarithmic utility and the power utility. For the power utility problem, we first derive the HJB equation to get the optimal strategy and then solve the HJB equation to derive a candidate for the value function. The candidates are given for two cases of asset processes: the case of a general drift and the case of a linear drift. We then present the verification result for the linear drift case.

Chapter 5 contains our extension of the central limit theorem for the relative difference between the discrete and continuous portfolios. We use this theorem to derive our portfolio volatility adjustment. Moreover, we prove a conditional limit theorem for the loss in the discrete portfolio conditional on a large loss in the continuous portfolio and establish, as a result, the conditional mean adjustment. Some numerical results to evaluate the quality of various approximations are given at the end of this chapter.

In Chapter 6, we define and interpret the temporal granularity followed by some examples.

In the final Chapter, we derive an approximation of the optimal strategy for discrete portfolios. We prove that it yields a larger utility compared to the optimal continuous portfolio strategy when the rebalancing period is sufficiently small. The main tool we use is Newton's method.

Chapter 2

The Wishart process

In this chapter we try to familiarize the reader with some basic concepts and properties about Wishart processes. We begin with the definition of the Wishart process in Section 2.1. Afterwards, some theorems about the existence and uniqueness of Wishart processes are given. In Section 2.3, we show that Wishart processes can be constructed as squares of matrix variate Ornstein-Uhlenbeck processes. Subsequently, some properties of Wishart processes like the affine property and the conditional distribution of Wishart processes are illustrated in Section 2.4, 2.5 and 2.6. Finally, we present a Wishart stochastic volatility model.

2.1 Definition

The Wishart process, which was originally introduced by Bru [10] in 1991, was extensively studied by Da Fonseca et al. [13, 14], Gourieroux & Sufana [25, 26] and by Gauthier & Possamai [20]. According to Gourieroux [23], the Wishart process can be defined through its diffusion representation.

Definition 2.1.1. *(Wishart Processes). Let $(W_t^\sigma)_{t\geq 0}$ denote a $d\times d$ matrix-valued Brownian motion, $Q \in GL_d(\mathbb{R})$ and $M \in \mathcal{M}_d(\mathbb{R})$ be an arbitrary matrix. The matrix-valued process $(\Sigma_t)_{t\geq 0}$ is said to be a Wishart process, if it is a strong solution of the following stochastic differential equation:*

$$d\Sigma_t = \left(\beta Q^T Q + M\Sigma_t + \Sigma_t M^T\right) dt + \Sigma_t^{1/2} dW_t^\sigma Q + Q^T \left(dW_t^\sigma\right)^T \Sigma_t^{1/2}, \quad \Sigma_0 = \sigma_0, \tag{2.1}$$

where $\sigma_0 \in S_d^+$ is a symmetric, strictly positive definite matrix and $\beta > d-1$ is a non-negative number.

Remark 2.1.2. *The condition $\beta > d-1$ ensures the existence and uniqueness of a weak solution of (2.1) on $\bar{S}_d^+(\mathbb{R})$ almost surely. If $\beta \geq d+1$ is imposed, the SDE (2.1) owns then a unique global strong solution on $S_d^+(\mathbb{R})$ a.s. These facts are illustrated in section 2.2 particularly.*

The Wishart process $(\Sigma_t)_{t\geq 0}$ has some desirable properties. First of all, it is shown in [41], if the eigenvalues of M only have negative real parts, i.e. $Re(\sigma(M)) \subseteq (-\infty, 0)$, the Wishart process $(\Sigma_t)_{t\geq 0}$ is mean reverting.

Furthermore, $(\Sigma_t)_{t\geq 0}$ is a process which never leaves $\bar{S}_d^+(\mathbb{R})$. It is obvious that the matrix Σ_t is real symmetric and owns the decomposition $\Sigma_t = O_t^T \Lambda_t O_t$, where O_t is an orthogonal matrix and Λ_t is real diagonal with the eigenvalues of Σ_t on the diagonal.

When $(\Sigma_t)_{t\geq 0}$, which starts at a symmetric, strictly positive definite matrix σ_0, hits the boundary of $\bar{S}_d^+(\mathbb{R})$ at time t, there exists a nonzero vector $a \in \mathbb{R}^d$ which satisfies $a^T \Sigma_t a = (O_t a)^T \Lambda_t (O_t a) = 0$. This implies that the diagonal entries of Λ_t are nonnegative with at least one zero. From [26, Appendix 1], the conditional variance of $d\left(a^T \Sigma_t a\right)$ given $\mathcal{F}_t$ is then:

$$V\left(d\left(a^T \Sigma_t a\right)\right) = 4\left(a^T \Sigma_t a\right)\left(a^T Q^T Q a\right) dt = 0,$$

where $(\mathcal{F}_t)_{t\geq 0}$ denotes the natural filtration of $(\Sigma_t)_{t\geq 0}$. Since $a^T \Sigma_t a = 0$ implies $\Sigma_t^{1/2} a = 0$ and $a^T \Sigma_t^{1/2} = 0$, one gets $\Sigma_t a = 0$ and $a^T \Sigma_t = 0$. Then there is

$$d\left(a^T \Sigma_t a\right) = \left(\beta a^T Q^T Q a + a^T M \Sigma_t a + a^T \Sigma_t M^T a\right) dt = \left(\beta a^T Q^T Q a\right) dt > 0$$

for $Q \in GL_d(\mathbb{R})$. Thus, it follows that $d\left(a^T \Sigma_t a\right)$ is deterministic and it goes to positivity immediately, when the boundary is reached. (See [26]) Thus, the zero eigenvalue is brought to be positive and we get that the process $(\Sigma_t)_{t\geq 0}$ stays always in $\bar{S}_d^+(\mathbb{R})$.

Remark 2.1.3. *The Wishart dynamics can be extended by considering the processes that satisfy the following stochastic differential equation*

$$d\Sigma_t = \left(\Omega\Omega^T + M\Sigma_t + \Sigma_t M^T\right) dt + \Sigma_t^{1/2} dW_t^\sigma Q + Q^T \left(dW_t^\sigma\right)^T \Sigma_t^{1/2}, \quad \Sigma_0 = \sigma_0, \qquad (2.2)$$

in which $\Omega \in GL_d(\mathbb{R})$ and the constraint $\Omega^T\Omega = \beta Q Q^T$ is not imposed.

2.2 Existence and uniqueness theorems

As an extension of the findings by Bru [10, Theorem 2], the conditions for the existence of a unique weak solution to the SDE (2.1) are given by Gauthier [20] in the following theorem:

Theorem 2.2.1. *(Existence and Uniqueness of the Wishart Process I). For every initial value $\sigma_0 \in S_d^+(\mathbb{R})$ and $Q \in GL_d(\mathbb{R})$, $M \in \mathcal{M}_d(\mathbb{R})$, the Wishart stochastic differential equation (2.1) has a unique weak solution in $\bar{S}_d^+(\mathbb{R})$, if $\beta > d-1$.*

Proof. See [10]. □

In the following, we discuss when the SDE (2.1) possesses a unique strong solution. It is known that the matrix square root function, $f : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}, \Sigma_t \to \Sigma_t^{1/2}$ is locally Lipschitz on the set of symmetric, strictly positive definite matrices $S_d^+(\mathbb{R})$ [44, Theorem 12.12], i.e. for $\forall N \in \mathbb{N}^+$ there exists a constant $K_N > 0$ such that for $\|\Sigma_s\| \leq N$ and $\|\Sigma_t\| \leq N$:

$$\|\Sigma_t^{1/2} - \Sigma_s^{1/2}\| \leq K_N \|\Sigma_t - \Sigma_s\|, \quad s, t \in [0, T].$$

Then one gets that all the coefficients in (2.1) are locally Lipschitz, until Σ_t hits the boundary of $\bar{S}_d^+(\mathbb{R})$. One can also easily get that all the coefficients in (2.1) are of linear growth, i.e. $\exists K > 0$,

$$\|\beta Q^T Q + M\Sigma_t + \Sigma_t M^T\| + \|\Sigma_t^{1/2} Q\| + \|Q^T \Sigma_t^{1/2}\| \leq K\|1 + \Sigma_t\|$$

Thus, as a result in [39], the SDE (2.1) has a unique strong solution, until the process $(\Sigma_t)_{t\geq 0}$ hits the boundary at the first time. This yields the following theorem.

Theorem 2.2.2. *(Existence and Uniqueness of the Wishart Process II). For every initial value $\sigma_0 \in S_d^+$, there exists a unique strong solution Σ_t of the Wishart stochastic differential equation (2.1) in S_d^+ up to the stopping time*

$$T = \inf\{t \geq 0 : \det(\Sigma_t) = 0\} > 0 \quad a.s.$$

Proof. See [41, Theorem 4.11]. □

Since it is difficult to show that for $d \geq 2$ there still exists a unique strong solution after the process hits the boundary, it has been focused on the study of conditions such that the process $(\Sigma_t)_{t\geq 0}$ stays in the set of strictly positive definite matrices $S_d^+(\mathbb{R})$. The sufficient conditions for it are given by the following theorem in [39]. In this way, one gets the unique strong solution for all $t \geq 0$.

Theorem 2.2.3. *(Existence and Uniqueness of the Wishart Process III). Let $B \in S_d(\mathbb{R})$ be a symmetric matrix, $Q \in \mathcal{M}_d(\mathbb{R})$, $M \in \mathcal{M}_d(\mathbb{R})$. If $B \succeq (d+1) Q^T Q$, then the following SDE*

$$d\Sigma_t = \left(B + M\Sigma_t + \Sigma_t M^T\right) dt + \Sigma_t^{1/2} dW_t^\sigma Q + Q^T \left(dW_t^\sigma\right)^T \Sigma_t^{1/2}, \quad \Sigma_0 = \sigma_0$$

has a unique strong solution on $S_d^+(\mathbb{R})$ and there is

$$T = \inf\{t \geq 0 : \det(\Sigma_t) = 0\} = \infty \quad a.s.$$

Proof. See [39, Theorem 2.2]. □

Remark 2.2.4. *Note that for $B = \beta Q^T Q$, $\beta > 0$, the condition for the existence of a unique strong solution to (2.1) for all $t \geq 0$ is $\beta \geq d+1$. For $B = \Omega\Omega^T$ as in (2.2), the condition is $\Omega\Omega^T \succeq (d+1) Q^T Q$.*

Remark 2.2.5. *For $d = 1$, Theorem 2.2.3 gives us a sufficient and necessary condition for the existence of a unique strong solution of the SDE (2.1). For $d \geq 2$, whether the condition $B \succeq (d+1) Q^T Q$ is necessary is still an open problem (See [39, section 5]).*

2.3 Construction of Wishart process

In the case that β is an integer, the following proposition links Ornstein-Uhlenbeck and Wishart processes.

Proposition 2.3.1. *Let $\beta \geq d+1$ be an integer, $Q \in GL_d(\mathbb{R})$, $M \in \mathcal{M}_d(\mathbb{R})$, $\sigma_0 \in S_d^+$ and let $\{X_{k,t}, t \geq 0\}$, $1 \leq k \leq \beta$ be independent vectorial Ornstein- Uhlenbeck processes in $\mathbb{R}^d$ with dynamic:*

$$dX_{k,t} = MX_{k,t}dt + Q^T dW_{k,t},$$

where $\{W_{k,t}, t \geq 0, 1 \leq k \leq \beta\}$ are independent vectorial Brownian motions. For $X_{k,0} = x_k$, $1 \leq k \leq \beta$, there is $\sum_{k=1}^{\beta} x_k x_k^T = \sigma_0$. Then

$$Y_t = \sum_{k=1}^{\beta} X_{k,t} X_{k,t}^T,$$

is the unique strong solution of dynamic (2.1).

Proof. See [41, Theorem 4.19]. □

The construction of Wishart process can be used as another way to define Wishart process (See [25, Definition 1] for details).

Remark 2.3.2. *One can also define Wishart process through its Laplace transforms, in this way the degrees of freedom β can be extended to be fractional.*

2.4 The affine property

Note that the dynamic of $(\Sigma_t)_{t\geq 0}$ in (2.1) admits drift and volatility functions which are affine functions of (Σ_t). Hence, the Wishart process (Σ_t) is an affine process with characteristic function and Laplace transform that are exponential affine dependent on (Σ_t). For the definition and analysis of affine processes, we refer to [18, 19].

The affine property of the Wishart process has been widely explored in the literature. The characteristic function and Laplace transform of Wishart and integrated Wishart processes are derived in e.g. [10, 20, 23]. Gnoatto and Grasselli have extended the original approach in [10] and shown in [22] the explicit formula for the joint Laplace transform of the Wishart process and its time integral. Moreover, it has been shown in [26] that the joint process of (Σ_t) and a given stock price is also an affine process. Since affine processes are generally introduced for the dynamics of interest rates and the so-called affine term structure models, the wishart affine property is also extensively studied in the setting of a Wishart affine term structure model (see e.g. [24, 25]).

Proposition 2.4.1. *(Characteristic function). Let $\Theta \in S_d(\mathbb{R})$ and $t, h \geq 0$ so that $I_d - 2iV(h)\Theta \in GL_d(\mathbb{C})$, where $I_d \in \mathbb{R}^{d\times d}$ is the identity matrix and $V(h)$ is defined below. Let $(\Sigma_t)_{t\geq 0}$ be the Wishart process solving (2.1), then, the characteristic function of Σ_{t+h} given $\Sigma_t = \Sigma \in S_d^+$ is:*

$$E^{t,\Sigma}\left[e^{iTr(\Theta\Sigma_{t+h})}\right] = \frac{\exp\left(iTr\left[\Theta\left(I_d - 2iV(h)\Theta\right)^{-1}\Delta(h)\Sigma\Delta(h)^T\right]\right)}{\left(det[I_d - 2iV(h)\Theta]\right)^{\beta/2}},$$

where

$$\Delta(h) = e^{hM}$$
$$V(h) = \int_0^h e^{sM}Q^TQe^{sM^T}ds.$$

Proof. See [20, Proposition 4]. □

Proposition 2.4.2. *(Laplace transform). Let $\Theta \in S_d(\mathbb{R})$ and $t, h \geq 0$ so that $I_d + 2V(h)\Theta \in GL_d(\mathbb{R})$. Let $(\Sigma_t)_{t\geq 0}$ be the Wishart process solving (2.1), then, the Laplace transform of Σ_{t+h} given $\Sigma_t = \Sigma \in S_d^+$ is:*

$$E^{t,\Sigma}\left[e^{-Tr(\Theta\Sigma_{t+h})}\right] = \frac{\exp\left(-Tr\left[\Theta\left(I_d + 2V(h)\Theta\right)^{-1}\Delta(h)\Sigma\Delta(h)^T\right]\right)}{\left(det[I_d + 2V(h)\Theta]\right)^{\beta/2}}, \tag{2.3}$$

where $\Delta(h)$ and $V(h)$ are defined as in Proposition 2.4.1.

Proof. See [23, Proposition 5] □

One can define a Wishart process through its Laplace transform. Thus, the result in Proposition 2.4.2 allows us to extend the definition of Wishart distributions to real values of β as long as $\beta > d-1$.

The joint Laplace transform of Σ_{t+h}, $h \geq 0$ and the integrated Wishart process, i.e.

$$E^{t,\Sigma}\left[\exp\left(Tr\left(C_1\Sigma_{t+h}+\int_t^{t+h} C_2\Sigma_u du\right)\right)\right] \tag{2.4}$$

is also exponential affine dependent on $\Sigma_t = \Sigma \in S_d^+$, where C_1 and C_2 are symmetric matrices for which the expression (2.4) makes sense.

There are several ways to get the explicit expression for (2.4). The first approach (the so called "matrix Cameron-Martin formula") is originally proposed by Bru in [10] and is then extended by Gnoatto and Grasselli in [22]. The following matrix Cameron-Martin formula can be found in [22]. To lighten some notations below, we define two matrix hyperbolic functions at first:

$$\cosh(A) = \frac{e^A + e^{-A}}{2}, \ \sinh(A) = \frac{e^A - e^{-A}}{2}, \quad A \in S_d(\mathbb{R}).$$

Theorem 2.4.3. *Let $(\Sigma_t)_{t\geq 0}$ be the Wishart process solving (2.1) with $Q \in GL_d(\mathbb{R})$, $M \in \mathcal{M}_d(\mathbb{R})$, $\beta \geq d+1$. Conditioned on $\Sigma_t = \Sigma \in S_d^+$, we denote the set of convergence of the Laplace transform (2.4) by $\mathcal{D}_h$. Assume*

$$M^T(Q^TQ)^{-1} = (Q^TQ)^{-1}M,$$

then (2.4) is given by:

$$\begin{aligned}
&E^{t,\Sigma}\left[\exp\left(Tr\left(C_1\Sigma_{t+h}+\int_t^{t+h} C_2\Sigma_u du\right)\right)\right]\\
&=\det\left(e^{-Mh}\left(\cosh\left(\sqrt{\tilde{C}_2}h\right)+\sinh\left(\sqrt{\tilde{C}_2}h\right)\kappa(h)\right)\right)^{\frac{\beta}{2}}\\
&\cdot\exp\left\{Tr\left[\left(\frac{Q^{-1}\sqrt{\tilde{C}_2}\kappa(h)Q^{-T}}{2}-\frac{M^T\left(Q^TQ\right)^{-1}}{2}\right)\Sigma\right]\right\},
\end{aligned}$$

where the matrices κ, $\tilde{C}_2$, $\tilde{C}_1$ are given by:

$$\begin{aligned}
\kappa(h) &= -\left(\sqrt{\tilde{C}_2}\cosh\left(\sqrt{\tilde{C}_2}h\right)+\tilde{C}_1\sinh\left(\sqrt{\tilde{C}_2}h\right)\right)^{-1}\cdot\\
&\quad\left(\sqrt{\tilde{C}_2}\sinh\left(\sqrt{\tilde{C}_2}h\right)+\tilde{C}_1\cosh\left(\sqrt{\tilde{C}_2}h\right)\right),\\
\tilde{C}_2 &= Q\left(-2C_2+M^TQ^{-1}Q^{-T}M\right)Q^T,\\
\tilde{C}_1 &= -Q\left(2C_1+\left(Q^TQ\right)^{-1}M\right)Q^T.
\end{aligned}$$

Moreover,

$$\mathcal{D}_h = \left\{(C_1, C_2) \in S_d : \sqrt{\tilde{C}_2}\cosh\left(\sqrt{\tilde{C}_2}s\right)+\tilde{C}_1\sinh\left(\sqrt{\tilde{C}_2}s\right) \in GL_d, \ \forall s \in [0,h]\right\}.$$

Note that the forth expression for (2.4) can be written as an exponential affine function:

$$E^{t,\Sigma}\left[\exp\left(Tr\left(C_1\Sigma_{t+h}+\int_t^{t+h}C_2\Sigma_u du\right)\right)\right]=\exp\left(\phi\left(h\right)+Tr[\psi\left(h\right)\Sigma]\right),$$

where the functions ϕ and ψ are given by:

$$\begin{aligned}\psi\left(h\right)&=\frac{Q^{-1}\sqrt{\tilde{C}_2}\kappa\left(h\right)Q^{-T}}{2}-\frac{M^T\left(Q^TQ\right)^{-1}}{2}\\ \phi\left(h\right)&=\frac{\beta}{2}\log\left(\det\left(e^{-Mh}\left(\cosh\left(\sqrt{\tilde{C}_2}h\right)+\sinh\left(\sqrt{\tilde{C}_2}h\right)\kappa\right)\right)\right).\end{aligned}$$

Proof. See [22, Theorem 1]. □

Remark 2.4.4. *a) Note that the condition $M^T(Q^TQ)^{-1}=(Q^TQ)^{-1}M$ is more general than the commutativity assumption $MQ=QM$ in [10]. For a discussion see [22].*

b) For the case of C_1 and C_2 being $d\times d$ zero matrices, we receive directly

$$E^{t,\Sigma}\left[\exp\left(Tr\left(C_1\Sigma_{t+h}+\int_t^{t+h}C_2\Sigma_u du\right)\right)\right]=1$$

for arbitrary $M\in\mathbb{R}^{d\times d}$, $Q\in\mathbb{R}^{d\times d}$. But from our theorem above we do not have $(C_1,C_2)=(\mathbf{0},\mathbf{0})\in\mathcal{D}_h$, $\forall M,Q\in\mathbb{R}^{d\times d}$, where $\mathbf{0}$ denotes the $d\times d$ zero matrix. It means that for $(\mathbf{0},\mathbf{0})\in\mathcal{D}_h$, it follows $\phi(h)=0$ and $\psi(h)=\mathbf{0}$ by our theorem. But in the case of $C_1=C_2=\mathbf{0}$, we get $E^{t,\Sigma}\left[\exp\left(Tr\left(C_1\Sigma_{t+h}+\int_t^{t+h}C_2\Sigma_u du\right)\right)\right]=1$ also for $(\mathbf{0},\mathbf{0})\notin\mathcal{D}_h$.

c)The matrix $\psi\left(h\right)$ is a symmetric matrix. To verify this property, one only needs to check if $\sqrt{\tilde{C}_2}\kappa\left(h\right)$ equals $\kappa^T\left(h\right)\sqrt{\tilde{C}_2}$. Employing

$$\left(\sqrt{\tilde{C}_2}\right)^{-1}e^{\sqrt{\tilde{C}_2}h}=e^{\sqrt{\tilde{C}_2}h}\left(\sqrt{\tilde{C}_2}\right)^{-1},$$

one gets $\sqrt{\tilde{C}_2}\kappa\left(h\right)=\kappa^T\left(h\right)\sqrt{\tilde{C}_2}$ by direct calculation.

Corollary 2.4.5. *Let $\left(\Sigma_t\right)_{t\geq 0}$ be the generalized Wishart process solving (2.2) with $\Omega\in GL_d\left(\mathbb{R}\right)$ and $M\in\mathcal{M}_d\left(\mathbb{R}\right)$. Then (2.4) is given explicitly by:*

$$E^{t,\Sigma}\left[\exp\left(Tr\left(C_1\Sigma_{t+h}+\int_t^{t+h}C_2\Sigma_u du\right)\right)\right]=\exp\left(\phi\left(h\right)+Tr\left(\psi\left(h\right)\Sigma\right)\right)$$

with the same $\mathcal{D}_h$, $\kappa(h)$, $\tilde{C}_2$, $\tilde{C}_1$ and $\psi(h)$ as in Theorem 2.4.3 and the following new $\phi(h)$:

$$\begin{aligned}\phi\left(h\right)=&-Tr\left(\Omega\Omega^T\frac{M^T\left(Q^TQ\right)^{-1}}{2}\right)h\\ &-\frac{1}{2}Tr\left(Q^{-T}\Omega\Omega^TQ^{-1}\log\left[\left(\sqrt{\tilde{C}_2}\right)^{-1}\left(\sqrt{\tilde{C}_2}\cosh\left(\sqrt{\tilde{C}_2}h\right)+\tilde{C}_1\sinh\left(\sqrt{\tilde{C}_2}h\right)\right)\right]\right).\end{aligned}$$

Proof. For the proof we refer to [22, Theorem 11]. □

Denote $K(s) = \sqrt{\tilde{C}_2}\cosh\left(\sqrt{\tilde{C}_2}s\right) + \tilde{C}_1\sinh\left(\sqrt{\tilde{C}_2}s\right)$, then $\mathcal{D}_h$ contains all $C_1 \in S_d$ and $C_2 \in S_d$, for which $K(s) \in GL_d(\mathbb{R})$, $\forall s \in [0, h]$. But sometimes we may be more interested in the set

$$\mathcal{D} = \left\{(C_1, C_2) \in S_d : K(s) \in GL_d,\ \forall s \geq 0\right\},$$

which is a subset of $\mathcal{D}_h$. To determine whether $(C_1,\ C_2) \in \mathcal{D}$ or not, we apply the following proposition.

Proposition 2.4.6. *$(C_1,\ C_2) \in \mathcal{D}$, i.e. $K(h) \in GL_d(\mathbb{R})$, $\forall h \geq 0$, if and only if*

$$Q^TQ \in GL_d(\mathbb{R}), \quad -2C_2 + M^TQ^{-1}Q^{-T}M \succ 0 \quad \textit{and} \quad \sqrt{\tilde{C}_2} + \tilde{C}_1 \succeq 0. \tag{2.5}$$

Proof. First of all, for the well-definedness of matrix square root of $\tilde{C}_2$, we need $\tilde{C}_2$ to be symmetric nonnegative definite, which is equivalent to

$$-2C_2 + M^TQ^{-1}Q^{-T}M \succeq 0. \tag{2.6}$$

Consider the situation $h = 0$, one gets $K(0) = \sqrt{\tilde{C}_2}$. Then $K(0) \in GL_d(\mathbb{R})$, if and only if $-2C_2 + M^TQ^{-1}Q^{-T}M \succ 0$.

Subsequently, we show the sufficiency of the conditions in (2.5) for $\forall h > 0$. For the sake of simplicity, we denote

$$K(h) := \frac{1}{2}\left(\sqrt{\tilde{C}_2} + \tilde{C}_1\right)e^{\sqrt{\tilde{C}_2}h} + \frac{1}{2}\left(\sqrt{\tilde{C}_2} - \tilde{C}_1\right)e^{-\sqrt{\tilde{C}_2}h}. \tag{2.7}$$

Note that for $\sqrt{\tilde{C}_2} \in S_d^+(\mathbb{R})$ and $h > 0$, there is $e^{\sqrt{\tilde{C}_2}h} \in S_d^+(\mathbb{R})$ and $e^{-\sqrt{\tilde{C}_2}h} = \left(e^{\sqrt{\tilde{C}_2}h}\right)^{-1} \in S_d^+(\mathbb{R})$. Moreover, expanding the matrix exponential functions as series, one gets $e^{\sqrt{\tilde{C}_2}h} - e^{-\sqrt{\tilde{C}_2}h} \succ 0$ for $\sqrt{\tilde{C}_2} \in S_d^+(\mathbb{R})$, $\forall h > 0$. Hence, one can write $e^{\sqrt{\tilde{C}_2}h} = e^{-\sqrt{\tilde{C}_2}h} + P(h)$ with some $P(h) \in S_d^+(\mathbb{R})$. Then there is

$$\begin{aligned}
K(h) =& \frac{1}{2}\left(\sqrt{\tilde{C}_2} + \tilde{C}_1\right)\left(e^{-\sqrt{\tilde{C}_2}h} + P(h)\right) + \frac{1}{2}\left(\sqrt{\tilde{C}_2} - \tilde{C}_1\right)e^{-\sqrt{\tilde{C}_2}h} \\
=& \sqrt{\tilde{C}_2}e^{-\sqrt{\tilde{C}_2}h} + \frac{1}{2}\left(\sqrt{\tilde{C}_2} + \tilde{C}_1\right)P(h),
\end{aligned}$$

which is always invertible if and only if $\det(K(h)) \neq 0$, $\forall h > 0$. Note that $\det(K(h)P^{-1}(h)) = \det(K(h))\det(P^{-1}(h))$ and $\det(P^{-1}(h)) > 0$, $\forall h > 0$, since $P(h) \in S_d^+(\mathbb{R})$ for all positive h, one gets $K(h) \in GL_d$ if and only if $\det(K(h)P^{-1}(h)) \neq 0$, $\forall h > 0$. Because of

$$\begin{aligned}
K(h)P^{-1}(h) =& \sqrt{\tilde{C}_2}e^{-\sqrt{\tilde{C}_2}h}\left(e^{\sqrt{\tilde{C}_2}h} - e^{-\sqrt{\tilde{C}_2}h}\right)^{-1} + \frac{1}{2}\left(\sqrt{\tilde{C}_2} + \tilde{C}_1\right) \\
=& \left(e^{2\sqrt{\tilde{C}_2}h}\left(\sqrt{\tilde{C}_2}\right)^{-1} - \left(\sqrt{\tilde{C}_2}\right)^{-1}\right)^{-1} + \frac{1}{2}\left(\sqrt{\tilde{C}_2} + \tilde{C}_1\right) \\
=& \underbrace{\left(2hI + 2h^2\sqrt{\tilde{C}_2} + \frac{4h^3}{3}\tilde{C}_2 + \dots\right)^{-1}}_{\succ 0} + \frac{1}{2}\left(\sqrt{\tilde{C}_2} + \tilde{C}_1\right),
\end{aligned}$$

it follows $\det\left(K(h)P^{-1}(h)\right)\neq 0$, $\forall h>0$ from $\sqrt{\tilde{C}_2}+\tilde{C}_1\succeq 0$.

Eventually, we show the necessity of the conditions. We assume $\sqrt{\tilde{C}_2}+\tilde{C}_1\not\succeq 0$. Note that for $h=0$, $K(C_1,C_2,0)=\sqrt{\tilde{C}_2}\succ 0$, which implies that all the eigenvalues of $K(C_1,C_2,0)$ are positive. If $\sqrt{\tilde{C}_2}+\tilde{C}_1\not\succeq 0$, i.e. $\sqrt{\tilde{C}_2}+\tilde{C}_1$ possesses at least one negative eigenvalue, one can identify the matrix $\left(\sqrt{\tilde{C}_2}+\tilde{C}_1\right)e^{\sqrt{\tilde{C}_2}h}$ owns also at least one negative eigenvalue through a matrix similarity transformation:

$$\sigma\left(\left(\sqrt{\tilde{C}_2}+\tilde{C}_1\right)e^{\sqrt{\tilde{C}_2}h}\right)=\sigma\left(\left(e^{\sqrt{\tilde{C}_2}h}\right)^{1/2}\left(\sqrt{\tilde{C}_2}+\tilde{C}_1\right)\left(e^{\sqrt{\tilde{C}_2}h}\right)^{1/2}\right).$$

Then for h large enough, it follows that $K(h)$ in (2.7) owns at least one negative eigenvalue. Since the spectrum of a matrix is a continuous function on the entries of the matrix [42], we conclude that $\exists h>0$ with $K(h)\notin GL_d(\mathbb{R})$, if $\sqrt{\tilde{C}_2}+\tilde{C}_1\not\succeq 0$. □

Remark 2.4.7. *Consider a special case of $\tilde{C}_1$ with $C_1=0$, if*

$$-M^T\left(Q^TQ\right)^{-1}\in S_d^+,$$

then $-2C_2+M^TQ^{-1}Q^{-T}M\succ 0$ implies $\sqrt{\tilde{C}_2}+\tilde{C}_1\succeq 0$ automatically, which means that we only need to check whether $-2C_2+M^TQ^{-1}Q^{-T}M\succ 0$. Note that a necessary condition for $-\left(Q^TQ\right)^{-1}M\in S_d^+$ is $Re\left(\sigma\left(M^T\left(Q^TQ\right)^{-1}\right)\right)\subseteq(-\infty,0)$.

Proposition 2.4.8. *Assume that the conditions in (2.5) hold and $\sqrt{\tilde{C}_2}+\tilde{C}_1\in GL_d(\mathbb{R})$, $\sqrt{\tilde{C}_2}-\tilde{C}_1\in GL_d(\mathbb{R})$, then, we have*

$$\sqrt{\tilde{C}_2}\kappa(h)\prec\sqrt{\tilde{C}_2},\quad\forall h\geq 0.$$

Proof. We first write the matrix function $\kappa(h)$ explicitly as:

$$\begin{aligned}
\kappa(h)=&-\left(\sqrt{\tilde{C}_2}\cosh\left(\sqrt{\tilde{C}_2}h\right)+\tilde{C}_1\sinh\left(\sqrt{\tilde{C}_2}h\right)\right)^{-1}\\
&\left(\sqrt{\tilde{C}_2}\sinh\left(\sqrt{\tilde{C}_2}h\right)+\tilde{C}_1\cosh\left(\sqrt{\tilde{C}_2}h\right)\right)\\
=&-\left(\sqrt{\tilde{C}_2}\left(e^{\sqrt{\tilde{C}_2}h}+e^{-\sqrt{\tilde{C}_2}h}\right)+\tilde{C}_1\left(e^{\sqrt{\tilde{C}_2}h}-e^{-\sqrt{\tilde{C}_2}h}\right)\right)^{-1}\\
&\left(\sqrt{\tilde{C}_2}\left(e^{\sqrt{\tilde{C}_2}h}-e^{-\sqrt{\tilde{C}_2}h}\right)+\tilde{C}_1\left(e^{\sqrt{\tilde{C}_2}h}+e^{-\sqrt{\tilde{C}_2}h}\right)\right)\\
=&-\left(\left(\sqrt{\tilde{C}_2}+\tilde{C}_1\right)e^{\sqrt{\tilde{C}_2}h}+\left(\sqrt{\tilde{C}_2}-\tilde{C}_1\right)e^{-\sqrt{\tilde{C}_2}h}\right)^{-1}\left(\sqrt{\tilde{C}_2}+\tilde{C}_1\right)e^{\sqrt{\tilde{C}_2}h}\\
&-\left(\left(\sqrt{\tilde{C}_2}+\tilde{C}_1\right)e^{\sqrt{\tilde{C}_2}h}+\left(\sqrt{\tilde{C}_2}-\tilde{C}_1\right)e^{-\sqrt{\tilde{C}_2}h}\right)^{-1}\left(\tilde{C}_1-\sqrt{\tilde{C}_2}\right)e^{-\sqrt{\tilde{C}_2}h}.
\end{aligned}$$

Denote the identity matrix by I, we get then

$$\kappa(h) = -\Bigg(I + \underbrace{e^{-\sqrt{\tilde{C}_2}h}\left(\sqrt{\tilde{C}_2}+\tilde{C}_1\right)^{-1}\left(\sqrt{\tilde{C}_2}-\tilde{C}_1\right)e^{-\sqrt{\tilde{C}_2}h}}_{=A_1(h)}\Bigg)^{-1}$$
$$+\Bigg(\underbrace{e^{\sqrt{\tilde{C}_2}h}\left(\sqrt{\tilde{C}_2}-\tilde{C}_1\right)^{-1}\left(\sqrt{\tilde{C}_2}+\tilde{C}_1\right)e^{\sqrt{\tilde{C}_2}h}}_{=A_2(h)}+I\Bigg)^{-1}.$$

The fact that the matrix $I + A_1(h)$ and $I + A_2(h)$ are invertible for $h \geq 0$ follows from the well-definedness of $\kappa(h)$ for $h \geq 0$ and our assumptions $\sqrt{\tilde{C}_2} + \tilde{C}_1 \in GL_d(\mathbb{R})$ as well as $\sqrt{\tilde{C}_2} - \tilde{C}_1 \in GL_d(\mathbb{R})$. Note that $A_1(h)\,A_2(h) = I$, thus, we get

$$\begin{aligned}
\kappa(h) &= -\left(A_1(h)\,A_2(h) + A_1(h)\right)^{-1} + \left(I + A_2(h)\right)^{-1} \\
&= -\left(A_2(h) + I\right)^{-1}\left(A_1(h)\right)^{-1} + \left(I + A_2(h)\right)^{-1} = \left(I + A_2(h)\right)^{-1}\left(I - A_2(h)\right) \\
&= \left(I + A_2(h)\right)^{-1}\left(I + A_2(h) - 2A_2(h)\right) = I - 2\left(I + A_1(h)\right)^{-1}.
\end{aligned}$$

Then it yields

$$\begin{aligned}
&\sqrt{\tilde{C}_2}\kappa(h) \\
=&\sqrt{\tilde{C}_2} - 2\sqrt{\tilde{C}_2}\left(I + A_1(h)\right)^{-1} \\
=&\sqrt{\tilde{C}_2} - 2\left(\left(I + A_1(h)\right)\left(\sqrt{\tilde{C}_2}\right)^{-1}\right)^{-1} \\
=&\sqrt{\tilde{C}_2} - 2\left(\left(\sqrt{\tilde{C}_2}\right)^{-1} + e^{-\sqrt{\tilde{C}_2}h}\left(\sqrt{\tilde{C}_2}+\tilde{C}_1\right)^{-1}\left(\sqrt{\tilde{C}_2}-\tilde{C}_1\right)\left(\sqrt{\tilde{C}_2}\right)^{-1}e^{-\sqrt{\tilde{C}_2}h}\right)^{-1} \\
=&\sqrt{\tilde{C}_2} - 2\left(e^{-\sqrt{\tilde{C}_2}h}A_3(h)e^{-\sqrt{\tilde{C}_2}h}\right)^{-1}
\end{aligned} \tag{2.8}$$

with

$$A_3(h) = \left(\sqrt{\tilde{C}_2}\right)^{-1}e^{2\sqrt{\tilde{C}_2}h} + \left(\sqrt{\tilde{C}_2}+\tilde{C}_1\right)^{-1}\left(\sqrt{\tilde{C}_2}-\tilde{C}_1\right)\left(\sqrt{\tilde{C}_2}\right)^{-1}.$$

Note that $A_3(h)$ is always symmetric invertible for $h \geq 0$ under our assumptions in the proposition. Furthermore, there is $A_3(h_1) \succ A_3(h_2)$ for $h_1 > h_2 \geq 0$, since

$$A_3(h_1) - A_3(h_2) = \left(\sqrt{\tilde{C}_2}\right)^{-1}\left(e^{2\sqrt{\tilde{C}_2}h_1} - e^{2\sqrt{\tilde{C}_2}h_2}\right) \succ 0$$

for $h_1 > h_2 \geq 0$. For $h = 0$, we have

$$\begin{aligned}
&A_3(0)\\
=&\left(\sqrt{\tilde{C}_2}\right)^{-1} + \left(\sqrt{\tilde{C}_2}+\tilde{C}_1\right)^{-1}\left(\sqrt{\tilde{C}_2}-\tilde{C}_1\right)\left(\sqrt{\tilde{C}_2}\right)^{-1}\\
=&\left(\sqrt{\tilde{C}_2}+\tilde{C}_1\right)^{-1}\left(\sqrt{\tilde{C}_2}+\tilde{C}_1\right)\left(\sqrt{\tilde{C}_2}\right)^{-1} + \left(\sqrt{\tilde{C}_2}+\tilde{C}_1\right)^{-1}\left(\sqrt{\tilde{C}_2}-\tilde{C}_1\right)\left(\sqrt{\tilde{C}_2}\right)^{-1}\\
=&2\left(\sqrt{\tilde{C}_2}+\tilde{C}_1\right)^{-1} \succ 0.
\end{aligned}$$

It implies then $A_3(h) \succ 0$ for all $h \geq 0$. Thus, we get our assertion from (2.9). □

Remark 2.4.9. *For $C_1 = 0$, Proposition 2.4.8 implies that*

$$\psi(h) = \frac{Q^{-1}\left(\sqrt{\tilde{C}_2}\kappa(h)+\tilde{C}_1\right)Q^{-T}}{2} \prec \frac{Q^{-1}\left(\sqrt{\tilde{C}_2}+\tilde{C}_1\right)Q^{-T}}{2}$$

on $[0,\infty)$, if the conditions in Proposition 2.4.8 are satisfied.

In [26], Gourieroux and Sufana have proposed a second approach to get the explicit exponential affine representation of (2.4). In this approach, the parameters are determined by a nonlinear matrix Riccati ODE.

Proposition 2.4.10. *Let $(\Sigma_t)_{t\geq 0}$ be the Wishart process solving (2.2) with $Q \in GL_d(\mathbb{R})$, $M \in \mathcal{M}_d(\mathbb{R})$, $\beta \geq d+1$. Conditioned on $\Sigma_t = \Sigma \in S_d^+$, the conditional Laplace transform (2.4) can be written as:*

$$E^{t,\Sigma}\left[\exp\left(Tr\left(C_1\Sigma_{t+h} + \int_t^{t+h} C_2\Sigma_u du\right)\right)\right] = \exp\left(\phi(h) + Tr[\psi(h)\Sigma]\right),$$

where

$$\frac{d\psi(h)}{dh} = \psi(h)M + M^T\psi(h) + 2\psi(h)Q^TQ\psi(h) + C_2, \tag{2.9}$$

$$\frac{d\phi(h)}{dh} = Tr[\psi(h)\Omega\Omega^T], \tag{2.10}$$

with initial conditions: $\psi(0) = C_1, \phi(0) = 0$. The closed-form solution for $\psi(h)$ is:

$$\begin{aligned}
\psi(h) = \psi^* &+ exp[\left(M+2Q^TQ\psi^*\right)h]^T\\
&\left\{(C_1-\psi^*)^{-1} + 2\int_0^h exp[\left(M+2Q^TQ\psi^*\right)u]Q^TQexp[\left(M+2Q^TQ\psi^*\right)u]^T du\right\}^{-1}\\
&exp[\left(M+2Q^TQ\psi^*\right)h],
\end{aligned}$$

where ψ^* is a real symmetric matrix which satisfies:

$$M^T\psi^* + \psi^*M + 2\psi^*Q^TQ\psi^* + C_2 = 0. \tag{2.11}$$

The closed-form solution for $\phi(h)$ is immediately deduced from the second differential equation:

$$\phi(h) = Tr\left[\int_0^h \psi(u)\,du\Omega\Omega^T\right].$$

Remark 2.4.11. *In [23, Appendix 4], Gourieroux wrote (2.11) as*

$$\left[\left(2Q^TQ\right)^{1/2}\psi^* + \left(2Q^TQ\right)^{-1/2}M\right]^T\left[\left(2Q^TQ\right)^{1/2}\psi^* + \left(2Q^TQ\right)^{-1/2}M\right] + C_2 - M^T\left(2Q^TQ\right)^{-1}M = 0$$

and it delivers a necessary condition for the existence of a solution ψ^* to equation (2.11):

$$C_2 \preceq M^T\left(2Q^TQ\right)^{-1}M.$$

Note that the necessity of this condition coincides with our previous statement (2.6).

To get the solutions of the ODEs system, Gourieroux solved the equation (2.9) for $C_2 = 0$ at first and then provided the general solution. Hence this approach is called the variation of constants method by Gnoatto and Grasselli in [22]. In [22], Gnoatto and Grasselli still introduced the third approach, i.e. the method of linearization of the matrix Riccati ODE, which is originally proposed by Grasselli and Tebaldi in [28]. For further details and the comparison of these methods, we refer to [22]. Thanks to Proposition 2.4.6, we have the conditions for the existence and uniqueness of the ODE system in Proposition 2.4.10. For readers who are interested in the existence and uniqueness conditions for a generalized matrix Riccati ODEs, we refer to [43].

2.5 Wishart process and Wishart distribution

In this section we consider the relationship between Wishart Processes and Wishart Distributions. We first introduce the definition of a non-central Wishart distribution:

Definition 2.5.1. *Consider β independent random vectors $X_1, \dots, X_\beta$ with values in $\mathbb{R}^d$, which are distributed according to a multivariate Gaussian distribution $N(0, V)$. Let $\mu_1, \dots, \mu_\beta$ be β non-random vectors in $\mathbb{R}^d$. The distribution of*

$$W = \sum_{i=1}^{\beta}(X_i + \mu_i)(X_i + \mu_i)^T,$$

is a non-central Wishart distribution with β degrees of freedom, denoted by $\mathcal{W}_\beta(\mu, V)$ where $\mu = \sum_{i=1}^{\beta}\mu_i\mu_i^T$.

It is known that the Laplace transform of the Wishart distribution $W \sim \mathcal{W}_\beta(\mu, V)$ is

$$E\left[e^{-Tr(\Theta W)}\right] = \frac{\exp\left(-Tr\left[\Theta\left(I_d + 2V\Theta\right)^{-1}\mu\right]\right)}{\left(det[I_d + 2V\Theta]\right)^{\beta/2}} \tag{2.12}$$

for $\Theta \in S_d(\mathbb{R})$ with $I_d + 2V\Theta \in GL_d(\mathbb{R})$. Comparing (2.3) and (2.12), one gets that the conditional distribution of Σ_t is a non-central Wishart distribution with bounded conditional moments (see [20, Theorem 4]).

Theorem 2.5.2. *If the process $(\Sigma_t)_{t\geq 0}$ has the dynamic (2.1), then conditional on Σ_t, Σ_{t+h} has the distribution $\mathcal{W}_\beta(\mu_t(h), V(h))$ where*

$$\mu_t(h) = e^{hM}\Sigma_t e^{hM^T}$$

$$V(h) = \int_0^h e^{sM}Q^TQe^{sM^T}ds.$$

The moments of a noncentral Wishart distribution can be classically obtained by differentiating its Laplace transform, then as a consequence of Theorem 2.5.2, the first two conditional moments of a Wishart process are given as follows (see [20, Proposition 9]):

Proposition 2.5.3. *If the process $(\Sigma_t)_{t\geq 0}$ has the dynamic (2.1), then we have $\forall 1 \leq i,j,k,l \leq d$*

$$\begin{aligned} E\left(\Sigma_{t+h}^{ij}\Sigma_{t+h}^{kl}|\Sigma_t\right) &= \mu_t(h)^{ij}\mu_t(h)^{kl} + \mu_t(h)^{il}V(h)^{kj} + \mu_t(h)^{ik}V(h)^{jl} \\ &+ \mu_t(h)^{jl}V(h)^{ik} + \mu_t(h)^{kj}V(h)^{il} + V(h)^{ij}V(h)^{kl}\beta^2 \\ &+ \left(V(h)^{il}V(h)^{kj} + V(h)^{ik}V(h)^{jl} + \mu_t(h)^{ij}V(h)^{kl} + \mu_t(h)^{kl}V(h)^{ij}\right)\beta, \end{aligned}$$

$$\begin{aligned} \beta\cdot Var[\Sigma_{t+h}|\Sigma_t] =& \left(\mu_t(h) + \beta V(h)\right)^2 - \mu_t(h)^2 + \left(\mu_t(h) + \beta\Sigma(h)\right)Tr\left(\mu_t(h) + \beta V(h)\right) \\ &- \mu_t(h)\,Tr\left(\mu_t(h)\right), \\ E[\Sigma_{t+h}|\Sigma_t] =& \mu_t(h) + \beta V(h), \end{aligned}$$

where

$$Var[\Sigma_{t+h}|\Sigma_t] := E\left[\left(\Sigma_{t+h} - E[\Sigma_{t+h}|\Sigma_t]\right)^2|\Sigma_t\right].$$

For the general moments of a conditional central and noncentral Wishart distribution we refer to e.g. [15, 45]. An analysis of the moments of Wishart processes and some functionals of Wishart processes can be found in [27].

2.6 Some other properties of Wishart processes

Lemma 2.6.1. *(Quadratic Variation of Wishart Processes). Let $(\Sigma_t)_{t\geq 0}$ be a strong solution of SDE (2.1) on $[0,T)$, where $T = \inf\{t \geq 0 : \det(\Sigma_t) = 0\} > 0$ a.s., then*

$$\frac{d\langle\Sigma_{lk},\Sigma_{pq}\rangle_t}{dt} = \Sigma_{lp}\left(Q^TQ\right)_{kq} + \Sigma_{pk}\left(Q^TQ\right)_{ql} + \Sigma_{lq}\left(Q^TQ\right)_{kp} + \Sigma_{kq}\left(Q^TQ\right)_{lp}.$$

As a special case of [39, Lemma 4.7], one gets the following Itô Formula, from which the Itô Formula for Wishart process follows directly:

Lemma 2.6.2. *(Itô's Formula). Let $(X_t)_{t\geq 0}$ be an S_d^+ -valued continuous semimartingale on the stochastic interval $[0,T)$ with $T = \inf\{t \geq 0 : \det(\Sigma_t) = 0\} > 0$ a.s. and $f : S_d^+ \to \mathbb{R}$ a twice continuously differentiable function, then $f(X_t)$ is a semimartingale on $[0,T)$ and*

$$f(X_t) = f(X_0) + Tr\left(\int_0^t \nabla f(X_s)^T dX_s\right) + \frac{1}{2}\sum_{i,j,k,l=1}^{d}\int_0^t \frac{\partial^2}{\partial x_{ij}\partial x_{kl}} f(X_s)\,d\langle X_{ij},X_{kl}\rangle_s,$$

where ∇ denotes the operator

$$\nabla := \frac{\partial}{\partial X_{ij}},\ 1 \leq i,j \leq d.$$

Lemma 2.6.3. *The infinitesimal generator associated with the Wishart process $(\Sigma_t)_{t\geq 0}$ in (2.1) is given by, for $\Sigma \in S_d^+$:*

$$\mathcal{A} = Tr\left[\left(\beta Q^T Q + M\Sigma + \Sigma M^T\right)\nabla + 2\Sigma\nabla Q^T Q\nabla\right],$$

where ∇ is defined as in Lemma 2.6.2, i.e. $\nabla = \left(\frac{\partial}{\partial \Sigma_{ij}}\right)_{1\leq i,j\leq d}$.

Proof. See [20, Proposition 3]. □

Lemma 2.6.4. *Let $(\mathbb{R}_{\geq 0} \times S_d \times F, \mathcal{B}([0,t]) \otimes \mathcal{B}(S_d) \otimes \mathcal{F}_t)$ and $\left(\mathbb{R}^{d\times d}, \mathcal{B}\left(\mathbb{R}^{d\times d}\right)\right)$ be measurable spaces. If $\Theta : \mathbb{R}_{\geq 0} \times S_d \times F \to \mathbb{R}^{d\times d}$, $(s,\Sigma,f) \to \Theta(s,\Sigma,f)$ is measurable and*

$$|\Theta\Theta^T(t,\Sigma,f)| \leq C_0(t,f)|\Sigma|$$

for all $(t,\Sigma,f) \in \mathbb{R}_{t\geq 0} \times S_d \times F$ with some bounded C_0 and $|\Sigma| := \sqrt{Tr(\Sigma\Sigma)}$ for $\Sigma \in S_d$. Then the process $Z_t := (Z_t)_{t\geq 0}$ defined by

$$Z_t := \exp\left(\int_0^t Tr\left(\Theta(s,\Sigma_s)dW_s^\sigma\right) + \frac{1}{2}\int_0^t Tr(\Theta\Theta^T)(s,\Sigma_s)ds\right)$$

is a martingale.

Proof. See [29, Lemma 4.2.]. □

2.7 A multivariate Wishart stochastic volatility model

It is well-known that the classical standard Black-Scholes model is not flexible enough to create the “smile effect” that is a U-shaped relationship between the implied Black-Scholes volatility and the strike prices for a given maturity. Furthermore, the standard Black-Scholes model does not show the “leverage effect” either.

To cover these shortages of the standard Black-Scholes model, C. Gouriéroux and R. Sufana have in 2004 presented a multivariate Wishart stochastic volatility model in [26]. The model introduced below possesses a generalized drift compared with Gouriéroux’s model.

In our model the market consists of one riskfree asset with price process $(S_t^0)_{t\geq 0}$ and d risky assets. The constant riskfree rate is r. The dynamic of the riskfree asset is

$$dS_t^0 = S_t^0 r dt,\ S_t^0 = 1.$$

We denote by $(S_{t,i})_{t\geq 0}$, $1 \leq i \leq d$ the price processes of the d risky assets and by $(S_t)_{t\geq 0} = (S_{t,1},\ldots,S_{t,d})_{t\geq 0}$ the assets vector process. The return of $(S_t)_{t\geq 0}$ follows a process represented by $\log(S_t)_{t\geq 0}$, which owns a Wishart stochastic volatility $(\Sigma_t)_{t\geq 0}$. The joint dynamics of $(\log(S_t))_{t\geq 0}$ and $(\Sigma_t)_{t\geq 0}$ are given by the following stochastic differential system:

$$d\log(S_t) = \left(\mu^0 + \mu(\Sigma_t)\right)dt + \Sigma_t^{1/2}dW_t^S, \tag{2.13}$$

$$d\Sigma_t = \left(\Omega\Omega^T + M\Sigma_t + \Sigma_t M^T\right)dt + \Sigma_t^{1/2}dW_t^\sigma Q + Q^T\left(dW_t^\sigma\right)^T\Sigma_t^{1/2}, \tag{2.14}$$

where $\left(W_t^S\right)_{t\geq 0}$, $(W_t^\sigma)_{t\geq 0}$ are a d dimensional Brownian motions vector and a $d\times d$ Brownian motions matrix respectively defined on the probability space $(\Omega,\mathcal{F},P)$. $(\mathcal{F}_t)_{t\geq 0}$ denotes the corresponding Brownian filtration. The entries between $\left(W_t^S\right)_{t\geq 0}$ and $\left(W_t^\sigma\right)_{t\geq 0}$ can be either uncorrelated or correlated. In the correlated situation, one represents $W_{t,k}^S = W_{t,k}^{(1)}$ and $W_{t,ij}^\sigma = \rho_{k,ij} W_{t,k}^{(1)} + \sqrt{1-\rho_{k,ij}^2} W_{t,ij}^{(2)}$, where $\left(W_{t,k}^{(1)}\right)_{t\geq 0}$ and $\left(W_{t,ij}^{(2)}\right)_{t\geq 0}$, $1 \leq k,i,j \leq d$ are independent Brownian motions. Then one gets $d\langle W_{t,k}^S, W_{t,ij}^\sigma\rangle = \rho_{k,ij} dt$, where $\langle \cdot \rangle$ denotes the quadratic covariation of two stochastic processes.

The parameter μ^0 is deterministic and $\mu : S_d^+(\mathbb{R}) \to \mathbb{R}^d$ is measurable, whereas Ω, M, Q are $d \times d$ matrices with Ω invertible. The parameter $\mu(\Sigma_t)$ can be interpreted as the risk premium for an investment of assets S_t and from

$$E_t[d\log S_{t,i}] = \left[\mu_i^0 + \mu_i(\Sigma_t)\right] dt, \quad i = 1, \ldots, d,$$

we expect that $\mu_i(\Sigma_t) > 0$, $i = 1, \ldots, d$ for risk averse agents.

Remark 2.7.1. *The asset return dynamic presented by C. Gouriéroux in [26] is a special case of (2.13) with*

$$\mu_i(\Sigma_t) = Tr(D_i \Sigma_t), \quad 1 \leq i \leq d,$$

where D_i, $1 \leq i \leq d$ are $d \times d$ matrices. In this case the risk premium factor, $Tr(D_i\Sigma_t)$, $1 \leq i \leq d$, is a linear function of $\Sigma_{t,ij}$, $1 \leq i,j \leq d$. The conditions that $\mu(\Sigma_t) > 0$ is satisfied if D_i, $1 \leq i \leq d$ are symmetric positive definite matrices (see [26]).

Example 2.7.2. *The Wishart stochastic volatility model introduced by Gouriéroux in [26] can be regarded as an extension of the Heston model to the multidimensional case. Recalling that the one-dimensional asset return process $(\log(s_t))_{t\geq 0}$ in Heston model is determined by the stochastic process:*

$$d\log(s_t) = \left(\mu - \frac{z_t}{2}\right) dt + \sqrt{z_t} dW_t^s,$$

whereas the volatility process $(z_t)_{t\geq 0}$ follows a Cox-Ingersoll-Ross process:

$$dz_t = \kappa(\theta - z_t) dt + \xi\sqrt{z_t} dW_t^z,$$

where W_t^s, W_t^z are Brownian motions with correlation ρ and μ, κ, θ and ξ are constants in $\mathbb{R}$. One can easily get that the dynamics above are specifications of (2.13) and (2.14) in the unidimensional case.

For further details of Heston model we refer to [31, Section 6.7.] and [30].

We denote $\Sigma_t = (\Sigma_{t,lk})_{1\leq l,k\leq d}$ and write the dynamic of $\Sigma_{t,lk}$ in (2.14) as

$$d\Sigma_{t,lk} = \sum_{i=1}^d \left(\Omega_{li}\Omega_{ik}^T + M_{li}\Sigma_{t,ik} + \Sigma_{t,li} M_{ik}^T\right) dt + \sum_{i,j=1}^d \Sigma_{t,li}^{1/2} (dW_t^\sigma)_{ij} Q_{jk} + \sum_{i,j=1}^d Q_{li}^T (dW_t^\sigma)_{ij}^T \Sigma_{t,jk}^{1/2}. \tag{2.15}$$

Moreover, using Itô's Formula, we write (2.13) as

$$dS_t = diag(S_t) B(\Sigma_t) dt + diag(S_t) \Sigma_t^{1/2} dW_t^S \tag{2.16}$$

with

$$B(\Sigma_t) = \mu^0 + \mu(\Sigma_t) + \frac{1}{2}\Sigma_t^{(1)},$$

where $\Sigma_t^{(1)}$ is a d dimensional vector with $\left(\Sigma_t^{(1)}\right)_{1\leq i\leq d} = \Sigma_{t,ii}$.

We assume that an agent can invest into this financial market and define the portfolio strategy process $(\tilde{\pi}_t)_{t\geq 0}$ as a $\mathbb{R}^{d+1}$-valued progressively measurable process with respect to $(\mathcal{F}_t)_{t\geq 0}$. We denote $\tilde{\pi}_t = (\pi_t^0, \pi_{t,1}, \ldots, \pi_{t,d})^T$, where π_t^0 represents the proportion of wealth invested in the riskfree asset S_t^0, i.e. $\pi_t^0 = \frac{\varphi_t^0}{X_t^\pi}$, where (X_t^π) denotes the portfolio wealth process and (φ_t^0) denotes the process of wealth amount invested in the riskfree asset S^0. In the same way, we define $\pi_{t,i}$ as the proportion of wealth invested in the i-th risky asset for $1 \leq i \leq d$. We denote the portfolio strategy of risky assets by

$$\pi_t = (\pi_{t,1}, \ldots, \pi_{t,d})^T$$

and one gets obviously $\pi_t^0 + \pi_t^T \mathbf{1} = 1$.

Then the portfolio wealth process (X_t^π) owns the following dynamic:

$$dX_t^\pi = X_t^\pi \pi_t^T \frac{dS_t}{S_t} + X^\pi \pi_t^0 \frac{dS_t^0}{S_t^0}. \tag{2.17}$$

Applying the dynamic (2.16), it yields

$$\begin{aligned} \frac{dX_t^\pi}{X_t^\pi} &= \left(\pi_t^T B\left(\Sigma_t\right) + \pi_t^0 r\right) dt + \pi_t^T \Sigma_t^{1/2} dW_t^S \\ &= \left(\pi_t^T \left(B(\Sigma_t) - \mathbf{r}\right) + r\right) dt + \pi_t^T \Sigma_t^{1/2} dW_t^S, \end{aligned} \tag{2.18}$$

with $X_0^\pi = x_0$. It is assumed that all coefficients of SDEs (2.14), (2.16) and (2.18) are progressively measurable with respect to the Brownian filtration $(\mathcal{F}_t)$ and that the SDEs have unique solutions. For (2.16) and (2.18) the latter requirement is met if [33, p. 54]

$$\int_0^T \left(\|B\left(\Sigma_s\right)\|_1 + \|\Sigma_s^{1/2}\|_2^2\right) ds < \infty \ a.s. \tag{2.19}$$

and

$$\int_0^T \left(|\pi_s^T \left(B\left(\Sigma_s\right) - \mathbf{r}\right) + r| + \|\pi_s^T \Sigma_s^{1/2}\|_2^2\right) ds < \infty \ a.s. \tag{2.20}$$

For (2.14) there exists a unique global strong solution on $S_d^+(\mathbb{R})$, if $\Omega\Omega^T \succeq (d+1) Q^T Q$ according to Remark 2.2.4. The solution of the portfolio wealth process (X_t^π) is given as follows:

$$X_T^\pi = X_t^\pi \exp\left(\int_t^T \left[\pi_s^T \left(B(\Sigma_s) - \mathbf{r}\right) + r - \frac{1}{2}\|\pi_s^T \Sigma_s^{1/2}\|_2^2\right] ds + \int_t^T \pi_s^T \Sigma_s^{1/2} dW_s^S\right). \tag{2.21}$$

Denote by $U : \mathbb{R}_+ \to \mathbb{R}$ a (strictly increasing, strictly concave) utility function, we define admissible portfolio strategies with respect to $U(\cdot)$ in the following sense:

Definition 2.7.3. *(admissible portfolio strategy) A portfolio strategy $(\pi_t)_{t\geq 0}$ is said to be admissible if the following conditions are satisfied:*

(i) $(\pi_t)_{t\geq 0}$ is a progressively measurable process with respect to $(\mathcal{F}_t)_{t\geq 0}$;

(ii) for all initial conditions $(t_0, x_0, \Sigma_0) \in [0,\infty)^2 \times S_d^+(\mathbb{R})$*, the wealth process* $(X_t^\pi)_{t\geq 0}$ *with* $X_{t_0}^\pi = x_0$ *given by (2.19) has a pathwise unique solution* $(X_t^\pi)_{t\geq 0}$*;*

(iii) $E^{t_0,x_0,\Sigma_0}\left(U\left(X_T^\pi\right)\right) < \infty$*;*

(iv) $X_t^\pi > 0$*.*

Note that under the assumption (2.20) the conditions (i), (ii) and (iv) for $(\pi_t)_{t\geq 0}$ to be admissible are satisfied.

Chapter 3

Optimal portfolio strategies in the uncorrelated Wishart volatility model

This and the next section are devoted to solve the optimal portfolio problems of maximizing utility function from terminal wealth with respect to a utility function U. The value function of the optimization problem reads as

$$\Phi(t,x) = \max_{(\pi_s)} E^{t,x}\left[U\left(X_T^\pi\right)\right]$$

with x being the portfolio wealth at t.

In this section we consider the optimal portfolio problem in the setting of a multivariate Wishart stochastic volatility model with uncorrelated $\left(W_t^S\right)$ and $\left(W_t^\sigma\right)$, i.e. the case of $\rho_{k,ij} = 0$, $1 \leq k,i,j \leq d$ (see Section 2.7 for the notations) and solve the optimal portfolio strategies for logarithmic utility

$$U(x) = \log(x)$$

and power utility

$$U(x) = \frac{x^\gamma}{\gamma}, \quad \gamma < 1,\ \gamma \neq 0,$$

respectively. We will proceed in Section 3 with the correlated case, i.e. $\exists 1 \leq k,i,j \leq d$, $\rho_{k,ij} \neq 0$.

3.1 Optimal portfolios in the model with one risk-free asset

We face now the Wishart model introduced in Section 2.7. The market consists of one riskfree asset with the price process (S_t^0) evolving as

$$dS_t^0 = S_t^0 r dt,\ S_t^0 = 1$$

and d risky assets. r denotes the constant riskfree rate. The price processes of the d risky assets are denoted by $\left(S_{t,i}\right)_{t\geq 0}$, $1 \leq i \leq d$ with dynamics as in (2.14) and (2.16). Then the portfolio wealth process (X_t^π) owns the dynamic (2.18). Assume that the conditions (2.19), (2.20) as well as

$$E^{t,x,\Sigma}\left[\int_t^T \|\pi_s^T \Sigma_s^{1/2}\|^2 ds\right] < \infty \tag{3.1}$$

are satisfied and $\Omega\Omega^T \succeq (d+1)Q^T Q$, it follows that the SDEs (2.14), (2.16) and (2.18) own unique strong solutions and $\int_t^T \pi_s^T \Sigma_s^{1/2} dW_s^S$ is a true martingale.

3.1.1 Optimal portfolios for logarithmic utility

For the utility function $U(x) = \log(x)$, the value function reads as

$$\Phi(t,x,\Sigma) = \max_{(\pi_s)} E^{t,x,\Sigma}\left(\log\left(X_T^\pi\right)\right). \tag{3.2}$$

Since (2.18) is assumed to possess a unique strong solution (2.21), we can replace X_T^π in (3.2) by (2.21) with $X_t^\pi = x$, it yields then

$$\begin{aligned}
&\max_{(\pi_s)} E^{t,x,\Sigma}\left(\log\left(X_T^\pi\right)\right) \\
=&\max_{(\pi_s)}\left(\log x + E^{t,x,\Sigma}\left[\int_t^T \pi_s^T\left(B\left(\Sigma_s\right)-\mathbf{r}\right)+r-\frac{1}{2}\|\pi_s^T\Sigma_s^{1/2}\|_2^2 ds\right]+\right. \\
&\left.+E^{t,x,\Sigma}\left[\int_t^T \pi_s^T \Sigma_s^{1/2} dW_s^S\right]\right) \qquad (3.3)\\
=&\log x + \max_{(\pi_s)}\left(E^{t,x,\Sigma}\left[\int_t^T \pi_s^T\left(B\left(\Sigma_s\right)-\mathbf{r}\right)+r-\frac{1}{2}\|\pi_s^T\Sigma_s^{1/2}\|_2^2 ds\right]\right) \\
=&\log x + E^{t,x,\Sigma}\left[\int_t^T \max_{\pi_s}\left(\pi_s^T\left(B\left(\Sigma_s\right)-\mathbf{r}\right)+r-\frac{1}{2}\|\pi_s^T\Sigma_s^{1/2}\|_2^2\right) ds\right], \qquad (3.4)
\end{aligned}$$

where $\max_{(\pi_s)}$ is a simplified representation of $\max_{(\pi_s)_{t\le s\le T}}$. Note that we have assumed that $\int_t^T \pi_s^T \Sigma_s^{1/2} dW_s^S$ is a true martingale, thus, we can cancel the last term in (3.3).

Proposition 3.1.1. *The optimal portfolio in the above model with logarithmic utility is*

$$\pi_s^* = \Sigma_s^{-1}\left(B(\Sigma_s) - \boldsymbol{r}\right) \tag{3.5}$$

given (π_s^*) *satisfying (3.1).*

Proof. Denote

$$H(\pi) = \pi^T\left(B(\Sigma) - \mathbf{r}\right) + r - \frac{1}{2}\|\pi^T\Sigma^{1/2}\|_2^2$$

and calculate the first derivative and the Hessian matrix of $H(\pi)$, we get

$$\frac{\partial H(\pi)}{\partial \pi} = B(\Sigma) - \mathbf{r} - \Sigma\pi$$

and

$$\frac{\partial^2}{\partial \pi^2} H(\pi) = -\Sigma.$$

The Hessian matrix is negative definite, since Wishart matrix Σ is positive definite. Thus, $H(\pi)$ attains its local maximum at π with $\partial H(\pi)/\partial\pi = 0$. For $\|\pi\|_2 \to \infty$, it is obvious that $H(\pi)$ goes to negative infinity. Thus, the optimal portfolio strategy π^* maximizing $H(\pi)$ is:

$$\pi_s^* = \Sigma_s^{-1}\left(B\left(\Sigma_s\right) - \mathbf{r}\right)$$

given π_s^* satisfying (3.1). □

For the case of $B(\Sigma) = r + \Sigma\mathbf{v}$ with a $\mathbf{v} \in \mathbb{R}^d$, it is evident that $\pi^* = \mathbf{v}$ and the condition (3.1) is satisfied.

Replacing the notation π_s in (3.4) by π_s^* and applying Fubini's theorem, it yields

$$\begin{aligned}
&\max_{(\pi_s)} E^{t,x,\Sigma}\left(\log\left(X_T^\pi\right)\right) \\
=&\log x + \int_t^T E^{t,x,\Sigma}\left(\left(\pi_s^*\right)^T\left(B\left(\Sigma_s\right) - \mathbf{r}\right) + r - \frac{1}{2}\|\left(\pi_s^*\right)^T\Sigma_s^{1/2}\|_2^2\right) ds \\
=&\log x + r\left(T-t\right) + \frac{1}{2}\int_t^T E^{t,x,\Sigma}\left(\left(B\left(\Sigma_s\right) - \mathbf{r}\right)^T\Sigma_s^{-1}\left(B\left(\Sigma_s\right) - \mathbf{r}\right)\right) ds.
\end{aligned}$$

with

$$B\left(\Sigma_s\right) = \mu^0 + \tilde{\mu}\left(\Sigma_s\right) = \mu^0 + \mu\left(\Sigma_s\right) + \frac{1}{2}\Sigma_s^{(1)}$$

as in Section 2.7, where $\Sigma^{(1)}$ is a d-dimensional vector with $\left(\Sigma^{(1)}\right)_{1\le i\le d} = \Sigma_{ii}$. For the further computation we need to calculate the conditional expectation of the function

$$f(\Sigma) := \left(B\left(\Sigma_s\right) - \mathbf{r}\right)^T\Sigma_s^{-1}\left(B\left(\Sigma_s\right) - \mathbf{r}\right), \quad \mathbb{R}^{d\times d} \to \mathbb{R}.$$

We refer to [17, 16, 38, 46] for the conditional expectation of the moments of real inverse Wishart distributed matrices and [45] for the moments of central and noncentral Wishart distributions.

3.1.2 Optimal portfolios for power utility

In this section we consider the optimization problem with power utility function $U(x) = \frac{x^\gamma}{\gamma}$, $\gamma < 1$, $\gamma \neq 0$. The value function of the optimization problem now reads as

$$\Phi\left(t,x,\Sigma\right) = \max_{(\pi_s)} E^{t,x,\Sigma}\left(\frac{\left(X_T^\pi\right)^\gamma}{\gamma}\right). \tag{3.6}$$

Replacing X_T^π in (3.6) by its solution (2.21) with $X_t^\pi = x$, we get

$$\begin{aligned}
\Phi\left(t,x,\Sigma\right) =& \max_{(\pi_s)} E^{t,x,\Sigma}\left[\frac{x^\gamma}{\gamma}\exp\left(\gamma\int_t^T\left[\pi_s^T\left(B(\Sigma_s) - \mathbf{r}\right) + r - \frac{1}{2}\|\pi_s^T\Sigma_s^{1/2}\|_2^2\right] ds\right.\right. \\
&\left.\left. + \gamma\int_t^T \pi_s^T\Sigma_s^{1/2} dW_s^S\right)\right].
\end{aligned}$$

From (2.20) it follows automatically that

$$P\left[\int_0^T \left\|\pi_s^T\Sigma_s^{1/2}\right\|^2 dt < \infty\right] = 1, \quad 0 \le T < \infty.$$

Let us define a Radon-Nikodym derivative

$$Z_t^\pi := \left.\frac{d\mathbb{Q}^\pi}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(\gamma\int_t^T \pi_s^T\Sigma_s^{1/2}dW_s^S - \frac{\gamma^2}{2}\int_t^T \|\pi_s^T\Sigma_s^{1/2}\|^2 ds\right) \tag{3.7}$$

and first assume that (Z_t^π) is a martingale, then, applying Girsanov theorem we get

$$\begin{aligned}
&\Phi\left(t,x,\Sigma\right)\\
=&\max_{(\pi_s)} E_{\mathbb{Q}^\pi}^{t,x,\Sigma}\left[\frac{x^\gamma}{\gamma}\exp\left(\gamma\int_t^T\left[\pi_s^T\left(B\left(\Sigma_s\right)-\mathbf{r}\right)+r-\frac{1}{2}\|\pi_s^T\Sigma_s^{1/2}\|_2^2+\frac{\gamma}{2}\|\pi_s^T\Sigma_s^{1/2}\|_2^2\right]ds\right)\right]\\
=&\max_{(\pi_s)} E_{\mathbb{Q}^\pi}^{t,x,\Sigma}\left[\frac{x^\gamma}{\gamma}\exp\left(\int_t^T\gamma\left[\pi_s^T\left(B\left(\Sigma_s\right)-\mathbf{r}\right)+r+\frac{\gamma-1}{2}\pi_s^T\Sigma_s\pi_s\right]ds\right)\right].
\end{aligned} \tag{3.8}$$

Remark 3.1.2. *Note that under $\mathbb{Q}^\pi$ the distribution of (Σ_t) is not changed, since (W_t^σ) and (W_t^S) are uncorrelated.*

Proposition 3.1.3. *The optimal portfolio in our model with power utility is achieved at π_s^* with*

$$\pi_s^* = \Sigma_s^{-1}\frac{\boldsymbol{r}-B(\Sigma_s)}{\gamma-1} \tag{3.9}$$

for $\gamma < 1$, $\gamma \neq 0$.

Proof. We first assume that (Z_t^π) is a martingale. Denote

$$F\left(\pi\right) = \gamma\left[\pi^T\left(B\left(\Sigma\right)-\mathbf{r}\right)+r+\frac{\gamma-1}{2}\pi_s^T\Sigma_s\pi_s\right],$$

we consider the extreme values of $F(\pi)$. The first derivative and the Hessian matrix of $F\left(\pi\right)$ are given as

$$\frac{\partial F\left(\pi\right)}{\partial\pi} = \gamma\left(B-\mathbf{r}+\left(\gamma-1\right)\Sigma\pi\right)$$

and

$$\frac{\partial^2 F\left(\pi\right)}{\partial\pi^2} = \gamma\left(\gamma-1\right)\Sigma$$

respectively. For $0<\gamma<1$, the Hessian matrix $\gamma\left(\gamma-1\right)\Sigma$ is negative definite, which implies that the multivariate quadratic function $F\left(\pi\right)$ attains its maximum at π with $\partial F\left(\pi\right)/\partial\pi = 0$. For negative γ, the Hessian matrix is positive definite, it follows that the multivariate quadratic function $F\left(\pi\right)$ owns a minimum point. Taking $\frac{x^\gamma}{\gamma}$ into consideration, we get

$$\frac{x^\gamma}{\gamma}\exp\left(\int_t^T F(\pi_s)ds\right) \leq \frac{x^\gamma}{\gamma}\exp\left(\int_t^T F(\pi_s^*)ds\right) \tag{3.10}$$

with

$$\pi_s^* = \Sigma_s^{-1}\frac{\mathbf{r}-B(\Sigma_s)}{\gamma-1}$$

for $\gamma < 1$, $\gamma \neq 0$. Note that by (3.8) we have

$$\Phi\left(t,x,\Sigma\right) = \max_{(\pi_s)} E^{t,x,\Sigma}_{\mathbb{Q}^\pi}\left[E^{t,x,\Sigma}_{\mathbb{Q}^\pi}\left[\frac{x^\gamma}{\gamma}\exp\left(\int_t^T F(\pi_s)ds\right)\middle|\left(W^\sigma_s\right)_{t\leq s\leq T}\right]\right]$$

and the inequality in (3.10) yields

$$\begin{aligned}&E^{t,x,\Sigma}_{\mathbb{Q}^\pi}\left[\frac{x^\gamma}{\gamma}\exp\left(\int_t^T F(\pi_s)ds\right)\middle|\left(W^\sigma_s\right)_{t\leq s\leq T}\right]\\ \leq &E^{t,x,\Sigma}_{\mathbb{Q}^\pi}\left(\frac{x^\gamma}{\gamma}\exp\left(\int_t^T F(\pi^*_s)ds\right)\middle|\left(W^\sigma_s\right)_{t\leq s\leq T}\right) = C^{W^\sigma}\end{aligned}$$

with $C^{W^\sigma}\in\mathbb{R}$. Thus, we get

$$\begin{aligned}\Phi\left(t,x,\Sigma\right) =&E^{t,x,\Sigma}_{\mathbb{Q}^\pi}\left[\frac{x^\gamma}{\gamma}\exp\left(\int_t^T F(\pi^*_s)ds\right)\right]\\ =&E^{t,x,\Sigma}\left[\frac{x^\gamma}{\gamma}\exp\left(\int_t^T\left(\gamma\left(\pi^*_s\right)^T\left(B\left(\Sigma_s\right)-\mathbf{r}\right)+\gamma r+\frac{\gamma^2-\gamma}{2}\left(\pi^*_s\right)^T\Sigma_s\pi^*_s\right)ds\right)\right]\\ =&E^{t,x,\Sigma}\left[\frac{x^\gamma}{\gamma}\exp\left(\int_t^T\left(\gamma r-\frac{\gamma}{2\left(\gamma-1\right)}\left(B\left(\Sigma_s\right)-\mathbf{r}\right)^T\Sigma_s^{-1}\left(B\left(\Sigma_s\right)-\mathbf{r}\right)\right)ds\right)\right].\end{aligned}$$

Note that $\left((\pi^*)^T_t\Sigma_t^{1/2}\right)$ and (W^S_t) are two independent processes, since (W^σ_t) and (W^S_t) are assumed to be independent, it yields then by [36, Section 6.2, Example 4] that the process $(Z^{\pi^*}_t)$ in (3.8) is a martingale indeed. This concludes the proof. □

For the special case $B\left(\Sigma\right)-\mathbf{r}=\Sigma\mathbf{v}$ with a $\mathbf{v}\in\mathbb{R}^d$, we have

$$\begin{aligned}\Phi\left(t,x,\Sigma\right) =& \frac{x^\gamma}{\gamma}E^{t,\Sigma}\left[\exp\left(\int_t^T\left(\gamma r-\frac{\gamma}{2\left(\gamma-1\right)}\mathbf{v}^T\Sigma_s\mathbf{v}\right)ds\right)\right]\\ =&\frac{x^\gamma e^{\gamma r(t-T)}}{\gamma}E^{t,\Sigma}\left[\exp\left(Tr\left[\int_t^T\Gamma^{(1)}\Sigma_s ds\right]\right)\right]\end{aligned}$$

with $\Gamma^{(1)}=-\frac{\gamma}{2(\gamma-1)}\mathbf{v}\mathbf{v}^T$. Denote

$$V\left(t,\Sigma\right)=E^{t,\Sigma}\left[\exp\left(Tr\left[\int_t^T\Gamma^{(1)}\Sigma_s ds\right]\right)\right], \tag{3.11}$$

we have the following proposition:

Proposition 3.1.4. *Let us denote* $h=T-t$ *and*

$$\begin{aligned}\kappa^{(1)}\left(h\right) =& -\left(\sqrt{C_2^{(1)}}\cosh\left(\sqrt{C_2^{(1)}}h\right)+C_1^{(1)}\sinh\left(\sqrt{C_2^{(1)}}h\right)\right)^{-1}\cdot\\ &\left(\sqrt{C_2^{(1)}}\sinh\left(\sqrt{C_2^{(1)}}h\right)+C_1^{(1)}\cosh\left(\sqrt{C_2^{(1)}}h\right)\right),\\ C_2^{(1)} =& Q\left(-2\Gamma^{(1)}+M^TQ^{-1}Q^{-T}M\right)Q^T,\\ C_1^{(1)} =& -Q\left(\frac{M^T\left(Q^TQ\right)^{-1}+\left(Q^TQ\right)^{-1}M}{2}\right)Q^T.\end{aligned}$$

Assume that

$$QQ^T \in GL_d(\mathbb{R}), \quad -2\Gamma^{(1)} + M^T Q^{-1} Q^{-T} M \succ 0, \quad \sqrt{C_2^{(1)}} + C_1^{(1)} \succeq 0 \quad \textit{and} \quad \Gamma^{(1)} \neq \mathbf{0},$$

then, the explicit solution of (3.11) is given by

$$V\left(t, \Sigma\right) = \exp\left(\phi^{(1)}\left(h\right) + Tr[\psi^{(1)}\left(h\right)\Sigma]\right)$$

for all $t \geq 0$, where the functions $\phi^{(1)}$ and $\psi^{(1)}$ are given by:

$$\psi^{(1)}\left(h\right) = \frac{Q^{-1}\sqrt{C_2^{(1)}}\kappa^{(1)}\left(h\right)Q^{-T}}{2} - \frac{M^T\left(Q^TQ\right)^{-1} + \left(Q^TQ\right)^{-1}M}{4}$$

$$\phi^{(1)}\left(h\right) = -Tr\left(\Omega\Omega^T\frac{M^T\left(Q^TQ\right)^{-1} + \left(Q^TQ\right)^{-1}M}{4}\right)h - \frac{1}{2}Tr\Bigg(Q^{-T}\Omega\Omega^TQ^{-1}\cdot$$

$$\log\left[\left(\sqrt{C_2^{(1)}}\right)^{-1}\left(\sqrt{C_2^{(1)}}\cosh\left(\sqrt{C_2^{(1)}}h\right) + C_1^{(1)}\sinh\left(\sqrt{C_2^{(1)}}h\right)\right)\right]\Bigg)$$

with $\psi^{(1)}\left(0\right) = \mathbf{0}$, $\phi^{(1)}\left(0\right) = 0$. $\mathbf{0}$ denotes a d by d zero matrix.

Proof. The proposition follows directly from Corollary 2.4.5 and Proposition 2.4.6. □

Remark 3.1.5. *For $\Gamma^{(1)} = \mathbf{0}$, we have obviously $V(t, \Sigma) = 1$. The reason that we treat this case separately is explained in Remark 2.4.4 a).*

3.2 Optimal portfolios in the model without risk-free assets

In this section we consider the uncorrelated Wishart volatility model without risk-free assets. We face a Wishart model with only d, $d \geq 2$ risky assets. The price processes of the risky assets are denoted again by $\left(S_{t,i}\right)_{t\geq 0}$, $1 \leq i \leq d$ with dynamics as in (2.14) and (2.16), i.e.

$$dS_t = diag\left(S_t\right)B\left(\Sigma_t\right)dt + diag\left(S_t\right)\Sigma_t^{1/2}dW_t^S,$$
$$d\Sigma_t = \left(\Omega\Omega^T + M\Sigma_t + \Sigma_tM^T\right)dt + \Sigma_t^{1/2}dW_t^\sigma Q + Q^T\left(dW_t^\sigma\right)^T\Sigma_t^{1/2}.$$

Let us denote our portfolio strategy by

$$\pi_t = \left(\pi_{t,1}, \ldots, \pi_{t,d}\right)^T,$$

one gets directly $\pi_t^T\mathbf{1} = 1$. Then the portfolio wealth process $\left(X_t^\pi\right)$ follows the following dynamic

$$\frac{dX_t^\pi}{X_t^\pi} = \pi_t^T\frac{dS_t}{S_t} = \pi_t^TB(\Sigma_t)dt + \pi_t^T\Sigma_t^{1/2}dW_t^S \tag{3.12}$$

with $X_0^\pi = x_0$. Recalling that in the previous model with one risk-free asset, the portfolio wealth process $\left(X_t^\pi\right)$ evolves as (2.18), i.e.

$$\frac{dX_t^\pi}{X_t^\pi} = \left(\pi_t^T\left(B(\Sigma_t) - \mathbf{1}r\right) + r\right)dt + \pi_t^T\Sigma_t^{1/2}dW_t^S.$$

We note that (3.12) is actually (2.18) with $r = 0$. Set

$$\mathcal{A} := \{ (\pi_t) | \, \pi_t \in \mathbb{R}^d, \; \pi_t^T \mathbf{1} = 1, \; \forall t \in [0, T] \},$$

then, our optimization problems read as

$$\max_{\pi \in \mathcal{A}} E^{t,x,\Sigma} \left(\log \left(X_T^\pi \right) \right)$$

for logarithmic utility and

$$\max_{\pi \in \mathcal{A}} E^{t,x,\Sigma} \left(\frac{(X_T^\pi)^\gamma}{\gamma} \right)$$

for power utility, respectively. We face now two stochastic control problems under the condition that there is no risk-free asset in the market. Note that such portfolio strategies take values in a special closed, convex subset of $\mathbb{R}^d$. Cvitanić and Karatzas [12] established existence results for optimal portfolios and discussed extensively several cases of optimization problems with portfolio strategies constrained to closed, convex subsets of $\mathbb{R}^d$, such as optimization problems with no short-selling and optimization problems in incomplete markets, etc.

As in Section 3.1.1 and Section 3.1.2, the optimal portfolios can be derived through replacing X_T^π by its unique solution, we get then

$$\max_{\pi \in \mathcal{A}} E^{t,x,\Sigma} \left(\log \left(X_T^\pi \right) \right) = \log x + E^{t,x,\Sigma} \left[\int_t^T \max_{\pi \in \mathcal{A}} \left(\pi_s^T B \left(\Sigma_s \right) - \frac{1}{2} \| \pi_s^T \Sigma_s^{1/2} \|_2^2 \right) ds \right]$$

and

$$\max_{\pi \in \mathcal{A}} E^{t,x,\Sigma} \left(\frac{(X_T^\pi)^\gamma}{\gamma} \right) = \max_{\pi \in \mathcal{A}} E_{\mathbb{Q}^\pi}^{t,x,\Sigma} \left[\frac{x^\gamma}{\gamma} \exp \left(\int_t^T \gamma \left[\pi_s^T B \left(\Sigma_s \right) + \frac{\gamma - 1}{2} \pi_s^T \Sigma_s \pi_s \right] ds \right) \right]$$

from (3.5) and (3.9) respectively. Thus, the optimal portfolios in the model without risk-free assets is achieved at

$$\pi_s^* = \pi^*(\Sigma_s) = arg \max_{\pi \in \mathcal{A}} \left(\pi_s^T B \left(\Sigma_s \right) - \frac{1}{2} \| \pi_s^T \Sigma_s^{1/2} \|_2^2 \right) \tag{3.13}$$

in the logarithmic utility case and

$$\pi_s^* = \pi^*(\Sigma_s) = arg \max_{\pi \in \mathcal{A}} E_{\mathbb{Q}^\pi}^{t,x,\Sigma} \left[\frac{x^\gamma}{\gamma} \exp \left(\int_t^T \gamma \left[\pi_s^T B \left(\Sigma_s \right) + \frac{\gamma - 1}{2} \pi_s^T \Sigma_s \pi_s \right] ds \right) \right] \tag{3.14}$$

in the power utility case with $\gamma < 1$, $\gamma \neq 0$ under the assumption that (3.7) is a martingale.

We will solve the problem (3.13) and (3.14) in the following lemmas. First let us consider the optimization problem (3.13).

Lemma 3.2.1. *Let us denote*

$$\Sigma_s^{d-1} := \left(\Sigma_{s,ij} \right)_{1 \le i,j \le d-1} \in \mathbb{R}^{(d-1) \times (d-1)}, \quad B^{d-1}(\Sigma_s) := \left(B_i(\Sigma_s) \right)_{1 \le i \le d-1} \in \mathbb{R}^{d-1},$$
$$\Sigma_s^-(d, :) := \left(\Sigma_{s,di} \right)_{1 \le i \le d-1} \in \left(\mathbb{R}^{d-1} \right)^T, \quad \Sigma_s^-(:, d) := \left(\Sigma_{s,id} \right)_{1 \le i \le d-1} \in \mathbb{R}^{d-1}.$$

The optimal portfolio with logarithmic utility in the model without risk-free assets, namely the solution of (3.13), is achieved at $\left(\pi^*_{s,i}\right)$, $1 \le i \le d$ *with*

$$\left(\pi^*_{s,1}, \ldots, \pi^*_{s,d-1}\right)^T =: \left(\pi^{d-1}\right)^*_s, \quad \pi^*_{s,d} := 1 - \left(\left(\pi^{d-1}\right)^*_s\right)^T \mathbf{1} \quad \textit{with} \quad \mathbf{1} \in \mathbb{R}^{d-1}$$

and $\left(\pi^{d-1}\right)^*_s$ *is determined by:*

$$\left(\pi^{d-1}\right)^*_s = \left(\Sigma_s^{d-1} + \Sigma_{s,dd}\mathbf{1}\mathbf{1}^T - \Sigma_s^-(:,d)\mathbf{1}^T - \mathbf{1}\Sigma_s^-(d,:)\right)^{-1} \cdot \left(B^{d-1}(\Sigma_s) - B_d(\Sigma_s)\mathbf{1} + \Sigma_{s,dd}\mathbf{1} - \Sigma_s^-(:,d)\right).$$

Proof. We note that the optimization problem

$$\max_{\pi, \pi^T\mathbf{1}=1} \left(\pi^T B\left(\Sigma\right) - \frac{1}{2}\|\pi^T\Sigma^{1/2}\|_2^2\right)$$

can be written as

$$\max_{\pi^{d-1}} P(\pi^{d-1})$$

with

$$\begin{aligned} P(\pi^{d-1}) := & \left(\pi^{d-1}\right)^T B^{d-1} + \left(1 - \left(\pi^{d-1}\right)^T \mathbf{1}\right) B_d - \frac{1}{2}\left(\pi^{d-1}\right)^T \Sigma^{d-1}\pi^{d-1} \\ & - \left(1 - \left(\pi^{d-1}\right)^T \mathbf{1}\right)\left(\pi^{d-1}\right)^T \Sigma^-(:,d) - \frac{1}{2}\left(1 - \left(\pi^{d-1}\right)^T \mathbf{1}\right)^2 \Sigma_{dd}. \end{aligned}$$

Calculating the first derivative and the Hessian matrix of $P(\pi^{d-1})$, we get

$$\begin{aligned} \frac{\partial P(\pi^{d-1})}{\partial \pi^{d-1}} = & B^{d-1} - \mathbf{1}B_d - \Sigma^{d-1}\pi^{d-1} - \Sigma^-(:,d) + \Sigma^-(:,d)\mathbf{1}^T\pi^{d-1} + \mathbf{1}\Sigma^-(d,:)\pi^{d-1} \\ & + \Sigma_{dd}\mathbf{1} - \Sigma_{dd}\mathbf{1}\mathbf{1}^T\pi^{d-1} \end{aligned}$$

and

$$\frac{\partial^2}{\partial \left(\pi^{d-1}\right)^2} P(\pi^{d-1}) = -\Sigma^{d-1} - \Sigma_{dd}\mathbf{1}\mathbf{1}^T + \Sigma^-(:,d)\mathbf{1}^T + \mathbf{1}\Sigma^-(d,:).$$

Let us denote $M := \Sigma^{d-1} + \Sigma_{dd}\mathbf{1}\mathbf{1}^T - \Sigma^-(:,d)\mathbf{1}^T - \mathbf{1}\Sigma^-(d,:)$ and represent Σ by block matrices

$$\Sigma = \begin{pmatrix} \Sigma^{d-1} & \Sigma^-(:,d) \\ \Sigma^-(d,:) & \Sigma_{dd} \end{pmatrix},$$

we get that M is the first component on the diagonal of the matrices product

$$\begin{pmatrix} I & -\mathbf{1} \\ -\mathbf{1}^T & 1 \end{pmatrix} \Sigma \begin{pmatrix} I & -\mathbf{1} \\ -\mathbf{1}^T & 1 \end{pmatrix}, \tag{3.15}$$

where I denotes the $(d-1) \times (d-1)$ identity matrix.

We know that for $\forall a \neq \mathbf{0} \in \mathbb{R}^{d-1}$, there exists a $b \in \mathbb{R}^{d-1}$ with

$$a^T \begin{pmatrix} I & -\mathbf{1} \\ -\mathbf{1}^T & 1 \end{pmatrix} \Sigma \begin{pmatrix} I & -\mathbf{1} \\ -\mathbf{1}^T & 1 \end{pmatrix} a = b^T\Sigma b > 0$$

for $\Sigma \in S_d^+$, thus, we get that (3.15) is a symmetric, strictly positive definite matrix. It implies then M is also symmetric, strictly positive definite, namely $M \in S_{d-1}^+(\mathbb{R})$.

Thus, it follows that the second derivative of $P(\pi^{d-1})$ is negative definite. Then $P(\pi^{d-1})$ attains its local maximum at $\left(\pi^{d-1}\right)^*$ with

$$\partial P\left(\left(\pi^{d-1}\right)^*\right)/\partial\pi = 0.$$

For $\|\pi^{d-1}\|_2 \to \infty$, it is obvious that $P(\pi^{d-1})$ goes to negative infinity. Thus, the optimal portfolio strategy $\left(\pi^{d-1}\right)^*$ maximizing $P\left(\pi^{d-1}\right)$ is:

$$\left(\pi^{d-1}\right)_s^* = \left(\Sigma_s^{d-1} + \Sigma_{s,dd}\mathbf{1}\mathbf{1}^T - \Sigma_s^-(:,d)\mathbf{1}^T - \mathbf{1}\Sigma_s^-(d,:)\right)^{-1} \cdot \left(B^{d-1}(\Sigma_s) - B_d(\Sigma_s)\mathbf{1} + \Sigma_{s,dd}\mathbf{1} - \Sigma_s^-(:,d)\right).$$

□

The solution of the optimization problem (3.14) is given by the following lemma.

Lemma 3.2.2. *Let us define*

$$\Sigma_s^{d-1}, \quad B^{d-1}(\Sigma_s), \quad \Sigma_s^-(d,:), \quad \Sigma_s^-(:,d), \quad \left(\pi^{d-1}\right)_s^* \quad \text{and} \quad \pi_{s,d}^*$$

as in Lemma 3.2.1. The optimal portfolio with power utility in the model without risk-free assets, namely the solution of (3.14), is achieved at π_s^* with $\left(\pi^{d-1}\right)_s^*$ given by:

$$\left(\pi^{d-1}\right)_s^* = \frac{1}{1-\gamma}\Upsilon_s^{-1} \cdot \left(B^{d-1}(\Sigma_s) - B_d(\Sigma_s)\mathbf{1} - (\gamma-1)\Sigma_{s,dd}\mathbf{1} + (\gamma-1)\Sigma_s^-(:,d)\right)$$

with

$$\Upsilon_s = \Sigma_s^{d-1} + \Sigma_{s,dd}\mathbf{1}\mathbf{1}^T - \Sigma_s^-(:,d)\mathbf{1}^T - \mathbf{1}\Sigma_s^-(d,:).$$

Proof. Denote

$$Q(\pi) = \gamma\left[\pi^T B(\Sigma) + \frac{\gamma-1}{2}\pi^T\Sigma\pi\right],$$

then, by $\pi_d = 1 - \left(\pi^{d-1}\right)^T\mathbf{1}$, we get

$$Q(\pi) = Q\left(\pi^{d-1}\right) = \gamma\Bigg[\left(\pi^{d-1}\right)^T B^{d-1} + \left(1-\left(\pi^{d-1}\right)^T\mathbf{1}\right)B_d + \frac{\gamma-1}{2}\left(\pi^{d-1}\right)^T\Sigma^{d-1}\pi^{d-1} + (\gamma-1)\left(1-\left(\pi^{d-1}\right)^T\mathbf{1}\right)\left(\pi^{d-1}\right)^T\Sigma^-(:,d) + \frac{\gamma-1}{2}\left(1-\left(\pi^{d-1}\right)^T\mathbf{1}\right)^2\Sigma_{dd}\Bigg].$$

The first derivative and the Hessian matrix of $Q(\pi_s^{d-1})$ are given by

$$\frac{\partial Q\left(\pi^{d-1}\right)}{\partial\pi^{d-1}} = \gamma\left(B^{d-1} - \mathbf{1}B_d + (\gamma-1)\Sigma^-(:,d) - (\gamma-1)\Sigma_{dd}\mathbf{1}\right) + \gamma(\gamma-1)\Upsilon\pi^{d-1}$$

and

$$\frac{\partial^2}{\partial\left(\pi^{d-1}\right)^2}Q\left(\pi^{d-1}\right) = \gamma(\gamma-1)\Upsilon$$

respectively. Recalling that $\Upsilon \in S_{d-1}^{+}$ shown in Lemma 3.2.1, we get that for $0 < \gamma < 1$, the Hessian matrix $\gamma(\gamma - 1)\Upsilon$ is negative definite and for negative γ, the Hessian matrix is positive definite. Thus, we have again

$$\frac{x^{\gamma}}{\gamma}\exp\left(\int_t^T Q\left(\pi_s^{d-1}\right) ds\right) \leq \frac{x^{\gamma}}{\gamma}\exp\left(\int_t^T Q\left(\left(\pi_s^{d-1}\right)^{*}\right) ds\right) \tag{3.16}$$

with

$$\left(\pi^{d-1}\right)_s^{*} = \frac{1}{1-\gamma}\Upsilon_s^{-1} \cdot \left(B^{d-1}(\Sigma_s) - B_d(\Sigma_s)\mathbf{1} - (\gamma - 1)\Sigma_{s,dd}\mathbf{1} + (\gamma - 1)\Sigma_s^{-}(:, d)\right).$$

for $\gamma < 1$, $\gamma \neq 0$.

Note that $\left(\left(\pi^{d-1}\right)^{*}\right)_t^{T}\Sigma_t^{1/2}\Big)$ and (W_t^S) are two independent processes, since (W_t^{σ}) and (W_t^S) are assumed to be independent, it yields that the process $(Z_t^{\pi^*})$ in (3.7) is a martingale. As in Proposition 3.1.3, it follows that $\left(\pi^{d-1}\right)_s^{*}$ solves (3.14). □

Chapter 4

Optimal portfolio strategies in the correlated Wishart volatility model

This section is devoted to the study of optimization problems in the correlated Wishart volatility model with one risk-free asset. The word "correlated" implies that the Brownian motions (W_t^S) and (W_t^σ) are correlated with correlation coefficients $\rho_{k,ij} = d\langle W_{t,k}^S, W_{t,ij}^\sigma \rangle / dt$. In our correlated model, the processes (S_t^0), (S_t) and (X_t^π) evolve as in Section 3.1.

4.1 Optimal portfolios for logarithmic utility

For the logarithmic utility function $U(x) = \log(x)$, we deal with the value function

$$\Phi(t, x, \Sigma) = \max_{\pi} E^{t,x,\Sigma} \left(\log \left(X_T^\pi \right) \right) \tag{4.1}$$

in the same way as in Section 3.1.1. Replacing X_T^π in (4.1) by its solution (2.21) with $X_t^\pi = x$, it yields

$$\max_{(\pi_s)} E^{t,x,\Sigma} \left(\log \left(X_T^\pi \right) \right) = \log x + E^{t,x,\Sigma} \left[\int_t^T \max_{(\pi_s)} \left(\pi_s^T \left(B\left(\Sigma_s\right) - \mathbf{r} \right) + r - \frac{1}{2} \| \pi_s^T \Sigma_s^{1/2} \|_2^2 \right) ds \right].$$

Denote

$$H(\pi) = \pi^T \left(B(\Sigma) - \mathbf{r} \right) + r - \frac{1}{2} \| \pi^T \Sigma^{1/2} \|_2^2$$

and calculate the first derivative and the Hessian matrix of $H(\pi)$, we get the optimal portfolio strategy:

Corollary 4.1.1. *The optimal portfolio in the forth model with logarithmic utility is*

$$\pi_s^* = \Sigma_s^{-1} \left(B(\Sigma_s) - \boldsymbol{r} \right) \tag{4.2}$$

given (π_s^*) satisfying (3.1).

4.2 Optimal portfolios for power utility

We assume that $\rho_{q,ij} = 0$ for $q \neq i$ and denote

$$(\rho_{q,qj})_{1\leq q,j\leq d} = (\rho_{qj})_{1\leq q,j\leq d} = \begin{pmatrix} \rho_{11} & \cdots & \rho_{1d} \\ \vdots & \vdots & \vdots \\ \rho_{d1} & \cdots & \rho_{dd} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\rho}^T \\ \vdots \\ \boldsymbol{\rho}^T \end{pmatrix} \tag{4.3}$$

with $\boldsymbol{\rho} = (\rho_{11} \ldots \rho_{1d})^T$ and $\boldsymbol{\rho}_j = \rho_{1j}$.

We apply the Hamilton-Jacobi-Bellman (HJB) principle to obtain explicitly the optimal portfolio and the value function in some parameter settings. We discuss specially the case that the assets drift is a linear function of the volatility function, i.e. $B(\Sigma) - \mathbf{r} = \Sigma\mathbf{v}$ with $\mathbf{v} \in \mathbb{R}^d$. In this case, a candidate of the value function is given by a Feynman-Kac representation and the conditions for its boundedness are studied and given in Proposition 4.2.7. Under these conditions we get the explicit expression of the value function. Finally, a verification result for the obtained value function candidate is presented.

4.2.1 The HJB equation and the optimal portfolio strategy

We consider now the optimization problem with power utility function $u(x) = \frac{x^\gamma}{\gamma}$, $\gamma < 1, \gamma \neq 0$. The optimization problem of the investor now reads as

$$\max_\pi E\left(\frac{(X_T^\pi)^\gamma}{\gamma}\right)$$

and its corresponding value function is denoted by

$$\Phi(t,x,\Sigma) = \max_\pi E^{t,x,\Sigma}\left(\frac{(X_T^\pi)^\gamma}{\gamma}\right) \tag{4.4}$$

with $\Sigma_t = \Sigma$ and $X_t = x$. Then we face a multidimensional control problem with state process $(t, X_t, \Sigma_t)_{t\geq 0}$.

We assume that $G(t,x,\Sigma)$ is a candidate for the value function, then from Lemma 2.6.2, one derives the Hamilton-Jacobi-Bellman equation as follows:

$$\partial_t G + Tr\left((\nabla G)\left(\Omega\Omega^T + M\Sigma + \Sigma M^T\right)\right) + rx\partial_x G + \sup_{\pi\in\mathbb{R}^d}\Bigg\{x\pi^T(B-\mathbf{r})\partial_x G +$$
$$\frac{1}{2}\frac{d\langle X\rangle}{dt}\partial_{x,x}G + \frac{1}{2}\sum_{l,k,p,q=1}^{d}\frac{d\langle\Sigma_{lk},\Sigma_{pq}\rangle}{dt}\partial_{\Sigma_{lk},\Sigma_{pq}}G + \sum_{l,k=1}^{d}\frac{d\langle\Sigma_{lk},X\rangle}{dt}\partial_{\Sigma_{lk},x}G\Bigg\} = 0, \tag{4.5}$$

where the terminal condition is $G(T,x,\Sigma) = \frac{1}{\gamma}x^\gamma$, $\gamma < 1, \gamma \neq 0$. The notation ∂ denotes the partial derivative, i.e. $\partial_t G = \frac{\partial}{\partial t}G(t,x,\Sigma)$ and we denote

$$\nabla := \left(\frac{\partial}{\partial\Sigma_{ij}}\right)_{1\leq i,j\leq d}.$$

Note that since Σ is symmetric, we have $G(t,x,\Sigma) = G(t,x,\Sigma^T)$. It follows $\nabla G = \nabla^T G$. Applying Lemma 2.6.1 and by direct calculation, one gets

$$\begin{aligned}
&\sum_{l,k,p,q=1}^{d} \frac{d\langle \Sigma_{lk}, \Sigma_{pq}\rangle}{dt} \partial_{\Sigma_{lk},\Sigma_{pq}} G \\
=& \sum_{l,k,p,q=1}^{d} \left(\Sigma_{lp} \left(Q^T Q\right)_{kq} + \Sigma_{pk} \left(Q^T Q\right)_{ql} + \Sigma_{lq} \left(Q^T Q\right)_{kp} + \Sigma_{kq} \left(Q^T Q\right)_{lp} \right) \partial_{\Sigma_{lk},\Sigma_{pq}} G \\
=& Tr\left(\Sigma \nabla (Q^T Q) \nabla\right) G + 2Tr\left(\Sigma \nabla (Q^T Q) \nabla\right) G + Tr\left(\Sigma \nabla (Q^T Q) \nabla\right) G \\
=& 4Tr\left(\Sigma \nabla (Q^T Q) \nabla\right) G
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{l,k=1}^{d} \frac{d\langle \Sigma_{lk}, X\rangle}{dt} \partial_{\Sigma_{lk},x} G \\
=& \sum_{l,k=1}^{d} \left(x \left(\pi^T \Sigma^{1/2} dW_t^S\right) \left(\Sigma^{1/2} dW_t^\sigma Q\right)_{lk} + x \left(\pi^T \Sigma^{1/2} dW_t^S\right) \left(Q^T \left(dW_t^\sigma\right)^T \Sigma^{1/2}\right)_{lk} \right) \frac{\partial_{\Sigma_{lk},x} G}{dt} \\
=& x \sum_{l,k=1}^{d} \left(\sum_{pqij=1}^{d} \pi_p \Sigma_{pq}^{1/2} dW_{t,q}^S \Sigma_{li}^{1/2} dW_{t,ij}^\sigma Q_{jk} + \pi_p \Sigma_{pq}^{1/2} dW_{t,q}^S Q_{il} dW_{t,ji}^\sigma \Sigma_{jk}^{1/2} \right) \frac{\partial_{\Sigma_{lk},x} G}{dt} \\
=& x \sum_{l,k=1}^{d} \left(\sum_{pqij=1}^{d} \pi_p \Sigma_{pq}^{1/2} \Sigma_{li}^{1/2} Q_{jk} \rho_{q,ij} + \pi_p \Sigma_{pq}^{1/2} Q_{il} \Sigma_{jk}^{1/2} \rho_{q,ji} \right) \partial_{\Sigma_{lk},x} G, \\
=& x \sum_{l,k=1}^{d} \left(\sum_{pqij=1}^{d} \pi_p \Sigma_{pq}^{1/2} \Sigma_{lq}^{1/2} Q_{jk} \boldsymbol{\rho}_j + \pi_p \Sigma_{pq}^{1/2} Q_{il} \Sigma_{qk}^{1/2} \boldsymbol{\rho}_i \right) \partial_{\Sigma_{lk},x} G, \\
=& 2x\pi^T \Sigma \nabla \left(Q^T \boldsymbol{\rho}\right) \partial_x G.
\end{aligned} \tag{4.6}$$

Then, the HJB equation (4.5) implies

$$\begin{aligned}
&\partial_t G + Tr\left((\nabla G)\left(\Omega\Omega^T + M\Sigma + \Sigma M^T\right)\right) + rx\partial_x G + 2\left[Tr\left(\Sigma \nabla (Q^T Q) \nabla\right) G\right] \\
&+ \sup_{\pi \in \mathbb{R}^d} \left\{ x\pi^T (B - \mathbf{r}) \partial_x G + \frac{x^2}{2} \pi^T \Sigma \pi \partial_{x,x} G + 2x\pi^T \Sigma \nabla \left(Q^T \boldsymbol{\rho}\right) \partial_x G \right\} = 0.
\end{aligned} \tag{4.7}$$

We use in the following the Ansatz

$$G(t,x,\Sigma) = \frac{x^\gamma}{\gamma} V(t,\Sigma), \quad \gamma < 1, \ \gamma \neq 0$$

together with the terminal condition $V(T,\Sigma) = 1$. Then plugging in the HJB equation (4.7) yields that $V(t,\Sigma)$ solves

$$\begin{aligned}
&\frac{1}{\gamma}\left[\partial_t V + 2Tr\left(\Sigma \nabla Q^T Q \nabla\right) V + Tr\left((\nabla V)\left(\Omega\Omega^T + M\Sigma + \Sigma M^T\right)\right)\right] \\
&+ rV + \sup_{\pi \in \mathbb{R}^d} \left\{ \underbrace{\pi^T (B - \mathbf{r}) V + \frac{\gamma - 1}{2} \pi^T \Sigma \pi V + 2\pi^T \Sigma \nabla \left(Q^T \boldsymbol{\rho}\right) V}_{=F(\pi)} \right\} = 0.
\end{aligned} \tag{4.8}$$

Proposition 4.2.1. *The maximizer of (4.8) is given by*

$$\pi^*(t,\Sigma_t) = \Sigma_t^{-1}\left(\frac{(B(\Sigma_t) - \boldsymbol{r})V(t,\Sigma_t) + N(\Sigma_t)}{(1-\gamma)V(t,\Sigma_t)}\right), \quad 0 \le t \le T \tag{4.9}$$

for $\gamma < 1,\ \gamma \neq 0$ *with*

$$N(\Sigma_t) = 2\Sigma_t \nabla \left(Q^T \boldsymbol{\rho}\right) V(t,\Sigma_t) \in \mathbb{R}^d. \tag{4.10}$$

Proof. To solve the optimization problem, we calculate the first derivative and the Hessian matrix of $F(\pi)$. By direct calculation we derive

$$\partial_\pi F(\pi) = V(B - \mathbf{r}) + (\gamma - 1)\Sigma\pi V + N$$

with

$$N = 2\partial_\pi\left(\pi^T \Sigma \nabla \left(Q^T \boldsymbol{\rho}\right) V\right) = 2\Sigma\nabla Q^T \boldsymbol{\rho} V \in \mathbb{R}^d.$$

The Hessian matrix of $F(\pi)$ is given by

$$\partial_{\pi,\pi} F(\pi) = (\gamma - 1)\Sigma V.$$

For $\|\pi\|_2 \to \infty$, it is obvious that $F(\pi)$ goes to negative infinity. Since the Hessian matrix of $F(\pi)$, i.e. $\partial_{\pi,\pi}F(\pi)$ is negative definite for $\gamma < 1,\ \gamma \neq 0$, it follows that the multivariate quadratic function $F(\pi)$ has a maximum point and the maximum is achieved at π^* satisfying $\partial_\pi F(\pi^*) = 0$. We have then

$$\pi^*(t,\Sigma) = \Sigma^{-1}\left(\frac{(B(t,\Sigma) - \mathbf{r})V(t,\Sigma) + N(t,\Sigma)}{(1-\gamma)V(t,\Sigma)}\right) \tag{4.11}$$

for $\gamma < 1,\ \gamma \neq 0$. Note that Σ_t is always invertible, since it is a symmetric positive definite matrix. □

Plugging $\pi^*(t,\Sigma)$ in (4.8), we get

$$\begin{aligned}
&\sup_\pi (F(\pi)) = F(\pi^*) \\
=& \underbrace{\frac{(B-\mathbf{r})^T V + N^T}{(1-\gamma)V}\Sigma^{-1}(B-\mathbf{r})V}_{I} + \underbrace{\frac{\gamma-1}{2}\frac{(B-\mathbf{r})^T V + N^T}{(1-\gamma)V}\Sigma^{-1}\frac{(B-\mathbf{r})V + N}{(1-\gamma)V}V}_{II} \\
&+ \underbrace{\frac{(B-\mathbf{r})^T V + N^T}{(1-\gamma)V}\Sigma^{-1}N}_{III} \\
=& \frac{1-\gamma}{2}\left(\frac{(B-\mathbf{r})^T V + N^T}{(1-\gamma)V}\Sigma^{-1}\frac{(B-\mathbf{r})V + N}{(1-\gamma)V}V\right).
\end{aligned} \tag{4.12}$$

The last equation in (4.12) is derived from $I + III = -2II$. Thus, (4.8) reduces to

$$\begin{aligned}
&\partial_t V + 2\left[Tr\left(\Sigma\nabla(Q^T Q)\nabla\right)V\right] + Tr\left((\nabla V)\left(\Omega\Omega^T + M\Sigma + \Sigma M^T\right)\right) \\
&+ \gamma r V + \frac{\gamma}{2(1-\gamma)V}\left((B-\mathbf{r})^T\Sigma^{-1}(B-\mathbf{r})V^2 + 2N^T\Sigma^{-1}(B-\mathbf{r})V + N^T\Sigma^{-1}N\right) = 0.
\end{aligned} \tag{4.13}$$

4.2.2 The case of the general asset process $(S_t)_{t\geq 0}$

In this section, we want to solve the partial differential equation (4.13) for the case that the assets process $(S_t)_{t\geq 0}$ owns assets dynamic (2.16). We assume that $\boldsymbol{\rho} = \hat{\rho}\mathbf{1}$ and the matrix Q admits

$$Q(i,:) = k_i \cdot Q(1,:), \quad k_i \in \mathbb{R}, \quad 2 \leq i \leq d. \tag{4.14}$$

Set $k_1 = 1$, we have

$$Q^T \boldsymbol{\rho}\boldsymbol{\rho}^T Q = \hat{\rho}^2 Q^T \mathbf{1}\mathbf{1}^T Q = \rho^2 Q^T Q$$

with

$$\rho^2 = \hat{\rho}^2 \frac{\left(\sum_{i=1}^d k_i\right)^2}{\sum_{i=1}^d k_i^2}, \tag{4.15}$$

since

$$\left(Q^T \mathbf{1}\mathbf{1}^T Q\right)_{pq} = \sum_{ij} Q_{ip} Q_{jq} = \sum_{ij} k_i k_j Q_{1p} Q_{1q}$$

and

$$\left(Q^T Q\right)_{pq} = \sum_i k_i^2 Q_{1p} Q_{1q}.$$

Then, by (4.10) one gets

$$\begin{aligned}
& N^T \Sigma^{-1} N \\
=& 4\boldsymbol{\rho}^T Q \nabla V \Sigma \cdot \nabla V Q^T \boldsymbol{\rho} \\
=& 4Tr\left(\boldsymbol{\rho}^T Q \nabla V \Sigma \cdot \nabla V Q^T \boldsymbol{\rho}\right) \\
=& 4\rho^2 Tr\left(\Sigma \nabla V Q^T Q \nabla V\right).
\end{aligned}$$

Plugging the above expression for $N^T\Sigma^{-1}N$ in (4.13), we get

$$\begin{aligned}
&\partial_t V + 2\left[Tr\left(\Sigma\nabla(Q^TQ)\nabla\right)V\right] + Tr\left((\nabla V)\left(\Omega\Omega^T + M\Sigma + \Sigma M^T\right)\right) \\
&+ \gamma r V + \frac{\gamma}{2(1-\gamma)V}\left((B-\mathbf{r})^T \Sigma^{-1}(B-\mathbf{r}) V^2 + 2N^T\Sigma^{-1}(B-\mathbf{r}) V\right) \\
&+ \frac{2\gamma\rho^2}{(1-\gamma)V} Tr\left(\Sigma\nabla V Q^T Q \nabla V\right) = 0.
\end{aligned} \tag{4.16}$$

In the following we make a further transformation by

$$V(t,\Sigma) = \nu(t,\Sigma)^{\delta}$$

for a parameter δ to be determined. Differentiating $V(t,\Sigma)$ yields

$$\begin{aligned}
\partial_t V = \delta \partial_t \nu \nu^{\delta-1}, \quad \partial_{\Sigma_{lk}} V = \delta \nu^{\delta-1}\partial_{\Sigma_{lk}}\nu, \\
\partial_{\Sigma_{lk},\Sigma_{pq}} V = \delta\nu^{\delta-1}\partial_{\Sigma_{lk},\Sigma_{pq}}\nu + \delta(\delta-1)\nu^{\delta-2}\partial_{\Sigma_{lk}}\nu \partial_{\Sigma_{pq}}\nu
\end{aligned}$$

for $1 \leq l, k, p, q \leq d$. Plugging the above derivatives in (4.16) leads to

$$\begin{aligned}
&\delta\partial_t \nu \nu^{\delta-1} + 2\delta\nu^{\delta-1}\left[Tr\left(\Sigma\nabla(Q^TQ)\nabla\right)\nu\right] + Tr\left(\left(\Omega\Omega^T + M\Sigma + \Sigma M^T + H\right)\nabla\nu\right)\delta\nu^{\delta-1} \\
&+ \left(\gamma r + \frac{\gamma}{2\left(1-\gamma\right)}\left(B-\mathbf{r}\right)^T\Sigma^{-1}\left(B-\mathbf{r}\right)\right)\nu^\delta \\
&+ \left(2\delta\left(\delta-1\right)\nu^{\delta-2} + \frac{2\gamma\rho^2\delta^2\nu^{\delta-2}}{\left(1-\gamma\right)}\right)\cdot Tr\left(\Sigma\nabla\nu Q^TQ\nabla\nu\right) = 0 \qquad (4.17)
\end{aligned}$$

with

$$H := \frac{\gamma}{(1-\gamma)}\left(Q^T\boldsymbol{\rho}(B-\mathbf{r})^T + (B-\mathbf{r})\boldsymbol{\rho}^TQ\right). \qquad (4.18)$$

The H in (4.18) comes from the term $\frac{\gamma}{2(1-\gamma)V}\cdot 2N^T\Sigma^{-1}\left(B-\mathbf{r}\right)V$ in (4.16) and the expression for H follows from the following computation:

$$\begin{aligned}
&\frac{\gamma}{(1-\gamma)}N^T\Sigma^{-1}\left(B-\mathbf{r}\right) \\
=&\frac{2\gamma}{(1-\gamma)}\left(\boldsymbol{\rho}^TQ\nabla V\right)\left(B-\mathbf{r}\right) \\
=&2Tr\left(\frac{\gamma}{(1-\gamma)}\nabla V Q^T\boldsymbol{\rho}(B-\mathbf{r})^T\right) \\
=&2Tr\left(\frac{\gamma}{(1-\gamma)}\nabla \nu Q^T\boldsymbol{\rho}(B-\mathbf{r})^T\right)\delta\nu^{\delta-1} \\
=&Tr\left(\frac{\gamma}{(1-\gamma)}\left[Q^T\boldsymbol{\rho}(B-\mathbf{r})^T + \left(Q^T\boldsymbol{\rho}(B-\mathbf{r})^T\right)^T\right]\nabla\nu\right)\delta\nu^{\delta-1} = Tr\left(H\nabla\nu\right)\delta\nu^{\delta-1}.
\end{aligned}$$

Multiplying both sides of (4.17) with $\frac{1}{\delta\nu^{\delta-1}}$ yields

$$\begin{aligned}
0 = &\partial_t\nu + 2Tr\left(\Sigma\nabla(Q^TQ)\nabla\right)\nu + Tr\left(\left(\Omega\Omega^T + M\Sigma + \Sigma M^T\right)\nabla\nu + H\nabla\nu\right) \\
&+ \left(\frac{\gamma r}{\delta} + \frac{\gamma}{2\left(1-\gamma\right)\delta}\left(B-\mathbf{r}\right)^T\Sigma^{-1}\left(B-\mathbf{r}\right)\right)\nu + \left(\frac{2\left(\delta-1\right)}{\nu} + \frac{2\gamma\rho^2\delta}{\left(1-\gamma\right)\nu}\right)Tr\left(\Sigma\nabla\nu Q^TQ\nabla\nu\right). \qquad (4.19)
\end{aligned}$$

If we set

$$\delta = \frac{(1-\gamma)}{(1-\gamma)+\gamma\rho^2}, \qquad (4.20)$$

then (4.19) becomes a linear partial differential equation, i.e. (4.19) reduces to

$$\begin{aligned}
&\partial_t\nu + 2\left[Tr\left(\Sigma\nabla(Q^TQ)\nabla\right)\nu\right] \\
&+ Tr\left(\left(\Omega\Omega^T + M\Sigma + \Sigma M^T + H\right)\nabla\nu\right) + \left(\frac{\gamma r}{\delta} + \frac{\gamma}{2\left(1-\gamma\right)\delta}\left(B-\mathbf{r}\right)^T\Sigma^{-1}\left(B-\mathbf{r}\right)\right)\nu = 0
\end{aligned}$$

with terminal condition $\nu(T,\Sigma) = 1$. Thus, we have shown:

Theorem 4.2.2. *Let $\boldsymbol{\rho} = \mathbf{1}\hat{\rho}$, Q satisfy (4.14) and ρ is given by (4.15), then if we use the transformation*

$$G\left(t,x,\Sigma\right) = \frac{x^\gamma}{\gamma}\nu^\delta\left(t,\Sigma\right), \quad \gamma < 1, \ \gamma \neq 0$$

with δ given by (4.20), the HJB equation for the optimization problem (4.4) reads as

$$\begin{aligned}&\partial_t \nu + 2\left[Tr\left(\Sigma\nabla(Q^T Q)\nabla\right)\nu\right]\\&+Tr\left(\left(\Omega\Omega^T + M\Sigma + \Sigma M^T + H\right)\nabla\nu\right) + \left(\frac{\gamma r}{\delta} + \frac{\gamma}{2(1-\gamma)\delta}(B-\mathbf{r})^T\Sigma^{-1}(B-\mathbf{r})\right)\nu = 0\end{aligned} \tag{4.21}$$

with terminal condition $\nu(T,\Sigma)=1$.

Recalling that in our correlated Wishart volatility model, the volatility process (Σ_t) evolves as

$$d\Sigma_t = \left(\Omega\Omega^T + M\Sigma_t + \Sigma_t M^T\right)dt + \Sigma_t^{1/2}dW_t^\sigma Q + Q^T\left(dW_t^\sigma\right)^T\Sigma_t^{1/2}$$

under the physical measure $\mathbb{P}$. We denote a new measure by $\tilde{\mathbb{P}}$ associating with the following Radon-Nikodym derivative:

$$Z_t := \left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(\int_t^T Tr\left(\theta^T(\Sigma_s)\,dW_s^\sigma\right) - \frac{1}{2}\int_t^T \|\theta(\Sigma_s)\|^2 ds\right) \tag{4.22}$$

with

$$\theta(\Sigma) = \frac{\gamma}{1-\gamma}\left(\Sigma^{1/2}\right)^{-1}\left(B(\Sigma)-\mathbf{r}\right)\boldsymbol{\rho}^T. \tag{4.23}$$

Then, we have the following proposition:

Proposition 4.2.3. *If the Radon-Nikodym derivative (Z_t) is an $(\mathcal{F}_t)$-martingale and if*

$$\tilde{\nu}(t,\Sigma) := \tilde{E}^{\Sigma,t}\left[\exp\left(\int_t^T\left(\frac{\gamma r}{\delta} + \frac{\gamma}{2(1-\gamma)\delta}\left(B(\Sigma_s)-\mathbf{r}\right)^T\Sigma_s^{-1}\left(B(\Sigma_s)-\mathbf{r}\right)\right)ds\right)\right] \in C^{1,2}(o) \tag{4.24}$$

with $o=[t,T]\times S_d(\mathbb{R})$, then, the solution $\nu(t,\Sigma)$ of (4.21) has the Feynman-Kac representation $\tilde{\nu}(t,\Sigma)$, where $\tilde{E}$ denotes the expectation under $\tilde{\mathbb{P}}$.

Proof. Let us first identify the dynamic of (Σ_t) under $\tilde{\mathbb{P}}$. From (4.22) we know that

$$\tilde{W}_t^\sigma = W_t^\sigma - \int_0^t \theta(\Sigma_s)\,ds$$

is a $d\times d$ Brownian motions matrix under $\tilde{\mathbb{P}}$ by the Girsanov theorem. We can also check that the equality

$$\Sigma^{1/2}\theta Q + Q^T\theta^T\Sigma^{1/2} = H \tag{4.25}$$

holds. Thus, it yields

$$\begin{aligned}d\Sigma_t &= \left(\Omega\Omega^T + M\Sigma_t + \Sigma_t M^T\right)dt + \Sigma_t^{1/2}dW_t^\sigma Q + Q^T\left(dW_t^\sigma\right)^T\Sigma_t^{1/2}\\&= \left(\Omega\Omega^T + M\Sigma_t + \Sigma_t M^T + H\right)dt + \Sigma_t^{1/2}d\tilde{W}_t^\sigma Q + Q^T\left(d\tilde{W}_t^\sigma\right)^T\Sigma_t^{1/2},\end{aligned} \tag{4.26}$$

i.e. the Wishart process $(\Sigma_t)_{t\geq 0}$ possesses the Wishart dynamic (4.26) under $\tilde{\mathbb{P}}$.

We get then for $f \in C^2\left(S_d^+(\mathbb{R})\right)$,

$$(\mathcal{A}f)(\Sigma) = 2\left[Tr\left(\Sigma\nabla(Q^TQ)\nabla\right)f\right] + Tr\left(\left(\Omega\Omega^T + M\Sigma + \Sigma M^T + H\right)\nabla f\right)$$

under $\tilde{\mathbb{P}}$ by Lemma 2.6.2. Then, the differential equation (4.21) can be written as

$$\mathcal{A}_t\nu = -\partial_t\nu - \left(\frac{\gamma r}{\delta} + \frac{\gamma}{2(1-\gamma)\delta}\left(B_t - \mathbf{r}\right)^T\Sigma_t^{-1}\left(B_t - \mathbf{r}\right)\right)\nu,$$
$$\nu(T,\Sigma) = 1.$$

Applying the theorem of Feynman-Kac representation in [33, Theorem 3.26.], we conclude that the representation in (4.24) is the solution of (4.21) with $\nu(T,\Sigma) = 1$ under proper conditions. □

Remark 4.2.4. *a) Note that for the cases that $B(\Sigma_s) - \mathbf{r} = 0$ and $B(\Sigma_s) - \mathbf{r} = \Sigma_s^{1/2}\mathbf{v}$ with $\mathbf{v} \in \mathbb{R}^d$, we can easily get that (4.22) is a martingale and (4.24) belongs to $C^{1,2}(o)$.*

b) The condition in (4.24) is not generally satisfied. We refer to Korn & Kraft [34, Proposition 3.4] for a counterexample in the setting of Heston model.

4.2.3 The case of $B(\Sigma) - \mathbf{r} = \Sigma\mathbf{v}$

Since for general $B(\Sigma)$, it is not possible to compute the expectation or verify the solution, we consider in this section a special case of $B(\Sigma)$ satisfying $B(\Sigma) - \mathbf{r} = \Sigma\mathbf{v}$ for a $\mathbf{v} \in \mathbb{R}^d$. We drop the former restrictions on Q and $\rho_{q,ij}$ (i.e. there is no $\boldsymbol{\rho} = \hat{\rho}\mathbf{1}$), but the correlation coefficients $\rho_{q,ij}$, $1 \leq q,i,j \leq d$ are still assumed to satisfy $\rho_{q,ij} = 0$ for $q \neq i$ and we still have (4.3). The asset dynamic $(S_t)_{t\geq 0}$ can now be written as

$$d\tilde{S}_t = diag\left(\tilde{S}_t\right)\left(\left(\mathbf{r} + \Sigma_t\mathbf{v}\right)dt + \Sigma_t^{1/2}dW_t^S\right). \tag{4.27}$$

The differential equation (4.13) now reads as

$$\partial_t V + 2Tr\left(\Sigma\nabla(Q^TQ)\nabla\right)V + Tr\left((\nabla V)\left(\Omega\Omega^T + M\Sigma + \Sigma M^T\right)\right) + \gamma r V$$
$$+ \frac{\gamma}{2(1-\gamma)V}\left(\mathbf{v}^T\Sigma\mathbf{v}V^2 + 2N^T\mathbf{v}V + N^T\Sigma^{-1}N\right) = 0. \tag{4.28}$$

with $N = 2\Sigma\nabla Q^T\boldsymbol{\rho}V$ given in (4.10) and $V(T,\Sigma) = 1$.

Theorem 4.2.5. *The partial differential equation (4.28) given initial value $V(T,\Sigma) = 1$ possesses the following solution in case that the expressions are finite:*

$$V(t,\Sigma) = \exp\left(\phi^{(2)}(T-t) + Tr[\psi^{(2)}(T-t)\Sigma]\right), \tag{4.29}$$

where $\phi^{(2)}(T-t) \in \mathbb{R}$ and $\psi^{(2)}(T-t) \in S_d$ are solutions of the following Riccati equations system:

$$-\frac{d\psi^{(2)}(T-t)}{dt} = \psi^{(2)}(T-t)\tilde{M} + \tilde{M}^T\psi^{(2)}(T-t) + 2\psi^{(2)}(T-t)\tilde{Q}^T\tilde{Q}\psi^{(2)}(T-t) + \tilde{\Gamma}, \tag{4.30}$$

$$-\frac{d\phi^{(2)}(T-t)}{dt} = Tr[\psi^{(2)}(T-t)\Omega\Omega^T] + \gamma r \tag{4.31}$$

with

$$\tilde{M} = M + \frac{\gamma}{(1-\gamma)} Q^T \boldsymbol{\rho} \mathbf{v}^T, \quad \tilde{Q}^T\tilde{Q} = Q^TQ + \frac{\gamma}{(1-\gamma)} Q^T \boldsymbol{\rho}\boldsymbol{\rho}^T Q, \quad \tilde{\Gamma} = \frac{\gamma}{2(1-\gamma)} \mathbf{v}\mathbf{v}^T$$

and the initial conditions: $\psi^{(2)}(0) = \mathbf{0}, \phi^{(2)}(0) = 0$. *The notation* $\mathbf{0}$ *denotes a d by d zero matrix.*

Proof. We simply verify that $V(t, \Sigma)$ given in (4.29) satisfies (4.28). We note that

$$\partial_t V = V \left(\frac{d\phi^{(2)}(T-t)}{dt} + Tr\left[\frac{d\psi^{(2)}(T-t)}{dt} \Sigma \right] \right),$$

$$\nabla V = V \psi^{(2)}(T-t)$$

and

$$\partial_{\Sigma_{lk}, \Sigma_{pq}} V = V \psi^{(2)}_{qp}(T-t) \psi^{(2)}_{kl}(T-t).$$

The left side of (4.28) reduces to

$$\begin{aligned}\Upsilon(t) =& \partial_t V + 2Tr\left(\Sigma\nabla(Q^TQ)\nabla\right)V + Tr\left((\nabla V)\left(\Omega\Omega^T + M\Sigma + \Sigma M^T\right)\right) + \gamma r V \\ &+ \frac{\gamma}{2(1-\gamma)V}\left(\mathbf{v}^T\Sigma\mathbf{v}V^2 + 2N^T\mathbf{v}V + N^T\Sigma^{-1}N\right)\end{aligned}$$

with

$$N = 2\Sigma\nabla\left(Q^T\boldsymbol{\rho}\right)V. \tag{4.32}$$

Plugging the derivatives into $\Upsilon(t)$, we get

$$\begin{aligned}\Upsilon(t) =& V\left(\frac{d\phi^{(2)}(T-t)}{dt} + Tr\left[\frac{d\psi^{(2)}(T-t)}{dt}\Sigma \right]\right) + 2Tr\left(\Sigma\psi^{(2)}(T-t)(Q^TQ)\psi^{(2)}(T-t)\right)V \\ &+ VTr\left(\psi^{(2)}(T-t)(\Omega\Omega^T + M\Sigma + \Sigma M^T)\right) + \gamma r V + \\ &+ \frac{\gamma V}{2(1-\gamma)}\left(\mathbf{v}^T\Sigma\mathbf{v} + 4\boldsymbol{\rho}^TQ\psi^{(2)}(T-t)\Sigma\mathbf{v} + 4\boldsymbol{\rho}^TQ\psi^{(2)}(T-t)\Sigma\psi^{(2)}(T-t)Q^T\boldsymbol{\rho}\right) \\ =& V\left(\frac{d\phi^{(2)}(T-t)}{dt} + Tr\left[\frac{d\psi^{(2)}(T-t)}{dt}\Sigma \right] + 2Tr\left(\Sigma\psi^{(2)}(T-t)(Q^TQ)\psi^{(2)}(T-t)\right)\right. \\ &+ Tr\left(\psi^{(2)}(T-t)(\Omega\Omega^T + M\Sigma + \Sigma M^T)\right) + \gamma r + \\ &\left. + \frac{\gamma}{2(1-\gamma)}Tr\left(\mathbf{v}^T\Sigma\mathbf{v} + 4\boldsymbol{\rho}^TQ\psi^{(2)}(T-t)\Sigma\mathbf{v} + 4\boldsymbol{\rho}^TQ\psi^{(2)}(T-t)\Sigma\psi^{(2)}(T-t)Q^T\boldsymbol{\rho}\right)\right).\end{aligned}$$

Note that

$$\begin{aligned}&\frac{\gamma}{2(1-\gamma)}Tr\left(\mathbf{v}^T\Sigma\mathbf{v}\right) = Tr\left(\tilde{\Gamma}\Sigma\right), \\ &\frac{\gamma}{2(1-\gamma)}Tr\left(4\boldsymbol{\rho}^TQ\psi^{(2)}(T-t)\Sigma\mathbf{v}\right) \\ =& \frac{\gamma}{(1-\gamma)}Tr\left(\mathbf{v}\boldsymbol{\rho}^TQ\psi^{(2)}(T-t)\Sigma\right) + \frac{\gamma}{(1-\gamma)}Tr\left(\psi^{(2)}(T-t)Q^T\boldsymbol{\rho}\mathbf{v}^T\Sigma\right)\end{aligned}$$

and

$$
\begin{aligned}
&\frac{\gamma}{2(1-\gamma)} Tr\left(4\boldsymbol{\rho}^T Q\psi^{(2)}(T-t)\Sigma\psi^{(2)}(T-t)Q^T\boldsymbol{\rho}\right) \\
=&\frac{2\gamma}{(1-\gamma)} Tr\left(\psi^{(2)}(T-t)Q^T\boldsymbol{\rho}\boldsymbol{\rho}^T Q\psi^{(2)}(T-t)\Sigma\right),
\end{aligned}
$$

we can get $\Upsilon(t) = 0$ for $t \in [0, T]$ by (4.30) and (4.31). To ensure $V(T, \Sigma) = 1$, we need the initial conditions $\psi^{(2)}(0) = \mathbf{0}$ and $\phi^{(2)}(0) = 0$. □

Remark 4.2.6. *It is known from Proposition 2.4.10 that (4.29) is the explicit solution of the following Laplace transform of the integrated Wishart process* $(\tilde{\Sigma}_t)$*:*

$$
V(\Sigma, t) = E^{\Sigma,t}\left[\exp\left(\gamma r(T-t) + \int_t^T Tr\left(\tilde{\Gamma}\tilde{\Sigma}_s\right) ds\right)\right] \tag{4.33}
$$

with

$$
d\tilde{\Sigma}_t = \left(\Omega\Omega^T + \tilde{M}\tilde{\Sigma}_t + \tilde{\Sigma}_t\tilde{M}^T\right) dt + \tilde{\Sigma}_t^{1/2} dW_t^\sigma \tilde{Q} + \tilde{Q}^T \left(dW_t^\sigma\right)^T \tilde{\Sigma}_t^{1/2}, \quad \tilde{\Sigma}_t = \Sigma \tag{4.34}
$$

under proper conditions. But the representation (4.33) satisfying the initial condition $\tilde{\Sigma}_t = \Sigma$ *is not obtainable in the setting of the HJB equation (4.28). Let us recall that in the HJB equation, we have defined*

$$
d\Sigma_t = \left(\Omega\Omega^T + M\Sigma_t + \Sigma_t M^T\right) dt + \Sigma_t^{1/2} dW_t^\sigma Q + Q^T \left(dW_t^\sigma\right)^T \Sigma_t^{1/2} \tag{4.35}
$$

under $\mathbb{P}$ *with the initial condition* $\Sigma_t = \Sigma$*. To get the Wishart process (4.34), we need to consider the process*

$$
\hat{\Sigma}_t = \tilde{Q}^T Q^{-T}\Sigma_t Q^{-1}\tilde{Q}
$$

and then apply the Girsanov theorem to the process $(\hat{\Sigma}_t)$ *to get* $(\tilde{\Sigma}_t)$*. It implies that for* $\tilde{Q} \neq Q$*, we can not have* $\tilde{\Sigma}_t = \Sigma = \Sigma_t$*, which means that* Σ *can not be the initial value of* $(\tilde{\Sigma}_t)$ *in the setting of the HJB equation (4.28) for* $\tilde{Q} \neq Q$*.*

Applying Corollary 2.4.5, Proposition 2.4.6 and Proposition 2.4.10, we can solve the Riccati equations system (4.30) and (4.31). Then, we get the following proposition:

Proposition 4.2.7. *Suppose* $\tilde{M}^T(\tilde{Q}^T\tilde{Q})^{-1} = (\tilde{Q}\tilde{Q})^{-1}\tilde{M}$*. Let* $h = T - t \in [0, T]$ *and we define*

$$
\begin{aligned}
\kappa^{(2)}(h) = &-\left(\sqrt{C_2^{(2)}}\cosh\left(\sqrt{C_2^{(2)}}h\right) + C_1^{(2)}\sinh\left(\sqrt{C_2^{(2)}}h\right)\right)^{-1} \cdot \\
&\left(\sqrt{C_2^{(2)}}\sinh\left(\sqrt{C_2^{(2)}}h\right) + C_1^{(2)}\cosh\left(\sqrt{C_2^{(2)}}h\right)\right), \\
C_2^{(2)} = &\;\tilde{Q}\left(-2\tilde{\Gamma} + \tilde{M}^T\tilde{Q}^{-1}\tilde{Q}^{-T}\tilde{M}\right)\tilde{Q}^T, \\
C_1^{(2)} = &-\tilde{Q}\tilde{M}^T(\tilde{Q}^T\tilde{Q})^{-1}\tilde{Q}^T
\end{aligned}
$$

with $\tilde{M}$*,* $\tilde{Q}$*,* $\tilde{\Gamma}$ *given in Theorem 4.2.5, then if*

$$
\tilde{Q}^T\tilde{Q} \in GL_d(\mathbb{R}), \quad -2\tilde{\Gamma} + \tilde{M}^T\tilde{Q}^{-1}\tilde{Q}^{-T}\tilde{M} \succ 0 \quad \text{and} \quad \sqrt{C_2^{(2)}} + C_1^{(2)} \succeq 0 \tag{4.36}
$$

are satisfied, the partial differential equation (4.28) given initial value $V(T,\Sigma)=1$ *possesses on* $[0,T]$ *the solution (4.29) with*

$$\psi^{(2)}(h) = \frac{\tilde{Q}^{-1}\sqrt{C_2^{(2)}}\kappa^{(2)}(h)\tilde{Q}^{-T}}{2} - \frac{\tilde{M}^T(\tilde{Q}^T\tilde{Q})^{-1}}{2}$$

$$\phi^{(2)}(h) = -Tr\left(\Omega\Omega^T\frac{\tilde{M}^T\left(\tilde{Q}^T\tilde{Q}\right)^{-1}}{2}\right)h + \gamma r h$$

$$-\frac{1}{2}Tr\left(\tilde{Q}^{-T}\Omega\Omega^T\tilde{Q}^{-1}\cdot\log\left[\left(\sqrt{C_2^{(2)}}\right)^{-1}\left(\sqrt{C_2^{(2)}}\cosh\left(\sqrt{C_2^{(2)}}h\right)+\tilde{C}_1\sinh\left(\sqrt{C_2^{(2)}}h\right)\right)\right]\right)$$

with $\psi^{(2)}(0)=\mathbf{0}$ *and* $\phi^{(2)}(0)=0$.

Remark 4.2.8. *a) Note that whether (4.33) is finite or not does not dependent on* Ω *from (4.36) . The last two conditions in (4.36) for the existence of the explicit solution of the Riccati equations system reduce to the single condition*

$$\tilde{\Gamma} \prec \tilde{M}^T\left(2\tilde{Q}^T\tilde{Q}\right)^{-1}\tilde{M}, \tag{4.37}$$

if $\tilde{M}^T\left(\tilde{Q}^T\tilde{Q}\right)^{-1}+\left(\tilde{Q}^T\tilde{Q}\right)^{-1}\tilde{M}$ *is negative semidefinite. Under this assumption,* $C_1^{(2)}$ *is positive semidefinite, thus the condition (4.37) implies the condition* $\sqrt{C_2^{(2)}}+C_1^{(2)}\succ 0$.

b) For the case of $d=1$ *the condition (4.36) coincide with the results in the Heston model in [35, Proposition 5.2]. Let us use the terminologies in Kraft [35] and denote for* $d=1$

$$M := -\frac{\kappa}{2},\quad \mathbf{v} := \bar{\lambda},\quad \boldsymbol{\rho} := \rho,\quad Q := \frac{\sigma}{2}\quad \tilde{M} = -\frac{\kappa}{2}+\frac{\gamma}{1-\gamma}\frac{\sigma}{2}\rho\bar{\lambda} := -\frac{\tilde{\kappa}}{2}.$$

In [35] it is assumed that $\sigma>0$, $\kappa>0$ *and* $\tilde{\kappa}>0$, *under these assumptions, it follows that* $\tilde{Q}^T\tilde{Q}=\frac{\sigma^2}{4}\left(1+\frac{\gamma}{1-\gamma}\rho^2\right)\in GL_d(\mathbb{R})$ *and* $\tilde{M}^T\left(\tilde{Q}^T\tilde{Q}\right)^{-1}+\left(\tilde{Q}^T\tilde{Q}\right)^{-1}\tilde{M}=-\frac{4\tilde{\kappa}}{\sigma^2\left(1+\frac{\gamma}{1-\gamma}\rho^2\right)}\le 0$. *The condition in (4.37) can be written as*

$$\frac{\gamma}{2(1-\gamma)}\bar{\lambda}^2 < \left(\kappa^2-\frac{2\gamma\bar{\lambda}\rho\sigma\kappa}{(1-\gamma)}+\frac{\gamma^2\bar{\lambda}^2\rho^2\sigma^2}{(1-\gamma)^2}\right)\cdot\frac{1}{2\sigma^2+\frac{2\gamma}{(1-\gamma)}\rho^2\sigma^2}.$$

Note that for $\gamma<1$, $\gamma\neq 0$, *the term* $\sigma^2+\frac{\gamma}{(1-\gamma)}\rho^2\sigma^2$ *is always positive, thus, multiplying both sides with this term, the inequality above can be simplified as*

$$\frac{\gamma\bar{\lambda}}{1-\gamma}\left(\frac{\bar{\lambda}}{2}+\frac{\rho\kappa}{\sigma}\right)<\frac{\kappa^2}{2\sigma^2},$$

which is condition (26) in Kraft [35].

Now we conclude that the HJB differential equation (4.13) has the finite solution $V(t,\Sigma)$ in (4.29) for the case $B(\Sigma)-r=\Sigma\mathbf{v}$ under conditions in (4.36). To get that the value function candidate

$$G(t,x,\Sigma)=\frac{x^\gamma}{\gamma}V(t,\Sigma),\quad \gamma<1,\ \gamma\neq 0$$

with $V(t,\Sigma)=1$ is indeed the value function of our optimization problem (4.4), we still need a verification result for $G(t,x,\Sigma)$.

4.2.4 The verification result for the case of $B(\Sigma) - \mathbf{r} = \Sigma\mathbf{v}$

We first identify the optimal portfolio strategy in the case of $B(\Sigma) - \mathbf{r} = \Sigma\mathbf{v}$. We still denote the optimal optimal strategy by $\pi^*(t, \Sigma)$. Recalling (4.9) and (4.32), we get

$$\pi^*(t,\Sigma) = \Sigma^{-1}\left(\frac{(B(\Sigma) - \mathbf{r})V + N}{(1-\gamma)V}\right) = \frac{\mathbf{v}}{1-\gamma} + \frac{\Sigma^{-1}N}{(1-\gamma)V} = \frac{\mathbf{v}}{1-\gamma} + \frac{2\psi^{(2)}(T-t)Q^T\boldsymbol{\rho}}{1-\gamma} \tag{4.38}$$

for $\gamma < 1,\ \gamma \neq 0$, since

$$\Sigma^{-1}N = \Sigma^{-1} \cdot 2\Sigma\nabla V Q^T\boldsymbol{\rho} = 2V\psi^{(2)}(T-t)Q^T\boldsymbol{\rho}$$

by the explicit expression of $V(t, \Sigma)$ in (4.29).

Remark 4.2.9. *Note that $\pi^*(t,\Sigma) = \pi_t^*$, i.e. the optimal strategy is purely deterministic and does not depend on Σ.*

Proposition 4.2.10. *Assume that the process $(\Sigma_t)_{t\geq 0}$ follows the dynamic (2.2). Let us denote*

$$Z_t := \exp\left(\int_t^T Tr\left(A_s\Sigma_s^{1/2}dW_s^\sigma\right) - \frac{1}{2}\int_t^T \|A_s\Sigma_s^{1/2}\|^2 ds\right),$$

where $(A_t)_{t\in[0,T]}$ is a deterministic process with values in $\mathbb{R}^{d\times d}$ and bounded by $A^* \in \mathbb{R}^{d\times d}$, then $(Z_t)_{t\in[0,T]}$ is a martingale.

Proof. By Lemma 2.6.4 we get that (Z_t) is a martingale, if there exists a constant $C^0 > 0$ such that

$$\sqrt{Tr\left(\theta(\Sigma)\theta^T(\Sigma)\theta(\Sigma)\theta^T(\Sigma)\right)} \leq C^0\sqrt{Tr(\Sigma\Sigma)} \tag{4.39}$$

with $\theta(\Sigma) = A\Sigma^{1/2}$. Consider the left-hand side of (4.39), it follows

$$\begin{aligned} &Tr\left(\theta(\Sigma)\theta^T(\Sigma)\theta(\Sigma)\theta^T(\Sigma)\right) \\ =&Tr(A\Sigma A^T A\Sigma A^T) \leq \lambda_{max} Tr\left(A\Sigma\Sigma A^T\right) = \lambda_{max} Tr\left(\Sigma A^T A\Sigma\right) \leq \lambda_{max}^2 Tr(\Sigma\Sigma), \end{aligned}$$

where λ_{max} is the largest eigenvalue of A^TA. The second last inequality follows from the fact that the trace of a matrix is the sum of its eigenvalues and

$$A\Sigma A^T A\Sigma A^T = A\Sigma O\Lambda O^T\Sigma A^T \preceq \lambda_{max} A\Sigma\Sigma A^T,$$

where $O\Lambda O^T$ is the spectral decomposition of A^TA. The last inequality follows in the same way, i.e.

$$\Sigma A^T A\Sigma = \Sigma O\Lambda O^T\Sigma \preceq \lambda_{max}\Sigma\Sigma.$$

Since $A_s \in \mathbb{R}^{d\times d}$, $0 \leq s \leq T$ are bounded on $[0,T]$, λ_{max} is also bounded on $[0,T]$ and we denote its upper bound by λ_{max}^*. Then one concludes that (4.39) is satisfied with $C^0 = \lambda_{max}^*$, which implies that (Z_t) is a martingale. □

Proposition 4.2.11. *Let us denote*

$$Z_t := \exp\left(\int_t^T A_s^T \Sigma_s^{1/2} dW_s^S - \frac{1}{2}\int_t^T \|A_s^T \Sigma_s^{1/2}\|^2 ds\right), \quad t \in [0,T]$$

where $(A_t)_{t\in[0,T]}$ is a deterministic process with values in $\mathbb{R}^d$, which is bounded by $A^* \in \mathbb{R}^d$. Then $(Z_t)_{t\in[0,T]}$ is a martingale.

Proof. Recalling the assumptions of $\rho_{q,ij}$ in (4.3), $\langle W_{t,k}^S, W_{t,kj}^\sigma \rangle = \boldsymbol{\rho}_j$ yields that $W_{t,k}^S \overset{\mathcal{D}}{=} \boldsymbol{\rho}_1 W_{t,k1}^\sigma + \sqrt{1-\boldsymbol{\rho}_1^2}\hat{W}_{t,k1}$, where $(\hat{W}_t)$ is a $d \times d$ Brownian motions matrix, independent of (W_t^σ). Let us denote $\varsigma := (\boldsymbol{\rho}_1, 0, \ldots, 0)^T \in \mathbb{R}^d$ and $\bar{\varsigma} := (\sqrt{1-\boldsymbol{\rho}_1^2}, 0, \ldots, 0)^T \in \mathbb{R}^T$, we get

$$A_s^T \Sigma_s^{1/2} dW_s^S \overset{\mathcal{D}}{=} Tr\left(\varsigma A_s^T \Sigma_s^{1/2} dW_s^\sigma\right) + Tr\left(\bar{\varsigma} A_s^T \Sigma_s^{1/2} d\hat{W}_s\right)$$

and

$$\|A_s^T \Sigma_s^{1/2}\|^2 = \left(\boldsymbol{\rho}_1^2 + 1 - \boldsymbol{\rho}_1^2\right)\|A_s^T \Sigma_s^{1/2}\|^2 = \|\varsigma A_s^T \Sigma_s^{1/2}\|^2 + \|\bar{\varsigma} A_s^T \Sigma_s^{1/2}\|^2.$$

Hence, we get

$$\begin{aligned}
Z_t =& \exp\left(\int_t^T Tr\left(\varsigma A_s^T \Sigma_s^{1/2} dW_s^\sigma\right) - \frac{1}{2}\int_t^T \|\varsigma A_s^T \Sigma_s^{1/2}\|^2 ds\right) \\
&\cdot \exp\left(\int_t^T Tr\left(\bar{\varsigma} A_s^T \Sigma_s^{1/2} d\hat{W}_s\right) - \frac{1}{2}\int_t^T \|\bar{\varsigma} A_s^T \Sigma_s^{1/2}\|^2 ds\right) \\
=& \mathcal{E}\left(\int_\cdot^T Tr\left(\varsigma A_s^T \Sigma_s^{1/2} d\hat{W}_s^\sigma\right)\right)_t \mathcal{E}\left(\int_\cdot^T Tr\left(\bar{\varsigma} A_s^T \Sigma_s^{1/2} d\hat{W}_s\right)\right)_t.
\end{aligned}$$

Now we obtain

$$\begin{aligned}
E[Z_t] =& E[E[Z_t|\mathcal{F}_T^{W^\sigma}]] \\
=& E\left[\mathcal{E}\left(\int_\cdot^T Tr\left(\varsigma A_s^T \Sigma_s^{1/2} d\hat{W}_s^\sigma\right)\right)_t E\left[\mathcal{E}\left(\int_\cdot^T Tr\left(\bar{\varsigma} A_s^T \Sigma_s^{1/2} d\hat{W}_s\right)\right)_t \middle| \mathcal{F}_T^{W^\sigma}\right]\right].
\end{aligned}$$

Since $(\hat{W}_t)$ and (W_t^σ) are independent, the inner conditional expectation is equal to 1 due to [36, Example 4]. From Proposition 4.2.10 we conclude that the remaining expression is also 1. □

Theorem 4.2.12. *(verification result I). Suppose the conditions in (4.36) are satisfied, then, given $\pi^*(t,\Sigma)$ as in (4.38), there is*

$$E^{t,x,\Sigma}\left[\frac{\left(X_T^{\pi^*}\right)^\gamma}{\gamma}\right] = G(t,x,\Sigma) = \frac{x^\gamma}{\gamma}\exp\left(\phi^{(2)}(T-t) + Tr[\psi^{(2)}(T-t)\Sigma]\right)$$

for $t \in [0,T]$, $x > 0$, $\Sigma \in S_d^+(\mathbb{R})$.

Proof. Recall that we have

$$X_T^\pi = X_t^\pi \exp\left(\int_t^T \left[\pi_s^T \Sigma_s \mathbf{v} + r - \frac{1}{2}\|\pi_s^T \Sigma_s^{1/2}\|_2^2\right] ds + \int_t^T \pi_s^T \Sigma_s^{1/2} dW_s^S\right)$$

with $X_t^\pi = x$. Let us denote

$$Z_t := \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\left(\gamma \int_t^T (\pi_s^*)^T \Sigma_s^{1/2} dW_s^S - \frac{\gamma^2}{2} \int_t^T \|(\pi_s^*)^T \Sigma_s^{1/2}\|^2 ds\right),$$

which is a martingale by Proposition 4.2.11. The application of Girsanov Theorem yields

$$\begin{aligned}
&E^{t,x,\Sigma}\left(\frac{(X_T^{\pi^*})^\gamma}{\gamma}\right) \\
=&\frac{x^\gamma}{\gamma} E^{t,\Sigma}\left(\exp\left(\gamma \int_t^T \left[(\pi_s^*)^T \Sigma_s \mathbf{v} + r - \frac{1}{2}\|(\pi_s^*)^T \Sigma_s^{1/2}\|_2^2\right] ds + \gamma \int_t^T (\pi_s^*)^T \Sigma_s^{1/2} dW_s^S\right)\right) \\
=&\frac{x^\gamma}{\gamma} E_{\mathbb{Q}}^{t,\Sigma}\left[\exp\left(\gamma \int_t^T \left[(\pi_s^*)^T \Sigma_s \mathbf{v} + r - \frac{1}{2}\|(\pi_s^*)^T (\Sigma_s)^{1/2}\|_2^2 + \frac{\gamma}{2}\|(\pi_s^*)^T (\Sigma_s)^{1/2}\|_2^2\right] ds\right)\right] \\
=&\frac{x^\gamma}{\gamma} E_{\mathbb{Q}}^{t,\Sigma}\left[\exp\left(\gamma \int_t^T \left[(\pi_s^*)^T \Sigma_s \mathbf{v} + r + \frac{\gamma-1}{2}(\pi_s^*)^T \Sigma_s \pi_s^*\right] ds\right)\right] \\
=&\frac{x^\gamma}{\gamma} E_{\mathbb{Q}}^{t,\Sigma}\left[\exp\left(\gamma r(T-t) + \int_t^T Tr\left[\left(\gamma \mathbf{v}(\pi_s^*)^T + \frac{\gamma(\gamma-1)}{2}\pi_s^*(\pi_s^*)^T\right)\Sigma_s\right] ds\right)\right]. \qquad (4.40)
\end{aligned}$$

In what follows let us denote the deterministic matrix-valued process

$$F_s := \left(\gamma \mathbf{v}(\pi_s^*)^T + \frac{\gamma(\gamma-1)}{2}\pi_s^*(\pi_s^*)^T\right).$$

Plugging π^* in F_S, we get

$$\begin{aligned}
F_s =&\gamma \mathbf{v}\left(\frac{\mathbf{v}}{(1-\gamma)} + \frac{2\psi^{(2)}(T-t)Q^T\boldsymbol{\rho}}{(1-\gamma)}\right)^T \\
&+ \frac{\gamma(\gamma-1)}{2}\left(\frac{\mathbf{v}}{(1-\gamma)} + \frac{2\psi^{(2)}(T-t)Q^T\boldsymbol{\rho}}{(1-\gamma)}\right)\left(\frac{\mathbf{v}}{(1-\gamma)} + \frac{2\psi^{(2)}(T-t)Q^T\boldsymbol{\rho}}{(1-\gamma)}\right)^T \\
=&\frac{\gamma \mathbf{v}\mathbf{v}^T}{2(1-\gamma)} + \frac{\gamma \mathbf{v}\boldsymbol{\rho}^T Q\psi^{(2)}(T-s)}{(1-\gamma)} + \frac{\gamma\psi^{(2)}(T-s)Q^T\boldsymbol{\rho}\mathbf{v}^T}{\gamma-1} + \frac{2\gamma\psi^{(2)}(T-s)Q^T\boldsymbol{\rho}\boldsymbol{\rho}^T Q\psi^{(2)}(T-s)}{(\gamma-1)}.
\end{aligned}$$

Replacing $\frac{\gamma \mathbf{v}\mathbf{v}^T}{2(1-\gamma)}$ above by the expression for $\tilde{\Gamma}$ derived in (4.30), we get

$$\begin{aligned}
F_s =&\frac{d\psi^{(2)}(T-s)}{d(-s)} - \psi^{(2)}(T-s)M - M^T\psi^{(2)}(T-s) - 2\psi^{(2)}(T-s)Q^TQ\psi^{(2)}(T-s) \\
&+ \frac{2\gamma\psi^{(2)}(T-s)Q^T\boldsymbol{\rho}\mathbf{v}^T}{\gamma-1} + \frac{4\gamma\psi^{(2)}(T-s)Q^T\boldsymbol{\rho}\boldsymbol{\rho}^T Q\psi^{(2)}(T-s)}{(\gamma-1)}.
\end{aligned}$$

Thus, it yields

$$\begin{aligned}
&\int_t^T Tr(F_s\Sigma_s)ds \\
=&\int_t^T Tr\left(\frac{d\psi^{(2)}(T-s)}{d(-s)}\Sigma_s\right) ds \\
&- \int_t^T Tr\Bigg(\Bigg(\psi^{(2)}(T-s)M + M^T\psi^{(2)}(T-s) + 2\psi^{(2)}(T-s)Q^TQ\psi^{(2)}(T-s) \\
&-\frac{2\gamma\psi^{(2)}(T-s)Q^T\boldsymbol{\rho}\mathbf{v}^T}{\gamma-1} - \frac{4\gamma\psi^{(2)}(T-s)Q^T\boldsymbol{\rho}\boldsymbol{\rho}^T Q\psi^{(2)}(T-s)}{(\gamma-1)}\Bigg)\Sigma_s\Bigg) ds. \qquad (4.41)
\end{aligned}$$

Next we compute $\int_t^T Tr(F_s\Sigma_s)ds$ under $\mathbb{Q}$. For this instance we need to identify the dynamic of (Σ_t) under $\mathbb{Q}$. Since dW^S and dW^σ are correlated, the application of Girsanov theorem has influence on (W^σ). We assume that

$$dW^\sigma_{ij} = \rho_{i,ij} dW^S_i + \sqrt{1-\rho^2_{i,ij}} dW_i, \tag{4.42}$$

where $W = (W_1, \ldots, W_d)^T$ is a d-dimensional standard Brownian motions vector, independent of W^S. This assumption ensures $dW^\sigma_{ij} dW^S_i = \rho_{i,ij} dt$. Recall that we have $\rho_{i,ij} = \boldsymbol{\rho}_j$, $1 \le i \le d$, then, (4.42) can be written as

$$dW^\sigma = dW^S \boldsymbol{\rho}^T + dW \tilde{\boldsymbol{\rho}}^T$$

with $\tilde{\boldsymbol{\rho}}_j = \sqrt{1-\rho^2_{i,ij}}$. Since under the new measure $\mathbb{Q}$, the process

$$d\hat{W}^S = dW^S - \gamma \Sigma^{1/2} \pi^* ds$$

is a standard Brownian motion, it follows that

$$d\hat{W}^\sigma = d\hat{W}^S \boldsymbol{\rho}^T + dW \tilde{\boldsymbol{\rho}}^T = dW^\sigma - \gamma \Sigma^{1/2} \pi^* \boldsymbol{\rho}^T ds$$

is also a standard Brownian motion. Thus, we get under $\mathbb{Q}$

$$\begin{aligned} d\Sigma_s &= \left(\Omega\Omega^T + M\Sigma_s + \Sigma_s M^T\right) ds + \Sigma_s^{1/2} dW^\sigma_s \tilde{Q} + \tilde{Q}^T \left(dW^\sigma_s\right)^T \Sigma_s^{1/2} \\ &= \left(\Omega\Omega^T + M\Sigma_s + \Sigma_s M^T + \gamma \Sigma_s \pi^* \boldsymbol{\rho}^T Q + \gamma Q^T \boldsymbol{\rho} (\pi^*)^T \Sigma_s\right) ds \\ &\quad + \Sigma_s^{1/2} d\hat{W}^\sigma_s Q + Q^T \left(d\hat{W}^\sigma_s\right)^T \Sigma_s^{1/2} \end{aligned}$$

is a Wishart process with drift

$$\begin{aligned} &\Omega\Omega^T + M\Sigma_s + \Sigma_s M^T + \gamma \Sigma_s \pi^* \boldsymbol{\rho}^T Q + \gamma Q^T \boldsymbol{\rho} (\pi^*)^T \Sigma_s \\ =&\Omega\Omega^T + M\Sigma_s + \Sigma_s M^T + \frac{\gamma}{1-\gamma} \Sigma_s \mathbf{v} \boldsymbol{\rho}^T Q + \frac{2\gamma}{1-\gamma} \Sigma_s \psi^{(2)}(T-s) Q^T \boldsymbol{\rho}\boldsymbol{\rho}^T Q \\ &+ \frac{\gamma}{1-\gamma} Q^T \boldsymbol{\rho} \mathbf{v}^T \Sigma_s + \frac{2\gamma}{1-\gamma} Q^T \boldsymbol{\rho}\boldsymbol{\rho}^T Q \psi^{(2)}(T-s) \Sigma_s. \end{aligned}$$

Due to the product rule and since $\psi^{(2)}(0) = \mathbf{0}$, $\Sigma_t = \Sigma$. we obtain under $\mathbb{Q}$

$$\begin{aligned} &\int_t^T Tr\left(\frac{d\psi^{(2)}(T-s)}{d(-s)} \Sigma_s\right) ds = -Tr\left(\int_t^T \Sigma_s \frac{d\psi^{(2)}(T-s)}{d(-s)} d(-s)\right) \\ =&-Tr\left(\int_t^T \Sigma_s d\psi^{(2)}(T-s)\right) = -Tr\left(\Sigma_T \psi^{(2)}(0) - \Sigma \psi^{(2)}(T-t) - \int_t^T \psi^{(2)}(T-s) d\Sigma_s\right) \\ =&Tr\left(\Sigma\psi^{(2)}(T-t) + \int_t^T \psi^{(2)}(T-s) d\Sigma_s\right) \\ =&Tr\left(\Sigma\psi^{(2)}(T-t)\right) + Tr\Bigg(\int_t^T \psi^{(2)}(T-s)\Big(\Omega\Omega^T + M\Sigma_s + \Sigma_s M^T + \frac{\gamma}{1-\gamma} \Sigma_s \mathbf{v} \boldsymbol{\rho}^T Q \\ &+ \frac{2\gamma}{1-\gamma} \Sigma_s \psi^{(2)}(T-s) Q^T \boldsymbol{\rho}\boldsymbol{\rho}^T Q + \frac{\gamma}{1-\gamma} Q^T \boldsymbol{\rho} \mathbf{v}^T \Sigma_s + \frac{2\gamma}{1-\gamma} Q^T \boldsymbol{\rho}\boldsymbol{\rho}^T Q \psi^{(2)}(T-s) \Sigma_s\Big) ds \\ &+ \psi^{(2)}(T-s) \Sigma_s^{1/2} d\hat{W}^\sigma_s Q + \psi^{(2)}(T-s) Q^T \left(d\hat{W}^\sigma_s\right)^T \Sigma_s^{1/2}\Bigg). \end{aligned} \tag{4.43}$$

Thus, plugging (4.43) in (4.41), it follows

$$\int_t^T Tr(F_s\Sigma_s)ds = Tr\left(\Sigma\psi^{(2)}(T-t)\right) + \int_t^T Tr\left(\psi^{(2)}(T-s)\Omega\Omega^T\right)ds$$
$$-\int_t^T Tr\left(2\psi^{(2)}(T-s)Q^TQ\psi^{(2)}(T-s)\Sigma_s\right)ds$$
$$+Tr\Bigg(\int_t^T \psi^{(2)}(T-s)\Sigma_s^{1/2}d\hat{W}_s^\sigma Q + \int_t^T \psi^{(2)}(T-s)Q^T\left(d\hat{W}_s^\sigma\right)^T\Sigma_s^{1/2}\Bigg).$$

Note that the differential equation (4.31) can be written as

$$\phi^{(2)}(T-t) = \int_t^T Tr\left(\psi^{(2)}(T-s)\Omega\Omega^T\right)ds + \gamma r(T-t),$$

we get then from (4.40):

$$x^{-\gamma}E^{t,\Sigma}\left((X_T^{\pi^*})^\gamma\right)$$
$$=E_{\mathbb{Q}}^{t,\Sigma}\left[\exp\left(\gamma r(T-t) + \int_t^T Tr(F_s\Sigma_s)ds\right)\right]$$
$$=E_{\mathbb{Q}}^{t,\Sigma}\Bigg[\exp\Bigg(Tr\left(\Sigma\psi^{(2)}(T-t)\right) + \phi^{(2)}(T-t) - \int_t^T Tr\left(2\psi^{(2)}(T-s)Q^TQ\psi^{(2)}(T-s)\Sigma_s\right)ds$$
$$+Tr\Bigg(\int_t^T \psi^{(2)}(T-s)\Sigma_s^{1/2}d\hat{W}_s^\sigma Q + \int_t^T \psi^{(2)}(T-s)Q^T\left(d\hat{W}_s^\sigma\right)^T\Sigma_s^{1/2}\Bigg)\Bigg)\Bigg]$$
$$=\exp\left(Tr\left(\Sigma\psi^{(2)}(T-t)\right) + \phi^{(2)}(T-t)\right)\cdot E_{\mathbb{Q}}^{t,\Sigma}\Bigg[\exp\Bigg(2Tr\Bigg(\int_t^T Q\psi^{(2)}(T-s)\Sigma_s^{1/2}d\hat{W}_s^\sigma\Bigg)$$
$$-\int_t^T Tr\left(2\psi^{(2)}(T-s)Q^TQ\psi^{(2)}(T-s)\Sigma_s\right)ds\Bigg)\Bigg].$$

Applying the Girsanov theorem again with the following Radon-Nikodym derivative

$$\tilde{Z}_t := \left.\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}}\right|_{\mathcal{F}_t}$$
$$=\exp\left(\sum_{i,j=1}^d\int_t^T\left(2Q\psi^{(2)}(T-s)\Sigma_s^{1/2}\right)_{ij}d\hat{W}_{s,ji}^\sigma - \frac{1}{2}\sum_{i,j=1}^d\int_t^T\left(2Q\psi^{(2)}(T-s)\Sigma_s^{1/2}\right)_{ij}^2ds\right)$$
$$=\exp\left(Tr\Bigg(\int_t^T 2Q\psi^{(2)}(T-s)\Sigma_s^{1/2}d\hat{W}_s^\sigma\Bigg) - Tr\left(\int_t^T 2Q\psi^{(2)}(T-s)\Sigma_s\psi^{(2)}(T-s)Q^Tds\right)\right),$$

which is a $\mathbb{Q}$-martingale by Proposition 4.2.10, we get finally

$$E^{t,x,\Sigma}\left(\frac{(X_T^{\pi^*})^\gamma}{\gamma}\right) = \frac{x^\gamma}{\gamma}\exp\left(Tr\left(\Sigma\psi^{(2)}(T-t)\right) + \phi^{(2)}(T-t)\right) = \frac{x^\gamma}{\gamma}V(t,\Sigma) = G(t,x,\Sigma).$$

□

We recall the definition of admissible strategies in Definition 2.7.3 and denote the set of admissible strategies by $\mathcal{B}$. We claim that $(\pi_t^*)_{t\in[0,T]} \in \mathcal{B}$ with the following statements.

Note that the conditions (i) and (iii) in Definition 2.7.3 are obviously satisfied by $(\pi_t^*)_{t\geq 0}$. For $B(\Sigma) - \mathbf{r} = \Sigma\mathbf{v}$ and the deterministic $(\pi^*)_{t\in[0,T]}$, we have that the assumption (2.20) is true. It follows that the second and the fourth conditions are also satisfied.

We define another set $\tilde{\mathcal{B}} \subset \mathcal{B}$, which includes all admissible strategies satisfying in addition

$$E\left(\int_t^T \pi_s^2 ds\right) < \infty.$$

Then we have clearly $(\pi_t^*)_{t\geq 0} \in \tilde{\mathcal{B}}$.

Proposition 4.2.13. *(verification result II). Assume $V(t,\Sigma) \in C^{1,2}[t,T]$, then we obtain*

$$E^{t,x,\Sigma}\left(\frac{(X_T^\pi)^\gamma}{\gamma}\right) \leq G(t,x,\Sigma)$$

for all $\pi \in \mathcal{B}$.

Proof. See [35, Proposition 4.3]. □

By Proposition 4.2.12 and Proposition 4.2.13, we can now conclude

$$\max_{\pi,\pi\in\mathcal{B}} E^{t,x,\Sigma}\left(\frac{(X_T^\pi)^\gamma}{\gamma}\right) = E^{t,x,\Sigma}\left(\frac{\left(X_T^{\pi^*}\right)^\gamma}{\gamma}\right) = G(t,x,\Sigma),$$

i.e. $G(t,x,\Sigma)$ is the value function of the optimization problem (4.4) for $B(\Sigma) - \mathbf{r} = \Sigma\mathbf{v}$ and (π^*) is the optimal portfolio strategy.

Remark 4.2.14. *The optimal portfolio strategy (π_t^*) in (4.38) can be decomposed into the Merton ration $\frac{\mathbf{v}}{1-\gamma}$ and the hedging demand given by*

$$\frac{2\psi^{(2)}(T-t)Q^T\boldsymbol{\rho}}{1-\gamma}.$$

In case there is no correlation between (W_t^S) and (W_t^σ), i.e. $\boldsymbol{\rho} = 0$, the optimal portfolio strategy reduces to the Merton ration and does not depend on time. In any case note that the optimal portfolio strategy does not depend on Ω.

4.2.5 An example for the case of $B(\Sigma) - \mathbf{r} = \Sigma\mathbf{v}$

In this subsection we give an example of the value function $G(t,x,\Sigma)$ for the case of $B(\Sigma) - \mathbf{r} = \Sigma\mathbf{v}$. By Theorem 4.2.5 we have that

$$G(t,x,\Sigma) = \frac{x^\gamma}{\gamma}\exp\left(\phi^{(2)}(T-t) + Tr[\psi^{(2)}(T-t)\Sigma]\right), \quad \gamma < 1,\ \gamma \neq 0,$$

where $\phi^{(2)}(T-t)$ and $\psi^{(2)}(T-t)$ are determined in Proposition 4.2.7.

Example 4.2.15. *In our example, we consider the financial market introduced in Section 4.2.3 with one riskfree asset and $d = 2$ risky assets. The parameters of the volatility process (Σ_t) are given by*

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}, \boldsymbol{\rho} = \begin{pmatrix} \rho \\ 0 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ 0 \end{pmatrix}$$

for $M_1, M_2, Q_1, Q_2, v_1 \in \mathbb{R}$. Then we arrive at the following $\tilde{M}$, $\tilde{Q}^T\tilde{Q}$ and $\tilde{\Gamma}$ that are defined in Theorem 4.2.5:

$$\tilde{M} = M + \frac{\gamma}{(1-\gamma)} Q^T \boldsymbol{\rho} \mathbf{v}^T = \begin{pmatrix} a & 0 \\ 0 & M_2 \end{pmatrix}, \quad \tilde{\Gamma} = \frac{\gamma}{2(1-\gamma)} \mathbf{v}\mathbf{v}^T = \begin{pmatrix} \frac{\gamma v_1^2}{2(1-\gamma)} & 0 \\ 0 & 0 \end{pmatrix},$$

$$\tilde{Q}^T\tilde{Q} = Q^T Q + \frac{\gamma}{(1-\gamma)} Q^T \boldsymbol{\rho}\boldsymbol{\rho}^T Q = \begin{pmatrix} Q_1^2 c & 0 \\ 0 & Q_2^2 \end{pmatrix}$$

with $a = M_1 + \frac{\gamma}{1-\gamma} v_1 Q_1 \rho$ and $c = 1 + \frac{\gamma}{1-\gamma}\rho^2$. We obtain $c > 0$ for $\gamma < 1$, $\gamma \neq 0$ and we assume that $Q_1, Q_2 \neq 0$. Then, we can take $\tilde{Q}$ simply as

$$\tilde{Q} = \begin{pmatrix} Q_1\sqrt{c} & 0 \\ 0 & Q_2 \end{pmatrix}$$

and

$$C_1^{(2)} = -\tilde{Q}\left(\frac{\tilde{M}^T(\tilde{Q}^T\tilde{Q})^{-1} + (\tilde{Q}^T\tilde{Q})^{-1}\tilde{M}}{2}\right)\tilde{Q}^T = -\tilde{M} = \begin{pmatrix} -a & 0 \\ 0 & -M_2 \end{pmatrix},$$

$$C_2^{(2)} = \tilde{Q}\left(-2\tilde{\Gamma} + \tilde{M}^T\tilde{Q}^{-1}\tilde{Q}^{-T}\tilde{M}\right)\tilde{Q}^T = \begin{pmatrix} b & 0 \\ 0 & M_2^2 \end{pmatrix}$$

with $b = a^2 - \frac{\gamma}{1-\gamma} v_1^2 Q_1^2 c$. By Proposition 4.2.7 we need to assume

$$\tilde{Q}^T\tilde{Q} \in GL_d(\mathbb{R}), \quad -2\tilde{\Gamma} + \tilde{M}^T\tilde{Q}^{-1}\tilde{Q}^{-T}\tilde{M} \succ 0 \quad \text{and} \quad \sqrt{C_2^{(2)}} + C_1^{(2)} \succeq 0, \tag{4.44}$$

which is satisfied if $M_2 \neq 0$, $\sqrt{b} - a \geq 0$ and $b > 0$. We obtain $\sqrt{C_2^{(2)}} = \begin{pmatrix} \sqrt{b} & 0 \\ 0 & |M_2| \end{pmatrix}$ and

$$\begin{aligned}\kappa^{(2)}(T-t) = &-\left(\sqrt{C_2^{(2)}}\cosh\left(\sqrt{C_2^{(2)}}(T-t)\right) + \tilde{C}_1\sinh\left(\sqrt{C_2^{(2)}}(T-t)\right)\right)^{-1} \cdot \\ &\left(\sqrt{C_2^{(2)}}\sinh\left(\sqrt{C_2^{(2)}}(T-t)\right) + C_1^{(2)}\cosh\left(\sqrt{C_2^{(2)}}(T-t)\right)\right) \\ = &\begin{pmatrix} -\left(\sqrt{b}\sinh(\sqrt{b}t) - a\cosh(\sqrt{b}t)\right)\Big/\left(\sqrt{b}\cosh(\sqrt{b}t) - a\sinh(\sqrt{b}t)\right) & 0 \\ 0 & \frac{|M_2|}{M_2} \end{pmatrix}.\end{aligned}$$

Then, we have $\psi^{(2)}(T-t)$ as

$$\begin{aligned}\psi^{(2)}(T-t) = &\frac{\tilde{Q}^{-1}\sqrt{C_2^{(2)}}\kappa^{(2)}(T-t)\tilde{Q}^{-T}}{2} - \frac{\tilde{M}^T(\tilde{Q}^T\tilde{Q})^{-1} + (\tilde{Q}^T\tilde{Q})^{-1}\tilde{M}}{4} \\ = &\begin{pmatrix} \frac{\gamma v_1^2}{2(1-\gamma)}\left(\sqrt{b}\coth(\sqrt{b}t) - a\right)^{-1} & 0 \\ 0 & 0 \end{pmatrix},\end{aligned}$$

and the solution of $\phi^{(2)}(T-t)$ can be deduced from (4.31) as

$$\phi^{(2)}(T-t) = Tr\left[\int_0^{T-t} \psi^{(2)}(u)\, du\Omega\Omega^T\right] = \left(\Omega_{11}^2 + \Omega_{12}^2\right)\int_0^{T-t} \frac{\gamma v_1^2}{2(1-\gamma)}\left(\sqrt{b}\coth(\sqrt{b}t) - a\right)^{-1} du.$$

The optimal portfolio strategy is here given by

$$\pi_t^* = \frac{1}{1-\gamma}\begin{pmatrix} v_1 \\ 0 \end{pmatrix} + \frac{\rho\gamma v_1}{(1-\gamma)^2}\begin{pmatrix} \left(\sqrt{b}\coth(\sqrt{b}t) - a\right)^{-1} \\ 0 \end{pmatrix}.$$

It implies that one would never invest in to the second asset in this example. Also note that due to $\sqrt{b} - a \geq 0$ *and* $\coth(\sqrt{b}t) \geq 1$, *the hedging demand in the first asset is positive if* $\rho\gamma > 0$ *and negative if* $\rho\gamma < 0$.

Chapter 5

Asymptotic error distribution and adjustments for discrete rebalancing with regular paths

The purpose of this chapter is to analyze the impact of incorporating "liquidity horizons" for less liquid assets. Given one discretely rebalanced (e.g. Δt periodic) portfolio $\hat{X}_{(N)}$ and one continuously rebalanced portfolio X_T, we want to find some relationships between the distributions of the two portfolios and as a result, approximate the distribution of $\hat{X}_{(N)}$ through X_T. More precisely, we will illustrate in Section 5.2 the limit distribution of the absolute error $\hat{X}_{(N)} - X_T$ and the relative error $\left(\hat{X}_{(N)} - X_T\right)/\hat{X}_{(N)}$ by letting the rebalancing period Δt go to zero and then establish a limit theorem. In Section 5.3 and Section 5.4, we will introduce two adjustments to correct the effect of discretely rebalancing, namely the volatility adjustment and the conditional mean adjustment. Finally, the limit theorem and the two adjustments will be tested in Section 5.5 by some examples. This chapter is an extension of Glasserman [21], in which the investigation focused on a model with constant portfolio weights and constant drift and volatility in the assets dynamics (See [21] for details).

5.1 Model dynamics

Let us consider now a market with d risky assets. We denote the price processes of the d risky assets by $(S_{t,i})_{t\geq 0}$, $i = 1, \ldots, d$ and the assets vector process by $(S_t)_{t\geq 0} = (S_{t,1}, \ldots, S_{t,d})_{t\geq 0}$. The dynamic of $(S_t)_{t\geq 0}$ is represented by the following stochastic differential equation

$$dS_t = diag(S_t)\mu_t dt + diag(S_t)\sigma_t dW_t^S \tag{5.1}$$

with $\mu_t \in \mathbb{R}^d$, $\sigma_t \in \mathbb{R}^{d\times d}$ and (W_t^S) being a d dimensional Brownian motions vector. We make the following assumptions to the drift vector process (μ_t) and the volatility coefficients process (σ_t):

1. (μ_t) and (σ_t) are bounded, càdlàg, deterministic vector processes with $\mu_t = (\mu_{t,i})_{1\leq i\leq d}$ and $\sigma_t = (\sigma_{t,ij})_{1\leq i,j\leq d}$,

2. the second-order right-sided derivatives of (σ_t) and (μ_t) with respect to t exist and are right-sided continuous in the interval $[0, T)$ with respect to time t.

We define the covariance matrix $\hat{\sigma}_t$ as

$$\hat{\sigma}_t = \sigma_t \sigma_t^T.$$

Let $(\pi_t)_{t\geq 0}$ be the fractional portfolio strategy process that is a $\mathbb{R}^d$-valued deterministic process. We denote the portfolio strategy by

$$\pi_t = (\pi_{t,1}, \ldots, \pi_{t,d})^T$$

with $\pi_t^T \mathbf{1} = 1$. $\pi_{t,k}$ represents the proportion of wealth invested into stock k at time t. We assume furthermore:

3. π_t is a continuous function of σ_t and μ_t.

The conditions **1**, **2** and **3** are called *regular conditions* in this thesis and paths satisfying regular conditions are called *regular paths*. We say that the above introduced processes (μ_t), (σ_t) and (π_t) have regular paths.

The portfolio wealth process of the continuously rebalanced portfolio (X_t) evolves as

$$\frac{dX_t}{X_t} = \sum_{i=1}^{d} \pi_{t,i} \frac{dS_{t,i}}{S_{t,i}} = \pi_t^T \mu_t dt + \pi_t^T \sigma_t dW_t^S = \pi_t^T \mu_t dt + \sqrt{\pi_t^T \sigma_t \sigma_t^T \pi_t} d\tilde{W}_t, \tag{5.2}$$

with $X_0 = 1$ and $\tilde{W}_t = \frac{1}{\sqrt{\pi_t^T \sigma_t \sigma_t^T \pi_t}} \pi_t^T \sigma_t W_t$ being a one-dimensional scalar Brownian motion.

Let T (e.g. 1 year) be the risk horizon and $\Delta t = T/N$ the rebalancing horizon. We introduce a discretely rebalanced (Δt periodic) portfolio with wealth process denoted by $(\hat{X}_t)$. The wealth process $(\hat{X}_t)$ evolves from $n\Delta t$ to $(n+1)\Delta t$, $0 \leq n \leq N-1$, as

$$\hat{X}_{(n+1)\Delta t} = \hat{X}_{n\Delta t} \left(\sum_{i=1}^{d} \pi_{t,i} \frac{S_{(n+1)\Delta t,i}}{S_{n\Delta t,i}} \right)$$

with $\hat{X}_0 = 1$.

We assume that (5.1) and (5.2) own unique strong solutions. Solving the stochastic differential equations, we get

$$S_{T,i} = S_{0,i} \exp\left(\int_0^T \left(\mu_{u,i} - \frac{1}{2} \|\sigma_u(i,:)\|^2 \right) du + \int_0^T \sigma_u(i,:) dW_u^S \right) \tag{5.3}$$

for $i = 1, \ldots, d$ and

$$X_T = \exp\left(\int_0^T \left(\pi_u^T \mu_u - \frac{1}{2} \|\pi_u^T \sigma_u\|^2 \right) du + \int_0^T \pi_u^T \sigma_u dW_u^S \right). \tag{5.4}$$

To lighten notation, let us denote $\hat{X}_{(n)} := \hat{X}_{n\Delta t}$, $X_{(n)} := X_{n\Delta t}$, $\sigma_{(n)} := \sigma_{n\Delta t}$, $\mu_{(n)} := \mu_{n\Delta t}$, $\pi_{(n)} := \pi_{n\Delta t}$ and $\Delta W_{(n)} := W_{(n+1)\Delta t} - W_{n\Delta t}$.

5.2 Asymptotic error

In this section we will study the asymptotic behavior of the relative difference between the continuously rebalanced and the discretely rebalanced portfolios at T. Such relative difference at T is called *portfolio error* and is defined as

$$\frac{\hat{X}_{(N)} - X_T}{X_T}.$$

We will approximate the portfolio error in Section 4.2.1 and establish in Subsection 5.2.2 a limit theorem illustrating the asymptotic behavior of the scaled portfolio error. Finally we will show that the limit theorem obtained in Subsection 5.2.2 applies also to some more general case.

5.2.1 Approximation of the errors

Note that for the portfolio error $(\hat{X}_{(N)} - X_T)/X_T$ the following holds:

$$\begin{aligned}\frac{\hat{X}_{(N)} - X_T}{X_T} &= \frac{\hat{X}_{(N)}}{X_T} - \frac{\hat{X}_{(N-1)}}{X_{(N-1)}} + \frac{\hat{X}_{(N-1)}}{X_{(N-1)}} - \frac{\hat{X}_{(N-2)}}{X_{(N-2)}} + \ldots + \frac{\hat{X}_{(1)}}{X_{(1)}} - \frac{\hat{X}_{(0)}}{X_{(0)}} \\ &= \sum_{n=1}^{N} \left(\frac{\hat{X}_{(n)}}{X_{(n)}} - \frac{\hat{X}_{(n-1)}}{X_{(n-1)}} \right). \end{aligned} \tag{5.5}$$

We have the following proposition concerning the portfolio error:

Proposition 5.2.1. *Assume that (σ_t), (μ_t) and (π_t) satisfy the regular conditions introduced in Section 4.1 and let*

$$\begin{aligned}\epsilon_n =& \frac{1}{2}\sum_{i=1}^{d} \pi_{(n-1),i} \left(\sigma_{(n-1)}(i,:)\Delta W^S_{(n-1)}\right)^2 - \frac{1}{2}\left(\pi^T_{(n-1)}\sigma_{(n-1)}\Delta W^S_{(n-1)}\right)^2 \\ &- \frac{1}{2}\sum_{i=1}^{d} \pi_{(n-1),i}\|\sigma_{(n-1)}(i,:)\|^2\Delta t + \frac{1}{2}\|\pi^T_{(n-1)}\sigma_{(n-1)}\|^2\Delta t,\end{aligned}$$

then

$$E\left[\left(\frac{\hat{X}_{(N)} - X_T}{X_T} - \sum_{n=1}^{N}\epsilon_n\right)^2\right] = o(\Delta t).$$

Proof. By (5.3) and (5.4) we get

$$S_{n\Delta t,i} = S_{(n),i} = S_{0,i}\exp\left(\int_0^{n\Delta t}\left(\mu_{u,i} - \frac{1}{2}\|\sigma_u\left(i,:\right)\|^2\right)du + \int_0^{n\Delta t}\sigma_u\left(i,:\right)dW^S_u\right)$$

and

$$X_{n\Delta t} = X_{(n)} = \exp\left(\int_0^{n\Delta t}\left(\pi^T_u\mu_u - \frac{1}{2}\|\pi^T_u\sigma_u\|^2\right)du + \int_0^{n\Delta t}\pi^T_u\sigma_u dW^S_u\right).$$

Then we define R_n and $\hat{R}_n$, $n = 1, \ldots, N$ as

$$R_n = \frac{X_{(n)}}{X_{(n-1)}} = \exp\left(\int_{(n-1)\Delta t}^{n\Delta t} \left(\pi_u^T \mu_u - \frac{1}{2}\|\pi_u^T \sigma_u\|^2\right) du + \int_{(n-1)\Delta t}^{n\Delta t} \pi_u^T \sigma_u dW_u^S\right)$$

and

$$\begin{aligned}\hat{R}_n &= \frac{\hat{X}_{(n)}}{\hat{X}_{(n-1)}} = \sum_{i=1}^{d} \pi_{(n-1),i} \frac{S_{(n),i}}{S_{(n-1),i}} \\ &= \sum_{i=1}^{d} \pi_{(n-1),i} \exp\left(\int_{(n-1)\Delta t}^{n\Delta t} \left(\mu_{u,i} - \frac{1}{2}\|\sigma_u(i,:)\|^2\right) du + \int_{(n-1)\Delta t}^{n\Delta t} \sigma_u(i,:) \, dW_u^S\right),\end{aligned}$$

respectively. Thus, we have

$$\frac{\hat{X}_{(N)}}{X_T} = \prod_{n=1}^{N} \frac{\hat{R}_n}{R_n} \tag{5.6}$$

and

$$\begin{aligned}\frac{\hat{R}_n}{R_n} = &\sum_{i=1}^{d} \pi_{(n-1),i} \exp\left(\int_{(n-1)\Delta t}^{n\Delta t} \left(\mu_{u,i} - \frac{1}{2}\|\sigma_u(i,:)\|^2 - \pi_u^T \mu_u + \frac{1}{2}\|\pi_u^T \sigma_u\|^2\right) du\right. \\ &\left. + \int_{(n-1)\Delta t}^{n\Delta t} \left(\sigma_u(i,:) - \pi_u^T \sigma_u\right) dW_u^S\right).\end{aligned}$$

We define a function $g_{N,n}(y)$ as follows:

$$\begin{aligned}g_{N,n}(y) = &\sum_{i=1}^{d} \pi_{(n-1),i} \exp\left(\int_{t_{n-1}}^{t_{n-1}+y^2} \left(\mu_{u,i} - \frac{1}{2}\|\sigma_u(i,:)\|^2 - \pi_u^T \mu_u + \frac{1}{2}\|\pi_u^T \sigma_u\|^2\right) du\right. \\ &\left. + \int_{t_{n-1}}^{t_{n-1}+y^2} \left(\sigma_u(i,:) - \pi_u^T \sigma_u\right) dW_u^S\right)\end{aligned} \tag{5.7}$$

with $t_{n-1} = (n-1)\Delta t$. We note that there is

$$\frac{\hat{R}_n}{R_n} = g_{N,n}\left(\sqrt{\Delta t}\right) \tag{5.8}$$

and $g_{N,n}(0) = 1$. Let

$$h_{u,i} := \mu_{u,i} - \frac{1}{2}\|\sigma_u(i,:)\|^2 - \pi_u^T \mu_u + \frac{1}{2}\|\pi_u^T \sigma_u\|^2 \tag{5.9}$$

and we use the following representation:

$$\int_{t_{n-1}}^{t_{n-1}+y^2} \left(\sigma_u(i,:) - \pi_u^T \sigma_u\right) dW_u^S = \sum_{j=1}^{d} \left(\int_{t_{n-1}}^{t_{n-1}+y^2} f_{u,ij} du\right)^{1/2} Z_{N,n,j}$$

with

$$f_{u,ij} := \left(\sigma_{u,ij} - \pi_u^T \sigma_u\left(:, j\right)\right)^2 \tag{5.10}$$

and $Z_{N,n,j}$ $j = 1, \ldots, d$ being independent standard normal distributed random variables. Thus, we obtain

$$g_{N,n}\left(y\right) = \sum_{i=1}^{d} \pi_{(n-1),i} \exp\left(\int_{t_{n-1}}^{t_{n-1}+y^2} h_{u,i} du + \sum_{j=1}^{d}\left(\int_{t_{n-1}}^{t_{n-1}+y^2} f_{u,ij} du\right)^{1/2} Z_{N,n,j}\right).$$

According to the assumptions of the paths of (σ_t) and (μ_t), the second-order right-sided derivatives of $h_{u,i}$ and $f_{u,ij}$, $i, j = 1, \ldots, d$ exist in the interval $u \in [(n-1)\Delta t, n\Delta t)$, $n = 1, \ldots, N$. Let F be the antiderivative of f, we get $F'_{u,ij} = f_{u,ij}$, $u \in [t_{n-1}, t_{n-1} + y^2]$, i.e.

$$\int_{t_{n-1}}^{t_{n-1}+y^2} f_{u,ij} du = F_{t_{n-1}+y^2,ij} - F_{t_{n-1},ij}. \tag{5.11}$$

The Leibniz integral rule yields

$$\left(\int_{t_{n-1}}^{t_{n-1}+y^2} f_{u,ij} du\right)' = f_{t_{n-1}+y^2,ij} \cdot 2y,$$

then we get the first derivative of $g_{N,n}\left(y\right)$:

$$\begin{aligned} g'_{N,n}(y) = &\sum_{i=1}^{d} \pi_{(n-1),i} \exp\left(\int_{t_{n-1}}^{t_{n-1}+y^2} h_{u,i} du + \sum_{j=1}^{d} \sqrt{\int_{t_{n-1}}^{t_{n-1}+y^2} f_{u,ij} du} Z_{N,n,j}\right) \\ &\left(h_{t_{n-1}+y^2,i} \cdot 2y + \sum_{j=1}^{d} \frac{f_{t_{n-1}+y^2,ij} \cdot y}{\sqrt{F_{t_{n-1}+y^2,ij} - F_{t_{n-1},ij}}} Z_{N,n,j}\right). \end{aligned} \tag{5.12}$$

Applying a Taylor expansion to $F_{t_{n-1}+y^2,ij}$ and write

$$F_{t_{n-1}+y^2,ij} = F_{t_{n-1},ij} + F'_{t_{n-1},ij} y^2 + \eta_{t_{n-1}+y^2,ij} y^2$$

with $\lim_{y\to 0} \eta_{t_{n-1}+y^2,ij} = 0$, we have

$$\lim_{y\to 0} \sqrt{\frac{y^2}{F_{t_{n-1}+y^2,ij} - F_{t_{n-1},ij}}} = \lim_{y\to 0} \sqrt{\frac{y^2}{F'_{t_{n-1},ij} y^2 + \eta_{t_{n-1}+y^2,ij} y^2}} = \frac{1}{\sqrt{f_{t_{n-1},ij}}}. \tag{5.13}$$

Thus, there is

$$\begin{aligned} g'_{N,n}\left(0\right) &= \sum_{i=1}^{d} \pi_{(n-1),i} \sum_{j=1}^{d} f_{t_{n-1},ij} Z_{N,n,j} \lim_{y\to 0} \sqrt{\frac{y^2}{F_{t_{n-1}+y^2,ij} - F_{t_{n-1},ij}}} \\ &= \sum_{i=1}^{d} \pi_{(n-1),i} \sum_{j=1}^{d} \sqrt{f_{t_{n-1},ij}} Z_{N,n,j} = \sum_{i=1}^{d} \pi_{(n-1),i} \left(\sigma_{t_{n-1}}\left(i, :\right) - \pi_{t_{n-1}}^T \sigma_{t_{n-1}}\right) Z_{N,n} = 0. \end{aligned}$$

Since (σ_t) and (μ_t) are assumed to be 2-times right-hand differentiable with respect to t in the interval $[(n-1)\Delta t, n\Delta t)$, it follows that $g_{N,n}(y)$ is 3-times right-hand differentiable with respect to y in $[(n-1)\Delta t, n\Delta t)$, which means that $\lim_{y\downarrow 0}\frac{g'_{N,n}(y)-g'_{N,n}(0)}{y}$ and $\lim_{y\downarrow 0}\frac{g''_{N,n}(y)-g''_{N,n}(0)}{y}$ exist. We have also assumed that the second-order right-sided derivatives of (μ_t) and (σ_t) are right continuous, thus, there is $\lim_{y\downarrow 0}\frac{g'_{N,n}(y)-g'_{N,n}(0)}{y} = \lim_{y\downarrow 0} g''_{N,n}(y)$ and $\lim_{y\downarrow 0}\frac{g''_{N,n}(y)-g''_{N,n}(0)}{y} = \lim_{y\downarrow 0} g'''_{N,n}(y)$. Because of $\sqrt{\Delta t} > 0$, we can apply a third order Taylor expansion to $g_{N,n}(\sqrt{\Delta t})$ at 0 under the assumption $g''_{N,n}(0) = \lim_{y\downarrow 0} g''_{N,n}(y)$ and $g'''_{N,n}(0) = \lim_{y\downarrow 0} g'''_{N,n}(y)$. Then for some $\xi_{N,n} \in [0, \sqrt{\Delta t}]$, there is

$$g_{N,n}(\sqrt{\Delta t}) = 1 + \frac{1}{2}g''_{N,n}(0)\Delta t + \frac{1}{6}g'''_{N,n}(0)\Delta t^{3/2} + c_{N,n}(\sqrt{\Delta t})\Delta t^{3/2} \tag{5.14}$$

with

$$c_{N,n}(\sqrt{\Delta t}) = \frac{1}{6}\left(g'''_{N,n}(\xi) - g'''_{N,n}(0)\right) \tag{5.15}$$

for a $\xi \in (0, \sqrt{\Delta t})$. By direct computation we get the following second-order derivative of $g_{N,n}(y)$, $0 \le y \le \sqrt{\Delta t}$ as follows:

$$\begin{aligned} g''_{N,n}(y) = &\sum_{i=1}^{d} \pi_{(n-1),i} \exp\left(\int_{t_{n-1}}^{t_{n-1}+y^2} h_{u,i}du + \sum_{j=1}^{d}\left(\int_{t_{n-1}}^{t_{n-1}+y^2} f_{u,ij}du\right)^{1/2} Z_{N,n,j}\right) \\ &\cdot\left\{\left(h_{t_{n-1}+y^2,i}2y + \sum_{j=1}^{d}\frac{f_{t_{n-1}+y^2,ij}\cdot y}{\sqrt{F_{t_{n-1}+y^2,ij}-F_{t_{n-1},ij}}}Z_{N,n,j}\right)^2 + h_{t_{n-1}+y^2,i}2\right. \\ &\left. + h'_{t_{n-1}+y^2,i}4y^2 + \sum_{j=1}^{d}\left(\frac{f_{t_{n-1}+y^2,ij}\cdot y}{\sqrt{F_{t_{n-1}+y^2,ij}-F_{t_{n-1},ij}}}\right)' Z_{N,n,j}\right\} \end{aligned}$$

with

$$\begin{aligned} &\left(\frac{f_{t_{n-1}+y^2,ij}\cdot y}{\sqrt{F_{t_{n-1}+y^2,ij}-F_{t_{n-1},ij}}}\right)'\Bigg|_{y=0} \\ &= \frac{f_{t_{n-1}+y^2,ij} + f'_{t_{n-1}+y^2,ij}\cdot 2y^2}{\sqrt{F_{t_{n-1}+y^2,ij}-F_{t_{n-1},ij}}}\Bigg|_{y=0} - \frac{f_{t_{n-1}+y^2,ij}\cdot y\cdot f_{t_{n-1}+y^2,ij}\cdot y}{\left(F_{t_{n-1}+y^2,ij}-F_{t_{n-1},ij}\right)^{3/2}}\Bigg|_{y=0} \\ &= \frac{f_{t_{n-1}+y^2,ij} + f'_{t_{n-1}+y^2,ij}\cdot 2y^2}{\sqrt{F_{t_{n-1}+y^2,ij}-F_{t_{n-1},ij}}}\Bigg|_{y=0} - \frac{\left(f_{t_{n-1}+y^2,ij}\right)^2}{\sqrt{F_{t_{n-1}+y^2,ij}-F_{t_{n-1},ij}}}\Bigg|_{y=0}\cdot\frac{1}{f_{t_{n-1},ij}} \\ &= \frac{f'_{t_{n-1}+y^2,ij}2y^2}{\sqrt{F_{t_{n-1}+y^2,ij}-F_{t_{n-1},ij}}}\Bigg|_{y=0} = 0. \end{aligned}$$

Thus, it follows

$$g''_{N,n}(0) = \sum_{i=1}^{d}\pi_{(n-1),i}\left\{\left(\sum_{j=1}^{d}\sqrt{f_{t_{n-1},ij}}Z_{N,n,j}\right)^2 + 2h_{t_{n-1},i}\right\}$$

and

$$\begin{aligned}\frac{1}{2}g^{''}_{N,n}(0)\,\Delta t =& \frac{1}{2}\sum_{i=1}^{d}\pi_{(n-1),i}\left\{\left[\left(\sigma_{(n-1)}(i,:)-\pi^{T}_{(n-1)}\sigma_{(n-1)}\right)\Delta W^{S}_{(n-1)}\right]^2\right.\\ &\left.+2\left(\mu_{(n-1),i}-\frac{1}{2}\|\sigma_{(n-1)}(i,:)\|^2-\pi^{T}_{(n-1)}\mu_{(n-1)}+\frac{1}{2}\|\pi^{T}_{(n-1)}\sigma_{(n-1)}\|^2\right)\Delta t\right\}\\ =&\frac{1}{2}\sum_{i=1}^{d}\pi_{(n-1),i}\left(\sigma_{(n-1)}(i,:)\,\Delta W^{S}_{(n-1)}\right)^2-\frac{1}{2}\left(\pi^{T}_{(n-1)}\sigma_{(n-1)}\Delta W^{S}_{(n-1)}\right)^2\\ &-\frac{1}{2}\sum_{i=1}^{d}\pi_{(n-1),i}\|\sigma_{(n-1)}(i,:)\|^2\Delta t+\frac{1}{2}\|\pi^{T}_{(n-1)}\sigma_{(n-1)}\|^2\Delta_t=\epsilon_n.\end{aligned}$$

It follows then

$$\sum_{n=1}^{N}\epsilon_n=\frac{1}{2}\sum_{n=1}^{N}g^{''}_{N,n}(0)\,\Delta t. \tag{5.16}$$

From (5.6), (5.8) and (5.14) we get

$$\frac{\hat{X}_{(N)}}{X_T}=\prod_{n=1}^{N}g_{N,n}\left(\sqrt{\Delta t}\right)=\prod_{n=1}^{N}\left[1+\frac{1}{2}g^{''}_{N,n}(0)\,\Delta t+\frac{1}{6}g^{'''}_{N,n}(0)\,\Delta t^{3/2}+c_{N,n}(\sqrt{\Delta t})\Delta t^{3/2}\right] \tag{5.17}$$

$$=1+\frac{1}{2}\sum_{n=1}^{N}g^{''}_{N,n}(0)\,\Delta t+\frac{1}{6}\sum_{n=1}^{N}g^{'''}_{N,n}(0)\,\Delta t^{3/2}+\sum_{n=1}^{N}c_{N,n}(\sqrt{\Delta t})\Delta t^{3/2}+r_N, \tag{5.18}$$

where the remainder r_N includes all other terms in the product. Then, the proof will be concluded from (5.16) , if we can show that the last three terms in (5.18) are negligible.

The Minkowski inequality yields

$$\left\|\frac{\hat{X}_{(N)}-X_T}{X_T}-\sum_{n=1}^{N}\epsilon_n\right\|\leq\left\|\frac{1}{6}\sum_{n=1}^{N}g^{'''}_{N,n}(0)\,\Delta t^{3/2}\right\|+\left\|\sum_{n=1}^{N}c_{N,n}(\sqrt{\Delta t})\Delta t^{3/2}\right\|+\|r_N\|\,. \tag{5.19}$$

We will show next that each term on the right side is $o\left(\sqrt{\Delta t}\right)$. Let us first consider $g^{'''}_{N,n}(y)$,

$0 < y < \sqrt{\Delta t}$, we compute

$$
\begin{aligned}
g_{N,n}^{'''}(y) = & \sum_{i=1}^{d} \pi_{(n-1),i} \exp\left(\int_{t_{n-1}}^{t_{n-1}+y^2} h_{u,i} du + \sum_{j=1}^{d} \sqrt{\int_{t_{n-1}}^{t_{n-1}+y^2} f_{u,ij} du} Z_{N,n,j} \right) \\
& \cdot \left\{ \left\{ \left(h_{t_{n-1}+y^2,i} \cdot 2y + \sum_{j=1}^{d} \frac{f_{t_{n-1}+y^2,ij} \cdot y}{\sqrt{F_{t_{n-1}+y^2,ij} - F_{t_{n-1},ij}}} Z_{N,n,j} \right)^2 \right.\right. \\
& \left. + h_{t_{n-1}+y^2,i} \cdot 2 + h_{t_{n-1}+y^2,i}^{'} \cdot 4y^2 + \sum_{j=1}^{d} \left(\frac{f_{t_{n-1}+y^2,ij} \cdot y}{\sqrt{F_{t_{n-1}+y^2,ij} - F_{t_{n-1},ij}}} \right)^{'} Z_{N,n,j} \right\} \\
& \cdot \left(h_{t_{n-1}+y^2,i} \cdot 2y + \sum_{j=1}^{d} \frac{f_{t_{n-1}+y^2,ij} \cdot y}{\sqrt{F_{t_{n-1}+y^2,ij} - F_{t_{n-1},ij}}} Z_{N,n,j} \right) \\
& + 2 \left(h_{t_{n-1}+y^2,i} \cdot 2y + \sum_{j=1}^{d} \frac{f_{t_{n-1}+y^2,ij} \cdot y}{\sqrt{F_{t_{n-1}+y^2,ij} - F_{t_{n-1},ij}}} Z_{N,n,j} \right) \\
& \cdot \left(h_{t_{n-1}+y^2,i} \cdot 2 + h_{t_{n-1}+y^2,i}^{'} \cdot 4y^2 + \sum_{j=1}^{d} \left(\frac{f_{t_{n-1}+y^2,ij} \cdot y}{\sqrt{F_{t_{n-1}+y^2,ij} - F_{t_{n-1},ij}}} \right)^{'} Z_{N,n,j} \right) \\
& \left. + 12y h_{t_{n-1}+y^2,i}^{'} + 8x^3 h_{t_{n-1}+y^2,i}^{''} + \sum_{j=1}^{d} \left(\frac{f_{t_{n-1}+y^2,ij} \cdot y}{\sqrt{F_{t_{n-1}+y^2,ij} - F_{t_{n-1},ij}}} \right)^{''} Z_{N,n,j} \right\}.
\end{aligned}
$$

with

$$
\begin{aligned}
& \left(\frac{f_{t_{n-1}+y^2,ij} \cdot y}{\sqrt{F_{t_{n-1}+y^2,ij} - F_{t_{n-1},ij}}} \right)^{''} \Bigg|_{y=0} \\
= & \left(\frac{f_{t_{n-1}+y^2,ij} + f_{t_{n-1}+y^2,ij}^{'} \cdot 2y^2}{\sqrt{F_{t_{n-1}+y^2,ij} - F_{t_{n-1},ij}}} - \frac{f_{t_{n-1}+y^2,ij} \cdot y \cdot f_{t_{n-1}+y^2,ij} \cdot y}{\left(F_{t_{n-1}+y^2,ij} - F_{t_{n-1},ij}\right)^{3/2}} \right)^{'} \Bigg|_{y=0} \\
= & \left(\frac{f_{t_{n-1}+y^2,ij}^{''} \cdot 4y^3 + 6y f_{t_{n-1}+y^2,ij}^{'}}{\sqrt{F_{t_{n-1}+y^2,ij} - F_{t_{n-1},ij}}} - \frac{\left(f_{t_{n-1}+y^2,ij}^{'} \cdot 2y^2 + f_{t_{n-1}+y^2,ij} \right) f_{t_{n-1}+y^2,ij} \cdot y}{\left(F_{t_{n-1}+y^2,ij} - F_{t_{n-1},ij}\right)^{3/2}} \right. \\
& \left. - \frac{4y^3 f_{t_{n-1}+y^2,ij}^{'} f_{t_{n-1}+y^2,ij} + 2 f_{t_{n-1}+y^2,ij}^2 \cdot y}{\left(F_{t_{n-1}+y^2,ij} - F_{t_{n-1},ij}\right)^{3/2}} + \frac{3 f_{t_{n-1}+y^2,ij}^2 \cdot y^2 f_{t_{n-1}+y^2,ij} \cdot y}{\left(F_{t_{n-1}+y^2,ij} - F_{t_{n-1},ij}\right)^{5/2}} \right) \Bigg|_{y=0} \\
= & \left(\frac{-3 f_{t_{n-1}+y^2,ij}^2 \cdot y}{\left(F_{t_{n-1}+y^2,ij} - F_{t_{n-1},ij}\right)^{3/2}} + \frac{3 f_{t_{n-1}+y^2,ij}^2 \cdot y f_{t_{n-1}+y^2,ij} \cdot y^2}{\left(F_{t_{n-1}+y^2,ij} - F_{t_{n-1},ij}\right)^{5/2}} \right) \Bigg|_{y=0},
\end{aligned}
$$

which reduces to

$$
\begin{aligned}
&\lim_{y\to 0}\frac{3f^2_{t_{n-1}+y^2,ij}\cdot y}{\left(F_{t_{n-1}+y^2,ij}-F_{t_{n-1},ij}\right)^{3/2}}\left(\frac{f_{t_{n-1}+y^2,ij}\cdot y^2-F_{t_{n-1}+y^2,ij}+F_{t_{n-1},ij}}{F_{t_{n-1}+y^2,ij}-F_{t_{n-1},ij}}\right)\\
=&\lim_{y\to 0}\frac{3f^2_{t_{n-1}+y^2,ij}\cdot y}{\left(F_{t_{n-1}+y^2,ij}-F_{t_{n-1},ij}\right)^{5/2}}\left(f_{t_{n-1}+y^2,ij}\cdot y^2-f_{t_{n-1},ij}\cdot y^2-f'_{t_{n-1},ij}\frac{y^4}{2}-h^{(1)}_{t_{n-1}+y^2,ij}\cdot y^4\right)\\
=&\lim_{y\to 0}\frac{3f^2_{t_{n-1}+y^2,ij}\cdot y}{\left(F_{t_{n-1}+y^2,ij}-F_{t_{n-1},ij}\right)^{5/2}}\left(f'_{t_{n-1},ij}\cdot y^4+h^{(2)}_{t_{n-1}+y^2,ij}\cdot y^4-f'_{t_{n-1},ij}\frac{y^4}{2}-h^{(1)}_{t_{n-1}+y^2,ij}\cdot y^4\right)\\
=&\frac{3}{2}\frac{f'_{t_{n-1},ij}}{\sqrt{f_{t_{n-1},ij}}}
\end{aligned}
$$

by a Taylor expansion to $F_{t_{n-1}+y^2,ij}$ and $f_{t_{n-1}+y^2,ij}$, where $h^{(1)}_{t_{n-1}+y^2,ij}$ and $h^{(2)}_{t_{n-1}+y^2,ij}$ are two functions with $\lim_{y\to 0}h^{(1)}_{t_{n-1}+y^2,ij}=0$ and $\lim_{y\to 0}h^{(2)}_{t_{n-1}+y^2,ij}=0$. Then, we get

$$
\begin{aligned}
g'''_{N,n}(0)=&\sum_{i=1}^{d}\pi_{(n-1),i}\left(\sum_{j=1}^{d}\sqrt{f_{t_{n-1},ij}}Z_{N,n,j}\right)^3+6\sum_{i=1}^{d}\pi_{(n-1),i}h_{t_{n-1}}\left(\sum_{j=1}^{d}\sqrt{f_{t_{n-1},ij}}Z_{N,n,j}\right)\\
&+\frac{3}{2}\sum_{i,j=1}^{d}\pi_{(n-1),i}\frac{f'_{t_{n-1},ij}}{\sqrt{f_{t_{n-1},ij}}}Z_{N,n,j}.
\end{aligned}
$$

By (5.10) we obtain

$$
f'_{t_{n-1},ij}=2\sqrt{f_{t_{n-1},ij}}\left(\sigma_{u,ij}-\pi_u^T\sigma_u(:,j)\right)',
$$

we get then

$$
\begin{aligned}
g'''_{N,n}(0)=&\sum_{i=1}^{d}\pi_{(n-1),i}\left(\sum_{j=1}^{d}\sqrt{f_{t_{n-1},ij}}Z_{N,n,j}\right)^3+6\sum_{i=1}^{d}\pi_{(n-1),i}h_{t_{n-1}}\left(\sum_{j=1}^{d}\sqrt{f_{t_{n-1},ij}}Z_{N,n,j}\right)\\
&+3\sum_{i,j=1}^{d}\pi_{(n-1),i}\left(\sigma_{u,ij}-\pi_u^T\sigma_u(:,j)\right)'Z_{N,n,j}.
\end{aligned}
$$

The assumption that (σ_t), (μ_t) and (π_t) satisfy the regular conditions yields that $E[g'''_{N,n}(0)]=0$ and $Var[g'''_{N,n}(0)], n=1,\dots,N$ are bounded. Thus, the first term on the right side of (5.19) is the norm of a sum of independent, but not identically distributed mean zero random variables. It follows that

$$
\left\|\frac{1}{6}\sum_{n=1}^{N}g'''_{N,n}(0)\Delta t^{3/2}\right\|^2=\frac{\Delta t^3}{36}E\left[\sum_{n=1}^{N}g'''_{N,n}(0)\right]^2=\frac{\Delta t^3}{36}\sum_{n=1}^{N}Var\left[g'''_{N,n}(0)\right]
$$

with

$$
\frac{\Delta t^3}{36}N\min_{1\le n\le N}Var\left[g'''_{N,n}(0)\right]\le\frac{\Delta t^3}{36}\sum_{n=1}^{N}Var\left[g'''_{N,n}(0)\right]\le\frac{\Delta t^3}{36}N\max_{1\le n\le N}Var\left[g'''_{N,n}(0)\right].
$$

Thus, we get $\left\| \frac{1}{6} \sum_{n=1}^{N} g_{N,n}^{'''}(0)\, \Delta t^{3/2} \right\| \in O(\Delta t)$.

For the next term in (5.19), the summands do not have zero mean, but the Minkowski inequality yields

$$\left\| \sum_{n=1}^{N} c_{N,n}(\sqrt{\Delta t}) \Delta t^{3/2} \right\| \leq \Delta t^{3/2} \sum_{n=1}^{N} \left\| c_{N,n}(\sqrt{\Delta t}) \right\|.$$

The norm of $c_{N,n}(\sqrt{\Delta t}) = \frac{1}{6}\left(g_{N,n}^{'''}(\xi) - g_{N,n}^{'''}(0)\right)$, $n = 1, \dots, N$ go to zero as $\sqrt{\Delta t} \to 0$, since the norm of $g_{N,n}^{'''}(\xi)$ and $g_{N,n}^{'''}(0)$ exist and $g_{N,n}^{'''}(\xi)$ is a continuous function of ξ for small ξ. Thus, we have

$$\left\| \sum_{n=1}^{N} c_{N,n}(\sqrt{\Delta t}) \Delta t^{3/2} \right\| = o\left(N \Delta t^{3/2}\right) = o\left(\sqrt{\Delta t}\right).$$

It remains to show $\|r_N\| = O(\Delta t)$ for the remainder in (5.19). Each term in r_N is a product of the four types of terms in (5.17). We group the terms in r_N according to the number J of factors different from 1, for $J = 2, 3, \dots, N$. The case $J = 0$ and $J = 1$ appear explicitly in (5.18). To lighten notation, we write

$$a_n := g_{N,n}^{''}(0)/2, \quad b_n := g_{N,n}^{'''}(0)/6, \quad c_n := c_{N,n}(\sqrt{\Delta t}),$$

for $n = 1, \dots, N$. Then,

$$r_N = \sum_{J=2}^{N} \sum_{k=0}^{J} \sum_{l=0}^{J-k} \sum_{n_1,\dots,n_J} c_{n_1} \dots c_{n_k} b_{n_{k+1}} \dots b_{n_{k+l}} a_{n_{k+l+1}} \dots a_{n_J} \Delta t^{3k/2} \Delta t^{3l/2} \Delta t^{J-k-l}. \tag{5.20}$$

Here, $n_1, \dots, n_J$ denote the J factors different from 1, and each n_i ranges from 1 to N. Note that the innermost sum $\sum_{n_1,\dots,n_J}$ is taken over sets of distinct indices $n_1, \dots, n_J$. In this expression for r_N, each power of Δt is determined by its degree in the Taylor expansion, i.e. the k factors $c_{n_1}, \dots, c_{n_k}$ contribute $\Delta t^{3k/2}$ and so on.

We note that with distinct indices, each product in (5.20) is a product of independent random variables drawn from up to $3N$ distributions. Thus, we have

$$\begin{aligned} &\left\| c_{n_1} \dots c_{n_k} b_{n_{k+1}} \dots b_{n_{k+l}} a_{n_{k+l+1}} \dots a_{n_J} \right\| \\ &= \|c_{n_1}\| \dots \|c_{n_k}\| \left\|b_{n_{k+1}}\right\| \dots \left\|b_{n_{k+l}}\right\| \left\|a_{n_{k+l+1}}\right\| \dots \|a_{n_J}\| \leq \rho^J, \end{aligned}$$

for some ρ not depending on J or N; e.g., if $\Delta t < 1$, we can take

$$\rho = \max_{1 \leq n \leq N} \left(\|a_n\|, \|b_n\|, \|c_n\|\right) < \infty.$$

Let J, k, l be fixed. We consider

$$Y = Y_{n_1,\dots,n_J} = c_{n_1} \dots c_{n_k} b_{n_{k+1}} \dots b_{n_{k+l}} a_{n_{k+l+1}} \dots a_{n_J}$$

with fixed $n_1, \dots, n_J$. Recall that each b_n is a linear combination of odd powers of normal distributions and each a_n is a centered sum of even powers. Thus, each $a_n b_n$ is a linear combination of odd powers of $Z_{N,n}$. It follows that $E[a_n b_n] = 0$. By the independence of the $Z_{N,n}, n = 1, \dots, N$, we also have

$$E[a_n a_m] = E[b_n b_m] = E[a_n b_m] = 0$$

whenever $n \neq m$. Thus, we have $E[Y] = 0$, except the one for which $k = J$. Now consider a term

$$Y' = Y_{n_1,\dots,n_k,m_{k+1},\dots,m_J} = c_{n_1} \dots c_{n_k} b_{m_{k+1}} \dots b_{m_{k+l}} a_{m_{k+l+1}} \dots a_{m_J},$$

different from Y. We could show that $E[YY'] = 0$. In the situation that an index $n \in \{n_{k+1}, \dots, n_J\}$ does not appear in $\{m_{k+1} \dots, m_J\}$, there exists also an index $m \in \{m_{k+1}, \dots, m_J\}$ not in $\{n_{k+1}, \dots, n_J\}$ with $n \neq m$, we can then pull the corresponding factors a_n (or b_n) and b_m (or a_m) out of the product to get $E[YY'] = 0$. Otherwise, there must be an index $n \in \{n_{k+1}, \dots, n_{k+l}\}$ that appears in $\{m_{k+l+1} \dots, m_J\}$ or else an index in $n \in \{n_{k+l+1}, \dots, n_J\}$ that appears in $\{m_{k+1} \dots, m_{k+l}\}$. In either case, we can pull a factor $a_n b_n$ out of the product of Y and Y' and again conclude that $E[YY'] = 0$. Thus, we have Y and Y' are uncorrelated.

The norm of a sum of M uncorrelated, mean zero random variables is smaller than $\sqrt{M}$ times the norm of the largest one of them. Using this property and then the triangle inequality with $T = N \cdot \Delta t$, $\Delta t < 1$, we get, for $k \leq J - 1$,

$$\begin{aligned}
&\left\| \sum_{n_1,\dots,n_J} c_{n_1} \dots c_{n_k} b_{n_{k+1}} \dots b_{n_{k+l}} a_{n_{k+l+1}} \dots a_{n_J} \Delta t^{3k/2} \Delta t^{3l/2} \Delta t^{J-k-l} \right\| \\
=& \left\| \sum_{n_1,\dots,n_k} c_{n_1} \dots c_{n_k} \sum_{n_{k+1},\dots,n_J} b_{n_{k+1}} \dots b_{n_{k+l}} a_{n_{k+l+1}} \dots a_{n_J} \right\| \Delta t^{J+k/2+l/2} \\
\leq& N^k \max_{n_1,\dots,n_k} \left\| c_{n_1} \dots c_{n_k} \sum_{n_{k+1},\dots,n_J} b_{n_{k+1}} \dots b_{n_{k+l}} a_{n_{k+l+1}} \dots a_{n_J} \right\| \Delta t^{J+k/2+l/2} \\
\leq& N^k N^{\frac{J-k}{2}} \max_{n_1,\dots,n_J} \| c_{n_1} \dots c_{n_k} b_{n_{k+1}} \dots b_{n_{k+l}} a_{n_{k+l+1}} \dots a_{n_J} \| \Delta t^{J+k/2+l/2} \\
\leq& \rho^J (1+T)^J \Delta t^{(J+l)/2} \\
\leq& \rho^J (1+T)^J \Delta t^{J/2}
\end{aligned} \tag{5.21}$$

for Δt small. We write $(1 + T)$ in the forth calculation to cover the possibility that $T < 1$. For $k = J$, we have

$$\left\| \sum_{n_1,\dots,n_J} c_{n_1} \dots c_{n_J} \right\| \Delta t^{3J/2} \leq N^J \| c_{n_1} \dots c_{n_J} \| \Delta t^{3J/2} = \rho^J T^J \Delta t^{J/2}, \tag{5.22}$$

so the bound in (5.21) applies in this case as well.

Now we return to (5.20). For each J there are

$$(1 + J) + J + \dots + 1 = (J + 2)(J + 1)/2 = \begin{pmatrix} J+2 \\ 2 \end{pmatrix}$$

combinations of values of $k = 0, 1, \dots, J$ and $l = 0, 1, \dots, J - k$ factors. Thus, we have

$$\|r_N\| \leq \sum_{J=2}^{N} \begin{pmatrix} J+2 \\ 2 \end{pmatrix} \rho^J (1+T)^J \Delta t^{J/2} \leq \sum_{J=2}^{N} (6\rho^2 (1+T)^2 \Delta t)^{J/2} = K\Delta t \frac{1 - (K\Delta t)^{\frac{N-1}{2}}}{1 - \sqrt{K\Delta t}}$$

with $K = 6\rho^2(1+T)^2$. Since

$$\lim_{\Delta t \to 0} K\Delta t \frac{1-(K\Delta t)^{\frac{N-1}{2}}}{1-\sqrt{K\Delta t}} \frac{1}{\Delta t} = K,$$

we get $r_N = O(\Delta t)$. This concludes the proof. □

5.2.2 Limit theorem

For ϵ_n given in Proposition 5.2.1, there is

$$\begin{aligned} Var[\epsilon_n] =& Var\left[\frac{1}{2}\sum_{i=1}^{d} \pi_{(n-1),i}\left(\sigma_{(n-1)}(i,:)\Delta W^S_{(n-1)}\right)^2 - \frac{1}{2}\left(\pi^T_{(n-1)}\sigma_{(n-1)}\Delta W^S_{(n-1)}\right)^2\right] \\ =& Var\left[\frac{1}{2}\sum_{i=1}^{d} \pi_{(n-1),i}\left(\left(\sigma_{(n-1)}(i,:) - \pi^T_{(n-1)}\sigma_{(n-1)}\right)\Delta W^S_{(n-1)}\right)^2\right]. \end{aligned} \tag{5.23}$$

Let

$$\hat{\sigma}^2_{L,n-1} := Var\left[\frac{1}{2}\sum_{i=1}^{d} \pi_{(n-1),i}\left(\left(\sigma_{(n-1)}(i,:) - \pi^T_{(n-1)}\sigma_{(n-1)}\right)\hat{Z}_{N,n}\right)^2\right],$$

where $\hat{Z}_{N,n}$ is a d-dimensional random vector with $\hat{Z}_{N,n} \sim N(0, I)$, then, we get

$$Var[\epsilon_n] = \hat{\sigma}^2_{L,n-1}\Delta t^2. \tag{5.24}$$

With Z being a d-dimensional standard normal vector and $A \in \mathbb{R}^{d\times d}$, there is

$$Var[Z^T A Z] = 2Tr(A^T A),$$

this equality yields

$$\begin{aligned} \hat{\sigma}^2_{L,n-1} =& Var\left[\frac{1}{2}\sum_{j,k=1}^{d} \hat{Z}^{(j)}_{N,n}\hat{Z}^{(k)}_{N,n}\right. \\ &\left.\cdot\left(\sum_{i=1}^{d} \pi_{(n-1),i}\left(\sigma_{(n-1),ij} - \pi^T_{(n-1)}\sigma_{(n-1)}(:,j)\right)\cdot\left(\sigma_{(n-1),ik} - \pi^T_{(n-1)}\sigma_{(n-1)}(:,k)\right)\right)\right] \\ =& \frac{1}{2}Tr(B_n^2), \end{aligned} \tag{5.25}$$

where B_n is a symmetric matrix with entries

$$B_{n,jk} = \sum_{i=1}^{d} \pi_{(n-1),i}\left(\sigma_{(n-1),ij} - \pi^T_{(n-1)}\sigma_{(n-1)}(:,j)\right)\cdot\left(\sigma_{(n-1),ik} - \pi^T_{(n-1)}\sigma_{(n-1)}(:,k)\right)$$

for $j,k=1,\dots,d$. To lighten notation, we omit the time parameter and let the parameters σ, $\hat{\sigma}$ and ω be taken at $(n-1)\Delta t$. Then there is

$$\begin{aligned}
&\hat{\sigma}^2_{L,n-1} = \frac{1}{2}\sum_{j,k=1}^{d} B^2_{n,jk}\\
=&\frac{1}{2}\sum_{jkip=1}^{d} \pi_i \left(\sigma_{ij}-\pi^T\sigma(:,j)\right)\cdot\left(\sigma_{ik}-\pi^T\sigma(:,k)\right)\pi_p\left(\sigma_{pj}-\pi^T\sigma(:,j)\right)\cdot\left(\sigma_{pk}-\pi^T\sigma(:,k)\right)\\
=&\frac{1}{2}\sum_{ip=1}^{d}\pi_i\pi_p\sum_{j=1}^{d}\left(\sigma_{ij}-\pi^T\sigma(:,j)\right)\cdot\left(\sigma_{pj}-\pi^T\sigma(:,j)\right)\sum_{k=1}^{d}\left(\sigma_{ik}-\pi^T\sigma(:,k)\right)\cdot\left(\sigma_{pk}-\pi^T\sigma(:,k)\right)\\
=&\frac{1}{2}\sum_{ip=1}^{d}\pi_i\pi_p\left(\hat{\sigma}_{ip}-\pi^T\hat{\sigma}(:,i)-\pi^T\hat{\sigma}(:,p)+\pi^T\hat{\sigma}\pi\right)^2\\
=&\frac{1}{2}\sum_{ip=1}^{d}\pi_i\pi_p\left(\hat{\sigma}_{ip}\hat{\sigma}_{ip}-2\pi^T\hat{\sigma}(:,p)\hat{\sigma}_{ip}-2\pi^T\hat{\sigma}(:,i)\hat{\sigma}_{ip}+2\pi^T\hat{\sigma}\pi\hat{\sigma}_{ip}+2\pi^T\hat{\sigma}(:,i)\pi^T\hat{\sigma}(:,p)\right.\\
&\left.-2\pi^T\hat{\sigma}\pi\pi^T\hat{\sigma}(:,p)-2\pi^T\hat{\sigma}\pi\pi^T\hat{\sigma}(:,i)+\pi^T\hat{\sigma}(:,i)\pi^T\hat{\sigma}(:,i)+\pi^T\hat{\sigma}(:,p)\pi^T\hat{\sigma}(:,p)+\left(\pi^T\hat{\sigma}\pi\right)^2\right)\\
=&\frac{1}{2}\left(\sum_{ip=1}^{d}\pi_i\pi_p\hat{\sigma}_{ip}\hat{\sigma}_{ip}-2\sum_{i=1}^{d}\pi^T\hat{\sigma}(:,i)\pi_i\hat{\sigma}(i,:)\pi+\left(\pi^T\hat{\sigma}\pi\right)^2\right)\\
=&\frac{1}{2}\left(\pi^T\left(\hat{\sigma}\circ\hat{\sigma}\right)\pi+\left(\pi^T\hat{\sigma}\pi\right)^2-2\pi^T\hat{\sigma}D\hat{\sigma}\pi\right),
\end{aligned}$$

where $\circ$ denotes elementwise multiplication and $D := diag(\pi)$. The entire expression for $\hat{\sigma}^2_{L,n-1}$ is then given by

$$\begin{aligned}
\hat{\sigma}^2_{L,n-1} =&\frac{1}{2}\left(\pi^T_{(n-1)}\left(\hat{\sigma}_{(n-1)}\circ\hat{\sigma}_{(n-1)}\right)\pi_{(n-1)}+\left(\pi^T_{(n-1)}\hat{\sigma}_{(n-1)}\pi_{(n-1)}\right)^2\right.\\
&\left.-2\pi^T_{(n-1)}\hat{\sigma}_{(n-1)}D_{(n-1)}\hat{\sigma}_{(n-1)}\pi_{(n-1)}\right).
\end{aligned} \tag{5.26}$$

Theorem 5.2.2. *(Limit theorem I) As $N\to\infty$,*

$$\left\{\sqrt{N}\left(\hat{X}_{(N)}-X_T,\frac{\hat{X}_{(N)}-X_T}{X_T}\right)\right\}\xrightarrow{\mathcal{D}}\left(X_TV,V\right), \tag{5.27}$$

where

$$V\sim N\left(0,T\int_0^T\hat{\sigma}^2_{L,u}du\right)$$

is independent of X_T with

$$\hat{\sigma}^2_{L,u}=\frac{1}{2}\left(\pi^T_u\left(\hat{\sigma}_u\circ\hat{\sigma}_u\right)\pi_u+\left(\pi^T_u\hat{\sigma}_u\pi_u\right)^2-2\pi^T_u\hat{\sigma}_uD_u\hat{\sigma}_u\pi_u\right),$$

where $\circ$ denotes elementwise multiplication and D_u denotes the diagonal matrix $diag(\pi_u)$.

Proof. We can obtain the assertion by applying multidimensional Lindeberg-Feller central limit theorem. The proof follows in 3 steps.
(i) *Fulfillment of the multidimensional Lindeberg-Feller condition:*

It is clear that $\epsilon_n,\ n = 1, \ldots, N$ given in Proposition 5.2.1 are independent distributed and $Var[\epsilon_n] < \infty$. We note that $E[\epsilon_n] = 0$, since

$$E\left[\frac{1}{2}\sum_{i=1}^{d}\pi_{(n-1),i}\left(\sigma_{(n-1)}(i,:)\Delta W^S_{(n-1)}\right)^2\right] = \frac{1}{2}\sum_{i=1}^{d}\pi_{(n-1),i}\|\sigma_{(n-1)}(i,:)\|^2\Delta t$$

and

$$E\left[\frac{1}{2}\left(\pi^T_{(n-1)}\sigma_{(n-1)}\Delta W^S_{(n-1)}\right)^2\right] = \frac{1}{2}\|\pi^T_{(n-1)}\sigma_{(n-1)}\|^2\Delta t.$$

Let

$$\gamma_n := \int_{(n-1)\Delta t}^{n\Delta t}\left(\pi_u\mu_u - \frac{1}{2}\left\|\pi^T_u\sigma_u\right\|^2\right)du + \int_{(n-1)\Delta t}^{n\Delta t}\pi^T_u\sigma_u dW^S_u$$

and

$$\tilde{\gamma}_n := \gamma_n - E[\gamma_n] = \int_{(n-1)\Delta t}^{n\Delta t}\pi^T_u\sigma_u dW^S_u$$

for $1 \le n \le N$, we have that $(\tilde{\gamma}_n)_{n\ge1}$ is a sequence of independent random variables with zero mean.

Let us consider in the following a sequence of independent $\mathbb{R}^2$-valued random vectors $E_n = \left(\tilde{\gamma}_n, \sqrt{N}\epsilon_n\right),\ 1 \le n \le N$ with zero mean. We want to show that the sequence E_n satisfies the multidimensional Lindeberg-Feller condition in [8, Corollary 18.2], i.e. for $\forall\delta > 0$, $\forall\theta_1, \theta_2 \in \mathbb{R}$,

$$\sum_{n=1}^{N}E\left[\left(\theta_1\tilde{\gamma}_n + \theta_2\sqrt{N}\epsilon_n\right)^2 \cdot \mathbf{1}_{\{\theta_1\tilde{\gamma}_n+\theta_2\sqrt{N}\epsilon_n>\delta\}}\right] \longrightarrow 0 \quad \textit{as } N \to \infty. \tag{5.28}$$

To show (5.28) we first write $\tilde{\gamma}_n$ as

$$\tilde{\gamma}_n = Z^{(1)}_{N,n}\left(\int_{(n-1)\Delta t}^{n\Delta t}\|\pi^T_u\sigma_u\|^2du\right)^{1/2} = K^{(1)}_{N,n}Z^{(1)}_{N,n}\sqrt{\Delta t}$$

by the mean value theorem with

$$\min_{0\le u\le T}\|\pi^T_u\sigma_u\| \le K^{(1)}_{N,n} \le \max_{0\le u\le T}\|\pi^T_u\sigma_u\|, \quad \forall N \in \mathbb{N}$$

and $Z^{(1)}_{N,n}$ being a one-dimensional standard normal distributed random variable. We write ϵ_n in the same way as

$$\epsilon_n = \sum_{i,j=1}^{d}K^{(2)}_{N,n,jk}Z^{(2)}_{N,n,j}Z^{(2)}_{N,n,k}\Delta t + \left(\sum_{j=1}^{d}K^{(3)}_{N,n,j}Z^{(3)}_{N,n,j}\right)^2\Delta t + K^{(4)}_{N,n}\Delta t,$$

where $K^{(2)}_{N,n,jk}, K^{(3)}_{N,n,j}, K^{(4)}_{N,n} \in \mathbb{R}$ are bounded for $j, k = 1, \ldots, d$, $\forall N \in \mathbb{N}$ and $Z^{(2)}_{N,n} = \left(Z^{(2)}_{N,n,j}\right)_{1\le j\le d}$, $Z^{(3)}_{N,n} = \left(Z^{(3)}_{N,n,j}\right)_{1\le j\le d}$ are two d-dimensional standard normal distributed random vectors. Then, we obtain

$$\lim_{N\to\infty}\left(E\left[\left(\mathbf{1}_{\{\theta_1\tilde{\gamma}_n+\theta_2\sqrt{N}\epsilon_n>\delta\}}\right)^2\right]\right)^{1/2}=0,\quad \forall\delta>0$$

and

$$\frac{\sqrt{E\left[\left(\theta_1\tilde{\gamma}_n+\theta_2\sqrt{N}\epsilon_n\right)^4\right]}}{\Delta t}$$
$$=\frac{\sqrt{E\left[\left(\theta_1 K^{(1)}_{N,n}Z^{(1)}_{N,n}+\theta_2\left[\sum_{i,j=1}^d K^{(2)}_{N,n,jk}Z^{(2)}_{N,n,j}Z^{(2)}_{N,n,k}+\left(\sum_{j=1}^d K^{(3)}_{N,n,j}Z^{(3)}_{N,n,j}\right)^2+K^{(4)}_{N,n}\right]\right)^4\right]\Delta t}}{\Delta t}$$
$$=\sqrt{E\left[\left(\theta_1 K^{(1)}_{N,n}Z^{(1)}_{N,n}+\theta_2\left[\sum_{i,j=1}^d K^{(2)}_{N,n,jk}Z^{(2)}_{N,n,j}Z^{(2)}_{N,n,k}+\left(\sum_{j=1}^d K^{(3)}_{N,n,j}Z^{(3)}_{N,n,j}\right)^2+K^{(4)}_{N,n}\right]\right)^4\right]}$$

is bounded. Employing Cauchy-Schwarz inequality, we get

$$\begin{aligned}&E\left[\left(\theta_1\tilde{\gamma}_n+\theta_2\sqrt{N}\epsilon_n\right)^2\cdot\mathbf{1}_{\{\theta_1\tilde{\gamma}_n+\theta_2\sqrt{N}\epsilon_n>\delta\}}\right]\\ &\le\left(E\left[\left(\theta_1\tilde{\gamma}_n+\theta_2\sqrt{N}\epsilon_n\right)^4\right]E\left[\left(\mathbf{1}_{\{\theta_1\tilde{\gamma}_n+\theta_2\sqrt{N}\epsilon_n>\delta\}}\right)^2\right]\right)^{1/2},\end{aligned}$$

thus, we conclude that $E\left[\left(\theta_1\tilde{\gamma}_n+\theta_2\sqrt{N}\epsilon_n\right)^2\cdot\mathbf{1}_{\{\theta_1\tilde{\gamma}_n+\theta_2\sqrt{N}\epsilon_n>\delta\}}\right]$ is $o(\sqrt{\Delta t})$ and (5.28) holds. The fulfillment of the Lindeberg-Feller condition yields then

$$\left(\sum_{n=1}^N\tilde{\gamma}_n,\sqrt{N}\sum_{n=1}^N\epsilon_n\right)\xrightarrow{\mathcal{D}}N(0,\hat{E})$$

with $\hat{E}=\lim_{N\to\infty}\sum_{n=1}^N Cov(E_n)\in\mathbb{R}^{d\times d}$ being the limit of the sum of the covariance matrix $Cov\,(E_n)$ over $n=1,\dots,N$.
(ii) *Computation of* $\hat{E}$:
We first show that $\tilde{\gamma}_n$ and ϵ_n are uncorrelated, namely we want to show

$$E\left[\left(\int_{(n-1)\Delta t}^{n\Delta t}\pi_u^T\sigma_u\,(:,j)\,dW^S_{u,j}\right)\Delta W^S_{(n-1),k}\cdot\Delta W^S_{(n-1),q}\right]=0 \tag{5.29}$$

for $j,k,q=1,\dots,d$. For the cases that j,k,q are not all the same, (5.29) follows easily from the independence of the entries of $\Delta W^S_{(n-1)}$. By [5, Lemma 48.2.], we get that for $j=k=q$,

$$\left(\int_{(n-1)\Delta t}^{n\Delta t}\pi_u^T\sigma_u\,(:,j)\,dW^S_{u,j},\int_{(n-1)\Delta t}^{n\Delta t}dW^S_{u,j},\int_{(n-1)\Delta t}^{n\Delta t}dW^S_{u,j}\right)\sim N(0,\Gamma)$$

with

$$\Gamma = \begin{pmatrix} \int_{(n-1)\Delta t}^{n\Delta t}(\pi_u^T\sigma_u(:,j))^2 du & \int_{(n-1)\Delta t}^{n\Delta t}\pi_u^T\sigma_u(:,j)\,du & \int_{(n-1)\Delta t}^{n\Delta t}\pi_u^T\sigma_u(:,j)\,du \\ \int_{(n-1)\Delta t}^{n\Delta t}\pi_u^T\sigma_u(:,j)\,du & \Delta t & \Delta t \\ \int_{(n-1)\Delta t}^{n\Delta t}\pi_u^T\sigma_u(:,j)\,du & \Delta t & \Delta t \end{pmatrix}.$$

Let $Z_{N,n}^{(4)}$ and $Z_{N,n}^{(5)}$ be two independent standard normal distributed random variables, we represent

$$\int_{(n-1)\Delta t}^{n\Delta t} dW_{u,j}^S := Z_{N,n}^{(4)}\sqrt{\Delta t} \quad \textit{and} \quad \int_{(n-1)\Delta t}^{n\Delta t}\pi_u^T\sigma_u(:,j)\,dW_{u,j}^S := \rho_{N,n}^{(1)}Z_{N,n}^{(4)} + \rho_{N,n}^{(2)}Z_{N,n}^{(5)}$$

with

$$\rho_{N,n}^{(1)} = \int_{(n-1)\Delta t}^{n\Delta t}\pi_u^T\sigma_u(:,j)\,du\frac{1}{\sqrt{\Delta t}}, \quad \rho_{N,n}^{(2)} = \sqrt{\int_{(n-1)\Delta t}^{n\Delta t}(\pi_u^T\sigma_u(:,j))^2 du - \left(\rho_{N,n}^{(1)}\right)^2}.$$

Thus we get

$$\begin{aligned} &E\left[\left(\int_{(n-1)\Delta t}^{n\Delta t}\pi_u^T\sigma_u(:,j)\,dW_{u,j}^S\right)\Delta W_{(n-1),j}^S\Delta W_{(n-1),j}^S\right] \\ =&E\left[\left(Z_{N,n}^{(4)}\right)^2\Delta t\left(\rho_{N,n}^{(1)}Z_{N,n}^{(4)} + \rho_{N,n}^{(2)}Z_{N,n}^{(5)}\right)\right] = 0. \end{aligned}$$

and (5.29) is shown. Recalling the representation in (5.24), we get the following covariance matrix of E_n:

$$Cov(E_n) = Cov\left(\left(\tilde{\gamma}_n, \sqrt{N}\epsilon_n\right)\right) = \begin{pmatrix} \int_{(n-1)\Delta t}^{n\Delta t}\|\pi_u^T\sigma_u\|^2 du & 0 \\ 0 & \hat{\sigma}_{L,n-1}^2\Delta t T \end{pmatrix}.$$

We arrive at then the limit of its sum over n:

$$\begin{aligned} \lim_{N\to\infty}\sum_{n=1}^N Cov(E_n) &= \lim_{N\to\infty}\sum_{n=1}^N \begin{pmatrix} \int_{(n-1)\Delta t}^{n\Delta t}\|\pi_u^T\sigma_u\|^2 du & 0 \\ 0 & \hat{\sigma}_{L,n-1}^2\Delta t T \end{pmatrix} \\ &= \begin{pmatrix} \int_0^T\|\pi_u^T\sigma_u\|^2 du & 0 \\ 0 & T\int_0^T\hat{\sigma}_{L,u}^2 du \end{pmatrix} \end{aligned}$$

with $\hat{\sigma}_{L,u}^2$ given in the theorem.
(iii) *Multidimensional Lindeberg-Feller central limit theorem and the concluding result*: From the first and the second step we have achieved

$$\left(\sum_{n=1}^N\tilde{\gamma}_n, \sqrt{N}\sum_{n=1}^N\epsilon_n\right) \xrightarrow{\mathcal{D}} (Y, V)$$

with

$$Y \sim N\left(0, \int_0^T\|\pi_u^T\sigma_u\|^2 du\right), \quad V \sim N\left(0, T\int_0^T\hat{\sigma}_{L,u}^2 du\right)$$

independent. Since $X_T = \exp\left(\sum_{n=1}^N \gamma_n\right) = \exp\left(\int_0^T \left(\pi_u^T \mu_u - \frac{1}{2}\left\|\pi_u^T \sigma_u\right\|^2\right) du + \sum_{n=1}^N \tilde{\gamma}_n\right)$ is a continuous function of $\sum_{n=1}^N \tilde{\gamma}_n$, by the continuous mapping theorem, we get

$$\left(X_T, \sqrt{N}\sum_{n=1}^N \epsilon_n\right) \xrightarrow{\mathcal{D}} (X_T, V),$$

where X_T and V are independent. The Proposition 5.2.1 yields

$$E\left[\left(\sqrt{N}\frac{\hat{X}_{(N)} - X_T}{X_T} - \sqrt{N}\sum_{n=1}^N \epsilon_n\right)^2\right] \to 0$$

as $N \to \infty$, thus, we conclude that

$$V_N := \sqrt{N}\frac{\hat{X}_{(N)} - X_T}{X_T} \tag{5.30}$$

inherits the limiting distribution of $\sqrt{N}\sum_{n=1}^N \epsilon_n$, it yields then

$$\left(X_T, \sqrt{N}\frac{\hat{X}_{(N)} - X_T}{X_T}\right) \xrightarrow{\mathcal{D}} (X_T, V)$$

by [9, Theorem 4.1], where X_T and V are independent. Then by the continuous mapping theorem, we get

$$\sqrt{N}\left(\hat{X}_{(N)} - X_T, \frac{\hat{X}_{(N)} - X_T}{X_T}\right) \xrightarrow{\mathcal{D}} (X_T V, V),$$

where X_T and V are independent. □

5.2.3 Unconditional limit theorem

Instead of the deterministic processes (σ_t) and (μ_t) introduced in Section 5.1, let us now consider two d-dimensional stochastic processes (Σ_t) and (Θ_t), which are both independent of (W_t^S), as the coefficients of the asset price process. We assume that the paths of (Σ_t) and (Θ_t) satisfy the regular conditions listed in Section 5.1.

In this section we will show that there is a similar limit theorem result as Theorem 5.2.2 for the asset price process with stochastic drift process (Θ_t) and stochastic volatility process (Σ_t).

Let us consider a market with d risky assets. The continuous time evolution of the assets prices are given by

$$dS_t = diag(S_t)\Theta_t dt + diag(S_t)\Sigma_t dW_t^S \tag{5.31}$$

with (W_t^S) being a d dimensional Brownian motions vector. The covariance matrix of S_t is denoted by $\Psi_t = \Sigma_t \Sigma_t^T$.

We still denote the fractional portfolio strategy process by $(\pi_t)_{t\geq 0}$ with

$$\pi_t = (\pi_{t,1}, \ldots, \pi_{t,d})^T$$

satisfying $\pi_t^T \mathbf{1} = 1$. The portfolio wealth process of the continuously rebalanced portfolio and the discretely rebalanced portfolio are still denoted by (X_t) and $(\hat{X}_t)$ respectively. We define $(\mathcal{F}_t^{\Sigma})$ as the natural filtration with respect to (Σ_t), namely

$$\mathcal{F}_t^{\Sigma} := \sigma\{\Sigma_u, 0 \le u \le t\}.$$

Then the natural filtration with respect to (Θ_t) can be defined in the same way and is denoted by $(\mathcal{F}_t^{\Theta})$.

Theorem 5.2.3. *(Limit theorem II) As $N \to \infty$,*

$$\sqrt{N}\frac{\hat{X}_{(N)} - X_T}{X_T} \xrightarrow{\mathcal{D}} \sqrt{T}\int_0^T \Psi_{L,u} dW_u \tag{5.32}$$

with $(W_u)_{u\ge 0}$ being a Brownian motion independent of $(\Psi_{L,u})$ and

$$\Psi_{L,u} := \sqrt{\frac{1}{2}\left(\pi_u^T\left(\Psi_u \circ \Psi_u\right)\pi_u + \left(\pi_u^T \Psi_u \pi_u\right)^2 - 2\pi_u^T \Psi_u D_u \Psi_u \pi_u\right)},$$

where $\circ$ denotes elementwise multiplication and $D_u = diag\,(\pi_u)$.

Proof. By Theorem 5.2.2 we get as $N \to \infty$,

$$\sqrt{N}\left(\frac{\hat{X}_{(N)} - X_T}{X_T}\middle| \mathcal{F}_T^{\Sigma}, \mathcal{F}_T^{\Theta}\right) \xrightarrow{\mathcal{D}} N\left(0, T\int_0^T \bar{\Psi}_{L,u}^2 du\right), \tag{5.33}$$

where

$$\bar{\Psi}_{L,u}^2 := \frac{1}{2}\left(\bar{\pi}_u^T\left(\bar{\Psi}_u \circ \bar{\Psi}_u\right)\bar{\pi}_u + \left(\bar{\pi}_u^T \bar{\Psi}_u \bar{\pi}_u\right)^2 - 2\bar{\pi}_u^T \bar{\Psi}_u D_u \bar{\Psi}_u \bar{\pi}_u\right)$$

with $(\bar{\Psi}_t)$ and $(\bar{\pi}_t)$ being paths of (Ψ_t) and (π_t), respectively.

It is known that $N\left(0, T\int_0^T \bar{\Psi}_{L,u}^2 du\right)$ owns the same distribution as the stochastic integral

$$\bar{V} := \sqrt{T}\int_0^T \bar{\Psi}_{L,u} dW_u.$$

Let

$$\bar{V}_N := \sqrt{N}\left(\frac{\hat{X}_{(N)} - X_T}{X_T}\middle| \mathcal{F}_T^{\Sigma}, \mathcal{F}_T^{\Theta}\right),$$

then we get

$$\bar{V}_N \xrightarrow{\mathcal{D}} \bar{V} \quad as \quad N \to \infty \tag{5.34}$$

by (5.33). We denote

$$C_b := \{h : \mathbb{R} \to \mathbb{R} : \ h \textit{ is continuous and bounded}\},$$

then by the Portmanteau Theorem in [9, Theorem 2.1], (5.34) yields

$$\lim_{N\to\infty} E[h(\bar{V}_N)] = E[h(\bar{V})], \quad \forall h \in C_b,$$

i.e. for $\forall h \in C_b$

$$\lim_{N\to\infty} E\left[h\left(\sqrt{N}\left(\frac{\hat{X}_{(N)}-X_T}{X_T}\right)\right)\middle|\mathcal{F}_T^\Sigma,\mathcal{F}_T^\Theta\right] = E\left[h\left(\sqrt{T}\int_0^T \bar{\Psi}_{L,u}dW_u\right)\middle|\mathcal{F}_T^\Sigma,\mathcal{F}_T^\Theta\right].$$

Taking expectation on both sides, we get for $\forall h \in C_b$

$$\begin{aligned} &E\left[\lim_{N\to\infty} E\left[h\left(\sqrt{N}\left(\frac{\hat{X}_{(N)}-X_T}{X_T}\right)\right)\middle|\mathcal{F}_T^\Sigma,\mathcal{F}_T^\Theta\right]\right] \\ =&E\left[E\left[h\left(\sqrt{T}\int_0^T \bar{\Psi}_{L,u}dW_u\right)\middle|\mathcal{F}_T^\Sigma,\mathcal{F}_T^\Theta\right]\right]. \end{aligned} \tag{5.35}$$

From $h \in C_b$, it follows that $|h(\cdot)| \le K$, $\exists K \in \mathbb{R}$. It implies then

$$E\left[\left|h\left(\sqrt{N}\left(\frac{\hat{X}_{(N)}-X_T}{X_T}\right)\right)\right|\middle|\mathcal{F}_T^\Sigma,\mathcal{F}_T^\Theta\right] \le K, \quad \forall N \in \mathbb{N}.$$

Applying the dominated convergence theorem to (5.35), we get

$$\begin{aligned} &E\left[\lim_{N\to\infty} E\left[h\left(\sqrt{N}\left(\frac{\hat{X}_{(N)}-X_T}{X_T}\right)\right)\middle|\mathcal{F}_T^\Sigma,\mathcal{F}_T^\Theta\right]\right] \\ =&\lim_{N\to\infty} E\left[E\left[h\left(\sqrt{N}\left(\frac{\hat{X}_{(N)}-X_T}{X_T}\right)\right)\middle|\mathcal{F}_T^\Sigma,\mathcal{F}_T^\Theta\right]\right] \\ =&\lim_{N\to\infty} E\left[h\left(\sqrt{N}\left(\frac{\hat{X}_{(N)}-X_T}{X_T}\right)\right)\right] \end{aligned}$$

Then we arrive at the equality

$$\lim_{N\to\infty} E\left[h\left(\sqrt{N}\left(\frac{\hat{X}_{(N)}-X_T}{X_T}\right)\right)\right] = E\left[h\left(\sqrt{T}\int_0^T \Psi_{L,u}dW_u\right)\right]$$

for $\forall h \in C_b$ and the proof is concluded by applying the Portmanteau Theorem again. □

Remark 5.2.4. *From the (5.25) and (5.26), we get that*

$$\frac{1}{2}\left(\pi_u^T\left(\bar{\Psi}_u \circ \bar{\Psi}_u\right)\pi_u + \left(\pi_u^T\bar{\Psi}_u\pi_u\right)^2 - 2\pi_u^T\bar{\Psi}_u D_u\bar{\Psi}_u\pi_u\right)$$

is always positive. It guarantees that $\Psi_{L,u}$ is well-defined.

5.3 A volatility adjustment for discrete rebalancing

In this subsection, we derive a "continuity correction" which adjusts the distribution of X_T to get an approximation of the distribution of $\hat{X}_N$. As in [21], we investigate the asymptotic covariance between the scaled portfolio error V_N and the logarithm of the continuously rebalanced portfolio $\log(X_T)$ at first and get afterwards the so-called volatility adjustment.

We face the model in Section 5.1 again, namely we have deterministic processes (σ_t), (μ_t) and (ω_t) satisfying the regular conditions proposed in Section 5.1.

5.3.1 Asymptotic covariance

Recall that the scaled portfolio error V_N is given by

$$V_N := \sqrt{N}\left(\frac{\hat{X}_{(N)} - X_T}{X_T}\right)$$

in (5.30). It can be rewritten as

$$\hat{X}_{(N)} = X_T\left(1 + \frac{V_N}{\sqrt{N}}\right).$$

Note that the discretely rebalanced portfolio $\hat{X}_{(N)}$ is not guaranteed to be positive, since the relative difference $V_N/\sqrt{N}$ could be negative. Thus, we can not directly take its logarithm to calculate a volatility. Set

$$\bar{X}_T := X_T \exp\left(\frac{V_N}{\sqrt{N}}\right), \tag{5.36}$$

it yields then

$$\bar{X}_T = X_T\left(1 + \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{V_N}{\sqrt{N}}\right)^n\right) = \hat{X}_{(N)} + O(1/N).$$

Concerning the asymptotic covariance between V_N and $\log(X_T)$, we have the following proposition:

Proposition 5.3.1. *(i) We have*

$$\sqrt{N}Cov\left[\log(X_T), V_N)\right] \to \frac{T}{2}\int_0^T b_u^{''}du + T\int_0^T \lambda_{L,u}du, \tag{5.37}$$

where

$$b_u^{''} := -\pi_u^T\sigma_u\sigma_u^T \lim_{h\downarrow 0}\frac{\pi_{u+h} - \pi_u}{h}$$

and

$$\lambda_{L,u} := \mu_u^T D_u\sigma_u\sigma_u^T\pi_u - \pi_u^T\mu_u \cdot \|\pi_u^T\sigma_u\|^2 + \|\pi_u^T\sigma_u\|^4 - \pi_u^T\sigma_u\sigma_u^T D_u\sigma_u\sigma_u^T\pi_u, \tag{5.38}$$

with $D_u = diag\left(\pi_u\right)$.

(ii) $E\left[\left(\bar{X}_T - \hat{X}_{(N)}\right)^2\right] = O(N^{-2})$ *and*

$$N\left(Var\left[\log\left(\bar{X}_T\right)\right] - Var\left[\log(X_T)\right]\right) \to T\int_0^T \sigma_{L,u}^2du + T\int_0^T b_u^{''}du + 2T\int_0^T \lambda_{L,u}du.$$

Proof. By (5.5) we have

$$V_N = \sqrt{N}\frac{\hat{X}_{(N)} - X_T}{X_T} = \sqrt{N}\sum_{n=0}^{N-1}\left(\frac{\hat{X}_{(n+1)}}{X_{(n+1)}} - \frac{\hat{X}_{(n)}}{X_{(n)}}\right).$$

Replacing X_T by its expression in (5.4) and plugging (5.30) yield

$$\begin{aligned}
&Cov\left[\log(X_T), V_N\right] \\
=&E\left[\log(X_T)V_N\right] - E\left[\log(X_T)\right]E\left[V_N\right] \\
=&E\left[\left(\int_0^T \left(\pi_u^T\mu_u - \frac{1}{2}\|\pi_u^T\sigma_u\|^2\right)du + \int_0^T \pi_u^T\sigma_u dW_u^S\right)\cdot V_N\right] - E\left[\log(X_T)\right]E\left[V_N\right] \\
=&\int_0^T \left(\pi_u^T\mu_u - \frac{1}{2}\|\pi_u^T\sigma_u\|^2\right)du \cdot E\left[V_N\right] + E\left[\int_0^T \pi_u^T\sigma_u dW_u^S \cdot V_N\right] \\
&-\int_0^T \left(\pi_u^T\mu_u - \frac{1}{2}\|\pi_u^T\sigma_u\|^2\right)du \cdot E\left[V_N\right] \\
=&\sqrt{N}\sum_{k=1}^{N}\sum_{n=0}^{N-1} E\left[\int_{(k-1)\Delta t}^{k\Delta t} \pi_u^T\sigma_u dW_u^S\left(\frac{\hat{X}_{(n+1)}}{X_{(n+1)}} - \frac{\hat{X}_{(n)}}{X_{(n)}}\right)\right].
\end{aligned} \tag{5.39}$$

We claim that

$$E\left[\int_{(k-1)\Delta t}^{k\Delta t} \pi_u^T\sigma_u dW_u^S\left(\frac{\hat{X}_{(n+1)}}{X_{(n+1)}} - \frac{\hat{X}_{(n)}}{X_{(n)}}\right)\right] = \left\{\begin{array}{cc} 0 & k \geq n+2, \\ \tilde{\lambda}_{L,N,n} + O(\Delta t^3) & k = n+1, \\ o(\Delta t^{7/2}) & k \leq n \end{array}\right. \tag{5.40}$$

with $\tilde{\lambda}_{L,N,n} \in \mathbb{R}$. The first case in (5.40) follows immediately from the independence of Brownian motion increments. For the second case in (5.40), we may write

$$\begin{aligned}
&E\left[\int_{n\Delta t}^{(n+1)\Delta t} \pi_u^T\sigma_u dW_u^S\left(\frac{\hat{X}_{(n+1)}}{X_{(n+1)}} - \frac{\hat{X}_{(n)}}{X_{(n)}}\right)\right] \\
=&E\left[\int_{n\Delta t}^{(n+1)\Delta t} \pi_u^T\sigma_u dW_u^S \frac{\hat{X}_{(n+1)}}{X_{(n+1)}}\right] - \underbrace{E\left[\int_{n\Delta t}^{(n+1)\Delta t} \pi_u^T\sigma_u dW_u^S \frac{\hat{X}_{(n)}}{X_{(n)}}\right]}_{=0} \\
=&E\left[\int_{n\Delta t}^{(n+1)\Delta t} \pi_u^T\sigma_u dW_u^S \frac{\hat{R}_{n+1}}{R_{n+1}}\frac{\hat{X}_{(n)}}{X_{(n)}}\right] \\
=&E\left[\int_{n\Delta t}^{(n+1)\Delta t} \pi_u^T\sigma_u dW_u^S \frac{\hat{R}_{n+1}}{R_{n+1}}\right]E\left[\frac{\hat{X}_{(n)}}{X_{(n)}}\right]
\end{aligned} \tag{5.41}$$

with $\frac{\hat{R}_{n+1}}{R_{n+1}}$ given in Proposition 4.1. Now

$$
\begin{aligned}
& E\left[\int_{n\Delta t}^{(n+1)\Delta t} \pi_u^T \sigma_u dW_u^S \frac{\hat{R}_{n+1}}{R_{n+1}}\right] \\
&= \sum_{i=1}^{d} \pi_{(n),i} E\left[\int_{n\Delta t}^{(n+1)\Delta t} \pi_u^T \sigma_u dW_u^S \exp\left(\int_{n\Delta t}^{(n+1)\Delta t} \left(\mu_{u,i} - \frac{1}{2}\|\sigma_u\left(i,:\right)\|^2 - \pi_u^T \mu_u\right.\right.\right. \\
&\left.\left.\left. + \frac{1}{2}\|\pi_u^T \sigma_u\|^2\right) du + \int_{n\Delta t}^{(n+1)\Delta t} \left(\sigma_u\left(i,:\right) - \pi_u^T \sigma_u\right) dW_u^S\right)\right] \\
&= \sum_{i=1}^{d} \pi_{(n),i} E\left[\int_{n\Delta t}^{(n+1)\Delta t} \pi_u^T \sigma_u dW_u^S \exp\left(-\int_{n\Delta t}^{(n+1)\Delta t} \frac{1}{2}\|\sigma_u\left(i,:\right) - \pi_u^T \sigma_u\|^2 du\right.\right. \qquad (5.42) \\
&\left. + \int_{n\Delta t}^{(n+1)\Delta t} \left(\sigma_u\left(i,:\right) - \pi_u^T \sigma_u\right) dW_u^S\right) \cdot \exp\left(\int_{n\Delta t}^{(n+1)\Delta t} \left(\frac{1}{2}\|\sigma_u\left(i,:\right) - \pi_u^T \sigma_u\|^2 + \mu_{u,i}\right.\right. \\
&\left.\left.\left. - \frac{1}{2}\|\sigma_u\left(i,:\right)\|^2 - \pi_u^T \mu_u + \frac{1}{2}\|\pi_u^T \sigma_u\|^2\right) du\right)\right]
\end{aligned}
$$

Applying the Girsanov theorem to the forth equation with the following Radon-Nikodym derivative

$$
\exp\left(-\frac{1}{2}\int_{n\Delta t}^{(n+1)\Delta t} \|\sigma_u(i,:) - \pi_u^T \sigma_u\|^2 du + \int_{n\Delta t}^{(n+1)\Delta t} \left(\sigma_u(i,:) - \pi_u^T \sigma_u\right)^T dW_u^S\right),
$$

which is a martingale following from the independence of the processes (σ_u) and (W_u^S) by Example 4 in [36, P221], we get that the Itô integral $\int_{n\Delta t}^{(n+1)\Delta t} \pi_u^T \sigma_u dW_u^S$ in (5.42) is

$$
\int_{n\Delta t}^{(n+1)\Delta t} \pi_u^T \sigma_u dW_u^S + \int_{n\Delta t}^{(n+1)\Delta t} \pi_u^T \sigma_u \left(\sigma_u(i,:) - \pi_u^T \sigma_u\right)^T du
$$

under the new measure. Thus, we get

$$
\begin{aligned}
E\left[\int_{n\Delta t}^{(n+1)\Delta t} \pi_u^T \sigma_u dW_u^S \frac{\hat{R}_{n+1}}{R_{n+1}}\right] &= \sum_{i=1}^{d} \pi_{(n),i} \int_{n\Delta t}^{(n+1)\Delta t} \pi_u^T \sigma_u \left(\sigma_u\left(i,:\right) - \pi_u^T \sigma_u\right)^T du \\
\cdot \exp\left(\int_{n\Delta t}^{(n+1)\Delta t} \left(\mu_{u,i} - \pi_u^T \mu_u + \|\pi_u^T \sigma_u\|^2 - \sigma_u\left(i,:\right) \sigma_u \pi_u\right) du\right). & \qquad (5.43)
\end{aligned}
$$

In our model we have assumed that (π_t) is a continuous function of (σ_t) and (μ_t). It implies that (π_t) is twice differentiable with respect to t, thus we have $\pi_{t_1} - \pi_{t_2} = O(t_1 - t_2)$. Expanding the right side of the forth equation (5.43), we get

$$
E\left[\int_{n\Delta t}^{(n+1)\Delta t} \pi_u^T \sigma_u dW_u^S \frac{\hat{R}_{n+1}}{R_{n+1}}\right] = \tilde{\lambda}_{L,N,n}^{(1)} + \tilde{\lambda}_{L,N,n}^{(2)} + R_N
$$

with

$$
\tilde{\lambda}_{L,N,n}^{(1)} = \sum_{i=1}^{d} \pi_{(n),i} \int_{n\Delta t}^{(n+1)\Delta t} \pi_u^T \sigma_u (\sigma_u(i,:) - \pi_u^T \sigma_u)^T du = \int_{n\Delta t}^{(n+1)\Delta t} \pi_u^T \sigma_u \sigma_u^T (\pi_{(n)} - \pi_u) du,
$$

$$\tilde{\lambda}^{(2)}_{L,N,n} = \sum_{i=1}^{d} \pi_{(n),i} \int_{n\Delta t}^{(n+1)\Delta t} \pi_u^T \sigma_u (\sigma_u(i,:) - \pi_u^T \sigma_u)^T du \cdot \int_{n\Delta t}^{(n+1)\Delta t} \left(\mu_{u,i} - \pi_u^T \mu_u + \|\pi_u^T \sigma_u\|^2 - \sigma_u(i,:)\sigma_u \pi_u\right) du$$

and R_N being the remainder of the expansion. It yields then $\tilde{\lambda}^{(1)}_{L,N,n} \in O(\Delta t^2)$, $\tilde{\lambda}^{(2)}_{L,N,n} \in O(\Delta t^2)$ and $R_N \in O(\Delta t^3)$ from the boundedness of (π_u), (σ_u) and (μ_u). Thus,

$$E\left[\int_{n\Delta t}^{(n+1)\Delta t} \pi_u^T \sigma_u^T dW_u^S \frac{\hat{R}_{n+1}}{R_{n+1}}\right] \in O(\Delta t^2).$$

The $\tilde{\lambda}_{L,N,n}$ in (5.40) is defined as

$$\tilde{\lambda}_{L,N,n} := \tilde{\lambda}^{(1)}_{L,N,n} + \tilde{\lambda}^{(2)}_{L,N,n} = \sum_{i=1}^{d} \pi_{(n),i} \int_{n\Delta t}^{(n+1)\Delta t} \pi_u^T \sigma_u (\sigma_u(i,:) - \pi_u^T \sigma_u)^T du \cdot \left(1 + \int_{n\Delta t}^{(n+1)\Delta t} \left(\mu_{u,i} - \pi_u^T \mu_u + \|\pi_u^T \sigma_u\|^2 - \sigma_u(i,:)\sigma_u \pi_u\right) du\right). \tag{5.44}$$

Consider the rest term in (5.41), i.e. $E\left[\frac{\hat{X}_{(n)}}{X_{(n)}}\right]$ and recall $\frac{\hat{X}_{(n)}}{X_{(n)}} = \prod_{k=1}^{n} \frac{\hat{R}_k}{R_k}$, we have

$$E\left[\frac{\hat{X}_{(n)}}{X_{(n)}}\right] = \prod_{k=1}^{n} E\left[\frac{\hat{R}_k}{R_k}\right] = \prod_{k=1}^{n} \left(1 + o(\Delta t^{3/2})\right) = 1 + o(\sqrt{\Delta t}).$$

Note that $E\left[\frac{\hat{R}_k}{R_k}\right]$ belongs to $1 + o(\Delta t^{3/2})$ from

$$E\left[\frac{\hat{R}_k}{R_k}\right] = E\left[1 + \frac{1}{2} g''_{N,n}(0)\,\Delta t + \frac{1}{6} g'''_{N,n}(0)\,\Delta t^{3/2} + c_{N,n}(\sqrt{\Delta t})\Delta t^{3/2}\right]$$

with

$$E[g''_{N,n}(0)] = 0, \quad E[g'''_{N,n}(0)] = 0 \quad \textit{and} \quad E\left[c_{N,n}(\sqrt{\Delta t})\Delta t^{3/2}\right] = o(\Delta t^{3/2})$$

by (5.17). Since Brownian motion increments are independent, we can write the last case in (5.40) as

$$\begin{aligned}
&E\left[\int_{(k-1)\Delta t}^{k\Delta t} \pi_u^T \sigma_u dW_u^S \left(\frac{\hat{X}_{(n+1)}}{X_{(n+1)}} - \frac{\hat{X}_{(n)}}{X_{(n)}}\right)\right] \\
=&E\left[\int_{(k-1)\Delta t}^{k\Delta t} \pi_u^T \sigma_u dW_u^S \left(\prod_{m=1}^{n+1} \frac{\hat{R}_m}{R_m} - \prod_{m=1}^{n} \frac{\hat{R}_m}{R_m}\right)\right] \\
=&E\left[\int_{(k-1)\Delta t}^{k\Delta t} \pi_u^T \sigma_u dW_u^S \left(\prod_{m=1}^{n} \frac{\hat{R}_m}{R_m}\right)\left(\frac{\hat{R}_{n+1}}{R_{n+1}} - 1\right)\right] \\
=&E\left[\int_{(k-1)\Delta t}^{k\Delta t} \pi_u^T \sigma_u dW_u^S \left(\prod_{m=1}^{k} \frac{\hat{R}_m}{R_m}\right)\left(\prod_{m=k+1}^{n} \frac{\hat{R}_m}{R_m}\right)\left(\frac{\hat{R}_{n+1}}{R_{n+1}} - 1\right)\right] \\
=&E\left[\frac{\hat{X}_{(k-1)}}{X_{(k-1)}}\right] E\left[\int_{(k-1)\Delta t}^{k\Delta t} \pi_u^T \sigma_u dW_u^S \frac{\hat{R}_k}{R_k}\right]\left(\prod_{m=k+1}^{n} E\left[\frac{\hat{R}_m}{R_m}\right]\right) E\left[\frac{\hat{R}_{n+1}}{R_{n+1}} - 1\right],
\end{aligned}$$

which is

$$(1+o(\sqrt{\Delta t}))\cdot O(\Delta t^2)\cdot(1+o(\sqrt{\Delta t}))\cdot o(\Delta t^{3/2}) = o(\Delta t^{7/2}).$$

Then the order relationship in (5.40) is shown and we apply this relationship to (5.39), it yields then

$$\begin{aligned}\sqrt{N}Cov[\log X_T, V_N] =& O(N^3)\cdot o(\Delta t^{7/2}) + N^2 O(\Delta t^3) + N\sum_{n=0}^{N-1}\tilde{\lambda}_{L,N,n}\\ =& N\sum_{n=0}^{N-1}\tilde{\lambda}_{L,N,n} + o(\sqrt{\Delta t}).\end{aligned} \tag{5.45}$$

We identify in the following the explicit expression of $\lim_{N\to\infty} N\sum_{n=0}^{N-1}\tilde{\lambda}_{L,N,n}$. Note that

$$\lim_{N\to\infty} N\sum_{n=0}^{N-1}\tilde{\lambda}_{L,N,n} = \lim_{N\to\infty} N\sum_{n=0}^{N-1}\left(\tilde{\lambda}^{(1)}_{L,N,n}+\tilde{\lambda}^{(2)}_{L,N,n}\right)$$

by (5.44) and $\tilde{\lambda}^{(2)}_{L,N,n}$ can be written as the sum of $\lambda_{L,N,n}$ with

$$\begin{aligned}\lambda_{L,N,n} :=& \sum_{i=1}^{d}\pi_{(n),i}\int_{n\Delta t}^{(n+1)\Delta t}\pi_u^T\sigma_u\sigma_u^T(:,i)du\\ &\cdot\int_{n\Delta t}^{(n+1)\Delta t}\left(\mu_{u,i}-\pi_u^T\mu_u+\|\pi_u^T\sigma_u\|^2-\sigma_u(i,:)\sigma_u\pi_u\right)du \in O(\Delta t^2)\end{aligned}$$

and a rest term

$$-\int_{n\Delta t}^{(n+1)\Delta t}\pi_u^T\sigma_u\sigma_u^T\pi_u du\sum_{i=1}^{d}\pi_{(n),i}\int_{n\Delta t}^{(n+1)\Delta t}\left(\mu_{u,i}-\pi_u^T\mu_u+\|\pi_u^T\sigma_u\|^2-\sigma_u(i,:)\sigma_u\pi_u\right)du \in O(\Delta t^3).$$

It yields then

$$\lim_{N\to\infty} N\sum_{n=0}^{N-1}\tilde{\lambda}_{L,N,n} = \lim_{N\to\infty} N\sum_{n=0}^{N-1}\left(\tilde{\lambda}^{(1)}_{L,N,n}+\lambda_{L,N,n}\right).$$

Let us denote

$$b_t := \int_{n\Delta t}^{t}\pi_u^T\sigma_u\sigma_u^T(\pi_n-\pi_u)du.$$

The Taylor expansion of $b_{(n+1)\Delta t}$ yields

$$\tilde{\lambda}^{(1)}_{L,N,n} = b_{(n+1)\Delta t} = b_{n\Delta t} + b'_{n\Delta t} + \frac{1}{2}b''_{n\Delta t}\Delta t^2 + o(\Delta t^2) = \frac{1}{2}b''_{n\Delta t}\Delta t^2 + o(\Delta t^2)$$

with

$$b''_{n\Delta t} = \left.\frac{\partial(\pi_t^T\sigma_t\sigma_t^T)}{\partial t}\right|_{t\downarrow n\Delta t}\pi_{(n)} - \left.\frac{\partial(\pi_t^T\sigma_t\sigma_t^T\pi_t)}{\partial t}\right|_{t\downarrow n\Delta t} = -\,\pi_t^T\sigma_t\sigma_t^T\left.\frac{\partial\pi_t}{\partial t}\right|_{t\downarrow n\Delta t}. \tag{5.46}$$

Then we get

$$\lim_{N\to\infty} N\sum_{n=1}^{N}\tilde{\lambda}^{(1)}_{L,N,n} = \lim_{N\to\infty}\frac{T}{2}\sum_{n=1}^{N}b''_{n\Delta t}\Delta t = \frac{T}{2}\int_0^T b''_u u$$

with $b''_u = -\pi_u^T \sigma_u \sigma_u^T \lim_{h \downarrow 0} \frac{\pi_{u+h} - \pi_u}{\partial h}$. In the same way we write

$$\lambda_{L,N,n} = \sum_{i=1}^{d} \pi_{(n),i} \left(\pi_{(n)}^T \sigma_{(n)} \sigma_{(n)}^T (:, i) \Delta t + o(\Delta t) \right) \\ \cdot \left(\left(\mu_{(n),i} - \pi_{(n)}^T \mu_{(n)} + \|\pi_{(n)}^T \sigma_{(n)}\|^2 - \sigma_{(n)}(i, :) \sigma_{(n)} \pi_{(n)} \right) \Delta t + o(\Delta t) \right).$$

It yields

$$\begin{aligned} &\lim_{N \to \infty} N \sum_{n=1}^{N} \tilde{\lambda}_{L,N,n} \\ =& \lim_{N \to \infty} N \sum_{n=1}^{N} \sum_{i=1}^{d} \pi_{(n),i} \left(\pi_{(n)}^T \sigma_{(n)} \sigma_{(n)}^T (:, i) \right) \\ & \cdot \left(\mu_{(n),i} - \pi_{(n)}^T \mu_{(n)} + \|\pi_{(n)}^T \sigma_{(n)}\|^2 - \sigma_{(n)}(i, :) \sigma_{(n)} \pi_{(n)} \right) \Delta t^2 \\ =& \lim_{N \to \infty} T \int_0^T \left(\sum_{i=1}^{d} \pi_{u,i} \mu_{u,i} \pi_u^T \sigma_u \sigma_u^T (:, i) - \pi_u^T \mu_u \cdot \|\pi_u^T \sigma_u\|^2 + \|\pi_u^T \sigma_u\|^4 \right. \\ & \left. - \sum_{i=1}^{d} \pi_{u,i} \sigma_u(i, :) \sigma_u \pi_u \pi_u^T \sigma_u \sigma_u^T (:, i) \right) du \\ =& T \int_0^T \lambda_{L,u} du. \end{aligned}$$

Thus we get

$$\lim_{N \to \infty} N \sum_{n=1}^{N} \tilde{\lambda}_{L,N,n} = \frac{T}{2} \int_0^T b''_u du + T \int_0^T \lambda_{L,u} du,$$

which implies the first assertion in the proposition.

For the part (ii) of the proposition, the Taylor expansion of $\exp(V_N/\sqrt{N})$ yields

$$\bar{X}_T - \hat{X}_{(N)} = X_T \left(\exp(V_N/\sqrt{N}) - 1 - \frac{V_N}{\sqrt{N}} \right) = X_T \cdot \frac{e^{\xi_N}}{2} \frac{V_N^2}{N}$$

for some ξ_N with $|\xi_N| \le \frac{|V_N|}{\sqrt{N}}$. To get $E\left[\left(\bar{X}_T - \hat{X}_{(N)} \right)^2 \right] = O\left(N^{-2}\right)$, we need to show that $E[X_T^2 e^{2\xi_N} V_N^4]$ is bounded for all $N \in \mathbb{N}$. Recalling X_T in (5.4), we have

$$\begin{aligned} E[X_T^2 e^{2\xi_N} V_N^4] =& E\left[\exp\left(2 \int_0^T \left(\pi_u^T \mu_u - \frac{1}{2} \|\pi_u^T \sigma_u\|^2 \right) du + 2 \int_0^T \pi_u^T \sigma_u dW_u^S \right) e^{2\xi_N} V_N^4 \right] \\ =& \exp\left(\int_0^T 2\pi_u^T \mu_u du + \int_0^T \|\pi_u^T \sigma_u\|^2 du \right) \cdot \\ & E\left[\exp\left(2 \int_0^T \pi_u^T \sigma_u dW_u^S - 2 \int_0^T \|\pi_u^T \sigma_u\|^2 du \right) e^{2\xi_N} V_N^4 \right] \\ =& \exp\left(\int_0^T 2\pi_u^T \mu_u du + \int_0^T \|\pi_u^T \sigma_u\|^2 du \right) \cdot \tilde{E}\left[e^{2\xi_N} V_N^4 \right] \end{aligned}$$

by a change of measure with the following Radon-Nikodym derivative

$$\exp\left(2\int_0^T \pi_u^T\sigma_u dW_u^S - 2\int_0^T \|\pi_u^T\sigma_u\|^2 du\right).$$

Employing the Cauchy-Schwarz inequality, we have

$$\tilde{E}\left[e^{2\xi_N}V_N^4\right] \leq \sqrt{\tilde{E}\left[e^{4\xi_N}\right]\tilde{E}\left[V_N^8\right]}.$$

Note that $\tilde{E}\left[e^{4\xi_N}\right]$ is bounded, since $|\xi_N| \leq \frac{|V_N|}{\sqrt{N}}$. Furthermore, the fact that V_N^8 is a continuous function of log-normal distributed variables implies that $\tilde{E}\left[V_N^8\right]$ is bounded for $N \in \mathbb{N}$ and goes to $\tilde{E}\left[V^8\right]$ as $N \to \infty$. Thus, we have $\tilde{E}\left[e^{2\xi_N}V_N^4\right]$ is bounded for all $N \in \mathbb{N}$.

For the last statement in the proposition, we have

$$\begin{aligned} N\left(Var[\log \bar{X}_T] - Var[\log X_T]\right) =& N\left(Var\left[\log\left(X_T \exp\left(V_N/\sqrt{N}\right)\right)\right] - Var[\log X_T]\right) \\ =& Var[V_N] + 2\sqrt{N}Cov[\log X_T, V_N] \end{aligned}$$

and

$$Var[V_N] \to Var[V] = T\int_0^T \hat{\sigma}_{L,u}^2 du$$

with $\hat{\sigma}_{L,u}^2$ as in the Theorem 5.2.2 Then, the statement follows directly from the first assertion. □

5.3.2 Volatility adjustment for $\hat{X}_{(N)}$

In this thesis the volatility of a portfolio is defined as the standard deviation of the portfolio's logarithm, divided by T, i.e. the volatility of $\bar{X}_T$ is given by $\sqrt{Var[\log \bar{X}_T]/T}$. By (ii) in Proposition 5.3.1, we have

$$\begin{aligned} &\sqrt{\frac{Var[\log \bar{X}_T]}{T}} \\ =& \frac{1}{\sqrt{T}}\left[\left(Var\left[\log(X_T)\right] + \Delta t\int_0^T \sigma_{L,u}^2 du + \Delta t\int_0^T b_u^{''} du + 2\Delta t\int_0^T \lambda_{L,u} du\right)\right]^{1/2} \\ \approx& \frac{1}{\sqrt{T}}\left[\left(\int_0^T \|\pi_u^T\sigma_u\|^2 du + \Delta t\int_0^T \sigma_{L,u}^2 du + \Delta t\int_0^T b_u^{''} du + 2\Delta t\int_0^T \lambda_{L,u} du\right)\right]^{1/2}. \end{aligned}$$

Definition 5.3.2. *(Volatility adjustment)*

Replacing the volatility of X_T, i.e. $\left(\int_0^T \|\pi_u^T\sigma_u\|^2 du/T\right)^{1/2}$ by a new volatility σ_{adj}, we get an approximation of the distribution of $\hat{X}_{(N)}$. This new volatility is called the volatility adjustment for $\hat{X}_{(N)}$ and is given by

$$\sigma_{adj} = \sqrt{\frac{1}{T}\left(\int_0^T \|\pi_u^T\sigma_u\|^2 du + \Delta t\int_0^T \sigma_{L,u}^2 du + \Delta t\int_0^T b_u^{''} du + 2\Delta t\int_0^T \lambda_{L,u} du\right)}.$$

Note that for X_T given in (5.4) the volatility adjustment implies that we should replace

$$\int_0^T \|\pi_u^T \sigma_u\|^2 du \quad by \quad T\sigma_{adj}^2$$

and

$$\int_0^T \pi_u^T \sigma_u dW_u^S \quad by \quad \frac{\sqrt{T}\sigma_{adj}}{\sqrt{\int_0^T \|\pi_u^T \sigma_u\|^2 du}} \int_0^T \pi_u^T \sigma_u dW_u^S$$

with $Z \sim N(0,1)$ to correct for the discrete rebalancing. Thus, the volatility adjustment yields the following approximation for $\hat{X}_{(N)}$:

$$X_{adj} \approx \exp\left(\int_0^T \pi_u^{'} \mu_u du - \frac{T}{2}\sigma_{adj}^2 + \frac{\sqrt{T}\sigma_{adj}}{\sqrt{\int_0^T \|\pi_u^T \sigma_u\|^2 du}} \int_0^T \pi_u^T \sigma_u dW_u^S\right).$$

5.4 A conditional mean adjustment for discrete rebalancing

As illustrated by Glasserman [21], it is necessary to supplement the central limit theorem with an approximation specifically focused on the tails, since the normal approximation (central limit theorem) loses typically accuracy in the extreme tails.

This phenomenon can be shown by the following figure. In Figure 1 we see three curves representing the density functions of the continuously rebalanced portfolio X_T, the discretely rebalanced portfolio $\hat{X}_{(N)}$ with $N = 16$ and the portfolio corrected by volatility adjustment X_{adj}, respectively. The figure in the left panel gives us a whole impression, whereas the figure in the right panel focuses on the tail. It is evident to see that the density functions go apart in the tail.

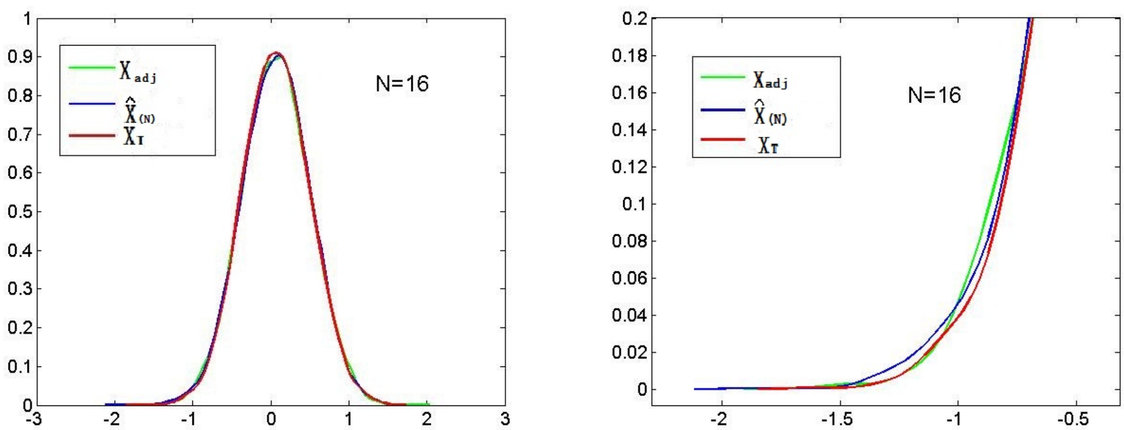


Figure 1: Density functions. The red curve corresponds to X_T, the blue curve corresponds to $\hat{X}_{(N)}$ with $N = 16$ and the green curve corresponds to X_{adj}.

In this section, we will identify a correction in approximating the distribution of $X_{(N)}$ with X_T in the tails. To this end, we will determine the distribution of $\hat{X}_{(N)}/X_T$ conditioned on an outcome of X_T in the tail of its distribution.

5.4.1 Conditioning on a large loss

Concerning the distribution of $\frac{\hat{X}_{(N)}}{X_T}$ conditioned on an extreme outcome of X_T, we have the following theorem:

Theorem 5.4.1. *Let $x_N = x\sqrt{N}$ and $y_N = \exp\left(x_N + \int_0^T \left(\pi_u^T\mu_u + \frac{1}{2}\|\pi_u^T\sigma_u\|^2\right) du\right)$, then*

$$\left(\frac{\hat{X}_{(N)}}{X_T}\middle| X_T = y_N\right) \xrightarrow{\mathcal{D}} \exp\left(\frac{1}{2}\beta_L x^2\right) \tag{5.47}$$

as $N \to \infty$, where

$$\beta_L := \frac{T\sum_{i=1}^d \int_0^T \pi_{u,i}\left[\Omega_u(i,:)\sigma_u\pi_u\right]^2 du}{\left(\int_0^T \|\pi_u^T\sigma_u\|^2 du\right)^2} = \frac{T\int_0^T \left(\pi_u^T\sigma_u\right)\tau_u\left(\pi_u^T\sigma_u\right)^T du}{\left(\int_0^T \|\pi_u^T\sigma_u\|^2 du\right)^2}, \tag{5.48}$$

with

$$\Omega_u(i,:) := \sigma_u(i,:) - \pi_u^T\sigma_u$$

and

$$\tau_u := \sum_i^d \pi_{u,i}\left(\sigma_u(i,:) - \pi_u^T\sigma_u\right)^T\left(\sigma_u(i,:) - \pi_u^T\sigma_u\right).$$

Proof. Using Lemma A.1. in Appendix A.1, we get

$$\left(\int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)dW_u^S \middle| W = a\right)$$
$$\sim N\left(\frac{a\cdot\int_{(n-1)\Delta t}^{n\Delta t}\Omega_u(i,:)\sigma_u\pi_u du}{\int_0^T \|\pi_u^T\sigma_u\|^2 du}, \int_{(n-1)\Delta t}^{n\Delta t}\|\Omega_u(i,:)\|^2 du - \frac{\left(\int_{(n-1)\Delta t}^{n\Delta t}\Omega_u(i,:)\sigma_u\pi_u du\right)^2}{\int_0^T \|\pi_u^T\sigma_u\|^2 du}\right).$$

Thus, we can represent the following conditional distribution

$$\left(\frac{\hat{X}_{(N)}}{X_T}\middle| X_T = y_N\right) = \left(\prod_{n=1}^N \frac{\hat{R}_n}{R_n}\middle| X_T = y_N\right) \tag{5.49}$$

as

$$\prod_{n=1}^N \sum_{i=1}^d \pi_{(n),i}\exp\left(\int_{(n-1)\Delta t}^{n\Delta t}\left(\mu_{u,i} - \frac{1}{2}\|\sigma_u(i,:)\|^2 - \pi_u^T\mu_u + \frac{1}{2}\|\pi_u^T\sigma_u\|^2\right) du \right.$$
$$\left. + \int_{(n-1)\Delta t}^{n\Delta t}\Omega_u(i,:)dM_u\right), \tag{5.50}$$

with $\int_{(n-1)\Delta t}^{n\Delta t}\Omega_u(i,:)dM_u$ given in (A.3) in Appendix A.1 and $a = x\sqrt{N}$. Thus, any limit of (5.50) is a weak limit of (5.49).

We transform the factors in (5.50) in following steps:

(i) replace $\int_{(n-1)\Delta t}^{n\Delta t}\Omega_u(i,:)dM_u$ by its definition in (A.3),

(ii) replace N by $T/\Delta t$ and a by $x\sqrt{N}$,

(iii) replace each $\int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:) dW_u^S$ by

$$\sum_{j=1}^{d} Z_{N,n,j} \sqrt{\int_{(n-1)\Delta t}^{n\Delta t} f_{u,ij} du}$$

and replace $\int_0^T \pi_u^T \sigma_u dW_u^S$ by

$$\sum_{j=1}^{d} Z_j \sqrt{\int_0^T \left(\pi_u^T \sigma_u(:,j)\right)^2 du},$$

with $f_{u,ij} := \Omega_u^2(i,:)$ as in (5.10) and Z, $Z_{N,n}$, $n = 1, \ldots, N$ are independent standard normal distributed random vectors in $\mathbb{R}^d$.

Then we get (5.50) can be written as $\prod_{n=1}^{N} G_{N,n}(\sqrt{\Delta t})$, where $G_{N,n}(s) : \mathbb{R} \to \mathbb{R}$ is a function of s with

$$G_{N,n}(s) := \sum_{i=1}^{d} G_{N,n,i}(s)$$

and

$$\begin{aligned} G_{N,n,i}(s) :=& \pi_{t_{n-1},i} \exp\left(\int_{t_{n-1}}^{t_{n-1}+s^2} \left(\mu_{u,i} - \frac{1}{2}\|\sigma_u(i,:)\|^2 - \pi_u^T \mu_u + \frac{1}{2}\|\pi_u^T \sigma_u\|^2 \right) du \right. \\ & \left. + \int_{t_{n-1}}^{t_{n-1}+s^2} \Omega_u(i,:) dM_u \right) \end{aligned}$$

with $t_{n-1} := (n-1)\Delta t$. Employing the transformations (i), (ii) and (iii) mentioned above, we get that $G_{N,n,i}(s)$ can be written as

$$\begin{aligned} G_{N,n,i}(s) =& \pi_{t_{n-1},i} \exp\left(\int_{t_{n-1}}^{t_{n-1}+s^2} h_{u,i} du + \sum_{j=1}^{d} Z_{N,n,j} \sqrt{\int_{t_{n-1}}^{t_{n-1}+s^2} f_{u,ij} du} \right. \\ & \left. + \int_{t_{n-1}}^{t_{n-1}+s^2} \Omega_u(i,:) \sigma_u \pi_u du \cdot \frac{\frac{x}{s}\sqrt{T} - \sum_{j=1}^{d} Z_j \sqrt{\int_0^T \left(\pi_u^T \sigma_u(:,j)\right)^2 du}}{\int_0^T \|\pi_u^T \sigma_u\|^2 du} \right) \end{aligned}$$

with $h_{u,i} := \mu_{u,i} - \frac{1}{2}\|\sigma_u(i,:)\|^2 - \pi_u^T \mu_u + \frac{1}{2}\|\pi_u^T \sigma_u\|^2$ as in (5.9). We can easily identify that $G_{N,n}(0) = 1$ and

$$G'_{N,n}(s) = \sum_{i=1}^{d} G_{N,n,i}(s) \cdot H_{N,n,i}(s)$$

with

$$H_{N,n,i}(s) := h_{t_{n-1}+s^2,i} 2s + \sum_{j=1}^{d} Z_{N,n,j} \frac{f_{t_{n-1}+s^2,ij} \cdot s}{\sqrt{\int_{t_{n-1}}^{t_{n-1}+s^2} f_{u,ij} du}} + \Omega_{t_{n-1}+s^2}(i,:) \cdot \sigma_{t_{n-1}+s^2} \pi_{t_{n-1}+s^2} \cdot 2s$$
$$\cdot \frac{\frac{x}{s}\sqrt{T} - \sum_{j=1}^{d} Z_j \sqrt{\int_0^T \left(\pi_u^T \sigma_u(:,j)\right)^2 du}}{\int_0^T \|\pi_u^T \sigma_u\|^2 du} - \frac{x\sqrt{T} \int_{t_{n-1}}^{t_{n-1}+s^2} \Omega_u(i,:)\sigma_u \pi_u du}{s^2 \int_0^T \|\pi_u^T \sigma_u\|^2 du}.$$

Note that for differentiable (σ_u) and (π_u), we have

$$\int_{t_{n-1}}^{t_{n-1}+s^2} \Omega_u(i,:)\sigma_u \pi_u du = \left(\sigma_{t_{n-1}}(i,:) - \pi_{t_{n-1}}^T \sigma_{t_{n-1}}\right) \sigma_{t_{n-1}} \pi_{t_{n-1}} s^2 + o(s^2)$$

and recalling (5.10), (5.11) and (5.12), we get

$$\lim_{s \downarrow 0} \frac{f_{t_{n-1}+s^2,ij} \cdot s}{\sqrt{\int_{t_{n-1}}^{t_{n-1}+s^2} f_{u,ij} du}} = \sqrt{f_{t_{n-1},ij}},$$

thus,

$$\begin{aligned}
G_{N,n}^{'}(0) =& \sum_{i=1}^{d} \pi_{t_{n-1},i} \left(\sum_{j=1}^{d} Z_{N,n,j} \sqrt{f_{t_{n-1},ij}} + \Omega_{t_{n-1}}(i,:)\sigma_{t_{n-1}} \pi_{t_{n-1}} \cdot \frac{2x\sqrt{T}}{\int_0^T \|\pi_u^T \sigma_u\|^2 du} \right. \\
& \left. - \lim_{s \to 0} \int_{t_{n-1}}^{t_{n-1}+s^2} \Omega_u(i,:)\sigma_u \pi_u du \cdot \frac{x\sqrt{T}}{s^2 \int_0^T \|\pi_u^T \sigma_u\|^2 du} \right) \\
=& \sum_{i=1}^{d} \pi_{t_{n-1},i} \left(\sum_{j=1}^{d} Z_{N,n,j} \sqrt{f_{t_{n-1},ij}} + \Omega_{t_{n-1}}(i,:)\sigma_{t_{n-1}} \pi_{t_{n-1}} \cdot \frac{2x\sqrt{T}}{\int_0^T \|\pi_u^T \sigma_u\|^2 du} \right. \\
& \left. - \Omega_{t_{n-1}}(i,:)\sigma_{t_{n-1}} \pi_{t_{n-1}} \cdot \frac{x\sqrt{T}}{\int_0^T \|\pi_u^T \sigma_u\|^2 du} \right) = 0.
\end{aligned}$$

By direct computation we get the second-order derivative of $G_{N,n}(s)$ as follows:

$$\begin{aligned}
G_{N,n}^{''}(s) =& \sum_{i=1}^{d} G_{N,n,i}(s) \left(H_{N,n,i}^2(s) + 2h_{t_{n-1}+s^2,i} + \sum_{j=1}^{d} Z_{N,n,j} \left(\frac{f_{t_{n-1}+s^2,ij} \cdot s}{\sqrt{\int_{t_{n-1}}^{t_{n-1}+s^2} f_{u,ij} du}} \right)^{'} \right. \\
& - \Omega_{t_{n-1}+s^2}(i,:)\sigma_{t_{n-1}+s^2} \pi_{t_{n-1}+s^2} \cdot \frac{2\sum_{j=1}^{d} Z_j \sqrt{\int_0^T \left(\pi_u^T \sigma_u(:,j)\right)^2 du}}{\int_0^T \|\pi_u^T \sigma_u\|^2 du} \\
& \left. - \left(\frac{\int_{t_{n-1}}^{t_{n-1}+s^2} \Omega_u(i,:)\sigma_u \pi_u du}{s^2} \right)^{'} \frac{x\sqrt{T}}{\int_0^T \|\pi_u^T \sigma_u\|^2 du} \right).
\end{aligned}$$

Recalling the computation of the second-order derivative of $g_{N,n}(s)$ in Proposition 5.2.1, we have

$$\sum_{j=1}^{d} Z_{N,n,j} \left(\frac{f_{t_{n-1}+s^2,ij} \cdot s}{\sqrt{\int_{t_{n-1}}^{t_{n-1}+s^2} f_{u,ij} du}} \right)^{'} = 0$$

and since $\Omega_t(i,:)$ is differentiable with respect to t, we get

$$\lim_{s\to 0}\left(\frac{\int_{t_{n-1}}^{t_{n-1}+s^2}\Omega_u(i,:)\sigma_u\pi_u du}{s^2}\right)'$$
$$=\lim_{s\downarrow 0}\frac{\left(\Omega_{t_{n-1}+s^2}(i,:)\sigma_{t_{n-1}+s^2}\pi_{t_{n-1}+s^2}2s\right)s^2}{s^4}-\lim_{s\downarrow 0}\frac{\int_{t_{n-1}}^{t_{n-1}+s^2}\Omega_u(i,:)\sigma_u\pi_u du2s}{s^4}$$
$$=\lim_{s\downarrow 0}\frac{2\left(\Omega_{t_{n-1}+s^2}(i,:)-\Omega_{t_{n-1}}(i,:)\right)}{s}-\frac{2\left(\left[\Omega_{t_{n-1}}(i,:)\sigma_{t_{n-1}}\pi_{t_{n-1}}\right]'\frac{s^4}{2}+\tilde{\eta}_\xi\frac{s^6}{6}\right)}{s^3}=0$$

with

$$\tilde{\eta}_\xi=\left(\Omega_\xi(i,:)\sigma_\xi\pi_\xi\right)''$$

for a $\xi\in[t_{n-1},t_{n-1}+s^2]$. Hence,

$$G''_{N,n}(0)=\sum_{i=1}^d\pi_{t_{n-1},i}\Bigg\{\left(\sum_{j=1}^d Z_{N,n,j}\sqrt{f_{t_{n-1},ij}}+\frac{\Omega_{t_{n-1}}(i,:)\sigma_{t_{n-1}}\pi_{t_{n-1}}\cdot x\sqrt{T}}{\int_0^T\|\pi_u^T\sigma_u\|^2du}\right)^2$$
$$+2h_{t_{n-1},i}-\Omega_{t_{n-1}}(i,:)\sigma_{t_{n-1}}\pi_{t_{n-1}}\cdot\frac{2\sum_{j=1}^d Z_j\sqrt{\int_0^T\left(\pi_u^T\sigma_u(:,j)\right)^2du}}{\int_0^T\|\pi_u^T\sigma_u\|^2du}\Bigg\}$$
$$=\sum_{i=1}^d\pi_{t_{n-1},i}\Bigg\{\left(\sum_{j=1}^d Z_{N,n,j}\sqrt{f_{t_{n-1},ij}}+\frac{\Omega_{t_{n-1}}(i,:)\sigma_{t_{n-1}}\pi_{t_{n-1}}\cdot x\sqrt{T}}{\int_0^T\|\pi_u^T\sigma_u\|^2du}\right)^2+2h_{t_{n-1},i}\Bigg\}.$$

We can write $G''_{N,n}(0)\Delta t/2$ as the sum of $Y_{N,n}$ and u_N/N with

$$Y_{N,n}:=\sum_{i=1}^d\pi_{t_{n-1},i}\Bigg(\left[\Omega_{t_{n-1}}(i,:)Z_{N,n}\right]^2\frac{\Delta t}{2}+\left(\|\pi_{t_{n-1}}^T\sigma_{t_{n-1}}\|^2-\|\sigma_{t_{n-1}}(i,:)\|^2\right)\frac{\Delta t}{2}$$
$$+\sum_{i=1}^d\pi_{t_{n-1},i}\Omega_{t_{n-1}}(i,:)Z_{N,n}\Omega_{t_{n-1}}(i,:)\sigma_{t_{n-1}}\pi_{t_{n-1}}\frac{x\sqrt{T}\Delta t}{\int_0^T\|\pi_u^T\sigma_u\|^2du}\Bigg)$$

and

$$u_n:=\frac{1}{2}\sum_{i=1}^d\pi_{t_{n-1},i}\left[\Omega_{t_{n-1}}(i,:)\sigma_{t_{n-1}}\pi_{t_{n-1}}\right]^2\frac{x^2T^2}{(\int_0^T\|\pi_u^T\sigma_u\|^2du)^2}.$$

Thus, by the Taylor expansion of $G_{N,n}(\sqrt{\Delta t})$, we have

$$G_{N,n}(\sqrt{\Delta t})=1+Y_{N,n}+\frac{u_n}{N}+r_{N,n},$$

where the remainder $r_{N,n}$ is of order $o(\Delta t)$. We write the logarithm of (5.50) as

$$\sum_{n=1}^N\log\left(1+Y_{N,n}+\frac{u_n}{N}+r_{N,n}\right). \tag{5.51}$$

We will show next that this sum converges to

$$\bar{u} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{u_n}{N} = \frac{x^2 T}{2\left(\int_0^T \|\pi_u^T \sigma_u\|^2 du\right)^2} \lim_{N\to\infty} \sum_{i=1}^{d} \sum_{n=1}^{N} \pi_{t_{n-1},i} \left[\Omega_{t_{n-1}}(i,:)\sigma_{t_{n-1}}\pi_{t_{n-1}}\right]^2 \Delta t$$
$$= \frac{x^2 T}{2\left(\int_0^T \|\pi_u^T \sigma_u\|^2 du\right)^2} \sum_{i=1}^{d} \int_0^T \pi_{u,i} \left[\Omega_u(i,:)\sigma_u \pi_u\right]^2 du$$

in probability. From

$$E\left[\left(\sum_{i=1}^{d} \pi_{t_{n-1},i}\Omega_{t_{n-1}}(i,:)Z_{N,n}\right)^2\right] = \sum_{i=1}^{d} \pi_{t_{n-1},i}\|\sigma_{t_{n-1}}(i,:)\|^2 - \|\pi_{t_{n-1}}^T \sigma_{t_{n-1}}\|^2$$

we identify

$$E[Y_{N,n}] = 0.$$

It is clear that $(Y_{N,n})_{n=1,\dots,N}$ for all $N \in \mathbb{N}$ are independent random variables with $E[Y_{N,n}N] = 0$ and $E\left[(Y_{N,n}N)^2\right]$ finite. Hence, there is

$$\sum_{n=1}^{N} \frac{Var(Y_{N,n})}{n^2} \leq \max_{1\leq n\leq N} Var(Y_{N,n}) \sum_{n=1}^{N} \frac{1}{n^2} < \infty, \quad N \to \infty.$$

Then by the Kolmogorov criterion of the law of large numbers, we have

$$\sum_{n=1}^{N} Y_{N,n} \overset{a.s.}{\to} 0,$$

it yields that $\sum_n Y_{N,n}$ converges to zero also in probability.

Since $Y_{N,n} \in O(\Delta t)$ and the remainder $r_{N,n} \in o(\Delta t)$, we get

$$\sum_{n=1}^{N} Y_{N,n}^2, \quad \sum_{n=1}^{N} r_{N,n}, \quad \sum_{n=1}^{N} r_{N,n}^2 \quad and \quad \sum_{n} Y_{N,n} r_{N,n}$$

all converge to zero in probability. It implies that

$$\sum_{n=1}^{N} \left(Y_{N,n} + \frac{u_n}{N} + r_{N,n}\right) \tag{5.52}$$

converges to $\bar{u}$ in probability and

$$\sum_{n=1}^{N} \left(Y_{N,n} + \frac{u_n}{N} + r_{N,n}\right)^2 \tag{5.53}$$

converges to zero in probability. Following from (5.53), we have

$$\max_{n=1,\dots,N} |Y_{N,n} + \frac{u_n}{N} + r_{N,n}| \tag{5.54}$$

also converge to zero in probability. Using the fact that, for all sufficiently small υ,

$$\upsilon - \upsilon^2 \leq \log(1+\upsilon) \leq \upsilon + \upsilon^2,$$

we have (5.51) converges to $\bar{u}$ in probability with $\upsilon = Y_{N,n} + \frac{u_n}{N} + r_{N,n}$. It follows that (5.50) converges in probability to $e^{\bar{u}}$ and thus that (5.49) converges in distribution to the same constant.

It remains to evaluate $\bar{u}$. From the definition of $\bar{u}$, we can write $\bar{u} = \beta_L x^2/2$ with

$$\beta_L = \frac{T \sum_{i=1}^{d} \int_0^T \pi_{u,i} \left[\Omega_u(i,:)\sigma_u \pi_u\right]^2 du}{(\int_0^T \|\pi_u^T \sigma_u\|^2 du)^2}.$$

Let

$$\tau_u := \sum_{i}^{d} \pi_{u,i} \Omega_u^T(i,:)\Omega_u(i,:),$$

we can write

$$\beta_L = \frac{T \int_0^T \pi_u^T \sigma_u \tau_u \left(\pi_u^T \sigma_u\right)^T du}{(\int_0^T \|\pi_u^T \sigma_u\|^2 du)^2}.$$

This concludes the proof. □

5.4.2 The conditional mean approximation $H(X_T)$

From Theorem 5.4.1, we get the approximation

$$\left(\hat{X}_{(N)} | X_T = y\right) \approx y \exp\left(\frac{1}{2}\beta_L x^2/N\right)$$

for large N with $x = \log y - \int_0^T \left(\pi_u^T \mu_u - \frac{1}{2}\|\pi_u^T \sigma_u\|^2\right) du$. If we define

$$H(y) = y \exp\left(\frac{1}{2}\beta_L \left(\log y - \int_0^T \left(\pi_u^T \mu_u - \frac{1}{2}\|\pi_u^T \sigma_u\|^2\right) du\right)^2 \Big/ N\right),$$

we have then $H(X_T) \approx (\hat{X}_{(N)} | X_T = y)$ for large N, it implies that $H(X_T) \approx \hat{X}_{(N)}$ for large N. Provided H is monotone near c, we get

$$P(\hat{X}_{(N)} \leq c) \approx P(X_T \leq H^{-1}(c)). \tag{5.55}$$

Similarly, we can approximate the VaR for the discretely rebalanced portfolio, $\widehat{VaR}_\alpha$ in terms of that for the continuously rebalanced portfolio, VaR_α, using

$$\widehat{VaR}_\alpha \approx H(VaR_\alpha),$$

if H is monotone near the VaR. Calculating $H'(y)$, we get that $H'(y)$ is positive for all sufficiently large N at each $y > 0$.

5.5 Examples

In this section, several examples will be given to test some important results in this thesis. We will first in Subsection 5.5.1 test the convergence result in Theorem 5.2.1 and then in subsection 5.5.2, we will compare the accuracies of the volatility adjustment, the conditional mean adjustment and the combined adjustment in different situations.

5.5.1 Test of the limit theorem

To illustrate the convergence in Theorem 4.2, we apply the method of Monto-Carlo simulation on our example:
Model 1: A portfolio consists of five assets with the volatility coefficients matrix $\sigma(Y_t) \in \mathbb{R}^{5\times 5}$ and the mean rate of return $\mu(Y_t) \in \mathbb{R}$ driven by an external deterministic process (Y_t) with state space $S = [1, 2, 3]$. In our example, we have

$$Y_t = \begin{cases} 1 & 0 \leq t < T_1 \\ 2 & T_1 \leq t < T_1 + T_2 \\ 3 & T_1 + T_2 \leq t \leq T \end{cases}$$

with $T_1 = 1$, $T_2 = 1/2$, $T_3 = 1/2$. Then, $T = T_1 + T_2 + T_3 = 2$ is the time horizon. The volatility coefficients matrix $\sigma(Y_t)$, $0 \leq t \leq T$ is given by

$$\sigma(Y_t) = \begin{pmatrix} .002 + .02Y_t & .001 + .2Y_t & .04 + .02Y_t & .03 + .02Y_t & .01 + .02Y_t \\ .03 + .2Y_t & .002 + .2Y_t & .001 + .02Y_t & .02 + .02Y_t & .04 + .02Y_t \\ .004 + .02Y_t & .003 + .02Y_t & .01 + .02Y_t & .01 + .02Y_t & .02 + .02Y_t \\ .003 + .2Y_t & .04 + .02Y_t & .02 + .02Y_t & .02 + .02Y_t & .03 + .02Y_t \\ .001 + .02Y_t & .002 + .02Y_t & .03 + .02Y_t & .01 + .02Y_t & .03 + .02Y_t \end{pmatrix}$$

and the mean rate of return $\mu(Y_t)$, $0 \leq t \leq T$ on the i-th asset is

$$\mu_i(Y_t) = 0.05 + \|\sigma(Y_t)(i,:)\|/4.$$

The portfolio weights are constant with $\pi = (-1, 0.5, 2, 0.5, -1)$. We measure the portfolio's leverage as the ratio of the total long position (the sum of the positive weights) divided by the initial capital (the sum of all the weights, which is simply 1). This yields a leverage ratio of 3.

In table 1, the first three columns of results show estimates of the standard deviation, skewness and kurtosis of the relative difference V_N, the next three columns show estimates for the absolute difference $\hat{X}_{(N)} - X_T$ and the last column shows estimates of the correlation between V_N and $log X_T$. The values in the table are estimated from one million replications; the last row shows theoretical values under the limiting distributions. We note that the moments of the relative difference V_N converge to the moments of the normal distribution as the number of rebalancing frequency N increases.

	Relative Difference V_N			Absolute Difference $\sqrt{N}(\hat{X}_{(N)} - X_T)$			
N	Std.	Skewness	Kurtosis	Std.	Skewness	Kurtosis	Correl.
8	0.20	-2.45	15.99	0.21	-1.29	11.90	0.2
100	0.17	-0.34	3.29	0.22	-0.49	6.45	0.06
200	0.17	-0.22	3.05	0.22	-0.29	6.01	0.05
∞	0.169	0	3	0.215	0	5.36	0

Table 1: Numerical illustration of the convergence of relative and absolute difference as the number of rebalancing dates N increases. The last row shows theoretical values for the limit.

Figure 2 compares the scatter plots of V_N against $logX_T$ at $N = 8$ and $N = 200$. At $N = 8$, a quadratic relation between V_N and $logX_T$ is reflected in the scatter plot in the left panel. The picture in the right panel shows the asymptotic independence.

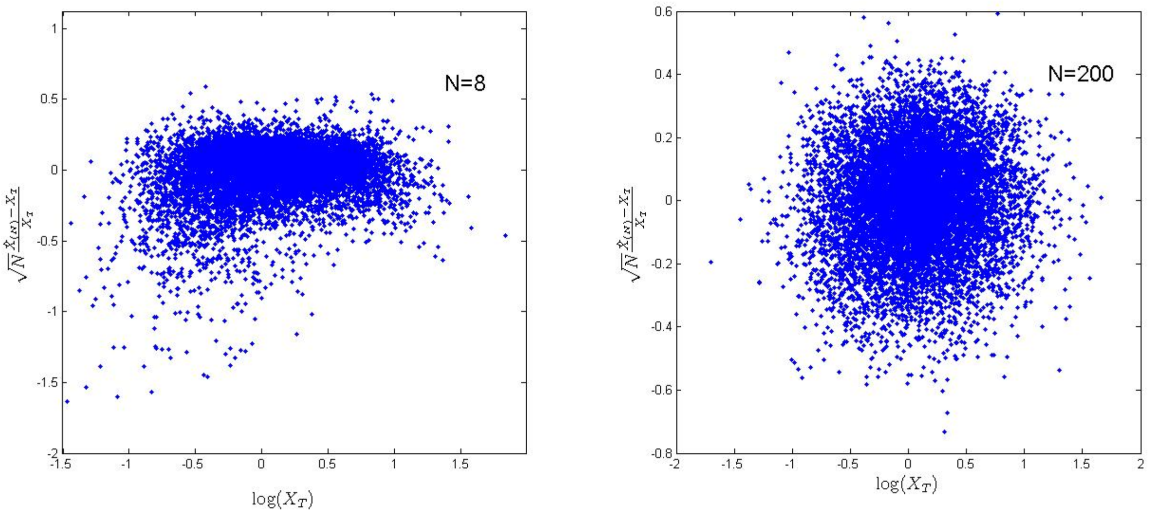


Figure 2: Scatterplots of $\sqrt{N}\frac{\hat{X}_{(N)}-X_T}{X_T}$ versus $log(X_T)$ for $N = 8$ and $N = 200$, illustrating the asymptotic independence.

Figure 3 shows QQ-plots (Quantile-Quantile plot) between the scaled relative rebalancing error $\sqrt{N}\frac{\hat{X}_{(N)}-X_T}{X_T}/\sqrt{T\int_0^T \sigma_{L,u}^2 du}$ and the standard normal distribution at $N = 8$ and $N = 200$. With the observations falling on a straight line in the right panel, we show that $\sqrt{N}\frac{\hat{X}_{(N)}-X_T}{X_T}/\sqrt{T\int_0^T \sigma_{L,u}^2 du}$ converges to the standard normal distribution as N increases.

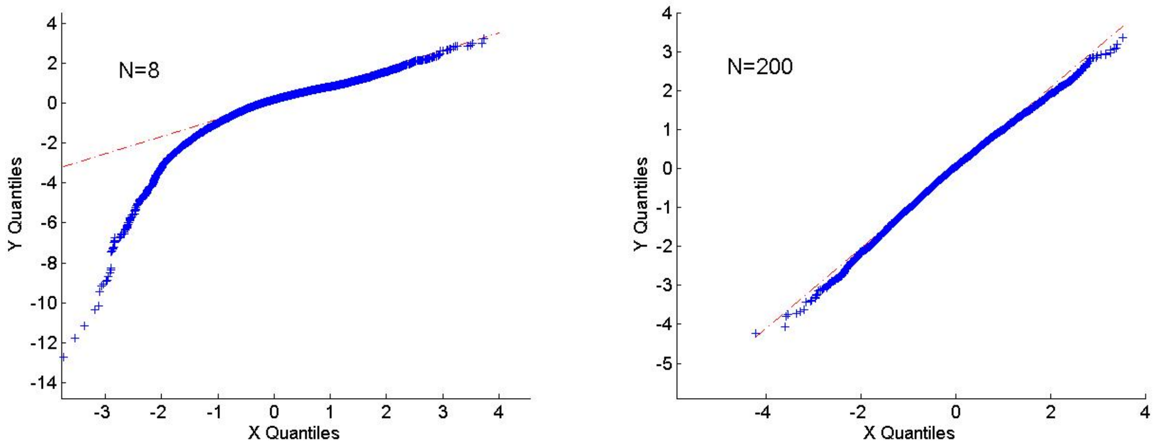


Figure 3: QQ-plots of $\sqrt{N}\frac{\hat{X}_{(N)}-X_T}{X_T}/\sqrt{T\int_0^T \sigma_{L,u}^2 du}$ for $N = 8$ and $N = 200$, illustrating the convergence to normality.

5.5.2 Test of the accuracy of the volatility adjustment

To measure the accuracy of the volatility adjustment, we use the concept of error reduction that is first introduced by Glasserman in [21]. The error reduction is defined as

$$1 - \frac{|\sigma_{adj} - \hat{\sigma}_N|}{|\sigma_\pi - \hat{\sigma}_N|},$$

where σ_{adj} is the adjusted volatility in Definition 5.3.2, $\hat{\sigma}_N$ denotes the estimated "volatility" of $\hat{X}_{(N)}$, i.e.

$$\hat{\sigma}_N = \sqrt{\frac{1}{T} Var\left(\log\left(\hat{X}_{(N)}\right)\right)}$$

and σ_π denotes the portfolio "volatility", i.e.

$$\sigma_\pi = \sqrt{\frac{1}{T}\int_0^T \|\pi_u^T \sigma_u\|^2 du}.$$

To get $\hat{\sigma}_{(N)}$, we calculate the standard deviation of $log\left(\hat{X}_{(N)}\right)$ across one million paths. Furthermore, to make our calculation always rational, we simply discard negative values of $\hat{X}_{(N)}$, which are rare in the examples we have tested, before taking logs.

We introduce two more models in the following. Both models have five assets.

Model 2: The volatility coefficients matrix $\sigma(Y_t)$, $0 \le t \le T$ is given by

$$\sigma(Y_t) = \begin{pmatrix} .02Y_t - .018 & .2Y_t - .199 & .02Y_t + .02 & .02Y_t + .01 & .02Y_t - .01 \\ .2Y_t - .17 & .2Y_t - .198 & .02Y_t - .019 & .02Y_t & .02Y_t + .02 \\ .02Y_t - .016 & .02Y_t - .017 & .02Y_t - .01 & .02Y_t - .01 & .02Y_t \\ .2Y_t - .197 & .02Y_t + .02 & .02Y_t & .02Y_t & .02Y_t + .01 \\ .02Y_t - .019 & .02Y_t - .018 & .02Y_t + .01 & .02Y_t - .01 & .02Y_t + .01 \end{pmatrix}$$

with (Y_t) as in Model 1; the mean rate of returns $\mu(Y_t)$ are as in Model 1; portfolio weights are $[-1, 1, 2, 1, -2]$ with a leverage ratio of 4;

Model 3: The volatility coefficients matrix $\sigma(Y_t)$ and the mean rate of return $\mu(Y_t)$ are given as in Model 1; portfolio weights are $[-0.2, 0.4, 0.4, 0.1, 0.3]$ with a leverage ratio of 1.2.

Table 2 shows that the portfolios with higher leverages and volatilities show a marked improvement in the error reduction as N increases, whereas for the portfolios with low leverage and low volatility, the adjusted volatility is nearly exact even at $N = 8$. This assertion extends the similar statement in Black-scholes model stated by Glasserman in [21].

	Model2				Model3			
	σ_π	σ_{adj}	$\hat{\sigma}_N$	error red.	σ_π	σ_{adj}	$\hat{\sigma}_N$	error red.
N=8	0.4615	0.4927	0.5136	59.9%	0.2710	0.2728	0.2727	95.3%
N=12	0.4615	0.4825	0.4903	73.0%	0.2710	0.2722	0.2719	64.6%
N=16	0.4615	0.4774	0.4790	90.1%	0.2710	0.2719	0.2715	31.6%

Table 2: Error reduction using the volatility adjustment across test models. The table shows the portfolio volatility σ_π, the adjusted volatility σ_{adj} and the volatility of the discretely rebalanced portfolio $\hat{\sigma}_N$. The columns "error red." show the error reduction achieved by the adjusted volatility at different rebalancing frequencies.

5.5.3 Test of the accuracy of the combined adjustment

In the following, we will illustrate the effects of the volatility adjustment, the conditional mean adjustment and the combined adjustment on different models through comparing the distribution functions of $\log X_T$, $\log \hat{X}_{(N)}$ and the different approximations of $\log \hat{X}_{(N)}$.

In a similar way as in [21], we first generate a sample of one million of X_T by simulating

$$\exp\left(\int_0^T \left(\pi_u^T \mu_u - \frac{1}{2}\|\pi_u^T \sigma_u\|^2\right) du + \sum_{i=1}^{d} \sqrt{\int_0^T \left(\pi_u^T \sigma_u(:,i)\right)^2 du} Z_i^{X_T}\right)$$

with $Z^{X_T} = \left(Z_i^{X_T}\right)_{1\le i\le d}$ being a d-dimensional standard normal distributed random vector and denote the distribution function of X_T by $F(c) := P(X_T \le c)$, $c \in \mathbb{R}^+$.

Then, we generate the sample of $\hat{X}_{(N)}$ through simulating

$$\prod_{n=1}^{N} \sum_{i=1}^{d} \pi_{(n-1),i} \exp\left(\int_{(n-1)\Delta t}^{n\Delta t} \left(\mu_{u,i} - \frac{1}{2}\|\sigma_u\,(i,:)\,\|^2\right) du + \sum_{j=1}^{d} \sqrt{\int_{(n-1)\Delta t}^{n\Delta t} \sigma_{u,ij}^2 du} Z_j^{\hat{X}_{(n)}}\right)$$

with $Z^{\hat{X}_{(N)}} = \left(Z_j^{\hat{X}_{(N)}}\right)_{1\le j\le d}$ being a d-dimensional standard normal distributed random vector for $n = 1, \dots, N$ and denote the distribution function of $\hat{X}_N$ by $\hat{F}(c) := P\left(\hat{X}_{(N)} \le c\right)$, $c \in \mathbb{R}$.

The volatility adjustment in Definition 5.3.2 yields the following approximation of $F(c)$

$$F_{adj}(c) := P\left(X_{adj} \le c\right) \approx P\left(\hat{X}_{(N)} \le c\right), \quad c \in \mathbb{R}^+$$

with

$$X_{adj} = \exp\left(\int_0^T \pi_u^{'} \mu_u du - \frac{T}{2}\sigma_{adj}^2 + \frac{\sqrt{T}\sigma_{adj}}{\sqrt{\int_0^T \|\pi_u^T \sigma_u\|^2 du}} \int_0^T \pi_u^T \sigma_u dW_u^S\right)$$

and the conditional mean adjustment in (5.56) provides the approximation $F_H(c)$ given by

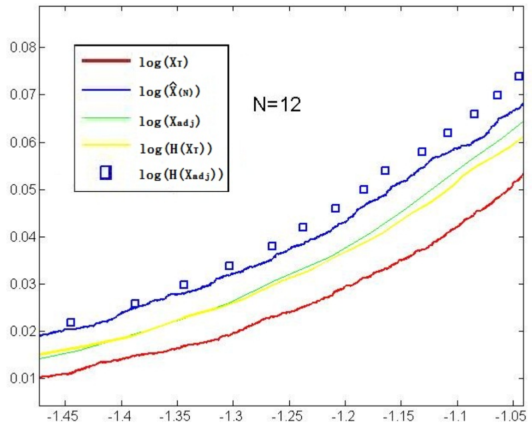
$$F_H(c) := P\left(H(X_T) \le c\right) \approx P\left(\hat{X}_{(N)} \le c\right)$$

with $H(y) : \mathbb{R} \to \mathbb{R}$ given in Section 5.4.2.

Finally, we define a combined approximation $F_{adj,H}(c)$ that use both adjustments, as

$$F_{adj,H}(c) := P\left(H\left(X_{adj}\right) \le c\right) \approx P\left(\hat{X}_{(N)} \le c\right).$$

In figure 4, the distribution functions of $\log X_T$, $\log \hat{X}_{(N)}$ and the different approximations of $\log \hat{X}_{(N)}$ are shown and we see that $F_{adj,H}$ provides the best approximation of the discretely rebalanced portfolio $\hat{X}_{(N)}$ on the region near probabilities of the order of $10^{-3} - 10^{-1}$. Figure 4 concentrates on this region, since it is important to estimate VaR at a 99.9% confidence, which is critical for the determination of the incremental risk charge (IRC).



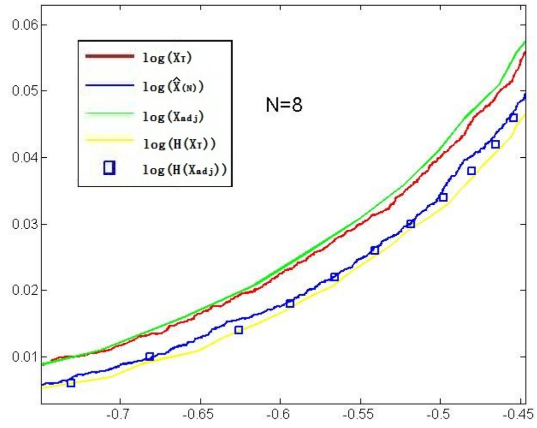


Figure 4: Distribution functions in model 2 (left panel) and model 3 (right panel). In each panel the red line plots $\log(X_T)$, the blue line plots $\log\left(\hat{X}_{(N)}\right)$ with $N = 4$, the green line is the approximation of $\log(X_{adj})$, the yellow line is the approximation of $H(X_T)$ and the square signs plot $\log(H(X_{adj}))$.

Chapter 6

Temporal granularity

The concept of temporal granularity is introduced in economics and finance to show the relationships between continuous-time and discrete-time models. The temporal granularity can be used to quantify the approximation errors that arise from discrete-time implementations of continuous-time models, for example the implementation of a continuous time delta-hedging strategy in discrete time. The discrepancy between the discrete-time and the continuous-time delta-hedging strategies is unusually called *tracking error* for delta-hedging.

Bertsimas, Kogan and Lo [6] have studied the temporal granularity in a derivative pricing model assuming that perfect replication of the derivative is not possible, namely, delta-hedging strategies exhibit tracking errors. In [6], it is shown that the normalized tracking error converges weakly to a particular stochastic integral and the *temporal granularity coefficient* is defined as the standard deviation of this stochastic integral. The temporal granularity in this thesis will be defined in a similar way.

6.1 Definition of temporal granularity

We consider the model in Section 5.2.3. The discretely rebalanced and continuously rebalanced portfolios are denoted still by $\hat{X}$ and X, respectively. Then, according to Theorem 5.2.3 (limit theorem *II*), we have as $N \to \infty$,

$$\sqrt{N}\frac{\hat{X}_{(N)} - X_T}{X_T} \stackrel{\mathcal{D}}{\to} \sqrt{T}\int_0^T \Psi_{L,u} dW_u$$

with $(W_u)_{u\geq 0}$ being a Brownian motion and

$$\Psi^2_{L,u} := \frac{1}{2}\left(\pi_u^T\left(\Psi_u \circ \Psi_u\right)\pi_u + \left(\pi_u^T \Psi_u \pi_u\right)^2 - 2\pi_u^T \Psi_u D_u \Psi_u \pi_u\right).$$

Then, the definition of temporal granularity can be given by:

Definition 6.1.1. *(Temporal granularity)*

The temporal granularity g is defined as the standard deviation of the limit distribution of $\sqrt{N}\left(\hat{X}_{(N)} - X_T/X_T\right)$, i.e.

$$g = \sqrt{\frac{T}{2}\int_0^T E\left(\pi_u^T\left(\Psi_u \circ \Psi_u\right)\pi_u + \left(\pi_u^T \Psi_u \pi_u\right)^2 - 2\pi_u^T \Psi_u D_u \Psi_u \pi_u\right) du} \tag{6.1}$$

with Ψ, π, D as in Section 5.2.3.

It is easy to see that $g = 0$ if (Σ_t) is constantly zero and g is merely determined by (π_t) and (Ψ_t). g is well-defined, since

$$E\left(\pi_u^T\left(\Psi_u \circ \Psi_u\right)\pi_u + \left(\pi_u^T \Psi_u \pi_u\right)^2 - 2\pi_u^T \Psi_u D_u \Psi_u \pi_u\right)$$

is always positive (See Remark 5.2.4).

6.2 Interpretation of Temporal granularity and some examples

The temporal granularity can be interpreted as a measure of the relative error between $\hat{V}_N$ and V_T. More formally, for large N, we can write

$$g \approx \sqrt{N}\sqrt{E\left[\left(\frac{\hat{V}_N - V(T)}{V(T)}\right)^2\right]}.$$

It yields then that the relative error between $\hat{V}_N$ and $V(T)$ can be approximated by $g/\sqrt{N}$ for large N, i.e.

$$\sqrt{E\left[\left(\frac{\hat{V}_N - V(T)}{V(T)}\right)^2\right]} \approx g/\sqrt{N}. \tag{6.2}$$

It implies that for a model with high granularity, a larger rebalancing frequency N, is required to achieve the same level of relative error as a model with low granularity. If we want the relative error of $\hat{V}_N$ and $V(T)$ to be within some small value δ, we have to require a rebalancing frequency $N \geq g^2/\delta^2$ by (6.2).

We want to give some examples of the temporal granularity in this subsection. Let us first consider a simple example, namely the temporal granularity of geometric Brownian motions.

Example 6.2.1. *Given the price process of an assets vector, which follows a multidimensional geometric Brownian motion with constant volatility coefficients matrix Σ and set $\Psi = \Sigma\Sigma$, then, the granularity g defined in (6.1) is given by:*

$$g = T\sqrt{\frac{\pi^T\left(\Psi \circ \Psi\right)\pi + \left(\pi^T \Psi \pi\right)^2 - 2\pi^T \Psi D \Psi \pi}{2}},$$

where π denotes the constant portfolio weights and $D = diag(\pi)$.

In a switch model we assume that the drift coefficients vector and the volatility coefficients matrix of a assets vector process, denoted by μ and Σ, are driven by an external Markov chain (Y_t). (Y_t) is stationary with state space $E = \{y(1), \ldots, y(l)\}$ and initial distribution Π. We still set $\Psi(Y_t) = \Sigma(Y_t) \cdot \Sigma(Y_t)$ and let $\pi(Y_t)$ denote the portfolio weights. Then, we have

Example 6.2.2. *In the switch model introduced above, the granularity g in (6.1) is given by*

$$g = T\sqrt{\frac{\sum_{i=1}^{l} \Pi(y(i))\Psi_{L,y(i)}}{2}}$$

with

$$\begin{aligned}\Psi_{L,y(i)} =& \pi^T \left(y(i)\right) \left(\Psi \left(y(i)\right) \circ \Psi \left(y(i)\right)\right) \pi \left(y(i)\right) + \left(\pi^T \left(y(i)\right) \Psi \left(y(i)\right) \pi \left(y(i)\right)\right)^2 \\ & - 2\pi^T \left(y(i)\right) \Psi \left(y(i)\right) D \left(y(i)\right) \Psi \left(y(i)\right) \pi \left(y(i)\right).\end{aligned}$$

The example above follows directly from Theorem 5.2.3, Definition 6.1.1 and the assumption that (Y_t) is stationary. We note that

$$T\sqrt{\frac{\Psi_{min}}{2}} \leq g \leq T\sqrt{\frac{\Psi_{max}}{2}}.$$

If the probability for a lower $\Psi_{L,y(i)}$ increases and the probability for a larger $\Psi_{L,y(i)}$ decreases, then, g becomes smaller.

Chapter 7

Optimal portfolios for discretely rebalanced portfolios in the Black-Scholes model

It is clear that the optimal portfolios for discretely rebalanced portfolios does not evolve the same as the optimal portfolios for continuously rebalanced portfolios. In Section 2 and 3 we have obtained explicit expressions for optimal strategies in Wishart volatility models regarding logarithmic utility and power utility, respectively.

The aim of this section is to present an approximate optimal strategy for discretely rebalanced portfolios in the Black-Scholes model. Let us denote by π^* the optimal strategy of a continuously rebalanced portfolio for logarithmic utility in the Black-Scholes model. It is known that the optimal strategy π^* remains constant in the entire time horizon T. In this section we familiarize the readers with a way to find some portfolio with an improved optimality for discretely rebalanced portfolios compared with π^*.

Since the terminal value of the discretely rebalanced portfolio $\hat{X}_N$ over N rebalancing periods does not own an "simple" explicit expression, we can not deal with the optimization problems by direct calculation as in Section 2.1.1. Hence, it is extremely difficult to get an explicit analytical result for the optimization problems for discretely rebalanced portfolios. Our objective is to find some portfolio, which shows a better optimality than π^* for Δt sufficient small.

There are several ways to approximate the optimal strategy. Our first intuition is to apply Theorem 4.2. (Limit theorem I), namely we approximate $\hat{X}_N$ by $V_T\left(1 + X\sqrt{\frac{\Delta t}{T}}\right)$ with X being the limit normal distribution in Theorem 4.2. This approach involves the computation of $E\left[\log\left(1 + X\sqrt{\frac{\Delta t}{T}}\right)\right]$, which is problematic due to the positive probability of negative $1 + X\sqrt{\frac{\Delta t}{T}}$. In this thesis we use another way to approximate the optimal strategy, namely the Newton's method.

7.1 Model dynamics and the portfolio problem

We consider a market consisting of d risky assets and a bond. The price process of the risky assets vector evolves as a geometric Brownian motion

$$dS_t = diag(S_t)\big(\mu dt + \sigma dW_t\big) \tag{7.1}$$

with $\mu = (\mu_1 \dots \mu_d)^T$ being the drift vector and $\sigma \in \mathbb{R}^{d\times d}$ being the volatility coefficients matrix. The dynamic of the riskfree asset is

$$dS_t^0 = S_t^0 r dt. \tag{7.2}$$

We denote by $W_t = (W_{1,t}, \dots, W_{d,t})$ a d-dimensional standard Brownian motion. The constant riskfree rate is denoted by r. The entries of the covariance matrix Σ are

$$\Sigma_{ij} = cov\left(\frac{dS_{i,t}}{S_{i,t}}, \frac{dS_{j,t}}{S_{j,t}}\right) = \sum_{k=1}^{d} \sigma_{ik}\sigma_{jk}, \quad i,j = 1,\dots,d,$$

i.e. $\Sigma = \sigma\sigma^T$.

The portfolio weights are denoted by $\pi = (\pi_1, \dots, \pi_d)^T$, which is constant over time. The optimal risky assets strategy of a continuously rebalanced portfolio with logarithmic utility in the above model is

$$\pi^* = \Sigma^{-1}\left(\mu - r\mathbf{1}\right). \tag{7.3}$$

We assume further that there exists no short-selling in the financial market, especially, we have $\left(\Sigma^{-1}\left(\mu - r\mathbf{1}\right)\right)_i \geq 0$ for $i = 1, \dots, d$ and $1 - \mathbf{1}^T\Sigma^{-1}\left(\mu - r\mathbf{1}\right) \geq 0$. We point out that the optimization problem for a continuously rebalanced portfolio in a market without short-selling can generally be solved. Cvitanić and Karatzas [12] have discussed the optimization problems, when the portfolio is constrained to take values in a given closed, convex, nonempty subset of $\mathbb{R}^d$ and treated the no short-selling constriction as a special case in Example 14.9.

The portfolio wealth process of the continuously rebalanced portfolio (X_t) evolves as

$$\frac{dX_t}{X_t} = \sum_{i=1}^{d} \pi_i \frac{dS_{i,t}}{S_{i,t}} + \left(1 - \pi^T\mathbf{1}\right) r dt.$$

Let T (e.g., 1 year) be the risk horizon and $\Delta t = T/N$ the rebalancing horizon. The wealth process of the discretely rebalanced portfolio $\left(\hat{X}_t\right)$ evolves from $n\Delta t$ to $(n+1)\Delta t$, $0 \leq n \leq N-1$, as

$$\hat{X}_{(n+1)\Delta t} = \hat{X}_{n\Delta t}\left(\sum_{i=1}^{d} \pi_i \frac{S_{i,(n+1)\Delta t}}{S_{i,n\Delta t}} + \left(1 - \pi^T\mathbf{1}\right) r\Delta t\right)$$

with $X_0 = \hat{X}_0 = 1$. To lighten notation, we denote $\hat{X}_{(n)} = \hat{X}_{n\Delta t}$, $X_{(n)} = X_{n\Delta t}$, $W_{(n)} = W_{n\Delta t}$, $S_{(n)} = S_{n\Delta t}$ and $\Delta W(n+1) = W_{(n+1)} - W_{(n)}$.

It is known that the optimal strategy π^* of (X_t) is not optimal for $\left(\hat{X}_t\right)$ in general. To get the optimal portfolio strategy for $\left(\hat{X}_t\right)$, we need to solve the following problem:

$$\begin{aligned}
&\max_{\pi} E\left[\log\left(\hat{X}_T\right)\right]\\
=&\max_{\pi} E\left[\log\left(\prod_{n=1}^{N}\frac{\hat{X}_{(n)}}{\hat{X}_{(n-1)}}\right)\right]\\
=&\max_{\pi}\sum_{n=1}^{N} E\left[\log\left(\sum_{i=1}^{d}\pi_i\exp\left(\left(\mu_i-\frac{1}{2}\|\sigma(i,:)\|^2\right)\Delta t+\sigma(i,:)\Delta W(n)\right)+\left(1-\pi^T\mathbf{1}\right)e^{r\Delta t}\right)\right]\\
=&N\max_{\pi} E\left[\log\left(\sum_{i=1}^{d}\pi_i\exp\left(\left(\mu_i-\frac{1}{2}\|\sigma(i,:)\|^2\right)\Delta t+\sigma(i,:)\Delta W(n)\right)+\left(1-\pi^T\mathbf{1}\right)e^{r\Delta t}\right)\right].
\end{aligned} \tag{7.4}$$

The last step follows from the fact that $\Delta W(n)$, $n=1,\ldots,N$ are identically distributed.

Since it is difficult to find the exact solution of problem (7.4). Our task is to find a strategy $\hat{\pi}(\Delta t)\in\mathbb{R}^d$, which owns a larger terminal logarithmic utility than π^* in (7.3) for small Δt. It is clear that $\hat{\pi}(\Delta t)$ should go to π^* as Δt goes to zero. To get such a $\hat{\pi}(\Delta t)$, we apply Newton's method with π^* being the initial value. Note that (7.4) is not always well-defined, since

$$\sum_{i=1}^{d}\pi_i\exp\left(\left(\mu_i-\frac{1}{2}\|\sigma(i,:)\|^2\right)\Delta t+\sigma(i,:)\Delta W(n)\right)$$

could be negative for arbitrary π_i. To make our optimization problem reasonable at least for small Δt, we assume that our optimal strategy of a continuously rebalanced portfolio π^* owns only positive components and the bond proportion $1-(\pi^*)^T\mathbf{1}$ is also positive. It implies that the components of $\hat{\pi}(\Delta t)$ and $1-(\hat{\pi}(\Delta t))^T\mathbf{1}$ are positive at least for small Δt and our optimization problem is well-defined under this condition for sufficient small Δt.

7.2 Optimality in the Black-Scholes model

Let us first denote

$$G_{N,n,i}:=\sum_{i=1}^{d}\pi_i\exp\left(\left(\mu_i-\frac{1}{2}\|\sigma(i,:)\|^2\right)\Delta t+\sigma(i,:)Z_{N,n}\sqrt{\Delta t}\right)+\left(1-\pi^T\mathbf{1}\right)e^{r\Delta t}$$

and

$$\bar{G}_{N,n,i}:=\sum_{i=1}^{d}\pi_i\exp\left(\left(\mu_i-\frac{1}{2}\|\sigma(i,:)\|^2\right)\Delta t+\sigma(i,:)\bar{Z}_{N,n}\sqrt{\Delta t}\right)+\left(1-\pi^T\mathbf{1}\right)e^{r\Delta t}, \tag{7.5}$$

where $Z_{N,n}$ is a d-dimensional standard normal distributed random vector with $Z_{N,n}=\Delta W(n)/\sqrt{\Delta t}$ and $\bar{Z}_{N,n}$ is a realization of $Z_{N,n}$.

Applying Taylor approximations to the exponential functions in (7.5), $G_{N,n,i}$ can be written as the following polynomial plus the rest term $R_{N,n}$, where $\bar{G}_{N,n,i}$ and $\bar{R}_{N,n}$ denote realizations of $G_{N,n,i}$ and $R_{N,n}$ respectively. Then we get

$$\begin{aligned}
\bar{G}_{N,n,i} =& \sum_{i=1}^{d} \pi_i \Bigg[1 + \left(\mu_i - \frac{1}{2}\|\sigma(i,:)\|^2\right)\Delta t + \sigma(i,:)\bar{Z}_{N,n}\sqrt{\Delta t} + \frac{1}{2}\left(\mu_i - \frac{1}{2}\|\sigma(i,:)\|^2\right)^2 \Delta t^2 \\
&+ \frac{1}{2}\left(\sigma(i,:)\bar{Z}_{N,n}\right)^2 \Delta t + \left(\mu_i - \frac{1}{2}\|\sigma(i,:)\|^2\right)\sigma(i,:)\bar{Z}_{N,n}\Delta t^{3/2} + \frac{1}{6}\left(\sigma(i,:)\bar{Z}_{N,n}\right)^3 \Delta t^{3/2} \\
&+ \frac{1}{24}\left(\sigma(i,:)\bar{Z}_{N,n}\right)^4 \Delta t^2 + \frac{1}{2}\left(\sigma(i,:)\bar{Z}_{N,n}\right)^2 \left(\mu_i - \frac{1}{2}\|\sigma(i,:)\|^2\right)\Delta t^2 \Bigg] \\
&+ \left(1 - \pi^T \mathbf{1}\right)\left(1 + r\Delta t + \frac{r^2 \Delta t^2}{2}\right) + \bar{R}_{N,n} \\
=& 1 + a_1\sqrt{\Delta t} + a_2 \Delta t + a_3 \Delta t^{3/2} + a_4 \Delta t^2 + \bar{R}_{N,n}
\end{aligned}$$

with

$$\begin{aligned}
a_1 &= \pi^T \sigma \bar{Z}_{N,n}, \quad a_2 = \pi^T \mu - \frac{\sum_{i=1}^{d} \pi_i \|\sigma(i,:)\|^2}{2} + \frac{1}{2}\sum_{i=1}^{d} \pi_i \left(\sigma(i,:)\bar{Z}_{N,n}\right)^2 + (1 - \pi^T \mathbf{1})r, \\
a_3 &= \sum_{i=1}^{d} \pi_i \left(\mu_i - \frac{1}{2}\|\sigma(i,:)\|^2\right)\sigma(i,:)\bar{Z}_{N,n} + \frac{\sum_{i=1}^{d} \pi_i \left(\sigma(i,:)\bar{Z}_{N,n}\right)^3}{6}, \\
a_4 &= \frac{\sum_{i=1}^{d} \pi_i \left(\mu_i - \frac{1}{2}\|\sigma(i,:)\|^2\right)^2}{2} + \frac{\sum_{i=1}^{d} \pi_i \left(\sigma(i,:)\bar{Z}_{N,n}\right)^4}{24} \\
&\quad + \frac{\sum_{i=1}^{d} \pi_i \left(\sigma(i,:)\bar{Z}_{N,n}\right)^2 \left(\mu_i - \frac{1}{2}\|\sigma(i,:)\|^2\right)}{2} + \left(1 - \pi^T \mathbf{1}\right)\frac{r^2}{2}
\end{aligned}$$

and

$$\begin{aligned}
\bar{R}_{N,n} :=& \frac{\sum_{i=1}^{d} \pi_i \left(\mu_i - \frac{1}{2}\|\sigma(i,:)\|^2\right)^3 \Delta t^3}{6} + \frac{\sum_{i=1}^{d} \pi_i \left(\sigma(i,:)\bar{Z}_{N,n}\right)\left(\mu_i - \frac{1}{2}\|\sigma(i,:)\|^2\right)^2 \Delta t^{5/2}}{2} \\
&+ \frac{\sum_{i=1}^{d} \pi_i \left(\mu_i - \frac{1}{2}\|\sigma(i,:)\|^2\right)^4 \Delta t^4}{24} + \frac{\sum_{i=1}^{d} \pi_i \left(\sigma(i,:)\bar{Z}_{N,n}\right)\left(\mu_i - \frac{1}{2}\|\sigma(i,:)\|^2\right)^3 \Delta t^{7/2}}{6} \\
&+ \frac{\sum_{i=1}^{d} \pi_i \left(\sigma(i,:)\bar{Z}_{N,n}\right)^3 \left(\mu_i - \frac{1}{2}\|\sigma(i,:)\|^2\right)\Delta t^{5/2}}{6} \\
&+ \frac{\sum_{i=1}^{d} \pi_i \left(\sigma(i,:)\bar{Z}_{N,n}\right)^2 \left(\mu_i - \frac{1}{2}\|\sigma(i,:)\|^2\right)^2 \Delta t^3}{4} \\
&+ \sum_{n=5}^{\infty} \frac{\sum_{i=1}^{d} \pi_i \left(\left(\mu_i - \frac{1}{2}\|\sigma(i,:)\|^2\right)\Delta t + \sigma(i,:)\bar{Z}_{N,n}\sqrt{\Delta t}\right)^n}{n!} + \sum_{n=3}^{\infty} \frac{(r\Delta t)^n}{n!}.
\end{aligned}$$

We obtain that $\bar{R}_{N,n} \in O(\Delta t^{5/2})$. In the following, we want to get an approximation of $\log(\bar{G}_{N,n,i})$ with exact values up to $O(\Delta t^2)$. From the Taylor series

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \quad for \quad -1 < x \leq 1$$

and the fact that $\left|a_1\sqrt{\Delta t}+a_2\Delta t+a_3\Delta t^{3/2}+a_4\Delta t^2+\bar{R}_{N,n}\right|<1$ for sufficiently small Δt, we get the following desired approximation:

$$\begin{aligned}\log\left(\bar{G}_{N,n,i}\right)=&\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\left(a_1\sqrt{\Delta t}+a_2\Delta t+a_3\Delta t^{3/2}+a_4\Delta t^2+\bar{R}_{N,n}\right)^n}{n}\\ =&\sum_{n=1}^{4}(-1)^{n+1}\frac{\left(a_1\sqrt{\Delta t}+a_2\Delta t+a_3\Delta t^{3/2}+a_4\Delta t^2\right)^n}{n}+\bar{R}^0_{N,n}\\ =&a_1\sqrt{\Delta t}+\left(a_2-\frac{a_1^2}{2}\right)\Delta t+\left(a_3-a_1a_2+\frac{a_1^3}{3}\right)\Delta t^{3/2}\\ &+\left(a_4-\frac{a_2^2}{2}-a_1a_3+a_1^2a_2-\frac{a_1^4}{4}\right)\Delta t^2+\bar{R}^{'}_{N,n},\end{aligned}$$

where $\bar{R}^0_{N,n}$ and $\bar{R}^{'}_{N,n}$ are obviously polynomials of a_1, a_2, a_3, a_4 and $\bar{R}_{N,n}$.

Let $R^{'}_{N,n}$ denote the random variable with realizations $\bar{R}^{'}_{N,n}$, we can easily get that $E\left[R^{'}_{N,n}\right]\in O(\Delta t^{5/2})$, since $R^{'}_{N,n}$ is a polynomial of $Z_{N,n}$ and thus the coefficients of Δt-terms are all bounded, hence, we get that

$$\begin{aligned}&E\left[\log\left[G_{N,n,i}\right]\right]\\ =&E\left[\left(a_2-\frac{a_1^2}{2}\right)\Delta t+\left(a_4-\frac{a_2^2}{2}-a_1a_3+a_1^2a_2-\frac{a_1^4}{4}\right)\Delta t^2+O(\Delta t^{5/2})\right]\\ =&\left(\pi^T\mu+(1-\pi^T\mathbf{1})r-\frac{\pi^T\Sigma\pi}{2}\right)\Delta t+\left(\frac{\sum_{i=1}^d\pi_i\mu_i^2}{2}+(1-\pi^T\mathbf{1})\frac{r^2}{2}-\frac{\left(\pi^T\mu+(1-\pi^T\mathbf{1})r\right)^2}{2}\right.\\ &-\frac{\sum_{i,j=1}^d\pi_i\pi_j\Sigma_{ij}^2}{4}-\sum_{i=1}^d\pi_i\mu_i\sigma(i,:)\sigma^T\pi+\pi^T\Sigma\pi\left(\pi^T\mu+(1-\pi^T\mathbf{1})r\right)\\ &\left.+\pi^T\sigma\sum_{i=1}^d\pi_i\sigma^T(:,i)\sigma(i,:)\sigma^T\pi-\frac{1}{4}\left(\pi^T\Sigma\pi\right)^2\right)\Delta t^2+O(\Delta t^{5/2}).\end{aligned}$$

For the convenience of our further computations, we rewrite

$$\begin{aligned}E\left[\log\left[G_{N,n,i}\right]\right]=&r\Delta t+\pi^TM_1-\pi^TM_2\pi+\pi^T\Sigma\pi\pi^T(\mu-r\mathbf{1})\Delta t^2\\ &+\pi^T\sigma\sum_{i=1}^d\pi_i\sigma^T(:,i)\sigma(i,:)\sigma^T\pi\Delta t^2-\frac{\left(\pi^T\Sigma\pi\right)^2\Delta t^2}{4}+O(\Delta t^{5/2})\end{aligned}\tag{7.6}$$

with

$$M_1:=\mu\Delta t-r\Delta t+\frac{\mu\circ\mu}{2}\Delta t^2-\frac{r^2\Delta t^2}{2}-(\mu-r\mathbf{1})r\Delta t^2$$

and

$$M_2:=\frac{\Sigma\Delta t}{2}+\frac{(\mu-r\mathbf{1})(\mu-r\mathbf{1})^T\Delta t^2}{2}+\frac{(\Sigma\circ\Sigma)\Delta t^2}{4}+\left(\mu\mathbf{1}^T\circ\sigma\right)\sigma^T\Delta t^2-\Sigma r\Delta t^2.$$

Theorem 7.2.1. *For Δt sufficiently small*

$$\hat{\pi}=\Sigma^{-1}(\mu-r\mathbf{1})-\tilde{H}^{-1}\tilde{J}$$

is a better portfolio strategy compared to

$$\pi^* = \Sigma^{-1}(\mu - r\mathbf{1})$$

for a Δt-periodic rebalanced portfolio, where $\tilde{H}$ is a $d\times d$ real matrix and $\tilde{J}$ is a d-dimensional real vector with

$$\begin{aligned}\tilde{H} =& -\Sigma - \frac{(\Sigma\circ\Sigma)\Delta t}{2} - 2\left(\mu\mathbf{1}^T\circ\sigma\right)\sigma^T\Delta t + 2\Sigma r\Delta t + 2\Sigma(\mu - r\mathbf{1})^T\Sigma^{-1}(\mu - r\mathbf{1})\Delta t \\ &+ (\mu - r\mathbf{1})(\mu - r\mathbf{1})^T\Delta t + 2\Sigma D\Sigma\Delta t + 2\Sigma diag(\mu - r\mathbf{1})\Delta t + 2diag(\mu - r\mathbf{1})\Sigma\Delta t \\ &- \Sigma(\mu - r\mathbf{1})^T\Sigma^{-1}(\mu - r\mathbf{1})\Delta t\end{aligned}$$

and

$$\begin{aligned}\tilde{J} =& \frac{\mu\circ\mu}{2}\Delta t - \frac{r^2\Delta t}{2} - \frac{\Sigma\circ\Sigma}{2}\Sigma^{-1}(\mu - r\mathbf{1})\Delta t - 2\left(\mu\mathbf{1}^T\circ\sigma\right)\sigma^T\Sigma^{-1}(\mu - r\mathbf{1})\Delta t \\ &+ r(\mu - r\mathbf{1})\Delta t + (\mu - r\mathbf{1})(\mu - r\mathbf{1})^T\Sigma^{-1}(\mu - r\mathbf{1})\Delta t + 2\Sigma D(\mu - r\mathbf{1})\Delta t \\ &+ (\mu - r\mathbf{1})\circ(\mu - r\mathbf{1})\Delta t\end{aligned}$$

with $D = diag(\pi^*)$.

Proof. We use the multidimensional Newton's method to prove the theorem. We may choose $\pi^* = \Sigma^{-1}(\mu - r\mathbf{1})$ to be the initial value of the Newton's method. This intuition comes from the fact that π^* optimizes the optimization problem (7.4) for $N \to \infty$, namely for large N, $\hat{\pi}$ is close to π^*. This guarantees that for N large enough the portfolio strategy $\hat{\pi}$ obtained by one Newton iteration, wins a larger utility compared with the initial value π^*.

To take one iteration, we need the first derivative of $E\left[\log\left[G_{N,n,i}\right]\right]$ with respect to π and its Hessian matrix. Let us denote

$$J := \frac{\partial}{\partial\pi}E\left[\log\left[G_{N,n,i}\right]\right]\Big|_{\pi=\pi^*} \quad \textit{and} \quad H := \frac{\partial^2}{\partial\pi^2}E\left[\log\left[G_{N,n,i}\right]\right]\Big|_{\pi=\pi^*}.$$

Then, by direct computation we get

$$\begin{aligned}J =& M_1 - 2M_2\pi^* + 2\Sigma\pi^*\left(\pi^*\right)^T(\mu - r\mathbf{1})\Delta t^2 + \left(\pi^*\right)^T\Sigma\pi^*(\mu - r\mathbf{1})\Delta t^2 \\ &+ 2\Sigma D\Sigma\pi^*\Delta t^2 + (\Sigma\pi^*)\circ(\Sigma\pi^*)\Delta t^2 - \left(\pi^*\right)^T\Sigma\pi^*\Sigma\pi^*\Delta t^2 \\ =& \frac{\mu\circ\mu}{2}\Delta t^2 - \frac{r^2\Delta t^2}{2} - \frac{\Sigma\circ\Sigma}{2}\Sigma^{-1}(\mu - r\mathbf{1})\Delta t^2 - 2\left(\mu\mathbf{1}^T\circ\sigma\right)\sigma^T\Sigma^{-1}(\mu - r\mathbf{1})\Delta t^2 \\ &+ r(\mu - r\mathbf{1})\Delta t^2 + (\mu - r\mathbf{1})(\mu - r\mathbf{1})^T\Sigma^{-1}(\mu - r\mathbf{1})\Delta t^2 + 2\Sigma D(\mu - r\mathbf{1})\Delta t^2 \\ &+ (\mu - r\mathbf{1})\circ(\mu - r\mathbf{1})\Delta t^2\end{aligned}$$

and

$$\begin{aligned}H =& -2M_2 + 2\Sigma\left(\pi^*\right)^T(\mu - r\mathbf{1})\Delta t^2 + 2\Sigma\pi^*(\mu - r\mathbf{1})^T\Delta t^2 + 2(\mu - r\mathbf{1})\left(\pi^*\right)^T\Sigma\Delta t^2 \\ &+ 2\Sigma D\Sigma\Delta t^2 + 2\Sigma diag(\Sigma\pi^*)\Delta t^2 + 2diag(\Sigma\pi^*)\Sigma\Delta t^2 - \Sigma\left(\pi^*\right)^T\Sigma\pi^*\Delta t^2 \\ &- 2\Sigma\pi^*\left(\pi^*\right)^T\Sigma\Delta t^2 \\ =& -2M_2 + \Sigma(\mu - r\mathbf{1})^T\Sigma^{-1}(\mu - r\mathbf{1})\Delta t^2 + 2(\mu - r\mathbf{1})(\mu - r\mathbf{1})^T\Delta t^2 + 2\Sigma D\Sigma\Delta t^2 \\ &+ 2\Sigma diag(\mu - r\mathbf{1})\Delta t^2 + 2diag(\mu - r\mathbf{1})\Sigma\Delta t^2.\end{aligned}$$

Following from one iteration of Newton's method, we get

$$\hat{\pi} = \Sigma^{-1}(\mu - r\mathbf{1}) + H^{-1}J.$$

The proof is concluded. □

Whether $\hat{\pi}$ shows a larger utility or not for a fixed Δt, depends on the effect of the Newton's method we used in the proof above, namely, considering a Δt-periodic rebalanced portfolio, a Newton's iteration started from π^* exhibits an enhancement of utility in our settings (7.4) and (7.6) only for Δt sufficiently small. For the investigation of the convergence of the Newton's method and the conditions for a nondecreasing iteration, we refer to [2, Lemma 9.19, Lemma 2.6.1].

7.3 Numerical examples

The optimality of the portfolio strategy $\hat{\pi}$ given in Theorem 7.2.1 will be illustrated by some numerical examples in this section. We compare three models in the following and apply the method of Monto-Carlo simulation on our models to get the logarithmic utilities and the optimal strategies $\hat{\pi}$ for different models. The Monto-Carlo simulations are performed across ten million paths. Since the received outcomes are not very stable (the values are too small), we perform every Monto-Carlo simulation 50 times and take the mean of the 50 outcomes as the final result. We first introduce our models.

Model 1: The portfolio consists of five risky assets with price vector processes as in (7.1). The volatility coefficients matrix is given by

$$\sigma = \begin{pmatrix} 1 & 0.05 & 0.2 & 0.15 & 0.05 \\ 0.15 & 2.5 & 0.05 & 0.1 & 0.2 \\ 0.2 & 0.15 & 1.5 & 0.05 & 0.1 \\ 0.15 & 0.2 & 0.1 & 2.5 & 0.3 \\ 0.05 & 0.1 & 0.15 & 0.05 & 3 \end{pmatrix}$$

The mean rate of return on the i-th asset is $\mu_i = 0.6 + \|\sigma(i,:)\|/4$ and the constant riskfree rate is $r = 0.5$. We choose portfolio weights for the continuously rebalanced portfolio by our principle in (7.3). This yields the portfolio strategy for risky assets

$$\pi^* = (0.2012, 0.0770, 0.1177, 0.0742, 0.0690)^T$$

and the weight for the riskfree asset $1 - \mathbf{1}^T\pi^* = 0.4609$.

Model 2: The model has five risky assets. The individual asset volatilities $\|\sigma(i,:)\|$, $i = 1, \ldots, 5$ range from 0.8 to 2.4 in increments of 0.4, namely $(\|\sigma(i,:)\|)_{i=1,\ldots,5} = (0.8, 1.2, 1.6, 2.0, 2.4)$ and all pairwise correlations of distinct assets are equal to 0.2. Since $\Sigma_{ij} = Cov(dS_i, dS_j)/dt$, $i, j = 1, \ldots, 5$ by its definition, we get $\Sigma_{ij} = \rho_{ij} \cdot \|\sigma(i,:)\|\|\sigma(j,:)\|$ for $i, j = 1, \ldots, 5$. This yields the following covariance matrix

$$\Sigma = \begin{pmatrix} 0.64 & 0.192 & 0.256 & 0.32 & 0.384 \\ 0.192 & 1.44 & 0.384 & 0.48 & 0.576 \\ 0.256 & 0.384 & 2.56 & 0.64 & 0.768 \\ 0.32 & 0.48 & 0.64 & 4 & 0.96 \\ 0.384 & 0.576 & 0.768 & 0.96 & 5.76 \end{pmatrix}.$$

The volatility coefficients matrix σ is given by the unique square root of Σ, i.e. $\sigma\sigma = \Sigma$. The mean rate of return on the i-th asset is $\mu_i = 0.65 + \|\sigma(i,:)\|/4$ and the constant riskfree rate is $r = 0.6$. The optimal portfolio strategies for continuously rebalanced portfolio is then

$$\pi^* = (0.2398, 0.1382, 0.0955, 0.0725, 0.0580)^T$$

and $1 - \mathbf{1}^T\pi^* = 0.396$.

Model 3: The mean rate of return on the i-th asset is changed to $\mu_i = \hat{\mu}_i + \|\sigma(i,:)\|/4$ with $(\hat{\mu}_i)_{i=1,\ldots,5} = (0.65, 0.65, 0.7, 0.65, 0.65)$. The other factors, i.e. $\|\sigma(i,:)\|$, $i = 1,\ldots,5$, ρ and Σ have the same form as in Model 2. This yields the following optimal portfolio strategies for continuously rebalanced portfolio

$$\pi^* = (0.2344, 0.1345, 0.1172, 0.0703, 0.0564)^T$$

and $1 - \mathbf{1}^T\pi^* = 0.3872$.

It is easy to see from Table 3 that in each model the risky assets strategies approach to their corresponding π^* as N increases. Especially, we note that although in Model 2 the mean rate of return increase gradually for $i = 1,\ldots,5$, the investment proportion of S_i is smaller than the investment proportion of S_{i-1}. This appears to against our intuition, since $\exp(\mu_i\Delta t)$, $i = 1,\ldots,5$ are exactly the expectations of the exponential functions in (7.4). This phenomenon may be explained by the fact that S_i owning a larger asset volatility $\|\sigma(i,:)\|$ is more risky than S_{i-1}. Furthermore, comparing the parameters in Model 2 and Model 3, we note that the mean rate of return on the third risky asset in Model 3 is a bit larger than it in Model 2 and all the other parameters are the same in both models. As we expected, in this situation the investment proportion on the third risky asset in Model 3 is larger than in Model 2.

	N=4	N=10
Model 1	(0.2721, 0.0513, 0.1272, 0.0477, 0.0328,)	(0.2309, 0.0675, 0.1239, 0.0637, 0.0527)
Model 2	(0.3096, 0.1632, 0.0969, 0.0605, 0.0388)	(0.2662, 0.1482, 0.0965, 0.0676, 0.0493)
Model 3	(0.3073, 0.1617, 0.1181, 0.0597, 0.0383)	(0.2616, 0.1452, 0.1178, 0.0659, 0.0480)
	N=20	N=30
Model 1	(0.2161, 0.0724, 0.1211, 0.0689, 0.0605)	(0.2112, 0.0740, 0.1201, 0.0707, 0.0633)
Model 2	(0.2528, 0.1433, 0.0961, 0.0700, 0.0535)	(0.2485, 0.1416, 0.0959, 0.0709, 0.0550)
Model 3	(0.2478, 0.1400, 0.1176, 0.0681, 0.0520)	(0.2433, 0.1381, 0.1175, 0.0689, 0.0534)

Table 3: Improved portfolio strategies on the risky assets for discretely rebalanced portfolios at $N = 4$, $N = 10$, $N = 20$ and $N = 30$ in different models.

	N=4		N=10		N=20		N=30	
	U_{π^*}	$U_{\hat{\pi}}$	U_{π^*}	$U_{\hat{\pi}}$	U_{π^*}	$U_{\hat{\pi}}$	U_{π^*}	$U_{\hat{\pi}}$
Model 1	0.1497	0.1527	0.0635	0.0637	0.0319	0.0320	0.0204	0.0205
Model 2	0.1753	0.1755	0.0717	0.0717	0.0358	0.0358	0.0236	0.0236
Model 3	0.1775	0.1776	0.0712	0.0713	0.0356	0.0356	0.0237	0.0237

Table 4: The logarithmic utilities for discretely rebalanced portfolios at $N = 4$, $N = 10$, $N = 20$ and $N = 30$ in different models. The columns of U_{π^*} indicate the utilities obtained by the portfolio strategy π^* and the columns of $U_{\hat{\pi}}$ indicate the utilities obtained by $\hat{\pi}$.

Table 4 illustrates the improvement on utilities for discretely rebalanced portfolios by investing $\hat{\pi}$ given in Theorem 7.2.1. The optimal portfolio strategy for continuously rebalanced portfolio π^* indicates already a lower utility at $N = 4$ compared to π^* and the superiority of $\hat{\pi}$ keeps in our example. It is self-evident that $U_{\hat{\pi}}$ converges to U_{π^*} as $N \to \infty$, hence, the differences between $U_{\hat{\pi}}$ and U_{π^*} tend to be smaller than 0.0001 at $N = 30$ in Model 2 and 3.

Appendix A

Proof of Lemma A.1.

Lemma A.0.1. *(Conditional Distribution of Brownian Increments)*
Let us denote

$$W := \int_0^T \pi_u^T \sigma_u dW_u^S \tag{A.1}$$

and

$$\Omega_u(i,:) := \sigma_u(i,:) - \pi_u^T \sigma_u, \quad 1 \leq i \leq d,$$

then, the increments $\int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)dW_u^S$, $1 \leq n \leq N$ *conditional on* $W = a$, $a \in \mathbb{R}$, *are jointly normal. Each* $\int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)dW_u^S$, $1 \leq n \leq N$ *has the conditional distribution*

$$\begin{aligned}&\left(\int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)dW_u^S | W = a\right)\\ \sim& N\left(\frac{a \cdot \int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)\sigma_u \pi_u du}{\int_0^T \|\pi_u^T \sigma_u\|^2 du}, \int_{(n-1)\Delta t}^{n\Delta t} \|\Omega_u(i,:)\|^2 du - \frac{\left(\int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)\sigma_u \pi_u du\right)^2}{\int_0^T \|\pi_u^T \sigma_u\|^2 du}\right)\end{aligned} \tag{A.2}$$

and each pair of increments $\int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)dW_u^S$, $\int_{(k-1)\Delta t}^{k\Delta t} \Omega_u(i,:)dW_u^S$, $n \neq k$, *has conditioned covariance*

$$-\frac{\left(\int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)\sigma_u \pi_u du\right)\left(\int_{(k-1)\Delta t}^{k\Delta t} \Omega_u(i,:)\sigma_u \pi_u du\right)}{\int_0^T \|\pi_u^T \sigma_u\|^2 du}.$$

The conditional joint distribution of $\int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)dW_u^S$, $1 \leq n \leq N$, *coincides with the unconditional joint distribution of*

$$\begin{aligned}\int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)dM_u :=& \int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)dW_u^S + \frac{a \cdot \int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)\sigma_u \pi_u du}{\int_0^T \|\pi_u^T \sigma_u\|^2 du}\\ &- \frac{\left(\int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)\sigma_u \pi_u du\right)\int_0^T \pi_u^T \sigma_u dW_u^S}{\int_0^T \|\pi_u^T \sigma_u\|^2 du}, \quad n = 1, \ldots, N\end{aligned} \tag{A.3}$$

with

$$dM_u = dW_u^S + \frac{a \cdot \sigma_u \pi_u du}{\int_0^T \|\pi_u^T \sigma_u\|^2 du} - \frac{\int_0^T \pi_u^T \sigma_u dW_u^S \cdot \sigma_u \pi_u du}{\int_0^T \|\pi_u^T \sigma_u\|^2 du}.$$

Proof. The first assertion follows from the fact that jointly normal random variables remain jointly normal when conditional on a linear combination. To derive the conditional means and covariance of the increments, we first compute the joint distribution of $\int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)dW_u^S$ and $\int_{(k-1)\Delta t}^{k\Delta t} \Omega_u(i,:)dW_u^S$, $n \neq k$,:

$$\begin{pmatrix} \int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)dW_u^S \\ \int_{(k-1)\Delta t}^{k\Delta t} \Omega_u(i,:)dW_u^S \\ W \end{pmatrix} \sim N(\bar{\mu}, \bar{\Sigma})$$

with $\bar{\mu}$ being a 3-dimensional zero vector and

$$\bar{\Sigma} = \begin{pmatrix} \int_{(n-1)\Delta t}^{n\Delta t} \|\Omega_u(i,:)\|^2 du & 0 & \int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)\sigma_u \pi_u du \\ 0 & \int_{(k-1)\Delta t}^{k\Delta t} \|\Omega_u(i,:)\|^2 du & \int_{(k-1)\Delta t}^{k\Delta t} \Omega_u(i,:)\sigma_u \pi_u du \\ \int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)\sigma_u \pi_u du & \int_{(k-1)\Delta t}^{k\Delta t} \Omega_u(i,:)\sigma_u \pi_u du & \int_0^T \|\pi_u^T \sigma_u\|^2 du \end{pmatrix}.$$

We write $\bar{\mu}$ and $\bar{\Sigma}$ as block matrices with the following components:

$$\bar{\Sigma}_{11} = \begin{pmatrix} \int_{(n-1)\Delta t}^{n\Delta t} \|\Omega_u(i,:)\|^2 du & 0 \\ 0 & \int_{(k-1)\Delta t}^{k\Delta t} \|\Omega_u(i,:)\|^2 du \end{pmatrix}, \bar{\Sigma}_{12} = \begin{pmatrix} \int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)\sigma_u \pi_u du \\ \int_{(k-1)\Delta t}^{k\Delta t} \Omega_u(i,:)\sigma_u \pi_u du \end{pmatrix},$$

$$\bar{\Sigma}_{21} = \bar{\Sigma}_{12}^T, \bar{\Sigma}_{22} = \int_0^T \|\pi_u^T \sigma_u\|^2 du, \quad \mu^{(1)} = (0\ 0)^T, \mu^{(2)} = 0.$$

By [1, Theorem 2.3.1.] we get the conditional joint distribution of $\int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)dW_u^S$ and $\int_{(k-1)\Delta t}^{k\Delta t} \Omega_u(i,:)dW_u^S$ given by

$$\left(\begin{matrix} \int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)dW_u^S \\ \int_{(k-1)\Delta t}^{k\Delta t} \Omega_u(i,:)dW_u^S \end{matrix} \middle| W = a \right) \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \tag{A.4}$$

where

$$\boldsymbol{\mu} = \mu^{(1)} + \bar{\Sigma}_{12}\bar{\Sigma}_{22}^{-1}(W - \mu^{(2)}) = \frac{a}{\int_0^T \|\pi_u^T \sigma_u\|^2 du} \begin{pmatrix} \int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)\sigma_u \pi_u du \\ \int_{(k-1)\Delta t}^{k\Delta t} \Omega_u(i,:)\sigma_u \pi_u du \end{pmatrix},$$

and

$$\boldsymbol{\Sigma} = \bar{\Sigma}_{11} - \bar{\Sigma}_{12}\bar{\Sigma}_{22}^{-1}\bar{\Sigma}_{21} = \begin{pmatrix} \int_{(n-1)\Delta t}^{n\Delta t} \|\Omega_u(i,:)\|^2 du & 0 \\ 0 & \int_{(k-1)\Delta t}^{k\Delta t} \|\Omega_u(i,:)\|^2 du \end{pmatrix}$$

$$- \frac{1}{\int_0^T \|\pi_u^T \sigma_u\|^2 du} \begin{pmatrix} \left(\int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)\sigma_u \pi_u^T du\right)^2 & \Lambda_{N,n,k} \\ \Lambda_{N,n,k} & \left(\int_{(k-1)\Delta t}^{k\Delta t} \Omega_u(i,:)\sigma_u \pi_u^T du\right)^2 \end{pmatrix}$$

with $\Lambda_{N,n,k} = \left(\int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)\sigma_u \pi_u^T du\right)\left(\int_{(k-1)\Delta t}^{k\Delta t} \Omega_u(i,:)\sigma_u \pi_u^T du\right)$.

The conditional distribution $(A.2)$ and the conditional covariance of $\int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)dW_u^S$ and $\int_{(k-1)\Delta t}^{k\Delta t} \Omega_u(i,:)dW_u^S$, $n \neq k$ can be directly read from this joint distribution.

For the last assertion, we can check $\int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)dM_u$, $1 \leq n \leq N$ defined in (A.3) is normal distributed with the following expectation, variance and covariance:

$$E\left[\int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)dM_u\right] = \frac{a \cdot \int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)\sigma_u \pi_u du}{\int_0^T \|\pi_u^T \sigma_u\|^2 du},$$

$$Var\left(\int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)dM_u\right) = \int_{(n-1)\Delta t}^{n\Delta t} \|\Omega_u(i,:)\|^2 du - \frac{\left(\int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)\sigma_u \pi_u du\right)^2}{\int_0^T \|\pi_u^T \sigma_u\|^2 du}$$

and

$$Cov\left(\int_{(k-1)\Delta t}^{k\Delta t} \Omega_u(i,:)dM_u, \int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)dM_u\right) = -\frac{\Lambda_{N,n,k}}{\int_0^T \|\pi_u^T \sigma_u\|^2 du}$$

respectively. Thus, the last assertion is concluded, since each pair of $\int_{(n-1)\Delta t}^{n\Delta t} \Omega_u(i,:)dM_u$, $1 \leq n \leq N$ has the joint distribution in $(A.4)$. □

Appendix B

Nomenclature

$\mathbb{N}^+ = \{1, 2, 3, \ldots\}$	set of positive integers
$\mathbb{R}$	set of real numbers
$\mathcal{M}_{d,n}(\mathbb{R})$	set of real $d \times n$ matrices
$\mathcal{M}_d(\mathbb{R})$	set of real $d \times d$ matrices
$GL_d(\mathbb{R})$	set of real invertible $d \times d$ matrices
$S_d(\mathbb{R})$	set of real symmetric matrices
$S_d^+(\mathbb{R})$	set of symmetric, strictly positive definite matrices
$\bar{S}_d^+(\mathbb{R})$	set of symmetric positive semidefinite matrices
Σ^T	transposed matrix of Σ
Σ^{-T}	the inverse matrix of $\Sigma^T \in GL_d(\mathbb{R})$, i.e. $\Sigma^{-T} = \left(\Sigma^T\right)^{-1} = \left(\Sigma^{-1}\right)^T$
$\Sigma^{1/2}$	the unique nonnegative symmetric matrix of $\Sigma \in \bar{S}_d^+(\mathbb{R})$ with $\Sigma^{1/2}\Sigma^{1/2} = \Sigma$
$Tr(\Sigma)$	trace of $\Sigma \in \mathcal{M}_d(\mathbb{R})$
$\sigma(\Sigma)$	spectrum of Σ for $\Sigma \in \mathcal{M}_d(\mathbb{R})$
$diag(S)$, $S \in \mathbb{R}^d$	the diagonal matrix with $(diag(S))_{ii} = S_i$, $1 \leq i \leq d$
$\mathbf{1}$	the d-dimensional vector with $\mathbf{1}_i = 1$, $1 \leq i \leq d$ and $\mathbf{r} = \mathbf{1}r$
$\mathbf{r}$	the d-dimensional vector with $\mathbf{r} = \mathbf{1}r$, $r \in \mathbb{R}$
$\mathbf{1}_A$	the indicator function of the set A
$\stackrel{\mathcal{D}}{\to}$	convergence in distribution
$\prec$	the order relation on $S_d(\mathbb{R})$ with $\Sigma_1 \prec \Sigma_2 \Leftrightarrow \Sigma_2 - \Sigma_1 \in S_d^+(\mathbb{R})$ for $\Sigma_1, \Sigma_2 \in S_d(\mathbb{R})$
$\preceq$	the order relation on $S_d(\mathbb{R})$ with $\Sigma_1 \preceq \Sigma_2 \Leftrightarrow \Sigma_2 - \Sigma_1 \in \bar{S}_d^+(\mathbb{R})$ for $\Sigma_1, \Sigma_2 \in S_d(\mathbb{R})$
$\lVert\Sigma\rVert_1 = \sum_{i=1}^d \sum_{j=1}^n E\lvert\Sigma_{ij}\rvert$	L_1 norm of a real random $d \times n$ matrix Σ and for $\Sigma \in \mathbb{R}^{d\times n}$ we have $\lVert\Sigma\rVert_1 = \sum_{i=1}^d \sum_{j=1}^n \lvert\Sigma_{ij}\rvert$
$\lVert\Sigma\rVert = \sqrt{\sum_{i=1}^d \sum_{j=1}^n E^2[\Sigma_{ij}]}$	L_2 norm of a real random $d \times n$ matrix Σ and for $\Sigma \in \mathbb{R}^{d\times n}$ we have $\lVert\Sigma\rVert = \sqrt{\sum_{i=1}^d \sum_{j=1}^n \Sigma_{ij}^2}$

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