Optimal price setting with observation and menu costs*

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Abstract

We model the optimal price setting problem of a firm in the presence of both information and menu costs. In this problem the firm optimally decides when to collect costly information on the adequacy of its price, an activity which we refer to as a price "review". Upon each review, the firm chooses whether to adjust its price, subject to a menu cost, and when to conduct the next price review. This behavior is consistent with recent survey evidence documenting that firms revise prices infrequently and that only a few price revisions yield a price adjustment. The goal of the paper is to study how the firm's choices map into several observable statistics, depending on the level and relative magnitude of the information vs the menu cost. The observable statistics are: the frequency of price reviews, the frequency of price adjustments, the size-distribution of price adjustments, and the shape of the hazard rate of price adjustments. We provide an analytical characterization of the firm decisions and a mapping from the structural parameters to the observable statistics. We compare these statistics with the ones obtained for the models with only one type of cost. The predictions of the model can, with suitable data, be used to quantify the importance of the menu cost vs. the information cost.

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1 Introduction

We model the optimal price setting problem of a firm in the presence of both information and menu costs. In this problem the firm optimally decides when to collect costly information on the adequacy of its price, an activity which we refer to as a price "review". Upon each review, the firm chooses whether to adjust its price, subject to a menu cost, and when to conduct the next price review. The goal of the paper is to study how the firm's choices concerning several observable statistics depend on the level and relative magnitude of the information vs the menu cost. Among the observable statistics we focus on are: the frequency of price reviews, the frequency of price adjustments, the size-distribution of price adjustments, and the shape of the hazard rate of price adjustments.

Our interest in this question is twofold. First, survey data indicates that firms review the adequacy of their prices infrequently, and that not all price reviews yield a price adjustment. Such a pattern cannot be accounted for by existing menu cost models, where price reviews occur continuously, nor by costly information models, where each price review is also a price adjustment. The model with both information and menu costs naturally accounts for the observed patterns. Second, menu cost and information cost models have different implications for the response to aggregate shocks. For instance models with only observation cost, such as Mankiw and Reis (2002, 2006, 2007), yield "time-dependent" rules, while models with only menu cost, as Golosov and Lucas (2007), yield "state-dependent" rules. Our theory makes a step towards understanding which of these mechanisms is more relevant: with suitable data the model can be used to quantify the magnitude of the menu vs. the information cost.

The starting point of the problem is the static cost of a decision maker, simply modeled as minimizing a quadratic loss function of the difference between their choice variable (which we interpret as the log of the current price) and its bliss point (or target price). We interpret this function as the cost of having a price at a value different from the one that maximizes current profits, i.e. as the negative of the second order expansion on the (log of the) profit function, where the target is the price that maximizes the static profits. We let the target value follow a random walk with drift. We interpret this variation as variation in the marginal cost and marginal benefit of the monopolist, where the drift is the inflation rate, and the innovations are idiosyncratic shocks. We refer to the difference between the current price and the target price as the price gap, which we denote by $\tilde{p}$. Thus the instantaneous loss for a firm is given by $B(\tilde{p}(t))^2$, where B is a parameter that measures the sensitivity of the price gap deviations in units of the objective function.We assume that the decision maker faces two fixed costs to adjust prices. The first is a standard menu cost that applies to any change in prices, which we denote by ψ. The second one is an information cost, that the decision maker must incur to find the value of the target, or equivalently the value of the price gap, which we denote as ϕ. [1] We assume that the decision maker minimizes expected present value of the quadratic losses plus the expected discounted sum of the fixed costs incurred. We consider two different cases on the assumption for the price changing technologies: the first is that an information cost has to be paid every time that there is a price adjustment, but that the agent can observe without changing the price -and hence without paying the menu cost. The second case allows for price changes to be done without observing the price gap - a form of indexation to inflation. We show that if the drift on the target (i.e. the inflation rate) is small enough, the decision rules are the same in both cases, i.e. there is no indexation. In both cases the optimal policy can be found by solving a dynamic programming problem with one state, namely the price gap right after a price review.

We spend most of the time in the characterization of the optimal decision rules and its implications for the case without drift, i.e. with zero inflation. In this case, right after an observation of the price gap $\tilde{p}$, the agent choses whether to pay the menu cost and adjust the price or not. Additionally the agent choses the time until the next observation. We give an analytical characterization for the solution as we let the discount rate and the fixed menu cost to be neglishable.[2] The decision of whether to adjust the price follows a sS rule: price adjustment are made if the price gap right after an observation is larger than a threshold $\bar{p}$, and in this case the price gap is set to zero. If the price gap is is smaller than the threshold $\bar{p}$, there is no price adjustment. The optimal time until the next observation, denoted by $\tau(\tilde{p})$ is also a function of the size of the price gap: in the range that it triggers a price change it is the longest time, denoted as $\hat{\tau}$. In the inaction range the function $\tau(\cdot)$ is an inverted U-shape: it peaks at a zero price gap, attaining $\hat{\tau}$ and is otherwise decreasing in the size of the price gap, namely $\tau(\tilde{p}) = \hat{\tau} - (\tilde{p}/\sigma)^2$, where σ is the standard deviation of the innovations on the idiosyncratic shock on the target price. This is quite intuitive: for price gaps close but smaller than $\bar{p}$, the firms realizes that it is likely to cross the threshold $\bar{p}$ and hence decides to monitor sooner. To summarize, the optimal decision rules can be expressed as a function of two parameters: $\bar{p}$ and $\hat{\tau}$. We show that in turn these two parameters can be expressed in essentially close form solution as simple functions of three structural parameters: the two normalized cost ϕ/B, ψ/B and the innovation variance σ^2. We use this characterization of the decision rules, as well as the dynamics of the price gap, adjustment and observations to compute four statistics: the expected number of price adjustment per unit of time, the expected number of price reviews per unit of time, the invariant distribution of price changes,

[1] If the monopolist faces an isoelastic demand curve and a random marginal cost, using a second order log approximation, the parameter B is a simple function of this elasticity and the fixed costs are interpreted as fraction of a period profit.

[2] We confirm the quality of the approximation with extensive numerical analysis.

and the shape of the hazard rate of price changes. We find these statistics interesting because they can be computed in using available survey and scanner micro data sets.

The model we develop embeds the two polar cases of menu cost and information cost only. We compare our analytical characterization of the mapping from the structural parameters to the firms decision, and to the observable statistics in each of three setups: menu cost only, information cost only, and the case with both costs.[3] Analyzing the outcomes of the different setups we show the aspects for which the predictions of the model with two cost are similar to a "weighted average" of the two polar cases, and the ones where the interaction of the two cost yields novel predictions for e.g. the size distribution of price adjustments and the shape of the hazard-rate function are obtained.

For comparison purposes we first describe the form of the decision rules as well as the implied statistics for the two models with only one cost in the benchmark case. In the information cost only case, price reviews and price adjustments coincide and take place at equally spaced time intervals of length $\hat{\tau}$. Thus, the frequency reviews and adjustment satisfy $n_a = n_r = 1/\hat{\tau}$, which have elasticity 1/2 with respect to $B\sigma^2/\phi$. Since upon review an adjustment takes place, with no drift the price gap is closed to zero, and hence price changes are normally distributed with variance $\sigma^2\phi/B$. In this case the average change in prices in absolute value, $\mathbb{E}[|\Delta p|]$, has an elasticity 1/4 with respect to $\sigma^2\phi/B$. Moreover, the instantaneous hazard rate $\boldsymbol{h}(t)$ as a function of the time t between price changes, is equal to zero until just before $t = 1/n_a$ at which the review and adjustment take place, where it jumps to infinity. The other extreme case is the one when price reviews have no cost, but there is a menu cost ψ to change prices. In this case price reviews can be considered to occur constantly or $\tau(\tilde{p}) = 0$, and hence have infinite frequency, i.e. $n_r = \infty$. The the optimal policy is a standard sS policy, with optimal return point equal to zero, due to the lack of drift. The threshold of the symmetric range of inaction is given by $\bar{p}$, which has elasticity 1/4 with respect to $\sigma^2\psi/B$. By using results from hitting times of a Brownian motion with two barriers we find adjustment frequency n_a as well as the hazard rate. The average number of adjustment per unit of time n_a has elasticity 1/2 with respect to $B/(\sigma^2\psi)$. The instantaneous hazard rate $\boldsymbol{h}(t)$ starts at zero, is strictly increasing, and it asymptote to a value with elasticity 1/2 with respect to $\sigma^2\psi/B$. In the case with menu cost only the time between price changes is random but price changes takes two values $\bar{p}$ and $-\bar{p}$, and hence $\mathbb{E}[|\Delta p|]$ also has elasticity 1/4 with respect to $B/(\sigma^2\psi)$.

The optimal decision rules for the model with two costs combine several of the elements and elasticities with respect to the structural parameters just described. Recall that with

[3] Most, but not all of the analysis in the case of one cost only is in the literature in different papers. We find it useful to collect the known result and to extend some of them in exactly the same framework so that the effect of the different frictions are easier to compare.

two costs, $\tau(\tilde{p}) = \hat{\tau} - (\tilde{p}/\sigma)^2$, in the range of inaction, and otherwise it equals $\hat{\tau}$. Thus the optimal decision rules can be described by two numbers: $\hat{\tau}$ and $\bar{p}$. We find it convenient to describe them first fixing the ratio of the menu cost to the information cost, which we denote as $\alpha = \psi/\phi$. Fixing α, the elasticity of $\hat{\tau}$ with respect to $\phi/(\sigma^2 B)$ is $1/2$, as in the observation cost model. Likewise, fixing α, the elasticity of $\bar{p}$ with respect to $\sigma^2\psi/B$ is also $1/2$, as in the menu cost model. On the other hand, increasing the ratio of the cost information to menu cost $\alpha = \psi/\phi$ increases the threshold $\bar{p}$, and decreases the time until an observation after an adjustment $\hat{\tau}$. We find these results the natural extension of each of the models with only one cost.

Once we have obtained a characterization of the decision rules as a function of the structural parameters, we move to characterize the "observable statistics". We characterize the invariant distribution of the price gap upon a review for a firm following the optimal decision rules. Using this distribution we characterize n_a, the expected number of adjustment per unit of time, as well as the the expected number of reviews per unit of time n_r . We also use this distribution to characterize the distribution of price changes. We use these statistics to make several point:

First, the baseline model with two cost naturally accounts for the observation that firms review the adequacy of their prices more often than they adjust them, since for the price reviews where the price gap fell in the inaction region there is no price adjustment. The feature that price review happen more often than price adjustment has been documented by Fabiani et al. (2007) for the Euro area.[4]

Second, the distribution of price changes has no mass between $-\bar{p}$ and $\bar{p}$, and outside these values it has a density with the same tails as a normal, but with higher values close to $\bar{p}$ or $\bar{p}$. Compare this with the the case of information cost only, which gives a normal distribution, and with the case of menu cost only, which gives a binomial distribution. Additionally, fixing the ratio of the menu to information cost $\alpha = \psi/\phi$, the average size of price changes $\mathbb{E}[|\Delta p|]$ has an elasticity of $1/2$ with respect to $\sigma^2 B/\phi$, or alternatively, $\sigma^2 B/\psi$. Notice that this is the same elasticity than in the case with only one cost. We find the shape of the distribution of price changes interesting because depending on the size of adjustment and observation costs, gives more flexibility to accommodate both relatively large average size of price changes and relatively small price changes, which seem to be in the data as displayed by Klenow and Kryvtsov (2008). Indeed Mankiw and Reis (2010) discuss some of the difficulties of models of price setting under either adjustment or observation costs to simultaneously account for these facts. However, a positive menu cost $\psi > 0$ prevents too small price changes from

[4]See also Greenslade and Parker (2008), et al. (1998) and Alvarez et al. (2006) for survey evidence about the U.K., U.S. and Canada.

occurring. This seems consistent with the evidence in Alvarez et al. (2006) and Cavallo (2009) about the distribution of price changes in different countries.

Third, we find that two ratio of "observable statistics" provide direct information about the ratio of the cost. First we show that n_r/n_a, the ratio between average frequencies of review and adjustment is a monotonic function that only depends on the relative size of observation and adjustment costs $\alpha = \psi/\phi$. This is quite intuitive if one consider the extreme cases, this ratio is 1 for the observation cost only, and infinity for the menu cost only. In addition we also that the ratio between the mode and the average of the size of price changes is a monotonic function of the relative size of observation and adjustment costs α. This result is also quite intuitive considering the extreme cases, since the distribution is normal for the observation cost only model, and binomial for the menu cost mode. These results provide two schemes to identify the size of the two costs using statistics that are, at least in principle, available.

Forth, we derive the instantaneous hazard rate $\boldsymbol{h}(t)$ of the baseline model with two cost. This function shares some properties with the observation cost only, like an initial value of zero for the hazard rate between times $t \in [0,\ \hat{\tau})$, and a spike (infinite) hazard rate at $t = \hat{\tau}$. But unlike this model, it has a finite continuos non-zero hazard rate for higher values of t. Indeed, loosely speaking, the shape of the hazard rate function has some periodicity, in that it looks like a series of non-monotone functions around duration that are multiples of $\hat{\tau}$. The reason for this monotonicity comes from the fact reviews happens at unequal length of time – given by the function $\tau(\cdot)$ and that adjustment happens depending on whether the price gap at time of review is larger than the threshold $\bar{p}$ at the time of an observation. The non-monotonicity of the hazard rate is unique to the model with both costs. We find this feature appealing because most of the studies fail to find evidence for increasing hazard rates, which is the implication for the extreme models with only one cost, as well as for most of the models in the literature.[5]

Most of our analysis is in the context of the model with $\pi = 0$, i.e. zero inflation rate. We briefly consider the effect of inflation on the rest of the decision rules, first keeping the assumption that any price change requires an observations. In this case the optimal return point will naturally be different form the static optimum, i.e. zero price gap, indeed once prices are adjusted the firm will set it to a higher value anticipating the effect of inflation: right after an adjustment the price gap will be positive and roughly equal to the inflation rate π times half of expected time until the next adjustment. To understand the forces that the

[5] Klenow and Kryvtsov (2008) estimate a flat hazard rate on U.S. CPI data, while Nakamura and Steinsson (2008) estimate a downward sloping hazard using a similar dataset but different methodology. Similar evidence is documented in the Euro area by Alvarez et al. (2005). Cavallo (2009) estimates an upward sloping or hump shaped hazard rate for four Latin America countries

drift in the target price brings relative to its volatility, we consider the extreme case where we fixed $\pi > 0$ and let $\sigma \downarrow 0$. In this extreme case, the range of inaction becomes asymmetric, as well as the function $\tau(\tilde{p})$ which becomes decreasing in the range of inaction, anticipating that the price gap has a negative drift equal to $-\pi$. Through numerical computation, we show that these features also show up for small inflation rate, albeit in a tempered form is π is small.

Finally we relax the assumption that price adjustment requires observation. Note that an adjustment without observation is a form of automatic indexation, or what has been called in the literature, a *price path* or a *price plan*. We show that fixing a value of $\psi > 0$, for positive but sufficiently small inflation rate all the adjustment will take place immediately after observation.

Notice that fixing πcase where the inflation rate is Our results are robust to several assumptions about the technology of reviewing and changing prices. In particular, we show that the main results of our paper hold under values of drift in inflation that are of the same order of magnitude of average inflation estimated in modern developed economies. However, we also characterize the optimal policy in the case of relatively large average inflation.

The paper is organized as follows. Section 2 gives a brief survey of the related literature. Section 3 discusses the recent survey evidence on the firms' decisions concerning the frequency of price reviews and price adjustments in a number of countries. Section 4 presents our model of the firm's price setting problem with information and menu costs. The section characterizes the firm optimal policy for each of the two the polar cases (menu or information cost only) and for the general case. An analytical solution for the firm's optimal decision rule in the case of zero inflation is given in Section 5. This rule is used in Section 6 to characterize analytically the model predictions for the frequency of price reviews, the frequency of price adjustments, the size-distribution of price adjustments, and the shape of the hazard rate of price adjustments. A comparison between the results of our model with those produced by the polar cases is given. We discuss how these result match against the recent micro evidence on the distribution of price changes and the shape of the hazard rate of price adjustments. The results developed so far, under the assumption of zero inflation, provide a good benchmark for analyzing cases where inflation is small. This is shown in Section 7, where the case of high inflation is also discussed. Section 8 explores the circumstances in which price adjustment can be made without finding out the information about the state. Proofs and documentation material is given in the Appendix; an set of Online Appendices provides even more details on the models discussed in this paper and the findings in the related literature.

2 Related literature

A vast body of research has studied the price setting decision by firms.[6] In fact, understanding the way firms set the price of their products is an important issue in the macroeconomic literature as, among other things, price setting behavior has direct consequences for the response of an economy to nominal shocks and, therefore, for monetary policy. Despite the large effort, the debate on which model better describes firms' price setting behavior is still very much open.[7] At the centerpiece of this debate is understanding why and how much product prices are rigid in response to economic shocks, and why prices change rather infrequently when compared with predictions of standard models under flexible prices.[8] Two types of frictions have been proposed as main causes behind price rigidity, namely adjustment costs and observation costs.[9]

The key element of the model of price setting under adjustment cost is the state dependence of individual decisions: the agent acts when the state crosses some critical threshold, balancing cost and benefits of adjustment. Within this literature, Dixit (1991) has studied the problem of a firm that faces a state given by a random walk process. Dixit (1991)' s results provide a natural framework to analyze the properties of this class of models. In fact, our model nests Dixit (1991) in the spacial case of no observation cost. Similarly to our model, the model with adjustment cost only implies that the firm adjusts its price only infrequently. In particular, the frequency of price adjustment decreases in the size of the adjustment threshold, while increases in the volatility of the state. However, from the perspective of models of price adjustment under 'menu' cost, our paper contributes to the existing literature on several important dimensions. First, not surprisingly, the model with both types of costs is able to generate infrequent price adjustments as well as infrequent price reviews, with the latter taking place more often than the former.[10] Second, quite interestingly, our model naturally generates shapes for the distribution of price changes as well as for the hazard rate of price adjustments that are more consistent with existing empirical evidence. [11]

[6] For a review of the literature see, for instance, Golosov and Lucas (2007), Klenow and Malin (2010), Mankiw and Reis (2010) and references therein.

[7] See, for instance, Klenow and Malin (2010) and Mackowiak and Smets (2008).

[8] See Christiano et al. (1998), Boivin et al. (2009) and Mackowiak et al. (2009) for empirical evidence.

[9] See Sheshinski and Weiss (1977, 1979, 1983), Barro (1972) and Dixit (1991) for a review of early results on price setting behavior under costly price adjustments;

[10] See Section 3 for survey evidence about the relationship between frequency of price adjustment and review.

[11] For instance, Klenow and Kryvtsov (2008) document that price changes are usually big in absolute terms (averaging around 10 percent), although a large subset are much smaller (5 percent or less). There is disagreement on the shape of the hazard rate: while Klenow and Kryvtsov (2008) report a flat hazard rate, by looking at the same data but with a different econometric methodology, Nakamura and Steinsson (2008)

On the other side, our model directly compares to the existing literature studying price setting decisions under imperfect information. This literature dates back to the seminal work by Phelps (1969) and Lucas (1972). In these models, the speed with which prices respond to changes in the state relates to the speed with which information about this shock is embedded in the price setting decision. This literature has recently been revisited by several authors looking for alternative models of price setting in order to account for the sluggish response of prices to nominal shocks. In particular, Reis (2006) and Mankiw and Reis (2002, 2006, 2007) have modeled imperfect information as arising from a fixed cost of observing the state.[12] In these models, the firm reviews the state infrequently and, due to the absence of any adjustment cost, adjust prices anytime the state is reviewed. There is no arrival of new information in between times of reviews. Reis (2006) shows that, in standard frameworks, the optimal rule is to review and, contemporaneously, adjust prices at a constant frequency. By including both observation and adjustment costs, our model contributes to this literature on several dimensions. First, consistently with the survey evidence, our model implies that not all price reviews yield an adjustment. Second, our model implies distribution and hazard rate of price adjustments that are more consistent with existing empirical evidence. In particular, our model avoids infinitesimally small price adjustments from occurring, and allows for richer shapes of the hazard rate of price adjustments than the model with observation cost only.[13] In addition, in our model the optimal time between reviews is state dependent.

More generally, our model with both observation and adjustment costs contributes to the existing literature by providing a natural framework within which evaluating and quantifying the impact of each type of cost on the price setting decision.

Finally, there is a recent literature that studies price setting decisions, and their aggregate consequences, within models that incorporates both some combination of imperfect information and sticky prices.[14] However, in most of these studies either the frequency of price

favors a downward sloping shape. See Alvarez et al. (2006) for equivalent evidence on the Euro area, and Cavallo (2009) for evidence on developing countries. Midrigan (2007) has proposed a multi-product setting in which firms face economies of scope in the technology of adjusting prices to justify contemporaneously larger average size of price adjustments but many small price changes; Nakamura and Steinsson (2008) allow for heterogeneity in the frequency of price adjustment across different products to account for the downward sloping hazard rate. See Section 5 for more details.

[12]Sims (2003), Woodford (2001) and Makowiak and Wiederholt (2009) model information flows as depending on a signal about the underlying state which realizes every period. In these models, what prevents the firm from perfectly observing the state in every period is limited information processing capabilities. Their framework implies that the firms sets the price on the basis of partial information on the current state, and changes the price every period. Therefore, while these models generate different speeds of price adjustment to different types of shocks, depending on how much information the firm processes about each of them, they cannot account, in absence of other frictions, for the infrequent price reviews, adjustments and, in particular, for the positive gap.

[13]See Section 5 for more details.

[14]See, for instance, Nimark (2008), Morris and Shin (2006), Angeletos and La'O (2009), Gertler and Leahy

adjustment or review is exogenously given. By endogenizing both decisions of reviewing the information and adjusting the price, our model allows to capture the interaction and existing complementarity between these two frictions. Similarly to this paper, Gorodnichenko (2008) studies a model where each firm has to pay a fixed cost to acquire information and a fixed cost to change the price. Differently from this paper, Gorodnichenko (2008) studies the price setting problem within a general equilibrium framework where firms observe the past realization of the aggregate price level at no cost. While this more general framework allows to address some important questions such as the response of inflation and output to nominal shocks, it implies that the average frequency of price adjustment is larger than the average frequency of information acquisition. The latter is at odds with survey evidence by Fabiani et al. (2006). Moreover, the simple framework of this paper allows for a quasi-analytical solution of the model which fully characterizes the mapping from observable statistics to the costs of acquiring information and adjusting the price.

3 Evidence on Price Adjustments vs. Price Reviews

Several recent studies measure two distinct dimensions of the firm's price management: the frequency of price reviews, or the decision of assessing the appropriateness of the price currently charged, and the frequency of price changes, i.e. the decision to adjust the price. The typical survey question asks firms: "In general, how often do you review the price of your main product (without necessarily changing it)?"; with possible choices yearly, semi-yearly, quarterly, monthly, weekly and daily. The same surveys contain questions on frequency of price changes too. Fabiani et al. (2007) survey evidence on frequencies of reviews and adjustments for different countries in the Euro area, and Blinder et al. (1998), Amirault et al. (2006), and Greenslade and Parker (2008) present similar evidence for US, Canada and UK. This section uses this survey data to document that the frequency of price reviews is larger than the frequency of price changes. We believe that the level of both frequencies, especially the one for reviews, are measured very imprecisely. Yet, importantly for the theory presented in this paper, we have found that in all countries, and in almost all industries in each country, and for almost all the firms in several countries, the frequency of review is consistently higher than the frequency of adjustment. We will argue that, given our understanding of the precision of the different surveys, the most accurate measures of the ratio of price reviews to price adjustments per year are between 1 and 2. In the rest of this section we document this fact, referring the interested reader to the Online Appendix D for more documentation.

The upper panel of Table 1 reports the median frequency of price reviews and the median

(2008) and Gorodnichenko (2008). See Mankiw and Reis (2010) for a review.

Table 1: Price-reviews and price-changes per year

	AT	BE	FR	GE	IT	NL	PT	SP	EURO	CAN	UK	US
							Medians					
Review	4	1	4	3	1	4	2	1	2.7	12	4	2
Change	1	1	1	1	1	1	1	1	1	4	2	1.4
							Mass of firms (%) with at least 4 reviews/changes					
Review	54	12	53	47	43	56	28	14	43	78	52	40
Change	11	8	9	21	11	11	12	14	14	44	35	15

Number of changes and reviews per year. The sources for the medians are Fabiani et al. (2007) 2003 Euro area survey, Amirault et al. (2006) 2003 Canadian survey, Greenslade and Parker (2008) 2008 UK survey. See the Online Appendix D for a discussion of the sources and the measurement issues involved.

frequency of price adjustments across all firms in surveys taken from various countries. The median firm in the Euro area reviews its price a bit less than three times a year, but changes its price only about once a year, and similar for UK and US. [15]

These surveys collect a wealth of information on many dimensions of price setting, well beyond the ones studied in this paper. Yet, for the questions that we are interested in, the survey data from several countries have some drawbacks. We think that, mostly due to the design of these surveys, the level of the frequencies of price review and price adjustments are likely subject to a large amount of measurement error. One reason is that in most of the surveys firms were given the following choices for the frequency of price reviews: yearly, quarterly, monthly and weekly (in some also semi-yearly and less than a year). It turns out that these bins are too coarse for a precise measurement, given where the medians of the responses are. For example, consider the case where in the population the median number of price reviews is exactly one per year, but where the median number of price changes is strictly larger than one per year. Then, in a small sample, the median for reviews will be likely 1 or 2 reviews a year, with similar likelihood. Instead the sample median for number of adjustments per year is likely to be one. From this example we remark that the median for price reviews is imprecisely measured, as its estimates fluctuates between two values that are one hundred percent apart. The configuration described in this example is likely to describe several of the countries in our surveys.[16] Another reason is that in some cases the sample

[15] This evidence about the frequency of price adjustment is roughly consistent with previous studies at the retail level. See Alvarez et al. (2006) for more details.

[16] For example in Portugal the median frequency of review is 2, but the fraction that reviews at one year or less is 47%, while for price adjustment the median is one and the fraction of firms adjusting exactly once a year is 49.5%, see Martins (2005). In the UK for 1995 the median price review is 12 times a year, but the fraction that reviews at most 4 times a years is about 46%, while for price adjustment the median is 2 and the fraction of firms adjusting 2 times or less a year is 66%, see Hall et al. (2000). Indeed, consistent with

size is small. While most surveys are above one thousand firms, the surveys for Italy has less than 300 firms and the one for the US has about 200 firms. Yet another difficulty with these measures is that several surveys use different bins to classify the frequency of price reviews and that one price changes. For instance in France and Italy firms are asked the average number of changes, instead of being given a set of bins, as is the case for the frequency of reviews.[17]

The bottom panel of Table 1 reports another statistic that is informative on the relative frequency of reviews and adjustments: the fraction of firms reviewing and changing respectively their price at least four times a year. We see this statistic as informative, and less subject to measurement error, because this frequency bin appears in the questionnaire for both the review and the adjustment decision for almost all countries. It shows that the mass of firms reviewing prices at least four times a year is substantially larger than the corresponding one for price changes.

Figure 1: Average industry frequency of price changes vs. adjustments

Note: data for each dot are the mean number of price changes and reviews in industry j in country i.

Figure 1 plots the average number of price reviews against the average number of price

our hypothesis of measurement error, in a similar survey for the UK for year 2007-2008, the median price review is 4 and the median price adjustment is 4, see Greenslade and Parker (2008).

[17]Furthermore for Germany firms where asked whether or not they adjusted the price in each of the preceding 12 months; this places an upper bound of 12 on the frequency of adjustments, while no such restriction applies to the number of reviews.

adjustments across a number of industries in six countries. This figure shows that in the six countries the vast majority of the industry observations lies above the 45 degree line, where the two frequencies coincide. Most of the industries for Belgium, Spain and the UK have a ratio of number of reviews per adjustment between 1 and 2 (i.e. lies between the two lower straight lines). The data for France, Italy and Germany has much higher dispersion in this ratio. We believe that the reason of the higher dispersion for Italy, Germany and France is due to the measurement error discussed above. Our belief is based on the fact that the questionnaire in the surveys for Belgium and Spain treat price reviews and price changes symmetrically and they record the average frequencies as an integer as opposed to a coarse bin.

For four countries Table 2 classifies the answers of each firm on three mutually exclusive categories: 1) those that change their prices more frequently than they review them, 2) those that change and review their prices at the same frequency, 3) and those that change their prices less frequently than they adjust them. Table 2 shows that most of the firms respond that they review their prices at frequencies greater or equal than the one in which change their prices. We conjecture that the percentage of firms in catergory 1, i.e. those changing the price more frequently than reviewing it, is actually even smaller than what is displayed in the table due to measurement error.

Table 2: Frequency of Price Changes and Reviews at Firm Level

	Belgium	France	Germany	Italy	Spain*
Percentage of Firms with:					
1) Change > Review	3	5	19	16	0
2) Change = Review	80	38	11	38	89
3) Change < Review	17	57	70	46	11
N of Observations (firms)	890	1126	835	141	194

* For Spain is only for firms that review four or more times a year. Sources: Table 17 in Aucremanne and Druant (2005) for Belgium, and our calculations based on the individual data described in Loupias and Ricart (2004), Stahl (2005), and Fabiani et al. (2004) for France, Germany, and Italy. For Spain from section 4.4 of Alvarez and Hernando (2005). See Online Appendix D for details.

4 A price setting problem

We consider a decision maker who faces a period return function given by $-B\ (p(t) - p^*(t))^2$ where $p(t)$ is a decision for the agent and $p^*(t)$ is random target, i.e. the optimal value that she will set with full knowledge of the state of the problem. The target changes stochastically, and we assume that the agent must pay a fixed cost ϕ to observe the state $p^*(t)$, and that she maximizes expected discounted values. The constant B measures of the strength of the cost of the deviations from the target relative to ϕ. Moreover, it is assumed that the agent faces a physical cost ψ associated with resetting the price (a "menu cost").

The target price $p^*(t)$ follows a random walk with drift π, with normal innovations with variance σ^2 per unit of time, or

$$p^*(T+t) = p^*(T) \ + \pi \ t \ + s \ \sigma \ \sqrt{t} \ , \tag{1}$$

where s is a standard normal, so that $\mathbb{E}_T \left[\ p^*(T+t) \ \right] = p^*(T) + \pi \ t$, $Var_T \left[\ p^*(T+t) \ \right] = \sigma^2 \ t$, and $\mathbb{E}_T \left(p(T) - p^*(T+t)\right)^2 = \left(p\left(T\right) - p^*\left(T\right) - \pi t\right)^2 + t \ \sigma^2$. In the case where the firm sets the price level $p(T)$ in nominal terms, the drift π can be interpreted as the inflation rate, since $p^*(t)$ denotes a real variable.[18]

We index the times at which the agent chooses to pay the cost ϕ and observe the state by T_i, with $0 = T_0 < T_1 < T_2 < \cdots$. After observing the value of the state at a date $t = T$, the agent decides whether to pay the cost ψ and adjust the value of its control variable $p(t)$ that will apply for the times $t \in (T_i, T_{i+1})$.

Appendix A discusses two simple cases that give rise to the quadratic return function given above: a linear demand case, and a second-order log approximation of the profit function. In these cases $p(t)$ is either the price or the log-price of a monopolistic firm, and $p^*(t)$ its corresponding static optimal level. In the linear demand case the changes in $p^*(t)$ are due to both shifts in the demand intercept a and changes in the marginal cost c. In the log-approximation of the widely used Dixit-Stiglitz monopolist problem with constant demand elasticity $\eta > 1$, they are due to changes in the (log of the) marginal cost c. The parameter B in the flow return is given by half of the second derivative of the profit function, so it is given by $B = b$ the absolute value of slope of the linear demand, and by $B = \frac{1}{2} \ \eta \ (\eta - 1)$, in the Dixit-Stiglitz case. In the linear case the fixed cost ϕ is an additive term to the objective function. In the case where the return function is measured in logs, the cost ϕ is measured as a proportion of profits per unit of time. In the Dixit-Stiglitz example, revenues are η times profits, a relationship that we use to quantify the fixed cost. Finally, to conserve on notation, our quadratic expression for the firm profits ignores the terms that do not depend on $p(t)$, since obviously these do not affect the firm's choice.[19]

We make two comments about our assumption about the 'technology' to change prices.

[18] Another interpretation is that π describes a drift in the target $p^*(t)$ due to, say, a trend in productivity or a product life-cycle.

[19] The simplification of using a quadratic approximation to the profit function has been used in the seminal work on price setting problem and aggregation by Caplin and Spulber (1987) and Caplin and Leahy (1991, 1997), as discussed by Stokey (2008).

First, for most of the paper we impose that at the time of an adjustment the agent must also observe, i.e. it must pay the observation cost. Most of the paper treats the case where the target price $p^*(t)$ has no drift, i.e. the no inflation case, in which even if agents could adjust without observing they will not do so. Instead if the drift is large enough relative to the volatility, it will be optimal to sometimes adjust without observing. We discuss this issue in Section 8. Second, we assume that, after paying the adjustment cost, firms set a price that stays fixed until the next adjustment. In Section 4.1 we briefly comment on the possibility that the firm sets a path for the prices rather than a value.

For a better understanding the nature of the problem with both observation and menu costs, and to relate to the literature, the next two subsection discuss the polar cases in which there is an observation cost only ($\phi > 0$, $\psi = 0$) and a menu cost only ($\phi = 0$, $\psi > 0$). We then discuss the problem in the presence of both costs.

4.1 Price Setting with observation cost

In the simpler case where the menu cost is zero ($\psi = 0$) the agent solves the following cost minimization problem:

$$V_0 \equiv \min_{\{T_i, p(T_i)\}_{i=0}^{\infty}} \mathbb{E}_0 \left[\sum_{i=0}^{\infty} e^{-\rho T_i} \left(\phi + B \int_{T_i}^{T_{i+1}} e^{-\rho(t-T_i)} \mathbb{E}_{T_i} \left(p(T_i) - p^*(t) \right)^2 dt \right) \right] \tag{2}$$

where, without loss of generality, we are starting at time $t = 0$ being an observation date, so that $T_0 = 0$.

The term $e^{-\rho T_i}\phi$ is the present value of the cost paid to observe the state $p^*(T_i)$ at time T_i. The term $\int_{T_i}^{T_{i+1}} e^{-\rho(t-T_i)} \mathbb{E}_{T_i} \left[B \left(p(T_i) - p^*(t) \right)^2 \right] dt$ is the integral of the present value of the expected profits after observing the state at time T_i and setting the price to $p(T_i)$ to be maintained from time T_i until the new observation date T_{i+1}. The conditioning set is all the information available up to T_i. The expectation outside the sum of the maximization problem conditions on the information available at time zero. The notation of this problem assumes that a price adjustment can only happen at the time of a price review. This is the assumption we employ in most of the paper, except for Section 8 were we relax it.

Given two arbitrary T_i and T_{i+1}, define:

$$x \equiv \frac{p(T_i) - p^*(T_i)}{\pi} \quad \text{and} \quad v(T_{i+1} - T_i) \equiv \min_x \int_0^{T_{i+1}-T_i} e^{-\rho t} \left(x - t \right)^2 dt \quad . \tag{3}$$

Two comments are in order. First, after observing the value of $p^*(T_i)$, the firm optimal pricing decision x concerns the new price in excess of $p^*(T_i)$ measured in units of the drift π.

Second, instead of setting a constant price we might consider letting the firm set a path for $p(t)$ for $t \in [T_i \ , \ T_{i+1})$. It is clear from the objective function that the firm would then choose $p(t) = p^*(T_i) + \pi t$, and hence the minimized objective function would be identically zero, i.e. $v(T_{i+1} - T_i) = 0$. Mechanically, this has the same effect on the solution of the problem in this section of setting the drift of the state to zero, i.e. setting the inflation rate to $\pi = 0$. Thus we can interpret the model with $\pi = 0$ as a problem in which the firm is allowed to set a path for $p(t)$. Alternatively, we can also consider the problem where the cost ϕ applies for a price review, where the firm finds out the value of $p^*(t)$, but that it has no cost of changing prices without gathering any new information. In this case the optimal policy is also to change the prices between reviews at the rate π per unit of time. Hence, in either of these alternative scenarios the solution of the firm's problem is equivalent to the one we present in this section, but setting the drift of prices to zero, i.e. $\pi = 0$. We will revisit this issue in Section 8, but we return to the assumption that prices can only be changed in a review.

The first order condition with respect to x of the function $v(\tau)$, defined in equation (3), gives the optimal reset price[20]

$$x^* = \int_0^\tau t \ \frac{\rho e^{-\rho t}}{1 - e^{-\rho \tau}} \ dt = \frac{-\tau e^{-\rho \tau}}{1 - e^{-\rho \tau}} + \frac{1}{\rho} \quad .$$

Using the function v we write the firm problem as:

$$V_0 \equiv \min_{\{T_i\}_{i=0}^\infty} \left[\sum_{i=0}^\infty e^{-\rho T_i} \left(\phi + B \ \pi^2 \ v(T_{i+1} - T_i) + B \ \int_0^{T_{i+1} - T_i} e^{-\rho t} \ t \ \sigma^2 dt \right) \right] \quad . \tag{4}$$

Comparing equation (2) with equation (4) we notice that in the second expression we have solved for the expected values, and we have also subsumed the choice of the price into the function v. We can write this problem in a recursive way by letting $\tau \equiv T_{i+1} - T_i$ be the time between successive observation-adjustment dates:

$$V = \min_\tau \ \left[\phi + B \ \pi^2 \ v(\tau) + B \ \int_0^\tau e^{-\rho t} \ t \ \sigma^2 dt \ + e^{-\rho \tau} V \right] \ ,$$

where we use the result that the history of the shocks up to that time is irrelevant for the optimal choice of τ, given our assumptions on $p^*(t)$. The optimal time between observations (and adjustments) solves:

$$V = B \ \sigma^2 \ \min_\tau \frac{\tilde{\phi} + \ v(\tau) \ \frac{\pi^2}{\sigma^2} + \ \int_0^\tau e^{-\rho t} \ t \ dt}{1 - e^{-\rho \tau}} \quad , \quad \text{where} \quad \tilde{\phi} \equiv \frac{\phi}{B \ \sigma^2} \quad , \tag{5}$$

[20] Using integration by parts gives $\int_0^\tau e^{-\rho t} \ t \ dt = \frac{-\tau e^{-\rho \tau}}{\rho} + \frac{1 - e^{-\rho \tau}}{\rho^2}$.

a problem defined by three parameters: $\tilde{\phi}$, $(\pi/\sigma)^2$ and ρ.

Using this setup, the next proposition provides an analytical characterization of the optimal length of the inaction period for the case of small discounting ($\rho \to 0$) and for the case without drift ($\pi = 0$). Both cases provide very accurate approximation of the solution with non-zero drift and discounting provided these are small, as is likely the case in our data.

Proposition 1. The optimal decision rule τ^* for the time between observations when $\rho \to 0$ is a function of two arguments, $\tau^*\left(\frac{\phi}{B\ \sigma^2}\ ,\ \frac{\pi^2}{\sigma^2}\right)$, with the following properties:

1. τ^* is increasing in the normalized cost $\phi/(B\sigma^2)$, decreasing in the normalized drift π/σ, and decreasing in the innovation variance σ^2

2. The elasticity of τ^* with respect to $\phi/(B\sigma^2)$ is:
 $1/2$ as $\phi/B \to 0$, or $\pi = 0$;
 $1/3$ for $\sigma = 0$.

3. For $\pi = 0$ and $\rho > 0$, then τ^* solves $\tilde{\phi} = \frac{1}{2}\tau^2 - \frac{1}{6}\rho\tau^3 + o(\rho^2\tau^3)$, so that τ^* is increasing in ρ provided that ρ or $\tilde{\phi}$ are small enough; for $\rho = 0$ we obtain the square root formula: $\tau^* = \sqrt{2\ \tilde{\phi}}$.

The proposition shows that for small values of the (normalized) observation cost $\tilde{\phi}$ the square root formula gives a good approximation, so that second order costs of observation gathering give rise to first order spells of inattention. Two special cases are worth mentioning. First the case with zero drift, i.e. $\pi = 0$, which gives a square root formula on the cost $\tilde{\phi}$ and with elasticity -1 on σ. As we discussed above, the $\pi = 0$ case can be interpreted as a setting where the firm is allowed to set a path for its price. The other special case of interest is when there is no uncertainty, i.e. $\sigma = 0$. We can think of this as a limiting case of the observation problem. When the uncertainty is tiny, the rule becomes even more inertial, switching from a square to a cubic root. Alternatively, this can be reinterpreted as a deterministic model with a physical menu cost of price setting (equal to ϕ), where the firm's price drifts away from the optimal level due to the inflation trend.

For future reference we notice that in the case of no drift ($\pi = 0$), and when we let $\rho \downarrow 0$, we have that the time between observations/adjustments τ^*, its reciprocal, the number of observations/adjustments per unit of time n, and the average size of price adjustments denoted by $\mathbb{E}[\ |\Delta p|\ ]$, and given by the absolute value of normal with zero mean and standard deviation $\sigma\sqrt{\tau^*}$

$$
\begin{aligned}
n \equiv \frac{1}{\tau^*} &= \sqrt{\frac{\sigma^2 B}{2\ \phi}} \ \text{ and } \ \mathbb{E}[\ |\Delta p|\ ] = \left[\frac{2\ \sigma^2\ \phi}{B}\right]^{\frac{1}{4}} \sqrt{2/\boldsymbol{\pi}}\ , \\
\boldsymbol{h}(t) &= 0 \text{ for } t \in [0, \tau^*], \text{ and } \boldsymbol{h}(\tau^*) = \infty\ .
\end{aligned} \tag{6}
$$

where $\boldsymbol{h}(t)$ is the instantaneous hazard rate of a price adjustment as a function of t, the time elapsed since the last price change.

While the nature of our approximation is different, several of the conclusions of this proposition confirm previous findings in Reis (2006). In particular, in his Proposition 4 the approximate optimal solution for the inaction interval follows a square root formula, just like we obtain under point 2. Also, as in his Proposition 5, the length of the inaction intervals is decreasing in the variance of the innovations, and increasing in the (normalized) cost of adjustment, as we find under point 1. Finally in this setup the length of the inaction spells is constant, as in the special case that Reis discusses in his Proposition 5. In both our and his model the reason behind this result is that the state follows a brownian motion and that the level of the value function upon adjustment is independent of the state.[21]

4.2 Price Setting with menu cost

In this section we assume the firm observes the state p^* without cost, i.e. $\phi = 0$, but that it must pay a fixed cost ψ to adjust prices. This is the standard menu cost model. The firm observes the underlying target value $p^*(t)$ continuously but acts only when the current price, $p(t)$, is sufficiently different from it, i.e. when the deviation $\tilde{p}(t) \equiv p(t) - p^*(t)$, is sufficiently large. Thus optimal policy is characterized by a range of inaction. Using $\hat{p} \equiv p(T_i) - p^*(T_i)$ to denote the optimal reset price at the time of adjustment, and using the law of motion for p^* in equation (1), the evolution of the price deviation is $\tilde{p} = \hat{p} - \pi\ t - s\sigma\sqrt{t}$.

The Hamilton-Jacobi-Bellman equation in the range of inaction $\tilde{p} \in [\underline{p}\ ,\ \bar{p}]$ is:

$$\rho V(\tilde{p}) = B\ \tilde{p}^2 - V'(\tilde{p})\ \pi + \frac{1}{2} V''(\tilde{p})\ \sigma^2\ . \tag{7}$$

The optimal return point is:

$$\hat{p} = \arg\ \min_{\tilde{p}} V(\tilde{p}) \quad \Longrightarrow \quad V'(\hat{p}) = 0\ , \tag{8}$$

[21] In the sense that the state only affects the value function level but not its shape, so that the optimal choice of the inaction interval does not depend on it.

and the boundary conditions are given by

$$V(\bar{p}) = V(\hat{p}) + \psi \quad \text{and} \quad V'(\bar{p}) = 0 \ , \tag{9}$$

$$V(\underline{p}) = V(\hat{p}) + \psi \quad \text{and} \quad V'(\underline{p}) = 0 \ . \tag{10}$$

While the standard way to solve this problem is to use the close form solution of the ODE and use the boundary conditions to obtain an implicit equation for $\bar{p}$, we pursue an alternative strategy that will be useful to compare with the solution for the problem with both cost. Since for $\pi = 0$ the value function is symmetric and attains a minimum at $\tilde{p} = 0$, we use the following fourth order approximation to $V(\cdot)$ around zero:

$$V(\tilde{p}) = V(0) + \frac{1}{2}V''(0) \ (\tilde{p})^2 + \frac{1}{4!}V''''(0) \ (\tilde{p})^4 \quad \text{for all } \tilde{p} \in [-\bar{p} \ , \ \bar{p}], \tag{11}$$

where the symmetry arond zero implies $V'(0) = V'''(0) = 0$. Note that the boundary conditions equation (8), equation (9), and equation (10), imply that $V(\cdot)$ is convex around $\tilde{p} = 0$ but concave around $-\bar{p}$ and $\bar{p}$. Thus a forth order approximation is the smaller order that we can use to capture this, with $V''(0) > 0$ and $V''''(0) < 0$. The approximation will be accurate for small values of ψ, since in this case the range of inaction is small.

Proposition 2. Given $\pi = 0$, the width of the range of inaction, $2 \ \bar{p}$, for small $\psi\sigma^2/B$ and ρ is approximately given by :

$$\bar{p} = \left(\frac{6 \ \sigma^2 \ \psi}{B} \right)^{1/4} \ . \tag{12}$$

The result for the quartic root is essentially the one in Dixit (1991), who obtained it through a different argument. The approximation of equation (12) is very accurate for a large range of values of the cost ψ.

In this case price adjustments take two values only, and hence the average size of price changes is $\mathbb{E}[\ |\Delta p|\ ] = \bar{p}$. Instead, the number of adjustments per unit of time is more involved, but it can be computed using the function defining the expected time until adjustment, $\mathcal{T}(\tilde{p})$, i.e. the expected value of the time until $\tilde{p}$ first reaches $\bar{p}$ or $\underline{p}$. The average number of adjustments, denoted by n, is then $1/\mathcal{T}(\hat{p})$. The function $\mathcal{T}(\tilde{p})$ satisfies the o.d.e.:

$$0 = 1 - \pi \ \mathcal{T}'(\tilde{p}) + \frac{\sigma^2}{2}\mathcal{T}''(\tilde{p}) \quad \text{and} \quad \mathcal{T}(\underline{p}) = \mathcal{T}(\bar{p}) = 0 \ . \tag{13}$$

For the case of $\pi = 0$ the solution is $\mathcal{T}(\tilde{p}) = \frac{\bar{p}^2 - \tilde{p}^2}{\sigma^2}$. Hence the average number of adjustments per unit of time n satisfies $n \equiv \frac{1}{\mathcal{T}(0)} = \frac{\sigma^2}{\bar{p}^2}$. Indeed, the distribution of the first times between subsequent price adjustment is known in closed form, and hence one can use it to characterized $\boldsymbol{h}(t)$, the instantaneous hazard rate of a price change as a function of t the time elapsed since the last price change (see Appendix B for details). Future reference we have

$$
\begin{aligned}
n &= \frac{\sigma^2}{\bar{p}^2} = \sqrt{\frac{\sigma^2 \ B}{6 \ \psi}} \ \ , \ \ \mathbb{E}[\ |\Delta p|\ ] = \left[\frac{6 \ \sigma^2 \ \psi}{B}\right]^{\frac{1}{4}} \ \text{ and } \ \boldsymbol{h}(0) = 0 \ , \\
\boldsymbol{h}'(t) &\geq 0, \ \boldsymbol{h}(\cdot) \text{ is convex-concave with } \lim_{t\to\infty} \boldsymbol{h}(t) = \frac{\boldsymbol{\pi}^2}{8} \frac{\sigma^2}{\bar{p}^2} = \frac{\boldsymbol{\pi}^2}{8} \sqrt{\frac{\sigma^2 \ B}{6 \ \ \psi}} \ .
\end{aligned}
$$

where we use the approximation for $\bar{p}$.

4.3 Price Setting with observation and menu costs

In this section we consider the problem where the decision maker faces two costs: an observation cost ϕ that is paid to observe the state $p^*(t)$, and a fixed cost ψ that is paid to change the price. We first write the corresponding sequence problem:

$$
\begin{aligned}
V(p_0) \ &\equiv \\
\min_{\{T_i, p(T_i), \chi_{T_i}\}_{i=0}^{\infty}} \mathbb{E}_0 \Bigg\{ \sum_{i=0}^{\infty} e^{-\rho T_i} \Big[\phi \ &+ \ (1-\chi_{T_i}) \ B \int_{T_i}^{T_{i+1}} e^{-\rho(t-T_i)} \mathbb{E}_{T_i} \left(p(T_{i-1}) - p^*(t)\right)^2 dt \\
&+ \ \chi_{T_i} \ \Big(\ \psi + B \int_{T_i}^{T_{i+1}} e^{-\rho(t-T_i)} \mathbb{E}_{T_i} \big(p(T_i) - p^*(t)\big)^2 \ dt \Big) \ \Big] \Bigg\}
\end{aligned}
\tag{14}
$$

where, as in Section 4.1, the sequence $\{T_i\}$ denotes the stopping times for the observation of the state $p^*(t)$, and where χ_{T_i} is an indicator that the agent will pay the menu cost ψ and adjust the price to $p(T_i)$ so that prices evolve according to:

$$
\begin{aligned}
p(t) \ &= \ p(T_{i-1}) \text{ for } t \in [T_i, T_{i+1}) \ \text{ if } \chi_{T_i} = 0 \ , \\
p(t) \ &= \ p(T_i) \quad \text{ for } t \in [T_i, T_{i+1}) \ \text{ if } \chi_{T_i} = 1 \ .
\end{aligned}
\tag{15}
$$

where, without loss of generality, we are starting at time $t = 0$ being an observation date, so that $T_0 = 0$. As in the problem of Section 4.1, the notation $\mathbb{E}_{T_i}(\cdot)$ denotes expectations conditional on the history of $\{p^*(s)\}$ up to $t = T_i$. The process for $\{p^*(t)\}$ follows a random walk with drift as in equation (1), so the current value of the process is a sufficient statistics for the distribution of its future realizations. Finally notice that the objective function is homogeneous of degree one in (B, ϕ, ψ). Thus the optimal policy will be a function only of the parameters $(\rho, \pi, \phi/B, \psi/B, \sigma^2)$, so that we can normalize $B = 1$ without loss of generality where we find it convenient.

Following the same steps as in Section 4.1, we write the Bellman equation for the firm's problem excluding all the terms that the firm cannot affect and in terms of the deviations $\tilde{p}(T+t) \equiv p(T) - p^*(T+t) = \hat{p}(T) - \pi\ t - s\sigma\sqrt{t}$. We measure the value function just after paying the observation cost and observing the state $\tilde{p}$:

$$V(\tilde{p}) = \min\left\{\bar{V}(\tilde{p})\ ,\ \hat{V}\right\}\ , \tag{16}$$

where $\bar{V}(\tilde{p})$ is the value function if the firm knows the state, but does not change the price, and $\hat{V}$ is the value function if the firm decides to set the optimal price $\hat{p}$. Thus

$$\hat{V} = \psi + \phi + \min_{\hat{p},\hat{\tau}} B \int_0^{\hat{\tau}} e^{-\rho t}\left[(\hat{p} - \pi t)^2 + \sigma^2 t\right] dt + e^{-\rho\hat{\tau}} \int_{-\infty}^{\infty} V\left(\hat{p} - \pi\hat{\tau} - s\sigma\sqrt{\hat{\tau}}\right) dN(s) \tag{17}$$

where $N(\cdot)$ is the CDF of a standard normal.[22] The value function conditional on the firm not changing the price is:

$$\bar{V}(\tilde{p}) = \phi + \min_{\tau} B \int_0^{\tau} e^{-\rho t}\left[(\tilde{p} - \pi t)^2 + \sigma^2 t\right] dt + e^{-\rho\tau} \int_{-\infty}^{\infty} V\left(\tilde{p} - \pi\tau - s\sigma\sqrt{\tau}\right) dN(s)\ \ . \tag{18}$$

The next proposition states that the operator defined by the right side of equation (16), equation (17) and equation (18) is a contraction in the space of bounded and continuous functions. We will use this for further characterizations. The argument is intuitive but non-standard, since the length of the time period is a decision variable, potentially making the problem a continuous time one. Since we assume that $\phi > 0$ revisions should be optimally spread out. We then have:

Proposition 3. The value function is uniformly bounded and continuos on $\tilde{p}$. If $\phi > 0$ then the optimal time between observations is uniformly bounded by $\underline{\tau} > 0$, and thus the operator defined by the right side of equation (16), equation (17) and equation (18) is a contraction of modulus $\exp(-\rho\underline{\tau})$.

We conjecture that the form of the optimal decision rule is that there will be two thresholds

[22] Equations (17) and (18) use that the expected value: $\int_{-\infty}^{\infty}\left(p - \pi t - s\sigma\sqrt{t}\right)^2 dN(s) = (p - \pi t)^2 + \sigma^2 t$.

for $\tilde{p}$ that will define a range of inaction, $[\underline{p},\ \bar{p}]$. These threshold are such that

$$\bar{V}(\tilde{p}) \ < \ \hat{V} \ \text{ if } \ \tilde{p} \in (\underline{p}\ , \bar{p}) \ \text{ and that } \ \bar{V}(\tilde{p}) > \hat{V} \ \text{ if } \ \tilde{p} < \underline{p} \ \text{ or } \ \tilde{p} > \bar{p}\ .$$

If upon paying the observation cost the firm discovers that the state $\tilde{p}$ is in the range of inaction, then the firm will decide not to pay the cost ψ and not to adjust the price; moreover the firm will decide to take a look again at time $\tau(\tilde{p})$. If otherwise, the state is outside the range of inaction, the firm will pay the cost ψ, set the optimal price, and also set the new interval at which to observe the state $\hat{\tau}$.

5 Optimal Decision rules for the case with no-drift

In this section we study the case where p^* has no drift ($\pi = 0$) which simplifies the problem. We show that the value function is symmetric around zero, with a minimum at $\tilde{p} = 0$. The optimal choice of p upon paying the cost ψ is $\hat{p} = 0$, $\tau(0) = \hat{\tau}$, and $\tau(\tilde{p})$ symmetric around zero. Moreover, $\tau(\cdot)$ has a maximum at $\tilde{p} = 0$, and has an inverted U-shape. The thresholds for the range of inaction satisfy: $\underline{p} = -\bar{p}$.

We begin by showing that $V(\cdot)$ is symmetric around $\tilde{p} = 0$ and increasing around it. The Bellman equations for $\tilde{p} \in (-\infty, \infty)$ are:

$$\bar{V}(\tilde{p}) \ = \ \phi + \min_{\tau} B \int_0^{\tau} e^{-\rho t} \left[\tilde{p}^2 + \sigma^2 t\right] dt + e^{-\rho \tau} \int_{-\infty}^{\infty} V\left(\tilde{p} - s\sigma\sqrt{\tau}\right) dN(s) \tag{19}$$

$$\hat{V} \ = \ \psi + \phi + \min_{\hat{\tau}, \hat{p}} B \int_0^{\hat{\tau}} e^{-\rho t} \left[\hat{p}^2 + \sigma^2 t\right] dt + e^{-\rho \hat{\tau}} \int_{-\infty}^{\infty} V\left(\hat{p} - s\sigma\sqrt{\hat{\tau}}\right) dN(s) \tag{20}$$

$$V(\tilde{p}) \ = \ \min\{\hat{V}\ ,\ \bar{V}(\tilde{p})\} \tag{21}$$

Proposition 4. Let $\pi = 0$. The value function V is symmetric around $\tilde{p} = 0$, and V is strictly increasing in $\tilde{p}$ for $0 < \tilde{p} < \bar{p}$. The optimal price level conditional on adjustment is $\hat{p} = 0$. The derivative of $\bar{V}(\tilde{p})$ for $0 \leq \tilde{p}$ is given by

$$0 \leq \bar{V}'(\tilde{p}) = 2\ B\ \tilde{p}\ \frac{1 - e^{-\rho\tau(\tilde{p})}}{\rho} + e^{-\rho\tau(\tilde{p})} \int_0^{\infty} V'(z)\ \frac{e^{-\frac{1}{2}\left(\frac{z-\tilde{p}}{\sigma\sqrt{\tau(\tilde{p})}}\right)^2} - e^{-\frac{1}{2}\left(\frac{z+\tilde{p}}{\sigma\sqrt{\tau(\tilde{p})}}\right)^2}}{\sigma\ \sqrt{\tau(\tilde{p})\ 2\ \boldsymbol{\pi}}}\ dz$$

with strict inequality if $\tau(\tilde{p}) > 0$, where $\tau(\cdot)$ is the optimal decision rule for time between observations. Thus $V'(0) = \bar{V}'(0) = 0$, $V''(0) > 0$, and $V'(\tilde{p}) = 0$, for $\tilde{p} > \bar{p}$ and hence V is not differentiable at $\tilde{p} = \bar{p}$.

Notice that in this case at the boundary of the range of inaction the value function has a kink, i.e. there is *no smooth pasting*. This differs from the model with menu cost only of Section 4.2, which featured the smooth pasting property, typical of continuous-time fixed-cost models, see e.g. Dixit (1993) and Stokey (2008).

Next, we compute an analytical approximation to the value function and optimal policies. The approximation relies on the fact that $V(\cdot)$ is symmetric around $\tilde{p} = 0$, i.e. $V(\tilde{p}) = V(-\tilde{p})$, and hence all the derivatives of odd order are zero. Hence we approximate

$$\bar{V}(\tilde{p}) = V(0) + \frac{1}{2}V''(0)\ (\tilde{p})^2 + o(|\tilde{p}|^3)$$

since $V'(0) = V'''(0) = 0$ and $V''(0) > 0$. The other source of approximation is that we let ρ converge to zero, although the second source is only to simplify the expressions. The quadratic approximation for the value function is globally accurate if the range of inaction, i.e. $[-\bar{p},\ \bar{p}]$, is small. Since $\bar{p}$ converges to zero as the menu cost ψ goes to zero, the approximation will be accurate for small values of ψ relatively to ϕ. We will discuss the accuracy of this approximation more in detail in the following paragraphs. Proposition 5 - Proposition 8 use these approximations to characterize the optimal decision rules, i.e. to characterize the values of $\bar{p}$, $\hat{\tau}$ and the function $\tau(\cdot)$ in $[-\bar{p}, \bar{p}]$.

Proposition 5. Let $\hat{\varphi} \equiv \frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}$, then $V''(0)$, $\bar{p}$, and $\hat{\tau}$ solve the recursive system:

$$\sigma^2\psi/B = f(\hat{\varphi})\ , \quad \sigma^2\ \hat{\tau} = h(\hat{\varphi})\ , \ \text{and} \ \ V''(0) = 2\ \frac{\psi}{\bar{p}^2}\ ,$$

where $f(\cdot)$ and $h(\cdot)$ are the following known functions of $\hat{\varphi}$ and of two parameters $(\sigma^2\frac{\phi}{B}, \sigma^2\frac{\psi}{B})$:

$$\sigma^2\frac{\psi}{B} = f(\hat{\varphi}) \equiv \frac{\hat{\varphi}^2\ [h(\hat{\varphi})]^2}{1 -\ 2\ \sqrt{h(\hat{\varphi})}\ \left(\int_0^{\hat{\varphi}}\ s^2\ dN(s)\right)} \tag{22}$$

$$\hat{\tau} = \frac{h(\hat{\varphi})}{\sigma^2} \equiv \sqrt{\frac{\phi}{\sigma^2 B}\ 2 + \frac{\psi}{\sigma^2 B}\ 4\ \left(1 - N(\hat{\varphi})\right)}\ . \tag{23}$$

Equation (22) and equation (23) can be thougth as the first order conditions for $\bar{p}$ and $\hat{\tau}$. The variable $\hat{\varphi}$ is the minimum size (in absolute value) of the innovation of a standard normal that is required to get out of the inaction region $[-\bar{p}, \bar{p}]$ after resetting the price to $\hat{p} = 0$. Alternatively, consider a firm that has just adjusted its price, and hence will review its price in $\hat{\tau}$ periods. The expression $2\,(1 - N(\hat{\varphi}))$ is the probability of adjusting the price at the end of this review period. An immediate corollary of Proposition 5 is that the optimal values of $\hat{\varphi}$ and $\sigma^2\hat{\tau}$ are only functions of two parameters $\sigma^2\phi/B$ and $\sigma^2\psi/B$. Notice also that the expression for $\hat{\tau}$ in equation (23) is the same as the square root formula in Proposition 1 for the problem with observation cost only, except that the cost ϕ has been replaced by the "expected" cost $\phi + \psi\ 2\,(1 - N(\hat{\varphi}))$.

The previous proposition gives a recursive system of equations, whose solution is the optimal value of $(\hat{\tau}, \bar{p})$. The next proposition gives a sufficient condition for the existence and uniqueness of the system, and also provides some comparative statics. In fact, it turns out that the approximations used in this section can only be used globally – i.e. for all $\tilde{p}$ – if $\hat{\varphi} \in (0, 1)$, as it will become clear after Proposition 8. Thus, the next proposition restricts attention to parameter settings so that there is a unique solution in this range.

Proposition 6. Let $\hat{\varphi} \equiv \bar{p}/(\sigma\sqrt{\hat{\tau}})$. Assume that $\frac{\phi}{\psi} \geq 1/2 - 2(1 - N(1)) \approx 0.1827$. Then there exist a unique value $\hat{\varphi}^* \in (0, 1)$ that solves $\sigma^2 \frac{\psi}{B} = f(\hat{\varphi}^*)$ defined in equation (22). Also let $\hat{\tau}^*$ be the solution of $\hat{\tau}^* = h(\hat{\varphi}^*)/\sigma^2$ defined in equation (23). Then,

1. $\hat{\varphi}^*$ is decreasing in ϕ, and $\hat{\tau}^*$ is increasing in ϕ,
2. $\hat{\varphi}^*$ is decreasing in $\frac{\sigma^2}{B}$, and $\sigma^2\hat{\tau}^*$ is increasing in $\frac{\sigma^2}{B}$ with an elasticity $\geq 1/2$,
3. $\partial\hat{\varphi}^*/\partial\frac{\sigma^2}{B} = 0$ evaluated at $\frac{\sigma^2}{B} = 0$,
4. $\partial\hat{\varphi}^*/\partial\psi > 0$ if $\sigma^2\frac{\psi}{B}$ is small relative to ϕ.

The assumption in this proposition is that the observation cost must be sufficiently large relative to the menu cost in order for the approximation to be globally valid –i.e. in order for $\hat{\varphi} < 1$– for arbitrary values of σ^2/B. We can relax this assumption at the cost of imposing a lower bound on the σ^2/B, which will depend on ψ and ϕ. The intuition for why we need a condition like $\phi/\psi > 0.1827$, is that the formulation of the problem presumes that after adjusting the price the firm waits for $\hat{\tau} > 0$ periods before the next review, for which we require a cost of observation, ϕ, non-negligible relative to the menu cost ψ. For instance when $\phi = 0$ the formulation of the problem is incorrect as the model becomes the menu cost of Section 4.2, where $\hat{\tau} = 0$ and price reviews happen continuously.

The comparative statics of parts 2 and 3 of Proposition 6 may seem contradictory. On the one hand, part 2 shows that, in general, $\hat{\varphi}^*$ is a decreasing function of σ^2/B. On the other hand, part 3 shows that, around σ^2/B equal to zero, the value $\hat{\varphi}^*$ is non-responsive to σ^2/B. It is interesting to consider in Proposition 6 the limit case of $\sigma \downarrow 0$. In this case, the value of $\hat{\varphi}^*$ converges to value in $(0, 1)$, but $\hat{\tau}^*$ goes to ∞. This is quite intuitive: as the target price can be predicted arbitrarily well, the time between observations is arbitrarily large, but given the menu cost, there is still a range of inaction. For future reference we let $\hat{\varphi}_0$ be the solution to equation (22) and equation (23) when $\sigma^2/B = 0$. Notice that in this case the solution is just a function of the ratio of the two costs $\alpha \equiv \psi/\phi$, which we write as:

$$1 \;=\; (\hat{\varphi}_0)^2 \left(\frac{2}{\alpha} + 4\left[1 - N\left(\hat{\varphi}_0\right)\right] \right), \text{ where } \alpha \equiv \frac{\psi}{\phi} \,. \tag{24}$$

We find that for the values of σ^2/B that we are interested in the function $\hat{\varphi}^*$ is approximately constant with respect to σ^2/B, or putting it differently we are interested in relatively small values of σ^2/B. To quantify what we mean by small, we display the values of $\hat{\varphi}^*$ as a function of the ratio $\alpha = \psi/\phi$, for three values of σ^2/B, namely 0, 0.2/20, and $100 \times 0.2/20$. Notice that by definition $\hat{\varphi}^* = \hat{\varphi}_0$ when $\sigma^2/B = 0$. Using the interpretation of the model as an approximation to the problem of a firm facing a constant demand elasticity, as explained in Section A of the Appendix, we think of the value $\sigma = 0.2$ as the annual standard deviation of cost, as well as a value of $B = 20$, which in this context is implied by a markup of about 15%, as of the righ order of magnitude. Then, the third value is just 100 times the values we think of as reasonable. As it can be seen in Figure 2, the values of $\hat{\varphi}^*$ under different σ^2/B are very similar for all values of α.[23] We summarize the result that, for the range of

Figure 2: $\hat{\varphi}^*$ as a function of α and σ^2/B

Note: in this example we let ϕ vary, while $\psi = 0.03$.

interesting parameter values, the function $\hat{\varphi}^*$ does not depend much on σ^2/B as follows:

[23]As discussed before, the accuracy of our approximation also relies on relative small values of ψ.

Proposition 7. For small values of $\sigma^2 \frac{\psi}{B}$, the normalized width of inaction $\hat{\varphi}(\sigma^2 \frac{\phi}{B}, \sigma^2 \frac{\psi}{B})$ is well approximated by $\hat{\varphi}_0(\alpha)$ which has elasticity:

$$\frac{\partial \log \hat{\varphi}_0}{\partial \log \alpha} = \frac{1}{2} \text{ at } \alpha = 0 \text{ and } \frac{\partial \log \hat{\varphi}_0}{\partial \log \alpha} < \frac{1}{2} \text{ for } \alpha > 0 \ . \tag{25}$$

The optimal values for the time until next revision after an adjustment, $\hat{\tau}^*$ and the width of the range of inaction, $\bar{p}$, are then given by

$$\hat{\tau}^* = \sqrt{\frac{\phi}{\sigma^2 \ B}} \quad \frac{\sqrt{\alpha}}{\hat{\varphi}_0(\alpha)} \quad > \quad \sqrt{\frac{\phi}{\sigma^2 \ B} \ 2} \ = \ \hat{\tau}^* \ |_{\psi=0} \tag{26}$$

$$\bar{p} = \left[\sigma^2 \frac{\psi}{B}\right]^{\frac{1}{4}} \ \sqrt{\hat{\varphi}_0(\alpha)} \ < \ \left[\sigma^2 \frac{\psi}{B} \ 6\right]^{\frac{1}{4}} \ = \ \bar{p} \ |_{\phi=0} \ . \tag{27}$$

Proposition 7 shows that the expressions in equation (26) and equation (27) are the generalizations of the corresponding formulas for the case in which there is only an observation cost and for the case in which there is only a menu cost respectively: fixing the ratio of the cost α, they have exactly the same functional form. Furthermore, everything else being equal, the length of time until the next revision, $\hat{\tau}^*$, is higher in the model with both costs than in the model with observation cost only, and the width of the inaction band, $\bar{p}$, is smaller than in the menu cost model. From equation (26) and equation (25) it follows that $\hat{\tau}^*$ is weakly increasing in the ratio of the cost α, with elasticity less than $1/2$.

Finally, notice that the width of the inaction band has elasticity $1/4$ with respect to σ^2/B, as in the menyu cost model, and that the time to the next review after a price adjustment has elasticity equal to $-1/2$ with respect to $B\sigma^2$. Using equation (25) into equation (26) and equation (27) we obtain the following elasticities with respects to the cost:

$$0 \leq \frac{\partial \log \hat{\tau}^*}{\partial \log \psi} = \frac{1}{2} - \frac{\partial \log \hat{\varphi}_0}{\partial \log \alpha} \leq \frac{1}{2} \text{ and } \ 0 < \frac{\partial \log \hat{\tau}^*}{\partial \log \phi} = \frac{\partial \log \hat{\varphi}_0}{\partial \log \alpha} \leq \frac{1}{2} \ , \tag{28}$$

$$0 \leq \frac{\partial \log \bar{p}}{\partial \log \psi} = \frac{1}{2} \left(\frac{1}{2} + \frac{\partial \log \hat{\varphi}_0}{\partial \log \alpha} \right) \leq \frac{1}{2} \ \text{ and } \ 0 > \frac{\partial \log \bar{p}}{\partial \log \phi} = -\frac{1}{2} \frac{\partial \log \hat{\varphi}_0}{\partial \log \alpha} \geq -\frac{1}{2} \ . \tag{29}$$

Equation (28) says that the time to the next review after a price adjustment is increasing in both costs. Equation (29) says that the width of the inaction band is increasing in ψ, with an elasticity smaller than in the menu cost model, and *decreasing* in ϕ, since at the time of an observation an agent facing a higher cost minimizes the chances of paying further observation cost by narrowing the range of inaction.

We now turn to characterizing the remaining object defining the decision rules, namely the function $\tau(\cdot)$ describing the time until the next review in the range of inaction.

Proposition 8. The optimal rule for the time to the next revision $\tau(\tilde{p})$ for prices in the range of inaction is given by:

$$\tau\left(\tilde{p}\right) = \hat{\tau} \; - \left(\frac{\tilde{p}}{\sigma}\right)^2 \; . \tag{30}$$

Figure 3: Policy rule $\tau(\tilde{p})$ from numerical solution of the model and from Proposition 8

Note: parameter values are $B = 20,\ \rho = 0.02,\ \sigma = 0.2, \phi = 0.06$, and $\psi = 0.03$; the approximated solution for $\tau(\tilde{p})$ is given by Proposition 8; the numerical solution for $\tau(\tilde{p})$ is obtained from solving the model on a grid of points for $\tilde{p}$.

Few comments are in order. First, the shape of the optimal decision rule depends only on σ, and not on the rest of the parameters for the model, i.e. $B,\ \phi$, and ψ. Second, if the agent find herself after a review with a price gap $\tilde{p} = 0$, she will set the time exactly as he whould just adjusted, since the optimal adjustment would have implied a post adjustment price gap of zero. Third, this function is decreasing in the size of the price gap $\tilde{p}$. If upon a review the agent finds the price gap at a value close to the boundary of range of inaction, then she will plan for a next review relatively soon, since it is more likely that the target will close the threshold $\bar{p}$. Fourth, the price gap is normalized by the standard deviation of the changes in the target price σ. This is also very natural, since the interest of the decision maker is on the likelihood that the price target will deviate and hit the barriers, so that for a lower σ she is prepared to wait more for the same price gap $\tilde{p}$. Figure 3 plots the policy rule $\tau(\tilde{p})$ implied by Proposition 8 against the policy rule obtained from the numerical solution of the model, both evaluated at a set of structural paramters which we think reasonable. The horizontal axis of Figure 3 displays the deviation of firm's price from the target, $\tilde{p}$. The two vertical bars at $-\bar{p}$ and $\bar{p}$ denote the threshold values that delimit the inaction region. The vertical bar at $\hat{p}$, inside the inaction region, denotes the optimal return point after an adjustment. As in the case of the approximation solutions for $\bar{p}$ and $\hat{\tau}$, the approximation for $\tau(\cdot)$ is very accurate, and we found this result to be robust to all the economically interesting parameter values.

We discuss briefly the type of error in the decision rules that is produced by our quadratic approximation. Recall that the nature of the approximation used in this section is that the value function $\bar{V}$ is assumed to be quadratic in $\tilde{p}$, and also that we let ρ converge to zero. Here we focus on the first feature, which turns out to be the most important. Since the function $\bar{V}$ is symmetric and has a minimum at $\tilde{p} = 0$, a quadratic approximation must be accurate around $\tilde{p} = 0$. Also, recall that Proposition 4 shows that the function is increasing for all $\tilde{p} < \bar{p}$. Thus, since $\bar{p}$ tends to zero as ψ goes to zero, the relevant range of $\bar{V}$, given by $[-\bar{p}, \bar{p}]$, is very small and hence the approximation very accurate if ψ is small. On the other hand, when ψ is large relative to ϕ, the quality of the approximation deteriorates. In particular, as ϕ goes to zero, the problem converges to the menu cost model analyzed in Section 4.2. The value function $\bar{V}$ in the menu cost model is convex close to $\tilde{p} = 0$, but then it must be concave around $\bar{p}$, to satisfy smooth pasting. As explained in Section 4.2 above, this implies that $V''''(0) < 0$. Thus, as ϕ becomes small relative to ψ, the value function $\bar{V}$ becomes closer to the one of the menu cost, and hence our quadratic approximation of the value function becames worse, especially for values of $\tilde{p}$ away from zero. In particular, since our quadratic approximation has $V''''(0) = 0$, it tends to be higher for values of $\tilde{p}$ away from zero, and consequently the value of $\bar{p}$ that we obtain tends to be smaller. In Appendix C.6 we compare our analytical quadratic approximation for the value function and the thresholds with numerical solutions obtained of these objects. Finally, we notice that for small values of ϕ/ψ the shape of $\tau(\cdot)$ will be different for values of $\tilde{p}$ away from zero. In particular, the approximation that $\tau(\cdot)$ is quadratic in the whole range will be less accurate for small values of ϕ/ψ. In this case the true value of τ will be larger than the in the quadratic approximation obtained in Proposition 8.

6 Statistics for the case with no-drift

In this section we characterize the implications for the following statistics of interest in the case of no drift: frequency of price revisions, frequency of price adjustment, distribution of price adjustment, and hazard rate for price changes.

6.1 Average frequencies of review and adjustment

First we turn to the development of expressions for statistics of interest implied by this model the frequency of price adjustments n_a, and the frequency of price reviews n_r. First, we turn to the characterization of the frequency of price adjustments. Let the function $\mathcal{T}(\tilde{p})$, describe the expected time needed for the price gap to get outside the range of inaction $[-\bar{p}\ ,\ \bar{p}]$, conditional on the state (right after a revision) equal to $\tilde{p}$. This function solves the recursion:

$$\mathcal{T}(\tilde{p}) = \tau(\tilde{p}) + \int_{\frac{\tilde{p}-\bar{p}}{\sigma\sqrt{\tau(\tilde{p})}}}^{\frac{\tilde{p}+\bar{p}}{\sigma\sqrt{\tau(\tilde{p})}}} \mathcal{T}\left(\tilde{p} - \sigma\sqrt{\tau(\tilde{p})}\ s\right) dN(s) \tag{31}$$

for $\tilde{p} \in [-\bar{p}\ ,\ \bar{p}]$. Since after a price adjustment the price gap $\tilde{p}$ is zero, then $\mathcal{T}(0)$ is the expected time between price adjustments. By the fundamental theorem of renewal theory, the average number of price adjustments per unit of time is given by $n_a = 1/\mathcal{T}(0)$.

In the next proposition we derive analytical approximation for the expected time between adjustments as function of only two arguments: $\hat{\varphi} \equiv \bar{p}/(\sigma\sqrt{\hat{\tau}})$ and $\hat{\tau}$. The nature of the approximation is that we replace the function $\mathcal{T}(\tilde{p})$, that is symmetric around $\tilde{p} = 0$ and has zero odd derivatives at $\tilde{p} = 0$, by a quadratic function; moreover, based on Proposition 8, we use $\tau(\tilde{p}) = \hat{\tau} - (\tilde{p}/\sigma)^2$.

Proposition 9. Let $\bar{\varphi} \equiv \hat{\varphi}/\sqrt{1-\hat{\varphi}^2}$ and let $\tau(\tilde{p})$ be given by Proposition 8. The frequency of price adjustments is $n_a = 1/\mathcal{T}(0)$, where $\mathcal{T}(0) = \hat{\tau}\ \mathcal{A}(\hat{\varphi})$, and:

$$\mathcal{A}(\hat{\varphi}) \approx \frac{(1-\hat{\varphi}^2)}{1.5 - N\left(2\bar{\varphi}\right)} \left(1 - \frac{1/2\ \left(\int_0^{2\bar{\varphi}} \left(\bar{\varphi}-s\right)^2 dN(s) - \bar{\varphi}^2\right)}{1 - N(\hat{\varphi}) + \int_0^{\hat{\varphi}} s^2 dN(s) + \hat{\varphi} n(\hat{\varphi})}\right) \tag{32}$$

where $\mathcal{A}(0) = 1$, the approximation for $\mathcal{A}(\hat{\varphi})$ is strictly increasing for $0 \leq \hat{\varphi} \leq \hat{\varphi}_{sup}$, $\hat{\varphi}_{sup} \approx 0.75$, and where $\mathcal{A}(\hat{\varphi}_{sup}) \approx 1.8$.

The value $\hat{\varphi}_{sup}$ delimits the range over which the approximation is accurate. Clearly Proposition 9 implies that keeping $\hat{\varphi}$ fixed, the expected time between price adjustments is increasing in $\hat{\tau}$. The higher $\hat{\varphi}$, the larger the expected time between price adjustments for given $\hat{\tau}$.

Next we turn to the expected time between reviews. As an intermediate step we write a recursion for the distribution of the price gaps upon review (and before adjustment), for which we use density $g(\tilde{p})$ and CDF G. The recursion is akin to a Kolmogorov forward equation:

$$g(\tilde{p}) = \int_{-\bar{p}}^{\bar{p}} \frac{1}{\sigma\sqrt{\tau(p)}}\, n\left(\frac{\tilde{p}-p}{\sigma\sqrt{\tau(p)}}\right)\ g(p)\ \ dp + \frac{1}{\sigma\sqrt{\hat{\tau}}}\, n\left(\frac{\tilde{p}}{\sigma\sqrt{\hat{\tau}}}\right)\ \left[1 - \int_{-\bar{p}}^{\bar{p}} g(p)dp\right] \tag{33}$$

for all $\tilde{p} \in [-\bar{p}\ , \bar{p}]$, where $n(\cdot)$ is the density of a standard normal.[24] The first term on the right side of this equation gives the mass of firms with values of the price gap p that in last review were in the inaction region, and drew shocks to transit from p to $\tilde{p} = p - s\sigma\sqrt{\tau(p)}$ during $\tau(p)$ periods. The second term has the mass of firms that in the last review were outside the inaction region, and hence started with a price gap of zero, so that $1 - \int_{-\bar{p}}^{\bar{p}} g(p)dp$ is the fraction of reviews that end up outside the range of inacion, and hence trigger an adjustment. For future reference, notice that equation (33) does not use the values of $g(\cdot)$ outside the range of inaction. With the knowledge of the distribution g, the expected time between price reviews is given by

$$\mathcal{T}_r = \int_{-\bar{p}}^{\bar{p}} \tau(p)g(p)dp + \left[1 - \int_{-\bar{p}}^{\bar{p}} g(p)dp\right]\hat{\tau} \quad . \tag{34}$$

Similarly to Proposition 9, we derive analytical approximation for the expected time between reviews as function of only $\hat{\varphi} \equiv \bar{p}/(\sigma\sqrt{\hat{\tau}})$ and $\hat{\tau}$. In particular, we replace the density $g(\tilde{p})$, that is symmetric around $\tilde{p} = 0$ and has zero odd derivatives at $\tilde{p} = 0$, by a quadratic function, and use $\tau(\tilde{p}) = \hat{\tau} - (\tilde{p}/\sigma)^2$. We establish the following:

Proposition 10. Let $\bar{\varphi} \equiv \hat{\varphi}/\sqrt{1-\hat{\varphi}^2}$ and let $\tau(\tilde{p})$ be given by Proposition 8. The average frequency of price reviews is $n_r = 1/\mathcal{T}_r$ where $\mathcal{T}_r = \hat{\tau}\ \mathcal{R}(\hat{\varphi})$, and:

$$\mathcal{R}(\hat{\varphi}) \approx 1 - 2\int_0^{\bar{\varphi}} \left(q(0) + \frac{1}{2}q''(0)\ \varphi^2\right)\frac{\varphi^2}{1+\varphi^2}d\varphi, \tag{35}$$

where $q(0)$ and $q''(0)$ are known functions of $\bar{\varphi}$ given in Appendix C.11, and where $\mathcal{R}(0) = 1$, the approximation for $\mathcal{R}(\hat{\varphi})$ strictly decreasing for $0 \leq \hat{\varphi} \leq \hat{\varphi}_{sup}$ where $\hat{\varphi}_{sup} \approx 0.65$, and $\mathcal{R}(\hat{\varphi}_{sup}) \approx 0.96$.

Proposition 10 implies that keeping $\hat{\varphi}$ fixed, the expected time between price reviews is increasing in $\hat{\tau}$. However, the higher $\hat{\varphi}$, the smaller $\mathcal{R}(\hat{\varphi})$, and the smaller the expected time between price reviews for given $\hat{\tau}$.

Figure 4 plots $\mathcal{A}(\hat{\varphi})$, $\mathcal{R}(\hat{\varphi})$ and $\frac{n_r}{n_a}$ as a function of $\hat{\varphi}$ as defined in equation (32) and equation (35), against their numerical counterparts. Several results are worth mentioning.

[24] Notice that $\frac{1}{\sigma\sqrt{\tau(p)}}\, n(\cdot)$ is the probability density of $\tilde{p}$ conditional on p, where $\tilde{p} = p - s\ \sigma\sqrt{\tau(p)}$.

Figure 4: $\mathcal{A}(\hat{\varphi})$, $\mathcal{R}(\hat{\varphi})$ and $\frac{\mathcal{A}(\hat{\varphi})}{\mathcal{R}(\hat{\varphi})}$ as a function of $\hat{\varphi}$

Note: $\mathcal{A}(\hat{\varphi})$ and $\mathcal{R}(\hat{\varphi})$ have been computed numerically on a grid, as desribed in Section C.10-Section C.11 of the appendix.

First, the approximations in equation (32) and equation (35) are very close to their numerical counterparts, and basically coincide for $\hat{\varphi}$ small enough. Second, $\mathcal{A}(\cdot)$ is strictly increasing in $\hat{\varphi}$, while $\mathcal{R}(\cdot)$ is strictly decreasing in $\hat{\varphi}$.[25]

An important implication of Proposition 9 and Proposition 10 is that the ratio of the average number of price adjustment to the average number of reviews per unit of time, i.e. $\frac{n_a}{n_r} = \frac{\mathcal{R}(\hat{\varphi})}{\mathcal{A}(\hat{\varphi})}$, only depends on $\hat{\varphi}$. This result is very useful because, according to our model, we can directly identify the values of $\hat{\varphi}$, and $\hat{\tau}$, from observations of the average frequency of price adjustment and the average frequency of price review. In addition, from Proposition 6 we know that $\hat{\varphi}$ is roughly determined by the relative cost of reviewing relative to adjusting prices, $\alpha = \frac{\psi}{\psi}$. Therefore, we have obtained a very tight relationship between the relative frequency of price review to adjustment and the relative cost of review to adjustment. By applying these results to the interpretation of observations in Figure D.2, we could attribute equal increases in the average frequencies of adjustment and review either to an increase in the volatility of the state, σ, or to an increase in the curvature coefficient of the profit function, B, or to a parallel decrease in the costs of review and adjustment, ψ and ϕ. However, changes in the ratio of the two frequencies could roughly only be due to changes in the ratio of the costs of review and adjustment.

[25]This monotonicity of $\mathcal{A}(\cdot)$ and $\mathcal{R}(\cdot)$ applies for all values of $\hat{\varphi} \leq 1$ that we computed numerically.

PROPOSITION 11. Assume that $\sigma^2 \ \psi/B$ is small so that approximating $\hat{\varphi}$ using $\hat{\varphi}_0$ is accurate. Then the elasticities of the number of price adjustments and revisions with respect to the cost satisfy:

$$\frac{\partial \log n_a(\alpha)}{\partial \log \phi} = -\frac{\partial \log \hat{\varphi}_0(\alpha)}{\partial \log \alpha}\left(1 - \hat{\varphi}_0(\alpha)\frac{\mathcal{A}'(\hat{\varphi}_0(\alpha))}{\mathcal{A}(\hat{\varphi}_0(\alpha))}\right) \geq -\frac{1}{2} \text{ and } = -\frac{1}{2} \text{ if } \alpha = 0 \ , \tag{36}$$

$$\frac{\partial \log n_a(\alpha)}{\partial \log \psi} = -\frac{1}{2} - \frac{\partial \log n_a(\alpha)}{\partial \log \phi} \ , \tag{37}$$

$$\frac{\partial \log n_r(\alpha)}{\partial \log \phi} = -\frac{\partial \log \hat{\varphi}_0(\alpha)}{\partial \log \alpha}\left(1 - \hat{\varphi}_0(\alpha)\frac{\mathcal{R}'(\hat{\varphi}_0(\alpha))}{\mathcal{R}(\hat{\varphi}_0(\alpha))}\right) \leq 0 \text{ and } = -\frac{1}{2} \text{ if } \alpha = 0 \ , \tag{38}$$

$$\frac{\partial \log n_r(\alpha)}{\partial \log \psi} = -\frac{1}{2} - \frac{\partial \log n_r(\alpha)}{\partial \log \phi} \ . \tag{39}$$

Notice that equation (37) and equation (39) imply that an increase of ϕ and ψ in the same percentage, decreases the number of adjustments and the number of reviews n_a and n_r in half of that percentage. Note that this one half elasticity is the one present in the models described in Section 4.1 and Section 4.2 that feature either information cost only or menu cost only. Furthermore, we remark that while an increase in the information cost ϕ decreases the ratio n_r/n_a, so each adjustment corresponds to fewer observations, the elasticity of n_a with respect to ϕ displayed in equation (36) is negative, showing that observations are complement to adjustments. In Section C.12 we report results about the elasticities of n_r and n_a with respect to ψ and ϕ evaluated numerically at different values of α. These results are consistent with Proposition 11.

6.2 Average size and distribution of price changes

Next we derive expressions for the average size of price change, $\mathbb{E}[\ |\Delta p|\ ]$, and for the distribution of price changes. The average size of price adjustment equals the average price-gap upon review, conditional on that average gap being outside the range of inaction. Thus, to compute this average we extend $g(\cdot)$, defined in equation (33), to the values outside the range of inaction. Since the distribution of price changes has to be conditional on a price change, and the price change happens with probability $[1 - \int_{-\bar{p}}^{\bar{p}} g(p)dp]$, then price adjustments have a density:

$$w(\Delta p) \equiv \frac{g(\Delta p)}{1 - \int_{-\bar{p}}^{\bar{p}} g(p)dp} = \frac{\int_{-\bar{p}}^{\bar{p}} \frac{1}{\sigma\sqrt{\tau(p)}}\ n\left(\frac{\Delta p - p}{\sigma\sqrt{\tau(p)}}\right) g(p)dp}{1 - \int_{-\bar{p}}^{\bar{p}} g(p)dp} + \frac{1}{\sigma\sqrt{\hat{\tau}}}\ n\left(\frac{\Delta p}{\sigma\sqrt{\hat{\tau}}}\right) , \tag{40}$$

for $|\Delta p| \in (\bar{p}, \infty)$. The average size of price change is then given by $\mathbb{E}[\ |\Delta p|\ ] = 2\int_{\bar{p}}^{\infty} z\, w(z)dz$. The distribution of price changes $w(\cdot)$ is composed of two terms: the first term is positive and strictly decreasing in $|\Delta p|$; the second term is the density of a normal distribution with zero mean and standard deviation equal to $\sigma\sqrt{\hat{\tau}}$. Therefore, conditional on adjusting prices, $w(\cdot)$ assigns relatively more weight to values of $|\Delta p|$ that are closer to $\bar{p}$ than the normal distribution. Moreover, $w(\cdot)$ is symmetric around zero, as both the normal distribution and the distribution of the price gaps upon review, $g(\cdot)$, are symmetric. The next proposition characterizes this distribution, and some related statistics, as a function of the parameters for the optimal decision rules.

Proposition 12. Consider an optimal decision rule described by two parameters $(\hat{\tau}, \hat{\varphi})$ and where, in the range of inaction, $\tau(p)$ is given by the approximation described in Proposition 8, namely $\tau(p) = \hat{\tau} - (p/\sigma)^2$. Define $\bar{\varphi} \equiv \hat{\varphi}/\sqrt{1-\hat{\varphi}}$, and let x denote the normalized price changes: $x \equiv \Delta p/(\sigma\sqrt{\hat{\tau}})$. Then

$$\mathbb{E}[\ |\Delta p|\ ] \;=\; \sigma\,\sqrt{\hat{\tau}}\;\mathbb{E}[\ |x|\ ;\ \bar{\varphi}]\ , \tag{41}$$

$$\mathbb{E}[\ |x|\ ;\ \bar{\varphi}] \;=\; 2\int_{\bar{\varphi}}^{\infty} x\ 2\ v(x\ ;\ \bar{\varphi})\ dx\ , \tag{42}$$

$$\frac{\min|\Delta p|}{\mathbb{E}[\ |\Delta p|\ ]} \;=\; \frac{\text{mode } |\Delta p|}{\mathbb{E}[\ |\Delta p|\ ]} = \frac{\bar{\varphi}}{2\int_{\bar{\varphi}}^{\infty} x\ v(x\ ;\ \bar{\varphi})\ dx} \in [0,1]\ . \tag{43}$$

where the density $v(x;\bar{\varphi})$ of the normalized price changes solves:

$$v(x;\bar{\varphi}) \;=\; \frac{\int_{-\bar{\varphi}}^{\bar{\varphi}}\left[q(0)+\frac{1}{2}q''(0)\varphi^2\right]\sqrt{1+\varphi^2}\ n\left(x\ \sqrt{1+\varphi^2}-\varphi\right)d\varphi}{1-\int_{-\bar{\varphi}}^{\bar{\varphi}}\left[q(0)+\frac{1}{2}q''(0)\varphi^2\right]d\varphi} \tag{44}$$
$$+\ n\,(x)\quad \text{for } |x| > \bar{\varphi},\ \text{ and} = 0 \text{ otherwise}\ .$$

This proposition, together with the previous characterization of $(\hat{\tau},\ \hat{\varphi})$ as functions of the parameters of the problem, completely characterizes the distribution of price changes. In particular, notice that using the approximations developed in Proposition 7, for $\hat{\varphi}$, and hence $\bar{\varphi}$, in equation (24) and for $\hat{\tau}$ in equation (26), we can write the average price change as:

$$\mathbb{E}[\ |\Delta p|\ ] = \left(\frac{\phi\ \sigma^2}{B}\right)^{\frac{1}{4}} \mathcal{S}(\alpha) \tag{45}$$

where $\mathcal{S}(\alpha) \equiv \frac{\sqrt{\alpha}}{\hat{\varphi}_0(\alpha)}\mathbb{E}\left[\ |x|\ ;\ \frac{\hat{\varphi}_0(\alpha)}{\sqrt{1-\hat{\varphi}_0(\alpha)^2}}\right]$. Notice that, holding constant the ratio of the two costs, α, this expression has the same comparative static with respect to a change in both cost (ψ, ϕ), and to a change in (B, σ^2) than the one for the models with observation cost only or with menu cost only. Finally, notice that equation (44) implies that the ratio of the mode to the average price change is given by

$$\frac{\min |\Delta p|}{\mathbb{E}[\ |\Delta p|\ ]} = \frac{\text{mode } |\Delta p|}{\mathbb{E}[\ |\Delta p|\ ]} = \mathcal{M}(\alpha). \tag{46}$$

where $\mathcal{M}(\alpha) \equiv \bar{\varphi}(\alpha) \bigm/ 2 \int_{\bar{\varphi}(\alpha)}^{\infty} x\ v(x\ ; \bar{\varphi}(\alpha))\ dx$, and where $\bar{\varphi}(\alpha) \equiv \hat{\varphi}_0(\alpha)/\sqrt{1 - \hat{\varphi}_0(\alpha)}$. Thus the "shape" of the distribution depends on the ratio of the two cost. This result provides an additional identification scheme for α from observation of statistics about the distribution of price changes.

Figure 5: Distribution of price changes, conditional on a price change

Note: Benchmark parameter values for the model with two cost are: $B = 20$, $\rho = 0.02$, $\sigma = 0.2$, $\psi = 0.03$, $\phi = 0.06$; in the model with menu cost only and observation cost only, we set $\psi = 0.06$, $\phi = 0$, and $\psi = 0$, $\phi = 0.18$ respectively, while leaving the other parameters unchanged, so to obtain the same frequency of price adjustment of the model with both costs, i.e. 1.5 adjustments a year.

Figure 5 plots the distribution of price changes implied by our model against the one predicted by the special cases of observation cost and menu cost only, under a parametrization of the model that we think of reasonable. Parameters are such that the three models imply the same frequency of price changes. In comparing the distributions of price changes from these models, several results are worth mentioning. First, our model assigns a zero probability to price changes smaller than $\bar{p}$ in absolute value, while the model with observation cost only assigns a positive probability mass also at arbitrary small price changes. Second, our model with both types of costs has a mode for the absolute size of price changes at $\bar{p}$, while the model with observation costs only has a mode at zero.

We note that the distribution of price changes in our model is obviously different from the one implied by the model with menu cost only, where price changes occur only at a size equal to $\bar{p}$. In fact, the plain menu cost model implies a bi-modal distribution of price changes. Our model predicts a distribution of price changes that resembles the bi-modal, as it puts a lot of weight to price changes close to $\bar{p}$ in absolute value, but also assigns some positive probability to price changes of size larger than $\bar{p}$. Hence, the average size of price changes in the model with both observation and menu costs is larger than $\bar{p}$.

The distribution of price changes implied by our model resembles in many aspects the distribution of price changes estimated by several empirical studies. For instance, Alvarez et al. (2006) estimate the distribution of price changes for the Euro area and, finding very few price adjustments smaller in size than 2.5 percent, with most of the mass of price changes concentrated around 10 percent in absolute value. Similarly, Cavallo (2009) finds that the distribution of the size of price changes in four Latin American countries is roughly bimodal, with few changes close to zero percent. While a bimodal distribution of price adjustment is consistent with a menu cost model, the latter is unable to explain price adjustments of size larger than the thresholds of the inaction range. Allowing for both menu and observation costs allows the model to generate different possible shapes in the distribution of price changes: everything else being equal, the larger the menu cost, the more the mass around the adjustment thresholds; the larger the observation cost, the more the mass to the right and to the left of the adjustment thresholds. In addition, Klenow and Kryvtsov (2008) estimate the distribution of regular price changes in U.S. CPI data and find that around 44 percent of price changes are smaller than 5 percent, with a mode smaller than 2.5 percent, despite an average absolute size of price adjustment of about 10 percent. According to our model, the relatively small ration of mode to average size of price change suggests that menu cost is relatively small compared to observation cost.[26]

6.3 The hazard rate

Finally we turn to the analysis of the hazard rate of price changes in the model with both review and menu costs. In Section 4.1 and Section 4.2 we have shown that, for the models

[26]See Section D.3 for more details on empirical evidence about the distribution of price changes.

with only one type of cost, the hazard rate of a price change is monotone increasing on the time elapsed since the last change. We now show that the hazard rate of price changes with both costs is *not* monotone on the time elapsed since the last change.

We develop some notation to describe the mechanics of the construction of the hazard rate of price adjustments. Let t denote the time elapsed since the last price adjustment, and let $S(t)$ be the survival probability, i.e. the fraction of spells of unchanged prices that are of length t or longer. The instantaneous hazard rate is then defined as $\boldsymbol{h}(t) = -S'(t)/S(t)$. We fix a value of σ and the decision rule described by $\bar{p}$ and the symmetric function $\tau(p)$ defined in $\tau : [0, \bar{p}] \to [\underline{\tau}, \hat{\tau}]$, where $\underline{\tau} = \tau(\bar{p})$. We construct the hazard rate for the interval $t \in [0, \hat{\tau} + \min\{\hat{\tau} + 2\underline{\tau}\}]$. We let $p(t') \equiv \tau^{-1}(t')$ be the inverse of $\tau(\cdot)$ so that $\tau^{-1}(t') : [\underline{\tau}, \hat{\tau}] \to [\bar{p}, 0]$.

To characterize the hazard rate we start with a large set of firms that have that just adjusted their price and describe the sequence of events that will take so that they adjust to a new price. First, notice that until $\hat{\tau}$ period no firm will review its price, and hence no adjustments will take place, so that $S(t) = 1$ and the hazard rate is zero for $t \in [0, \hat{\tau})$. At $\hat{\tau}$ all the firms review their prices, and a fraction of them adjusts. This fraction is $2(1 - N(\hat{\varphi}))$, i.e. the probability that after the review the target is outside the range of inaction. Thus, there is a jump down in the survivor function to $S(\hat{\tau}) = 2\ N(\hat{\varphi}) - 1$, and thus the instantaneous hazard rate is infinite at this point. For the remaining firms that have review but not adjusted their price, the time of the next review depends on the current price gap $\tilde{p}$. The next review among these firms occurs $\underline{\tau}$ periods after the first review, these are the firms that have a price gap inside the range of inaction but arbitrarily close to its boundary, i.e. very close to $\bar{p}$ or $-\bar{p}$. In general, we describe the number of firms that change prices in their second review, between times $t = \hat{\tau} + t$ and $t + \Delta$, as approximately $\partial S'(t' + \hat{\tau})/\partial dt \times\ \Delta$, satisfying:

$$\frac{\partial S(\hat{\tau} + t')}{\partial t} = \left[1 - N\left(\frac{\bar{p} - p(t')}{\sigma\sqrt{t'}}\right) + N\left(\frac{-\bar{p} - p(t')}{\sigma\sqrt{t'}}\right)\right]\ 2\ \frac{\partial p(t')}{\partial t}\ \frac{n\left(\frac{p(t')}{\sigma\sqrt{\hat{\tau}}}\right)}{\sigma\sqrt{\hat{\tau}}}\ ,$$

for $\underline{\tau} < t' < \hat{\tau}$. The first term in square brackets is the fraction of those firms that had price gap $\tilde{p} > 0$ at time $\hat{\tau}$ and that after the second review are outside the range of inaction, and hence adjust their price. [27] The remaining term counts the number of firms that have a price gap $\tilde{p} = p(t')$ so that they will adjust their price at $\hat{\tau} + t'$. This, in turn, is made of two terms. The second ratio is the density of innovations from time zero to time $\hat{\tau}$ necessary to end up in the required value of the price gap $p(t')$. The derivative, $\partial p(t')/\partial t$, comes from a change of variables formula, to convert the density of prices into a density expressed with respect to

[27] This expression is multiplied by 2 to include the firms with $\tilde{p} < 0$ at time $\hat{\tau}$.

times. If $\hat{\tau} + \underline{\tau} > 2\hat{\tau}$, the expression for $\partial S'(t' + \hat{\tau})/\partial dt$ is valid for all $t \in [\hat{\tau} + \underline{\tau}, 2\hat{\tau}]$. In this case, since the symmetry of $\tau(\cdot)$ implies that $\partial \tau(0)/\partial p = 0$, then $\partial p(\hat{\tau})/\partial t = \infty$, and thus the hazard rate tends to infinity at the end of this interval, and reverts to zero afterwards. If this conditions is not satisfied, the expression for the derivative of S for values higher than $\hat{\tau} + 2\underline{\tau}$ is more complex because a price change can occur at exactly the same time after two or three reviews. We summarize these results in the next proposition:

Proposition 13. The hazard rate of price adjustments starts flat at zero, it jumps to infinity at $t = \hat{\tau}$, it returns flat to zero in the segment $t \in (\hat{\tau}, \hat{\tau} + \underline{\tau})$, it jumps to a positive value at $\hat{\tau}+\underline{\tau}$. In the segment $[\hat{\tau}+\underline{\tau}, \hat{\tau}+\min\{\hat{\tau}, 2\underline{\tau}\})$ stays strictly positive and, if $\underline{\tau} > (1/2)\hat{\tau}$, tends to infinity at the end of this segment and returns to zero:

$$
\begin{aligned}
\boldsymbol{h}(t) &= 0, \text{ for } t \in [0, \hat{\tau}), \\
\boldsymbol{h}(\hat{\tau}) &= \infty, \\
\boldsymbol{h}(t) &= 0, \text{ for } t \in (\hat{\tau}, \hat{\tau} + \underline{\tau}), \\
\lim_{t \downarrow \ \hat{\tau}+\underline{\tau}} \boldsymbol{h}(t) &> 0, \\
0 < \boldsymbol{h}(t) &< \infty, \text{ for } t \in [\hat{\tau} + \underline{\tau} \ , \hat{\tau} + \min\{\hat{\tau}, 2\underline{\tau}\}) \\
\text{if } \hat{\tau} &\leq 2\underline{\tau} : \lim_{t \uparrow \ 2\hat{\tau}} \boldsymbol{h}(t) = \infty, \\
\text{if } \hat{\tau} &< 2\underline{\tau} : \boldsymbol{h}(t) = 0, \text{ for } t \in (2\hat{\tau}, \hat{\tau} + 2\underline{\tau}) \ .
\end{aligned}
$$

The previous proposition does not characterize the survivor function or the hazard rate when durations are longer than $\hat{\tau} + \min\{\hat{\tau}, 2\underline{\tau}\}$. While an expression can be developed for the remaining values of elapsed times it becomes increasingly complex because, for higher values of the elapsed time t, a price change can happen after several combinations of previous reviews. Indeed the larger the value of t, the larger the number of combinations of different duration of previous reviews that can happen. The effect of this feature is that the hazard rate for larger values of elapsed time t will tend to be lower but without the 'holes', i.e. stretches with zero values that we have identified for low values of t in Proposition 13. We illustrate these features in Figure 6 that displays the *weakly* hazard rate based on a large number of simulations of the decision rule for the model. The red step-function like is the weakly hazard rate obtained through simulations. In the simulations we compute a weakly hazard rate, i.e. the fraction of price changes that occur during a week. We use this period because it is the shortest time unit that can be estimated in actual price data, and that

Figure 6: Hazard Rate of Price Changes, weakly rates

Note: Benchmark parameter values for the model with two cost are: $B = 20$, $\rho = 0.02$, $\sigma = 0.2$, $\psi = 0.03$, $\phi = 0.06$; in the model with menu cost only and observation cost only, we set $\psi = 0.06$, $\phi = 0$, and $\psi = 0$, $\phi = 0.18$ respectively, while leaving the other parameters unchanged, so to obtain the same frequency of price adjustment of the model with both costs, i.e. 1.5 adjustments a year.

avoids the infinite hazard rate at $\hat{\tau}$. [28] The black line is the analytical counterpart of the hazard developed in Proposition 13 rate and expressed at weekly frequencies. We include dotted vertical lines every $\hat{\tau}$ periods to note that every $\hat{\tau}$ periods there is a "wave" of price adjustments. At $t = \hat{\tau}$ all the adjustments occur simultaneously, so the hazard rate has a spike. In the subsequent waves, i.e. for larger values of t, they are less concentrated around a single value, and hence the hazard rates have smaller spikes, but are higher in a wider range.[29] We compare the shape of the hazard rate function with the ones of the models with observation cost only, which is given by a single spike (in green) and the model with menu cost only (in blue).[30] The latter is characterized by an increasing hazard rate in the first weeks, but quickly converging to a constant rate.

There is no consensus in the existing literature about the shape of the hazard function. For instance, both Klenow and Kryvtsov (2008) and Nakamura and Steinsson (2008) show that the hazard rate computed on U.S. CPI data is downward sloping, being at odd with menu cost models of price adjustment. Similar evidence is reported by Alvarez et al. (2005) for the Euro area. However, when adjusting for heterogeneity Klenow and Kryvtsov (2008) and Nakamura and Steinsson (2008) reach opposite conclusions. The former finds a flat hazard rate, while the latter finds a downward sloping hazard despite adjusting for heterogeneity. Finally, a recent study by Cavallo (2009) shows that the hazard rate is upward sloping in four Latin America countries.[31] As shown above, a model with menu cost only produces an upward sloping hazard. However, heterogeneity could account for the observed downward sloping hazard rate if data was generated by this model. Our model with both observation and menu costs predicts a non-monotone hazard. Among other things, this implies that an econometrician estimating the product level hazard rate on data generated from our model could obtain an upward sloping or a downward sloping hazard rate depending on the length of the sample, and on the frequency of observations within that sample, even in absence of heterogeneity. For instance, an econometrician observing prices every $\hat{\tau}$ units of time would estimate a downward sloping hazard rate. However, if for instance the econometrician were only observing prices at any frequency between $\hat{\tau} + \underline{\tau}$ and $2\hat{\tau}$ he would obtain an upward sloping hazard.

[28]Since at $\hat{\tau}$ there is a fraction of firms adjusting all at that instant, in any finite interval, such as a week, the hazard rate will be finite.

[29]We note that if the hazard rate where computed for changes during a month, as opposed to changes during a week, the differences will be smaller.

[30]The examples plotted for the three models have the same average number of 1.5 price changes per year.

[31]See Section D.4 for more details on empirical evidence about the hazard rate.

7 The case of positive inflation

In this section we briefly discuss the effect of inflation $\pi > 0$, the drift of the (log of the) target price p^* into the solution of the problem studied above. First we establish that in the presence of inflation, immediately after an adjustment the price gap will be positive, i.e. the agent will set a price higher than value of the target at the time of the adjustment. This is quite intuitive, since the agent expect to adjust in the future and hence it compensate for the forecastable part of the deviation due to inflation. Indeed, the optimal value for $\hat{p}$ is to exceed the current value of the target p^* by an amount that is approximately equal to the expected value that $p^*(t)$ will take at approximately half of the expected time until its next adjustment.

For the next proposition set, without loss of generality, the current time at $t = 0$, and assume that the agent will adjust the price at this time. Also, without loss of generality we set $p^* = 0$. Let $\bar{T}$ be a stopping time indicating the time of the next price adjustment. Notice that $\bar{T}$ is the sum of the time elapsed between consecutive reviews with not adjustment plus the time until review and adjustment.

Proposition 14. The optimal price set after an adjustment exceeds the value of the target p^* if and only if the drift $\pi > 0$. Moreover, for small ρ the optimal value of the initial price gap is equal to one half of the inflation rate times the expected time until next adjustment, weighted by length of the adjustment times, or:

$$\hat{p} \;=\; \pi \, \mathbb{E}\left[\left(\frac{\bar{T}}{\mathbb{E}\left(\bar{T}\right)}\right)\frac{\bar{T}}{2}\right] \; . \tag{47}$$

The terms in equation (47) are intuitive if you first consider the case where the time to the next adjustment is deterministic, so that $\bar{T}$ has a degenerate distribution. In this case we obtain: $\hat{p} = \pi \; \bar{T}/2$, so that the initial gap is equal to the value that the target will have exactly at half of the time until the next adjustment.

7.1 "Large" Inflation

In this section we consider the case where π is large relative to σ. I particular, we consider the limit when $\sigma \downarrow 0$ and $\pi > 0$. We study this case as an approximation of the solution when inflation is large relative to the volatility of the idiosyncratic shocks.

In such a case, i.e. when there is no uncertainty the optimal policy is to review *once*, and after the first price adjustment the optimal policy is to adjust prices every τ^* periods, by exactly the same amount. This is version of the classical price adjustment model by Sheshinski and Weiss (1977). The optimal policy can also be written in terms of the price gap, following a sS band, adjusting the price when a lower barrier $\underline{p}$ is reached, to a value given by $\hat{p}$. We can also add a function $\tau(\tilde{p})$ function indicating the time of the next review when the current price gap is $\tilde{p}$.

PROPOSITION 15. Let $\sigma = 0$ and $\pi > 0$. As we let $\rho \downarrow 0$ the optimal decision rule satisfy:

$$\begin{aligned} n_a &= \left(\frac{B\ \pi^2}{6\ (\psi + \phi)}\right)^{1/3},\ \underline{p} = -\frac{1}{2}\left(\frac{6\ \pi\ (\psi+\phi)}{B}\right)^{1/3},\ \hat{p} = \frac{1}{2}\left(\frac{6\ \pi\ (\psi+\phi)}{B}\right)^{1/3}, \\ \frac{\hat{p} - \underline{p}}{\bar{p} - \hat{p}} &= \sqrt{3}\ ,\ \tau(\tilde{p}) = \frac{\tilde{p} - \underline{p}}{\pi}\ \text{ if } \tilde{p} \in [\underline{p},\ \bar{p}] \text{ and } \tau(\tilde{p}) = 1/n_a \text{ otherwise.} \end{aligned} \tag{48}$$

In the deterministic case with $\phi > 0$, the reviews will never be conducted unless there is an adjustment, and this is precisely the logic behind the linear decreasing shape of the function $\tau(\cdot)$ in the proposition. In the range of inaction, $\tau(\tilde{p})$ is exactly the time it takes until the next adjustment. This is in stark contrast with the symmetric shape of $\tau(\cdot)$ in the case with no drift. In this deterministic case the cost of each adjustment is given by the sum $\phi + \psi$, due to our assumption that a review has to be conducted at the time of an adjustment. In the limit case of Proposition 15 the frequency of adjustments has an elasticity of $1/3$ with respect to the cost $\phi + \psi$. This is different from the the case of zero drift, where the number of adjustment have $1/2$ elasticity with respect to an equal proportional change in the costs ϕ and ψ. The optimal return point is strictly positive, and increasing in inflation, an application of the general result of Proposition 14. The optimal return point $\hat{p}$ and the lower bound of the range of inaction $\underline{p}$ are at the same distance of the value that minimizes the static price gap, i.e. $\hat{p} = -\underline{p}$. This feature is very intuitive, since for $\rho = 0$ the agent gives the same weight to the deviations that occur just after adjusting as those just before the next adjustment. This also differs from the optimal return of zero of the model with no drift. The boundaries of the range of inaction are asymmetric: $(\hat{p} - \underline{p})/(\bar{p} - \hat{p}) = \sqrt{3} > 1$. This asymmetry is due to the fact that $\hat{p} > 0$ already takes into account the effect of positive inflation, and hence at $\bar{p}$ the deviation from the static optimum value of $\tilde{p}$ are very large.

Our interest in the function $\tau(\cdot)$ and the thresholds $\bar{p}, \hat{p}$ and $\underline{p}$ in this limit is that it should be informative about the shape of $\tau(\cdot)$ for very small, but strictly positive, values of σ, and in general for the forces that operates in the general case of π and σ strictly positive.

To evaluate the extent to which the features discussed in this limit are present in the case of strictly positive π and σ Figure 7 displays three panels where, for fixed values of all

Figure 7: Policy rule $\tau(\tilde{p})$ under different values of drift, $\pi = 0.05, 0.2, 0.6$

Note: Benchmark parameter values are: $B = 20$, $\sigma = 0.2$, $\psi = 0.03$, $\rho = 0.02$, $\phi = 0.06$; each panel shows optimal $\tau(\tilde{p})$ under a given value of π.

the parameters –including $\sigma = 0.2$–, we display the shape of the optimal decision rules for three strictly positive levels of the inflation rate. It is clear that for inflation rate of 4% the difference with the shape for zero inflation is very small, but already we can see that the range of inaction starts being asymmetric, that the optimal return point is positive, and that $\tau(\cdot)$ is asymmetric, with a peak at the right of the optimal return point, and with $\tau(\underline{p}) < \tau(\bar{p})$. These features are more apparent for 20 % inflation rate, and very descriptive for 80 % annual inflation rate.

8 Price adjustments with no observations

In this section we relax the assumption on the 'technology' to change prices that at the time of a price adjustment agents must observe the state. We show that in the case where the state has no drift it is indeed optimal to observe every time a price is adjusted. On the other hand, as the volatility goes to zero relatively to the drift it is optimal to do some adjustments without observing the state.

We consider an extension of the problem where upon paying a cost ϕ, and finding the value of the price gap $\tilde{p}$ in a review the firm can decide whether to immediately change the price, paying the cost ψ or not. The extension consist of allowing the firm to consider to adjust several times it price until the next review. For these adjustments the firm pays only the cost ψ. In this case the Bellman equations for a firm just after finding the value of price gap $\tilde{p}$ satisfy:

$$
\begin{aligned}
V_n(\tilde{p}) = \phi + & \min_{\tau_0, \{\hat{p}_i, \tau_i\}_{i=1}^n} \int_0^{\tau_0} e^{-\rho t} B \ \left[(\tilde{p} - \pi t)^2 + \sigma^2 t\right] dt \\
& + \sum_{i=1}^{n} \left[e^{-\rho \tau_{i-1}} \psi + \int_{\tau_{i-1}}^{\tau_i} e^{-\rho t} B \left[(\hat{p}_i - \pi t)^2 + \sigma^2 t\right] dt \right] \\
& + e^{-\rho \tau_n} \int_{-\infty}^{\infty} V \left(\hat{p}_n - \pi \tau_n - s\sqrt{\tau_n}\sigma\right) dN(s) \ , \qquad (49) \\
V(\tilde{p}) = & \min_{n \geq 0} V_n(\tilde{p}) \qquad (50)
\end{aligned}
$$

where $0 \leq \tau_i \leq \tau_{i+1}$. The function $V_n(\tilde{p})$ gives the optimal conditional on making n price adjustment before the next review, which will take place τ_n units of time from the current review. The first term in the right hand side of $V_n(\tilde{p})$ correspond to the loses incurred if there is no immediate price adjustment. Notice that if the firm chooses $\tau_0 = 0$, the time for which this loss is incurred is nil, and hence a price adjustment is conducted immediately. The next sum contains the losses corresponding to the n price adjustments. These n price adjustments are conducted using only the information of the initial price gap $\tilde{p}$. Notice that $V_0(\tilde{p})$ is the value of conducting a review and no price adjustment.[32] The value function V minimizes the cost by choosing whether to have an immediate price adjustment or not, and by minimizing on the number of price adjustments conducted between reviews. If we were to restrict this problem to have to have $n = 1$ if $\tau_0 = 0$, we will obtain exactly the same value functions as considered in the previous sections.

We remark that as $\psi \downarrow 0$, then the firm will be adjusting infinitely often, so that the optimal number of adjustment between reviews will have $n \to \infty$ and the adjustment will be converging to $\hat{p}_i = \pi \tau_i$. In this case the value function will converge to the value function for the problem with observation cost only of Section 4.1 and no drift on the state, i.e. $\pi = 0$. Thus, as the menu cost is small, and the drift is large, the optimal policy will involve more adjustments than reviews, contrary to what is found in the survey evidence discussed in Section 3.

The next proposition shows that if π is sufficiently close to zero, there is at most one price adjustment between reviews, i.e. the firm will not find optimal to adjust without finding the price gap. The logic of the result is simplest in the case of $\pi = 0$. In this case, it is immediate to see that the value function V is symmetric around zero. Using this

[32] In the previous section we refer to this as $\bar{V}$.

symmetry it is easy to see that, if a price adjustment will take place in the problem for $V_n(\tilde{p})$, the price gap will be adjusted to zero.[33] Thus, as long as the menu cost ψ is strictly positive, if $\pi = 0$ it is optimal to have at most one price adjustments between reviews, i.e. $V_n(\tilde{}) > \min\{V_0(\tilde{p}), V_1(\tilde{p})\}$. Moreover fixing $\psi > 0$, using continuity of the return function and law of motion with respect to the drift rate π one can make an argument along the lines of the theorem of the maximum, that if π is small, the value function will be close to the one corresponding to $\pi = 0$. On the other hand, having more than one adjustment between reviews will increase the cost by a discrete amount unrelated to π if $\psi > 0$, so the argument extend to this case. We have sketched the proof of the following proposition:

Proposition 16. *Fix all the parameters, including $\psi > 0$. Then there is a strictly positive inflation rate $\bar{\pi}$ such that for all inflation rates $|\pi| < \bar{\pi}$ it is optimal to adjust prices at most once between successive reviews.*

We note that if $0 < \pi < \bar{\pi}$, there is at most one adjustment between observations, but the decision rules are not necesarily the same as the ones for the problem with $\pi = 0$. In particular what can be different is that immediate adjustment after an observation, as the one considered in $\bar{V}$ of the previous section, may not take place. Instead, there will be an observation and no adjustment, followed by an adjustment without observation. To see why this may be better, consider the case of an observation o the price gap $\tilde{p}$ outside the range of inaction, an immediate adjustment to $\hat{p}$, and an subsequent observation $\hat{\tau}$ periods from now. This has an immediate cost $\phi + \psi$ and a period return

$$B\int_0^{\hat{\tau}} e^{-\rho t}[(\hat{p} - \pi t)^2 + \sigma^2]dt + e^{-\rho\hat{\tau}}\int_{-\infty}^{\infty} V(\hat{p} - \hat{\tau}\pi - s\sqrt{\hat{\tau}}\sigma)dN(s)$$

Now set $0 < \tau_0 < \tau_1 = \hat{\tau}$ then we have

$$B\int_0^{\tau_0} e^{-\rho t}[(\tilde{p} - \pi t)^2 + \sigma^2]dt + B\int_{\tau_0}^{\hat{\tau}} e^{-\rho t}[(\hat{p} - \pi t)^2 + \sigma^2]dt + e^{-\rho\hat{\tau}}\int_{-\infty}^{\infty} V(\hat{p} - \hat{\tau}\pi - s\sqrt{\hat{\tau}}\sigma)dN(s)$$

[33] This can be seen as a straightforward modification of the argument in the proof of Proposition 14.

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A Price Setting in the frictionless case

We consider two simple static problems for a monopolist firm. The first has a linear system, the second is a log approximation to a general case.

A.1 Linear Demand

Let the demand for a product be $q(p) = a - b\ p$ where q are quantities and p is the price, a the intercept, b is the slope. Assume that the marginal cost is constant at the value of c. Thus profits are $\Pi(p) = (a - b\ p)(p - c)$, and the static monopoly price is given by

$$p^* = \frac{a + c\ b}{2b} \ ,$$

and the maximized profits can be written as $\Pi^* = b\ (p^*)^2 - c\ a$, or

$$\Pi(p) = \Pi^* + \frac{1}{2}(-2\ b)(p - p^*)^2 = b\ (p^*)^2 - c\ a - b\ (p - p^*)^2 \ .$$

A.2 Log approximation of monopolist profit function

Let $\Pi(P)$ be the profit function of the monopolist as a function of the price P. Let P^* be the optimal price, satisfying $\Pi_P(P^*) = 0$. Consider a second order approximation of the log of Π around $P = P^*$ obtaining:

$$\log \Pi(P) = \log \Pi(P^*) + \frac{1}{2} \frac{\partial^2\ \log \Pi(P)}{\partial(\log P)^2}\bigg|_{P=P^*} (\log P - \log P^*)^2 + o\left((\log P - \log P^*)^2\right) \ .$$

A useful example for this approximation is the case with a constant elasticity of demand equal to $\eta > 1$ where: $q(P) = E\ P^{-\eta}$ where E is a demand shifter and where the monopolist faces a constant marginal cost c, where:

$$P^* = \frac{\eta}{\eta - 1}\ c \ \text{ and } \ \frac{\partial^2\ \log \Pi(P)}{\partial(\log P)^2}\bigg|_{P=P^*} = -\eta\ (\eta - 1)$$

So, letting $B = -\eta\ (\eta - 1)$, $p = \log P$ and $p^* = \log P^*$ we obtain the problem in the body of the paper.

B Hazard Rate of Menu Cost Model

In this appendix we described the details for the characterization of the hazard rate of price adjustments of the menu cost model of Section 4.2. Section 2.8.C formula (8.24) of Karatzas and Shreve (1991) displays the density of the distribution for the first time that a brownian motion hit either of two barriers, starting from an arbitrary point inside the barriers. In our case, the initial value is the price gap after adjustment, namely zero, and the barriers are symmetric, given by $-\bar{p}$ and $\bar{p}$. We found more useful for the characterization of the hazard rate to use a transformation of this density, obtained in Kolkiewicz (2002), section 3.3, as the sum of expressions (15) and (16). In our case we set the initial condition $x_0 = 0$ and the barriers $a < x_0 < b$ are thus given by $a = -\bar{p}$ and $b = \bar{p}$, thus obtaining the density $\boldsymbol{f}(t)$:

$$\boldsymbol{f}(t) = \frac{\boldsymbol{\pi}}{2\ (\bar{p}/\sigma)^2} \sum_{j=0}^{\infty} (2j+1)(-1)^j \ \exp\left(-\frac{(2j+1)^2 \boldsymbol{\pi}^2}{8\ (\bar{p}/\sigma)^2}\ t\right) . \tag{A-1}$$

The hazard rate $\boldsymbol{h}(t)$ is then defined as:

$$\boldsymbol{h}(t) = \frac{\boldsymbol{f}(t)}{\int_t^{\infty} \boldsymbol{f}(s) ds} . \tag{A-2}$$

Notice that since equation (A-1) is a sum of exponentials evaluated at the product of $-t$ times a positive quantity, each of them larger. Thus, for large values of t the first term in the sum dominates, and hence the expression for $\boldsymbol{f}(t)$ becomes

$$\boldsymbol{f}(t) \approx \frac{\boldsymbol{\pi}}{2\ (\bar{p}/\sigma)^2} \exp\left(-\frac{\boldsymbol{\pi}^2}{8\ (\bar{p}/\sigma)^2}\ t\right) \ \text{ for large } \ t\ . \tag{A-3}$$

and hence $\lim_{t\to\infty} \boldsymbol{h}(t) = \frac{\boldsymbol{\pi}^2}{8\ (\bar{p}/\sigma)^2}$. Indeed, the shape of this hazard rate is independent of $\bar{p}/\sigma$, this value only scales it up and down. Moreover, as Figure A-1 shows, the asymptote is approximately attained well before the expected value of the time.

Figure A-1: Hazard Rate of Menu Cost Model

Note: $B = 25, \sigma = 0.1$ and $\psi = 0.04$.

C Proofs

C.1 Proof of Proposition 1.

Proof. Note that as $\rho \to 0$ we have:

$$\frac{\rho e^{-\rho t}}{1-e^{-\rho\tau}} \to \frac{1}{\tau} \text{ so } x^* \to \frac{1}{\tau}\int_0^\tau t\, dt = \frac{\tau}{2} \quad , \text{ and } \int_0^\tau e^{-\rho t}\, t\, dt \to \tau \int_0^\tau t\, \frac{1}{\tau} dt = \frac{\tau^2}{2} \; .$$

$$v(\tau) \to \int_0^\tau \left(\frac{\tau}{2} - t\right)^2 dt = \tau \int_0^\tau \left(\frac{\tau}{2} - t\right)^2 \frac{1}{\tau} dt = \tau \frac{\tau^2}{12} = \frac{\tau^3}{12} \; .$$

Thus

$$\lim_{\rho\to 0} \rho V(\tau) = \lim_{\rho \to 0} \left(\frac{\pi^2}{\sigma^2} \frac{\rho\tau^3}{12\,(1-e^{-\rho\tau})} + \frac{\rho\, \tau^2}{2\,(1-e^{-\rho\tau})} + \frac{\tilde{\phi}\rho}{(1-e^{-\rho\tau})} \right) = \frac{\pi^2}{\sigma^2}\frac{\tau^2}{12} + \frac{\tau}{2} + \frac{\tilde{\phi}}{\tau}$$

The f.o.c. for τ is:

$$\frac{\tilde{\phi}}{(\tau^*)^2} = \frac{\pi^2}{\sigma^2}\frac{\tau^*}{6} + \frac{1}{2} \quad \text{or} \quad \tilde{\phi} = \frac{\pi^2}{\sigma^2}\frac{(\tau^*)^3}{6} + \frac{(\tau^*)^2}{2}$$

From here we see that the optimal inaction interval τ^* is a function of 2 arguments, it is increasing in the normalized cost $\tilde{\phi}$, and decreasing in the normalized drift $(\pi/\sigma)^2$. Keeping the parameters B, ϕ and π constant we can write:

$$\frac{\phi}{B} = \pi^2 \, \frac{(\tau^*)^3}{6} + \sigma^2 \frac{(\tau^*)^2}{2}$$

which implies that τ is decreasing in σ.

Note that for $\pi = 0$ we obtain a square root formula on the cost ϕ and with elasticity minus one on σ:

$$\tau^* = \sqrt{2\,\tilde{\phi}} = \sqrt{2\, \frac{\phi}{B\, \sigma^2}} = \sqrt{\frac{2}{B}}\; \phi^{1/2}\, \frac{1}{\sigma} \; .$$

The total differential of the foc for τ gives:

$$\tau^*(\tilde{\phi}) \left(\frac{\pi^2}{\sigma^2} \frac{\tau^*(\tilde{\phi})}{2} + 1 \right) \left(\frac{\partial \tau^*(\tilde{\phi})}{\partial \tilde{\phi}} \right) = 1$$

since $\lim_{\tilde{\phi}\to 0} \tau^*(\tilde{\phi}) = 0$, then one obtains the same expression than in the case of $\pi = 0$, and thus the elasticity is 1/2, or: $\lim_{\tilde{\phi}\to 0} \frac{\tilde{\phi}}{\tau^*(\tilde{\phi})} \frac{\partial \tau^*(\tilde{\phi})}{\partial \tilde{\phi}} = \frac{1}{2}$. To see the result for $\sigma = 0$, let us write:

$$\tilde{\phi}\; \sigma^2 \equiv \frac{\phi}{B} = \pi^2 \, \frac{(\tau^*)^3}{6} + \sigma \frac{(\tau^*)^2}{2} \; ,$$

then we let $\sigma^2 \to 0$ to get $\frac{\phi}{B} = \pi^2 \, \frac{\tau^3}{6}$ which implies a cubic root formula on the cost ϕ and with elasticity $-2/3$ on π:

$$\tau^* = \left(\frac{6\ \phi}{B\ \pi^2}\right)^{1/3} = \left(\frac{6}{B}\right)^{1/3}\ \phi^{1/3}\ \pi^{-2/3}.$$

Now consider the case when $\rho > 0$ and $\pi = 0$ then $\rho V(\tau)$ equals

$$\rho V(\tau) = \frac{\rho\tilde{\phi}}{1-e^{-\rho\tau}} - \frac{\tau e^{-\rho\tau}}{1-e^{-\rho\tau}} + \frac{1}{\rho}\ .$$

The first order condition with respect to τ implies that the optimal choice satisfies:

$$\tilde{\phi} = \frac{\rho\tau^* - 1 + e^{-\rho\tau^*}}{\rho^2}. \tag{A-4}$$

A third order expansion of the right hand side of equation (A-4), gives:

$$\tilde{\phi} = \frac{1}{2}\tau^2 - \frac{1}{6}\rho\tau^3 + o(\rho^2\tau^3).$$

The expression shows that if $\rho = 0$ we obtain a square root formula: $\tau^* = \sqrt{2\ \tilde{\phi}}$, and that the optimal τ^* is increasing in ρ provided ρ or $\tilde{\phi}$ are small enough. ■

C.2 Proof of Proposition 2.

Proof. Differentiating the Bellman equation and evaluating it at zero we obtain:

$$\rho V''(0) = 2B + \sigma^2/2V''''(0) \tag{A-5}$$

and evaluating this expression for $\rho = 0$ we have

$$V''''(0) = -\frac{2B}{\sigma^2/2}\ . \tag{A-6}$$

Differentiating the quartic approximation equation (11), evaluating at $\bar{p}$ and imposing the smooth pasting equation (9) we obtain:

$$0 = V''(0)\bar{p} + \frac{1}{3\ 2}V''''(0)\ \bar{p}^3\ . \tag{A-7}$$

Replacing into this equation the expression for $V''''(0)$ in equation (A-6) and solving for $V''(0)$ we obtain

$$V''(0) = -\frac{1}{3\ 2}V''''(0)\bar{p}^2 = \frac{1}{3\ 2}\frac{2B}{\sigma^2/2}\bar{p}^2\ . \tag{A-8}$$

Using the quartic approximation into the (levels) of equation (9) we obtain:

$$\psi = \frac{1}{2}V''(0)\bar{p}^2 + \frac{1}{4!}V''''(0)\bar{p}^4\ , \tag{A-9}$$

replacing into this equation $V''(0)$ from equation (A-8) and $V''''(0)$ from equation (A-6) we obtain

$$\psi = \left(\frac{1}{2\ 3\ 2} - \frac{1}{4\ 3\ 2} \right) \frac{2B}{\sigma^2/2}\ \bar{p}^4 = \frac{B}{6\ \sigma^2}\ \bar{p}^4\ , \tag{A-10}$$

thus solving for $\bar{p}$ we obtain the desired expression. ■

C.3 Proof of Proposition 3.

Proof. We start with a simple preliminary result to set up the analysis showing that the fixed point of V coincides with the solution of the sequence problem. Denote by $V_{\text{info},\phi+\psi}$ the solution of the problem with observation cost only of Section 4.1, but where the observation cost has been set to $\phi + \psi$, and denote the optimal value of the time between observations and price changes as τ_i. We interpret this as the value of following the feasible policy of observing and adjusting ($\chi_{T_i} = 1$ all T_i) every τ periods, and hence $V(\tilde{p}) \leq V_{\text{info},\phi+\psi}$. Also denote by $V_{\text{menu},\psi}(\cdot)$ the solution of the problem with menu cost ψ and no observation cost analyzed in Section 4.2. Since the observation cost is set to zero, this provides a lower bound for the value function: $V(\tilde{p}) \geq V_{\text{menu},\psi}(\tilde{p})$. Letting $\mathbf{T}$ the operator defined by the right side of equation (16), equation (17) and equation (18), and by $\mathbf{T}^n V_0$ the outcome of n successive applications of $\mathbf{T}$ to an initial function V_0, we have that for each $\tilde{p}$:

$$V_{\text{menu},\psi}(\tilde{p}) \leq \mathbf{T}^n V_{\text{menu},\psi}(\tilde{p}) \leq \mathbf{T}^n V_{\text{info},\phi+\psi}(\tilde{p}) \leq V_{\text{info},\phi+\psi}(\tilde{p})\ ,$$

and the two sequence of functions converge pointwise. Since they converge to a finite value, their limit must be the same, by an adaptation of Theorem 4.14 in Stokey and Lucas (1989). Thus:

$$V(\tilde{p}) = \lim_{n\to\infty} \mathbf{T}^n V_{\text{info},\phi+\psi}(\tilde{p}) = \lim_{n\to\infty} \mathbf{T}^n V_{\text{menu},\psi}(\tilde{p})\ .$$

pointwise. Furthermore, since $V_{\text{info},\phi+\psi}(\tilde{p})$ is a constant function, i.e. independent of $\tilde{p}$, and since $V_{\text{menu},\psi}(\tilde{p}) > 0$ for all $\tilde{p}$, then we have that the value function $V(\tilde{p})$ is uniformly bounded.

We sketch the argument to show that the value function V is continuous on $\tilde{p}$. Suppose not, that there is jump down at $\bar{p}$ so that $V(\bar{p}) > \lim_{p\downarrow\bar{p}} V(\bar{p})$. Then, fixed the policies that correspond to a value of $p > \bar{p}$ in terms of stopping times and prices $\hat{p}(\cdot)$ as defined in the sequence formulation of equation (14). Thus, the agent will observe, starting with $\bar{p}$, after the same cumulated value of the innovations on p^*, and it will adjust to the value Let $\epsilon = p - \bar{p}$ the difference between these two prices at time zero, and denote by $\{p(t)\}, \{\bar{p}(t)\}$ the stochastic process for the price that follow from the two initial prices. Notice that $\bar{p}(t) - p(t) = \epsilon$ for all $t > 0$. By following that policy when the initial price is $\bar{p}$, the expected discounted value of fixed cost paid are exactly the same for the two initial prices. Thus, the difference of the value function at $p(0)$ and at $\bar{p}(0)$ is given by the

$$\frac{\epsilon^2}{\rho} + \epsilon\ \mathbb{E}_0\Big[\int_0^\infty e^{-\rho t}\left(p(T_i) - p^*(t)\right) dt\ \Big]\ .$$

Since this difference is clearly continuous on ϵ and goes to zero as $\epsilon \downarrow 0$, then the value function is continuous.

Now we use the fact that V is bounded above uniformly and continuous to show that for

$\phi > 0$, then the optimal policy for $\tau(\tilde{p})$, is uniformly bounded away from zero. Here is a sketch of the argument. Suppose, as a way of contradiction, that for any $\epsilon > 0$, we can find a $\tilde{p}$ for which $\tau(\tilde{p}_\epsilon) < \epsilon$. We will argue that for small enough ϵ it is cheaper to double $\tau(\tilde{p}_\epsilon)$. The main idea is that this decreases the fixed by $e^{-\rho\epsilon}\phi$, and increases the cost due to different information gathering in a quantity that is a continuous function of ϵ. The reason why the second part is continuous as function of ϵ is that the distribution of value of $p^*(t+\epsilon)$ and $p^*(t+2\epsilon)$ have most of the mass concentrated in a neighborhood of $p^*(t)$. The effect due to the increase in cost due to evaluation of the period return objective function is small, since it is the expected value of the integral of a bounded function between 0 and ϵ or between 0 and 2ϵ. Thus, for small ϵ this difference is small. The effect on the value function of the mass $p^*(t)$ is small, because the value function is uniformly bounded and this probability is small, i.e. goest to zero as $\epsilon \downarrow 0$. For the mass that is in the neighborhood of $p^*(t)$, the effect in the value function is small because the value function is continuous.

Using that $\inf \tau(\tilde{p}) \equiv \underline{\tau} > 0$, by Blackwell's sufficient conditions, we obtain that $\mathbf{T}$ is a contraction of modulus $\exp(-\underline{\tau}\rho)$ in the space of continuous and bounded functions. ■

C.4 Proof of Proposition 4.

Proof. Under the conjecture that $\hat{p} = 0$ and that $V(\cdot)$ is symmetric around zero, and by the symmetry of the normal density, we can rewrite the Bellman equations (17) and (18) using only the positive range for $\tilde{p} \in [0\ ,\ \infty)$ as:[34]

$$\bar{V}(\tilde{p}) = \phi + \min_{\tau} B \int_0^{\tau} e^{-\rho t}\left[\tilde{p}^2 + \sigma^2 t\right] dt \ + \\ e^{-\rho\tau} \int_{-\tilde{p}/(\sigma\sqrt{\tau})}^{\infty} V\left(\tilde{p} + s\sigma\sqrt{\tau}\right) dN(s) \quad + e^{-\rho\tau} \int_{\tilde{p}/(\sigma\sqrt{\tau})}^{\infty} V\left(-\tilde{p} + s\sigma\sqrt{\tau}\right) dN(s) \tag{A-11}$$

$$\hat{V} = \psi + \phi + \min_{\hat{\tau}} B \int_0^{\hat{\tau}} e^{-\rho t}\left[\sigma^2 t\right] dt + e^{-\rho\hat{\tau}}\ 2 \int_0^{\infty} V\left(s\sigma\sqrt{\hat{\tau}}\right) dN(s) \tag{A-12}$$

We use the corollary of the contraction mapping theorem. First, notice that if the V in the right side of equation (20) is symmetric around $\tilde{p} = 0$, with a minimum at $\tilde{p} = 0$, then it is optimal to set $\hat{p} = 0$. Second, notice that if the function V in the right side of equation (19) is symmetric with a minimum at $\tilde{p} = 0$, then the value function in the left side of this equation is also symmetric, and hence V in equation (21) is symmetric. Third, using the symmetry, we show that if $V(\tilde{p})$ is weakly increasing, then the right side of equation (21) is weakly increasing. It suffices to show that $\bar{V}(\tilde{p})$ given by the right side of (A-11) is increasing in $\tilde{p}$ for a fixed arbitrary value of τ. We do this in two steps. The first step is to notice that the expression containing $\tilde{p}^2$ in (A-11) is obviously increasing in $\tilde{p}$. For the second step, without loss of generality, we assume that V is differentiable almost everywhere and compute the derivative with respect to $\tilde{p}$ of the remaining two terms involving the expectations of $V(\cdot)$ in

[34]The second line in equation (A-11) uses that $\int_{-\infty}^{\infty} V(p-s)dN(s) = \int_{-p}^{\infty} V(p+s)dN(s) + \int_{p}^{\infty} V(-p+s)dN(s)$.

(A-11). This derivative is:

$$
\begin{aligned}
& \frac{e^{-\rho\tau}}{\sigma\sqrt{\tau}}\left[V(0)\ dN\left(\frac{-\tilde{p}}{\sigma\sqrt{\tau}}\right) - V(0)\ dN\left(\frac{\tilde{p}}{\sigma\sqrt{\tau}}\right)\right] \\
+ \ & e^{-\rho\tau}\left[\int_{-\tilde{p}/(\sigma\sqrt{\tau})}^{\infty} V'(\tilde{p}+s(\sigma\sqrt{\tau}))dN(s) - \int_{\tilde{p}/(\sigma\sqrt{\tau})}^{\infty} V'(-\tilde{p}+s(\sigma\sqrt{\tau}))dN(s)\right] \\
= \ & e^{-\rho\tau}\left[\int_0^{\infty} V'(z)\frac{1}{\sigma\sqrt{\tau}}\left(dN\left(\frac{z-\tilde{p}}{\sigma\sqrt{\tau}}\right) - dN\left(\frac{z+\tilde{p}}{\sigma\sqrt{\tau}}\right)\right)\right] \\
= \ & e^{-\rho\tau}\left[\int_0^{\infty} V'(z)\frac{1}{\sigma\sqrt{\tau}\sqrt{2\ \pi}}\left(e^{-\frac{1}{2}\left(\frac{z-\tilde{p}}{\sigma\sqrt{\tau}}\right)^2} - e^{-\frac{1}{2}\left(\frac{z+\tilde{p}}{\sigma\sqrt{\tau}}\right)^2}\right)\ dz\right] \geq 0
\end{aligned}
$$

where the term involving $V(0)$ is zero due symmetry of $dN(s)$, and where the inequality follows since $e^{-\frac{1}{2}\left(\frac{x-\tilde{p}}{\sigma\sqrt{\tau}}\right)^2} - e^{-\frac{1}{2}\left(\frac{x+\tilde{p}}{\sigma\sqrt{\tau}}\right)^2} > 0$ for $x > 0$ and $\tilde{p} > 0$. Notice that the inequality is strict if $\tilde{p} > 0$ and $V'(x) > 0$ in a segment of strictly positive length. If $\tilde{p} = 0$, then the slope is zero.

Finally, differentiating the value function twice, and evaluating at $\tilde{p} = 0$ we get

$$
V''(0) = 2\ B\ \frac{1 - e^{-\rho\hat{\tau}}}{\rho} + 2\ \frac{e^{-\rho\hat{\tau}}}{\sigma\sqrt{\hat{\tau}}}\int_0^{\bar{p}} V'(z)\ z\ \frac{e^{-\frac{1}{2}\frac{z^2}{\sigma^2\hat{\tau}}}}{\sigma\sqrt{\hat{\tau}\ 2\ \pi}}\ dz > 0\ .
$$

■

C.5 Proof of Proposition 5.

Proof. First we notice that using the quadratic approximation into the definition of $\bar{p}$ given by $\bar{V}(\bar{p}) = \hat{V}$ implies

$$
\psi = \frac{1}{2}V''(0)\,(\bar{p})^2\ . \tag{A-13}
$$

Second we derive equation (23) as the first order condition for $\hat{\tau}$. To this end, use the Bellman equation (20) for a fixed $\hat{\tau} > 0$ evaluated at the optimal $\hat{p} = 0$, the symmetry of $V(\tilde{p})$, and the approximation

$$
V(\tilde{p}) = \min\{\hat{V}\ ,\ V(0) + \frac{1}{2}V''(0)\ (\tilde{p})^2\}
$$

to write:

$$
\begin{aligned}
V(0) = \hat{V} - \psi &= \phi + B\sigma^2 \int_0^{\hat{\tau}} e^{-\rho t} t \; dt + e^{-\rho \hat{\tau}} \int_{-\infty}^{\infty} V\left(s\sigma\sqrt{\hat{\tau}}\right) dN(s) \\
&= \phi + B\sigma^2 \int_0^{\hat{\tau}} e^{-\rho t} t \; dt + e^{-\rho \hat{\tau}} \; V(0) + \psi \; e^{-\rho \hat{\tau}} \; 2 \left[1 - N\left(\frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}\right)\right] \\
&+ e^{-\rho \hat{\tau}} \; \frac{V''(0)}{2} \; \int_{-\frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}}^{\frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}} \left(\sigma^2 \hat{\tau} s^2\right) \, dN(s) \\
&= \phi + B\sigma^2 \int_0^{\hat{\tau}} e^{-\rho t} t \; dt + e^{-\rho \hat{\tau}} \; V(0) + \psi \; e^{-\rho \hat{\tau}} \; 2 \left[1 - N\left(\frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}\right)\right] \\
&+ e^{-\rho \hat{\tau}} \; V''(0) \; \sigma^2 \hat{\tau} \; \int_0^{\frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}} s^2 \; dN(s)
\end{aligned}
$$

Thus

$$
\rho V(0) = \frac{\phi + B\sigma^2 \int_0^{\hat{\tau}} e^{-\rho t} t \; dt + \psi \; e^{-\rho \hat{\tau}} \; 2 \left[1 - N\left(\frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}\right)\right] + e^{-\rho \hat{\tau}} \; V''(0) \; \sigma^2 \hat{\tau} \; \int_0^{\frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}} s^2 \; dN(s)}{\left(1 - e^{-\rho \hat{\tau}}\right)/\rho}
$$

letting $\rho \downarrow 0$ gives

$$
\lim_{\rho \downarrow 0} \; \rho V(0) = \frac{\phi + \psi \; 2 \left[1 - N\left(\frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}\right)\right]}{\hat{\tau}} + B\sigma^2 \frac{\hat{\tau}}{2} + V''(0) \; \sigma^2 \; \int_0^{\frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}} s^2 \; dN(s) 98
$$

Maximizing the right side of this expression gives

$$
\begin{aligned}
0 = -\frac{\phi + \psi \; 2 \left[1 - N\left(\frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}\right)\right]}{\hat{\tau}^2} + \frac{B\sigma^2}{2} + \left(\psi \; 2 \; n\left(\frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}\right) \; \left(\frac{\bar{p}}{\sigma}\right) (\hat{\tau})^{-3/2}\right) \frac{1}{\hat{\tau}} \\
- \; V''(0) \; \sigma^2 \; \left(\frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}\right)^2 n\left(\frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}\right) \left(\frac{\bar{p}}{\sigma}\right) (\hat{\tau})^{-3/2}
\end{aligned}
$$

where we use $n(\cdot)$ for the density of the standard normal. Using that $V''(0) = 2 \; \psi \; / \; \bar{p}^2$, this expression simplifies to

$$
0 = -\frac{\phi + \psi \; 2 \left[1 - N\left(\frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}\right)\right]}{\hat{\tau}^2} + \frac{B\sigma^2}{2}
$$

rearranging and using the definition of $\hat{\varphi}$ gives $\sigma^2 \hat{\tau} = h(\hat{\varphi})$ of equation (23).

Third, we obtain an expression for $V''(0)$. Differentiating the value function twice, and evaluating it at $\tilde{p} = 0$ we get

$$
V''(0) = 2 \; B \; \frac{1 - e^{-\rho \hat{\tau}}}{\rho} + 2 \; \frac{e^{-\rho \hat{\tau}}}{\sigma\sqrt{\hat{\tau}}} \int_0^{\bar{p}} V'(z) \; z \; \frac{e^{-\frac{1}{2} \frac{z^2}{\sigma^2 \hat{\tau}}}}{\sigma\sqrt{\hat{\tau} \; 2 \; \boldsymbol{\pi}}} \; dz
$$

With a change in variable $s = z/(\sigma\sqrt{\hat{\tau}})$ we have:

$$V''(0) = 2\ B\ \frac{1-e^{-\rho\hat{\tau}}}{\rho} + 2\ e^{-\rho\hat{\tau}} \int_0^{\frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}} V'\left(\sigma\sqrt{\hat{\tau}}\ s\right)\ s\ \frac{e^{-\frac{1}{2}s^2}}{\sqrt{2\ \boldsymbol{\pi}}}\ ds\ .$$

Using the third order approximation $V(\tilde{p}) = V(0) + \frac{1}{2}V''(0)\,(\tilde{p})^2$ around $\tilde{p} = 0$ we obtain:

$$V''(0) = 2\ B\ \frac{1-e^{-\rho\hat{\tau}}}{\rho} + e^{-\rho\hat{\tau}}\ V''(0)\ 2\ \sigma\sqrt{\hat{\tau}}\ \int_0^{\frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}} s^2\ \frac{e^{-\frac{1}{2}s^2}}{\sqrt{2\ \boldsymbol{\pi}}}\ ds$$

or collecting terms:

$$V''(0) = \frac{2\ B\ \frac{1-e^{-\rho\hat{\tau}}}{\rho}}{1 - e^{-\rho\hat{\tau}}\ 2\ \sigma\sqrt{\hat{\tau}}\ \int_0^{\frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}} s^2\ \frac{e^{-\frac{1}{2}s^2}}{\sqrt{2\ \boldsymbol{\pi}}}\ ds}$$

and letting $\rho \downarrow 0$, using the definition of $\hat{\varphi}$ and N for the CDF of a standard normal:

$$V''(0) = \frac{2\ B\ \hat{\tau}}{1 -\ 2\ \sigma\sqrt{\hat{\tau}}\ \int_0^{\hat{\varphi}} s^2\ dN(s)}\ . \tag{A-14}$$

Using equation (A-13) to replace $V''(0)$ into equation (A-14), using the definition of $\hat{\varphi}$, and using $\sigma^2\hat{\tau} = h(\hat{\varphi})$ to replace $\hat{\tau}$ and $\sqrt{\hat{\tau}}$ we obtain equation (22). ■

C.6 Numerical evaluation of Proposition 5.

In this section we evaluate the accuracy of the approximated solution in Proposition 5. To do so, we solve the model numerically on a grid for $\tilde{p}$ and obtain the numerical counterparts to the policy rule derived in Proposition 5. In doing so we approximate $\bar{V}(\cdot)$ through either a cubic spline or a sixth order polynomial. Results are invariant to the latter.

As Figure A-2 - Figure A-4 show, the solution for $\bar{p}$ (and as a consequence for $\hat{\varphi}$) in Proposition 5 diverges from its numerical counterpart the more, the larger the ratio $\alpha \equiv \frac{\psi}{\phi}$ is. In particular, the approximated solution tend to understate the value of $\bar{p}$ relatively to the numerical solution.

This discrepancy is due to the nature of our approximation which relies on a second order approximation of $\bar{V}(\cdot)$, while higher orders (the fourth one in particular) become more relevant as ψ/ϕ increases, causing the inaction range to widen. To document this effect, we show the following computations. We solved the model numerically, assuming a polynomial of order sixth for $\bar{V}(\cdot)$, on a grid of values for $\tilde{p}$ for values of $\alpha = 0.1$ and $\alpha = 2$. We used the symmetry property of $\bar{V}(\cdot)$ to set the value of all the odd derivatives evaluated at $\tilde{p} = 0$ equal to zero. We then compared the numerical solution for $\bar{V}(\cdot)$ with the approximated one given by Proposition 5, but having an intercept (i.e. $\bar{V}(0)$) equal to the constant term in the sixth order polynomial. As Figure A-5 shows, the quadratic approximation for $\bar{V}(\cdot)$ works better for low values of $\tilde{p}$, and more generally for low values of α. The second order approximation for $\bar{V}(\cdot)$ tends to overstate the value of the function for values away from zero, as it ignores the fourth derivative $V''''(0)$, which is negative. While the difference in the approximation will also affect the value of $\hat{V}$, we find this effect much smaller in our computations. Therefore,

Figure A-2: Numerical and approximated $\hat{\varphi}$ as a function of $\alpha \equiv \frac{\psi}{\phi}$

Note: parameter values are $B = 20,\ \rho = 0.02,\ \sigma = 0.2, \psi = 0.03$. We let ϕ to vary.

Figure A-3: Numerical and approximated $\hat{\tau}$ as a function of $\alpha \equiv \frac{\psi}{\phi}$

Note: parameter values are $B = 20,\ \rho = 0.02,\ \sigma = 0.2, \psi = 0.03$. We let ϕ to vary.

Figure A-4: Numerical and approximated $\bar{p}$ as a function of $\alpha \equiv \frac{\psi}{\phi}$

Note: parameter values are $B = 20,\ \rho = 0.02,\ \sigma = 0.2, \psi = 0.03$. We let ϕ to vary.

the quadratic approximation tends to understate the inaction range, i.e. to produce values of $\bar{p}$ that are smaller. This is consistent with the values displayed in Figure A-4.

Figure A-5: Numerical and approximated $\bar{V}$ as a function of $\alpha \equiv \frac{\psi}{\phi}$

Note: parameter values are $B = 20,\ \rho = 0.02,\ \sigma = 0.2, \psi = 0.03$. We let ϕ to vary.

C.7 Proof of Proposition 6.

Proof. Begin defining

$$\begin{aligned} \hat{f}(\hat{\varphi}) &= \frac{\hat{h}(\hat{\varphi})\ \hat{\varphi}^2}{1 - \ 2\sqrt{h(\hat{\varphi})}\int_0^{\hat{\varphi}} s^2 dN(s)}\ , \\ \hat{h}(\hat{\varphi}) &= 2\ \ (\phi +\ 2\psi\ (1-N(\hat{\varphi})))\ \text{, so that} \\ \hat{f}(\hat{\varphi}) &= \frac{2\ \hat{\varphi}^2\ \ (\phi +\ 2\psi\ (1-N(\hat{\varphi})))}{1 - \ 2\left[2\sigma^2\frac{\phi}{B} +\ 4\sigma^2\frac{\psi}{B}\ (1-N(\hat{\varphi}))\right]^{1/4}\int_0^{\hat{\varphi}} s^2 dN(s)}\ , \end{aligned}$$

and noting that $\psi = \hat{f}(\varphi)$ is the same as the solution of equation (22) and equation (23).

First we turn to the existence and uniqueness of the solution. We show that it follows from an appliction of the intermediate function theorem, toghether with monotonicity. We show that if $\frac{\phi}{\psi} > 1/2 - 2(1-N(1)) \approx 0.1827$ then: there is a value $0 < \hat{\varphi}' \leq 1$ so that: i) the function $\hat{f}$ is continuous and increasing in $\hat{\varphi} \in [0, \hat{\varphi}')$, ii) $\hat{f}(0) = 0$, iii) $\hat{f}(\hat{\varphi}') > \psi$, iv) $\hat{f}(\hat{\varphi}) < 0$ for $\hat{\varphi} \in (\hat{\varphi}', 1]$.

The value of $\hat{\varphi}'$ is given by the minimum of 1 or the solution to

$$1 =\ 2\left[2\sigma^2\frac{\phi}{B} +\ 4\sigma^2\frac{\psi}{B}\ (1-N(\hat{\varphi}'))\right]^{1/4}\int_0^{\hat{\varphi}'} s^2 dN(s)\ , \tag{A-15}$$

so that if $\hat{\varphi}' < 1$, the function $\hat{f}$ as a discontinuity going from being positive and tending to $+\infty$ to being negative and tending to $-\infty$.

The rest of the proof fills in the details: Step (1): Show that $\hat{h}(\hat{\varphi})^2 \cdot (\hat{\varphi})^2$ is increasing in $\hat{\varphi}$ if $\phi / > \psi 0.1667$ for $\hat{\varphi} < 1$. Step (2): Show that $\sqrt{h(\hat{\varphi})} \cdot \int_0^{\hat{\varphi}} s^2 dN(s)$ is increasing in $\hat{\varphi}$ if $\hat{\varphi} < 1$. Step (3): Using (1) and (2) the function $\hat{f}$ is increasing in $\hat{\varphi}$ for values of φ that are smaller than 1, provided that its denominator is positive.

Step (1) follows from totally differentiating $h(\hat{\varphi})^2 \cdot (\hat{\varphi})^2$ with respect to $\hat{\varphi}$. Collecting terms we obtain that the derivative is proportional to $\phi + 2 \cdot \psi(1 - N(\hat{\varphi}) - \hat{\varphi} \cdot N'(\hat{\varphi}))$. Since the function $1 - N(\hat{\varphi}) - \hat{\varphi} \cdot N'(\hat{\varphi})$ is positive for small values of $\hat{\varphi}$ and negative for large values, we evaluate it at its upper bound for the relevant region, obtaining: $\phi + 2\psi(1 - N(1) - N'(1)) > 0$ or $\phi > \psi 2[N(1) + N'(1) - 1] \approx \psi\ 0.1667$. But notice that this condition is implied by our previous restriction $\phi > \psi[1/2 - 2(1 - N(1))] \approx \phi\ 0.1827$.

Step (2) follows from totally differentiating $\sqrt{h(\hat{\varphi})} \cdot \int_0^{\hat{\varphi}} s^2 dN(s)$ with respect to $\hat{\varphi}$. Collecting terms we obtain that the derivative is proportional to $\hat{\varphi}^2 - \int_0^{\hat{\varphi}} s^2 dN(s)\ \psi/2(\phi + \psi\ (1 - N(\hat{\varphi})))$. This expression is greater than $\hat{\varphi}^2 - \int_0^{\hat{\varphi}} s^2 dN(s)\ /2((1 - N(\hat{\varphi})))$, which is obtained by setting ϕ to zero. This integral is positive for the values of $\hat{\varphi}$ in $(0, 1)$.

Now we turn to the comparative statics results. That $\hat{\varphi}^*$ is decreasing in ϕ follows since $\hat{f}$ is increasing in ϕ. That $\sigma^2 \hat{\tau}^*$ is increasing it follows from the previous result and inspection of h. That $\hat{\varphi}^*$ is decreasing in σ^2/B follows since $\hat{f}$ is increasing in σ^2/B. That $\sigma^2 \hat{\tau}^*$ is increasing it follows from the previous result and inspection of h. That $\partial \hat{\varphi}^* / \partial \frac{\sigma^2}{B} = 0$ at $\sigma^2/B = 0$ follows from differentiating $\hat{f}$ with respect to σ^2/B and verifying that that derivative is zero when evaluated at at $\sigma^2/B = 0$. That $\hat{\varphi}^*$ is strictly increasing in ψ when σ^2/B is small relative to

ϕ it follows from differenting $\hat{f}/\psi$ with respect to ψ. That derivative is strictly negative and continuous on the parameters, when evaluated at $\phi > 0$ and $\sigma^2/B = 0$. ■

C.8 Proof of Proposition 7.

Proof. We rewrite the solution of $\bar{p}$ and $\hat{\tau}$ using the solution of equation (24) into equation (22) and equation (23). The approximation in equation (24) is based on the zero derivative found in Proposition 6 in item 3. We now characterize the elasticity of $\hat{\varphi}_0$. First we write the equation defining $\hat{\varphi}_0$ as

$$\tilde{\alpha} = 2\tilde{\varphi}(\tilde{\alpha}) + \log\left(2 + e^{\tilde{\alpha}} 4N\left(e^{\tilde{\varphi}(\tilde{\alpha})}\right)\right) \tag{A-16}$$

where we let $\tilde{\alpha} = \log \alpha$ and $\tilde{\varphi} = \log \hat{\varphi}_0$. Differentiating $\tilde{\varphi}$ this expression with respect to $\tilde{\alpha}$ and collecting terms we obtain:

$$\frac{\partial \tilde{\varphi}}{\partial \tilde{\alpha}} \equiv \frac{\partial \log \hat{\varphi}_0}{\partial \log \alpha} = \frac{1 - \frac{4\alpha(1-N(\hat{\varphi}_0))}{2+4\alpha(1-N(\hat{\varphi}_0))}}{2 - \frac{4\alpha(1-N(\hat{\varphi}_0))}{2+4\alpha(1-N(\hat{\varphi}_0))} \frac{n(\hat{\varphi}_0)\hat{\varphi}_0}{(1-N(\hat{\varphi}_0))}} \tag{A-17}$$

Since, $\hat{\varphi}_0 \to 0$ as $\alpha \to 0$, then $\frac{\partial \log \hat{\varphi}_0}{\partial \log \alpha} \to 0$. For values of $\alpha > 0$, we have that

$$\frac{\partial \log \hat{\varphi}_0}{\partial \log \alpha} < \frac{1}{2} \iff \frac{n(\hat{\varphi}_0)\hat{\varphi}_0}{(1-N(\hat{\varphi}_0))} < 2, \tag{A-18}$$

which is a property of the normal distribution for values of $\hat{\varphi}_0 < 1$. Finally, the first inequality follows because $2\ \hat{\varphi}_0/\alpha = [1 - \hat{\varphi}_0^2 4(1-N(\hat{\varphi}_0))] < 1$. The second inequality follows becasue $\hat{\varphi}_0 < 1$. ■

C.9 Proof of Proposition 8.

Proof. The expression is based on a second order expansion of τ around $\tilde{p} = 0$. The first order condition for τ can be written as:

$$F(\tau;\tilde{p}) \equiv e^{-\rho\tau}\left(B\left(\tilde{p}^2 + \sigma^2\tau\right) - \rho\int_{-\infty}^{\infty} V\left(\tilde{p} - s\sigma\sqrt{\tau}\right) dN(s) + \int_{-\infty}^{\infty} V'\left(\tilde{p} - s\sigma\sqrt{\tau}\right)\ \frac{-s\ \sigma}{2\ \sqrt{\tau}}\ dN(s)\right) \quad .$$

At a minimum $F(\tau(\tilde{p})\ ;\ \tilde{p}) = 0$ and $F_\tau(\tau(\tilde{p})\ ;\ \tilde{p}) \geq 0$. We have $\left.\frac{\partial \tau(\tilde{p})}{\partial \tilde{p}}\right|_{\tilde{p}=0} = -\frac{F_{\tilde{p}}}{F_\tau} = 0$. That $\partial\tau/\partial\tilde{p} = 0$ follows from the symmetry of $\tau(\cdot)$ around $\tilde{p}$, which is verified directly by checking that $F_{\tilde{p}} = 0$ (see below). Totally differentiating $F_\tau \tau' + F_{\tilde{p}}$ we obtain:

$$0 = F_{\tau\tau}(\tau')^2 + F_{\tau\tilde{p}}\tau' + F_\tau\tau'' + F_{\tilde{p}\tau}\tau' + F_{\tilde{p}\tilde{p}}\ ,$$

using that $\tau' = 0$ we get the second derivative:

$$\left.\frac{\partial^2 \tau(\tilde{p})}{(\partial \tilde{p})^2}\right|_{\tilde{p}=0} = -\frac{F_{\tilde{p}\tilde{p}}}{F_\tau} = 0$$

To compute this second derivative we first compute:

$$F_\tau(\tau;\tilde p) = -\rho F(\tau;\tilde p) + e^{-\rho\tau}\left(B\sigma^2 - \rho\int_{-\infty}^{\infty} V'\left(\tilde p - s\sigma\sqrt{\tau}\right)\frac{-s\sigma}{2\sqrt{\tau}}dN(s)\right.$$
$$\left. - \int_{-\infty}^{\infty} V'\left(\tilde p - s\sigma\sqrt{\tau}\right)\ \frac{-s\ \sigma\tau^{-3/2}}{4}\ dN(s) + \int_{-\infty}^{\infty} V''\left(\tilde p - s\sigma\sqrt{\tau}\right)\ \frac{s^2\sigma^2}{4\ \tau}\ dN(s)\right)$$

Taking $\rho \downarrow 0$, using that at the optimum $F = 0$, that in the approximation $\bar V'(\tilde p) = V''(0)\ \tilde p$ and that $\bar V''(\tilde p) = V''(0)$ we obtain:

$$\begin{aligned} F_\tau(\tau;0) &= B\ \sigma^2 - \int_{-\infty}^{\infty} V'\left(-s\sigma\sqrt{\tau}\right)\ \frac{-s\ \sigma\ \tau^{-3/2}}{4}\ dN(s) + \int_{-\infty}^{\infty} V''\left(-s\sigma\sqrt{\tau}\right)\ \frac{s^2\sigma^2}{4\ \tau}\ dN(s) \\ &= B\ \sigma^2 - \int_{-\infty}^{\infty} V''(0)\ \frac{s^2\sigma^2}{4\ \tau}\ dN(s) + \int_{-\infty}^{\infty} V''(0)\frac{s^2\sigma^2}{4\ \tau}\ dN(s) \\ &= B\ \sigma^2\ . \end{aligned} \tag{A-19}$$

We also have:

$$F_{\tilde p}(\tau;\tilde p) = e^{-\rho\tau}\left(2B\tilde p - \rho\int_{-\infty}^{\infty} V'\left(\tilde p - s\sigma\sqrt{\tau}\right)dN(s) + \int_{-\infty}^{\infty} V''\left(\tilde p - s\sigma\sqrt{\tau}\right)\ \frac{-s\ \sigma}{2\ \sqrt{\tau}}\ dN(s)\right)$$
$$F_{\tilde p\tilde p}(\tau;\tilde p) = e^{-\rho\tau}\left(2B - \rho\int_{-\infty}^{\infty} V''\left(\tilde p - s\sigma\sqrt{\tau}\right)dN(s) + \int_{-\infty}^{\infty} V'''\left(\tilde p - s\sigma\sqrt{\tau}\right)\ \frac{-s\ \sigma}{2\ \sqrt{\tau}}\ dN(s)\right)$$

Evaluating $F_{\tilde p\tilde p}$ at $\tilde p = 0$ for $\rho \downarrow 0$ and the approximation with $V'''(0) = 0$ gives:

$$F_{\tilde p\tilde p}(\tau;0) = 2B\ . \tag{A-20}$$

Expanding $\tau(\cdot)$ around $\tilde p = 0$, using that its first derivative is zero, and that the second derivative is the negative of the ratio of the expressions in equation (A-19) and equation (A-20) we obtain:

$$\tau\left(\tilde p\right) = \tau\left(0\right) + \tau'\left(0\right)\left(\tilde p\right) + \frac{1}{2}\tau''\left(0\right)\left(\tilde p\right)^2 = \hat\tau - \frac{1}{2}\frac{F_{\tilde p\tilde p}}{F_\tau}\ \left(\tilde p\right)^2 = \hat\tau - \left(\frac{\tilde p}{\sigma}\right)^2\ .$$

which appears in the proposition. ■

C.10 Proof of Proposition 9.

Proof. We start by computing the second derivative of $\mathcal{T}(p)$ at $p = 0$ (for notation simplicity we use p in the place of $\tilde p$). To further simplify notation rename the extreme of integration as:

$$s_1 \equiv \frac{p - \bar p}{\sigma\sqrt{\tau(p)}} \quad , \quad s_2 \equiv \frac{p + \bar p}{\sigma\sqrt{\tau(p)}}$$

which depend on p with derivatives

$$\frac{\partial s_1}{\partial p} = \frac{\sqrt{\tau(p)} - (p-\bar{p})\frac{\tau'(p)}{2\sqrt{\tau(p)}}}{\sigma\tau(p)} \quad , \quad \frac{\partial s_2}{\partial p} = \frac{\sqrt{\tau(p)} - (p+\bar{p})\frac{\tau'(p)}{2\sqrt{\tau(p)}}}{\sigma\tau(p)} \quad .$$

The first order derivative is:

$$\mathcal{T}'(p) = \tau'(p) + \int_{s_1}^{s_2} \mathcal{T}'\left(p - s\sigma\sqrt{\tau(p)}\,\right)\left(1 - \frac{\sigma\tau'(p)}{2\sqrt{\tau(p)}}\ s\right) dN(s)$$
$$- \mathcal{T}(\bar{p})\ n\left(s_1\right)\frac{\partial s_1}{\partial p} + \mathcal{T}(-\bar{p})\ n\left(s_2\right)\frac{\partial s_2}{\partial p}$$

where $n(s)$ denotes the density of the standard normal. The second order derivative is:

$$\begin{aligned}\mathcal{T}''(p) = \tau''(p) &+ \int_{s_1}^{s_2} \mathcal{T}''\left(p - s\sigma\sqrt{\tau(p)}\right)\left(1 - \frac{\sigma\tau'(p)}{2\ \sqrt{\tau(p)}}\ s\right)^2 dN(s) \\ &+ \int_{s_1}^{s_2} \mathcal{T}'\left(p - s\sigma\sqrt{\tau(p)}\right)\frac{s\sigma}{2\tau(p)}\left(-\tau''(p)\sqrt{\tau(p)} + \frac{(\tau'(p))^2}{2\sqrt{\tau(p)}}\right) dN(s) \\ &- \mathcal{T}'(\bar{p})\left(1 - \frac{\sigma\tau'(p)}{2\sqrt{\tau(p)}}\ s_1\right)\ n(s_1)\ \frac{\partial s_1}{\partial p} + \mathcal{T}'(-\bar{p})\left(1 - \frac{\sigma\tau'(p)}{2\sqrt{\tau(p)}}\ s_2\right)\ n(s_2)\ \frac{\partial s_2}{\partial p} \\ &- \mathcal{T}(\bar{p})\left(n'(s_1)\ \left(\frac{\partial s_1}{\partial p}\right)^2 + n(s_1)\ \frac{\partial^2 s_1}{(\partial p)^2}\right) + \mathcal{T}(-\bar{p})\left(n'(s_2)\ \left(\frac{\partial s_2}{\partial p}\right)^2 + n(s_2)\ \frac{\partial^2 s_2}{(\partial p)^2}\right) \quad .\end{aligned}$$

To evaluate this expression note that at $p = 0$ we have $\tau'(0) = 0$, $\tau(0) = \hat{\tau}$, $-s_1 = s_2 = \hat{\varphi}$, $\frac{\partial s_1}{\partial p} = \frac{\partial s_2}{\partial p} = \frac{1}{\sigma\sqrt{\hat{\tau}}}$ (recall the notation already used above $\hat{\varphi} \equiv \frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}$). Hence the second to last line in the previous formula is $-2\mathcal{T}'(\bar{p})\ n(s_1)\frac{\partial s_1}{\partial p}$ by the symmetry of $\mathcal{T}(p)$. Note moreover that at $p = 0$

$$\frac{\partial^2 s_1}{(\partial p)^2} = -\frac{\partial^2 s_2}{(\partial p)^2} = \frac{\bar{p}\ \tau''(0)\ \sqrt{\hat{\tau}}}{2\sigma\ \hat{\tau}^2}$$

Thus we get:

$$\begin{aligned}\mathcal{T}''(0) = \tau''(0) + \int_{-\hat{\varphi}}^{\hat{\varphi}} \mathcal{T}''\left(-s\sigma\sqrt{\hat{\tau}}\right) dN(s) - \frac{\sigma\tau''(0)}{2\sqrt{\hat{\tau}}}\int_{-\hat{\varphi}}^{\hat{\varphi}} \mathcal{T}'\left(-s\sigma\sqrt{\hat{\tau}}\right)\ s\ dN(s) \\ - 2\mathcal{T}'(\bar{p})\ \frac{n\left(\hat{\varphi}\right)}{\sigma\sqrt{\hat{\tau}}} - 2\mathcal{T}(\bar{p})\left(-\frac{n'(\hat{\varphi})}{\sigma^2\ \hat{\tau}} + n(\hat{\varphi})\frac{\bar{p}\ \tau''(0)\ \sqrt{\hat{\tau}}}{2\sigma\ \hat{\tau}^2}\right)\end{aligned} \tag{A-21}$$

Using that $\tau''(0) = -2/\sigma^2$ (from Proposition 8), the last term in the previous equation can be rewritten as $-2\frac{\mathcal{T}(\bar{p})}{\sigma^2\hat{\tau}}\left(-n'(\hat{\varphi}) - \hat{\varphi}\ n(\hat{\varphi})\right)$, which is zero since $n'(x) + x\ n(x) = 0$ for a standard normal density.

Given that $\mathcal{T}'(0) = \mathcal{T}'''(0) = 0$, and $\mathcal{T}''(0) < 0$, we approximate $\mathcal{T}(p)$ with a quadratic function on the interval $[-\bar{p}\ ,\ \bar{p}]$:

$$\mathcal{T}(p) = \mathcal{T}(0) + \frac{1}{2}\mathcal{T}''(0)\,(p)^2 \,.$$

Using the first and the second derivative of this quadratic approximation into the right hand side of equation (A-21), and $\tau''(0) = -2/\sigma^2$, gives:

$$\mathcal{T}''(0) = -2/\sigma^2 + \mathcal{T}''(0)\,(2\ N(\hat{\varphi}) - 1) - 2\mathcal{T}''(0)\int_0^{\hat{\varphi}} s^2 dN(s) - 2\ \mathcal{T}''(0)\ \hat{\varphi}\ n(\hat{\varphi})$$

or

$$\mathcal{T}''(0) = \frac{-1/\sigma^2}{1 - N(\hat{\varphi}) + \int_0^{\hat{\varphi}} s^2 dN(s) + \hat{\varphi} n(\hat{\varphi})} \quad . \tag{A-22}$$

To solve for $\mathcal{T}(0)$ let us evaluate $\mathcal{T}(p)$ at $\bar{p}$ obtaining:

$$\mathcal{T}(\bar{p}) = \tau(\bar{p}) + \int_0^{2\bar{\varphi}} \mathcal{T}\left(\bar{p} - s\sigma\sqrt{\tau(\bar{p})}\right) dN(s) \quad \text{, where} \quad \bar{\varphi} \equiv \frac{\bar{p}}{\sigma\sqrt{\tau(\bar{p})}} = \frac{\hat{\varphi}}{\sqrt{1-\hat{\varphi}^2}} \,.$$

Using the quadratic approximation for $\mathcal{T}$ in the previous equation we get

$$\begin{aligned}\mathcal{T}(\bar{p}) &= \tau(\bar{p}) + \int_0^{2\bar{\varphi}} \left(\mathcal{T}(0) + \frac{1}{2}\mathcal{T}''(0)\left(\bar{p} - s\sigma\sqrt{\tau(\bar{p})}\right)^2\right) dN(s) \\ &= \tau(\bar{p}) + \left(N\left(2\bar{\varphi}\right) - \frac{1}{2}\right)\mathcal{T}(0) + \frac{1}{2}\mathcal{T}''(0)\int_0^{2\bar{\varphi}} \left(\bar{p} - s\sigma\sqrt{\tau(\bar{p})}\right)^2 dN(s)\end{aligned}$$

Replacing $\mathcal{T}(\bar{p}) = \mathcal{T}(0) + \frac{1}{2}\mathcal{T}''(0)\ \bar{p}^2$ on the left hand side, and collecting terms gives

$$\begin{aligned}\mathcal{T}(0) =& \frac{\tau(\bar{p}) + \frac{1}{2}\mathcal{T}''(0)\left(\int_0^{2\bar{\varphi}}\left(\bar{p} - s\sigma\sqrt{\tau(\bar{p})}\right)^2 dN(s) - \bar{p}^2\right)}{1.5 - N\left(2\bar{\varphi}\right)} \\ &= \tau(\bar{p})\ \frac{1 + \frac{\sigma^2}{2}\ \bar{\varphi}^2\ \mathcal{T}''(0)\left(\int_0^{2\bar{\varphi}}\left(1 - \frac{s}{\bar{\varphi}}\right)^2 dN(s) - 1\right)}{1.5 - N\left(2\bar{\varphi}\right)} \\ &= \hat{\tau}\left(1 - \hat{\varphi}^2\right)\ \frac{1 + \frac{\sigma^2}{2}\ \bar{\varphi}^2\ \mathcal{T}''(0)\left(\int_0^{2\bar{\varphi}}\left(1 - \frac{s}{\bar{\varphi}}\right)^2 dN(s) - 1\right)}{1.5 - N\left(2\bar{\varphi}\right)}\end{aligned} \tag{A-23}$$

where the last line uses the equality $\tau(\bar{p}) = \hat{\tau} - \left(\frac{\bar{p}}{\sigma}\right)^2 = \hat{\tau}\,(1 - \hat{\varphi}^2)$. Substituting equation (A-22) into equation (A-23) gives

$$\mathcal{T}(0) = \hat{\tau}\frac{(1-\hat{\varphi}^2)}{1.5 - N\left(2\bar{\varphi}\right)}\ \left(1 - \frac{1/2\ \left(\int_0^{2\bar{\varphi}}\left(\bar{\varphi} - s\right)^2 dN(s) - \bar{\varphi}^2\right)}{1 - N(\hat{\varphi}) + \int_0^{\hat{\varphi}} s^2 dN(s) + \hat{\varphi} n(\hat{\varphi})}\right) \tag{A-24}$$

which gives the approximation for the expression $\mathcal{T}(0) = \hat{\tau} \cdot \mathcal{A}(\hat{\varphi})$ in the proposition. A numerical study of the function $\mathcal{A}(\hat{\varphi})$ shows that $\mathcal{A}(0) = 1$, and that the function approximation is accurate and increasing for $\hat{\varphi} \in (0, 0.75)$, that $\mathcal{A}(0.75) \cong 1.78$ and decreasing thereafter.

Next we show that, given equation (30), the average frequency of price adjustment can be written as $n_a = 1/\mathcal{T}(0)$ where $\mathcal{T}(0) = \hat{\tau}\tilde{\mathcal{A}}(\hat{\varphi})$. Rewrite equation (31) in terms of $\varphi(p) = \frac{p}{\sigma\sqrt{\tau(p)}}$,

$$\mathcal{T}(\tilde{p}) = \tilde{\mathcal{T}}(\tilde{\varphi}) = \tau(\tilde{\varphi}) + \int_{-\bar{\varphi}}^{\bar{\varphi}} \tilde{\mathcal{T}}(\varphi) n\left(\varphi\frac{\sqrt{\tau(\varphi)}}{\sqrt{\tau(\tilde{\varphi})}} - \tilde{\varphi}\right)\frac{\sqrt{\tau(\varphi)}}{\sqrt{\tau(\tilde{\varphi})}}d\varphi, \tag{A-25}$$

$$= \frac{\hat{\tau}}{1+\tilde{\varphi}^2} + \int_{-\bar{\varphi}}^{\bar{\varphi}} \tilde{\mathcal{T}}(\varphi) n\left(\varphi\frac{\sqrt{1+\tilde{\varphi}^2}}{\sqrt{1+\varphi^2}} - \tilde{\varphi}\right)\frac{\sqrt{1+\tilde{\varphi}^2}}{\sqrt{1+\varphi^2}}d\varphi, \tag{A-26}$$

where the first equality follows from strict monotonicity of $\varphi(p)$. Then we can write $\mathcal{T}(0) = \tilde{\mathcal{T}}(0) = \hat{\tau}\tilde{\mathcal{A}}(0,\hat{\varphi}) \equiv \mathcal{A}(\hat{\varphi})$, where

$$\tilde{\mathcal{A}}(\tilde{\varphi},\hat{\varphi}) = \frac{1}{1+\tilde{\varphi}^2} + \int_{-\bar{\varphi}}^{\bar{\varphi}} \tilde{\mathcal{A}}(\varphi,\hat{\varphi}) n\left(\varphi\frac{\sqrt{1+\tilde{\varphi}^2}}{\sqrt{1+\varphi^2}} - \tilde{\varphi}\right)\frac{\sqrt{1+\tilde{\varphi}^2}}{\sqrt{1+\varphi^2}}d\varphi. \tag{A-27}$$

We use a grid of values for $\tilde{\varphi}$ to solve recursively for $\tilde{\mathcal{A}}(\tilde{\varphi},\hat{\varphi})$. ■

C.11 Proof of Proposition 10.

Proof. Let $q(\varphi)$ and $Q(\varphi)$ be respectively the density and CDF of

$$\varphi(p) \equiv \frac{p}{\sigma\sqrt{\tau(p)}}\ .$$

Notice that $\varphi(p)$ is a monotonic transformation of p, $\frac{d\varphi}{dp} = \frac{1}{\sigma\sqrt{\tau(p)}} - \varphi\frac{\tau'(p)}{2\tau(p)} > 0$. Notice that using $p(\varphi)$ to denote the inverse function, we compute

$$\tau(p) = \hat{\tau} - \left(\frac{p}{\sigma}\right)^2 = \frac{\hat{\tau}}{1+(\varphi(p))^2}$$

which, abusing notation, defines the new function

$$\tau(\varphi) = \frac{\hat{\tau}}{1+\varphi^2} \tag{A-28}$$

The monotonicity of the transformation also gives that $Q(\varphi(p)) = G(p)$ at all p, implying

$$g(p)\frac{dp}{d\varphi}d\varphi = q(\varphi(p))d\varphi \tag{A-29}$$

Using equation (A-29) and the change of variables from p to φ in equation (33) we write:

$$
\begin{aligned}
g(\tilde{p}) &= \int_{-\bar{\varphi}}^{\bar{\varphi}} g(p(\varphi))\ n\left(\frac{\tilde{p}-p(\varphi)}{\sigma\sqrt{\tau(p(\varphi))}}\right)\frac{1}{\sigma\sqrt{\tau(p(\varphi))}}\frac{dp}{d\varphi}\ d\varphi + \left[1-\int_{-\bar{\varphi}}^{\bar{\varphi}} g(p(\varphi))\frac{dp}{d\varphi}d\varphi\right]\ n\left(\frac{\tilde{p}}{\sigma\sqrt{\hat{\tau}}}\right)\frac{1}{\sigma\sqrt{\hat{\tau}}}\\
&= \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi)\ n\left(\frac{\tilde{p}}{\sigma\sqrt{\tau(p(\varphi))}}-\varphi\right)\frac{1}{\sigma\sqrt{\tau(p(\varphi))}}d\varphi + \left[1-\int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi)d\varphi\right] n\left(\frac{\tilde{p}}{\sigma\sqrt{\hat{\tau}}}\right)\frac{1}{\sigma\sqrt{\hat{\tau}}}.
\end{aligned}
$$

For clarity, rewrite the previous equation using the density of $\tilde{p}$ conditional on φ:

$$
f(\tilde{p}|\varphi) \equiv n\left(\frac{\tilde{p}}{\sigma\sqrt{\tau(p(\varphi))}}-\varphi\right)\frac{1}{\sigma\sqrt{\tau(p(\varphi))}}
$$

This gives:

$$
g(\tilde{p}) = \int_{-\bar{\varphi}}^{\bar{\varphi}} f(\tilde{p}|\varphi)\ q(\varphi)d\varphi + \left[1-\int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi)d\varphi\right]\ f(\tilde{p}|0). \tag{A-30}
$$

Now consider the following monotone transformation of the random variable $\tilde{\varphi}$: $\Phi(\tilde{\varphi},\varphi) \equiv \tilde{\varphi}\frac{\sqrt{\tau(\tilde{\varphi})}}{\sqrt{\tau(\varphi)}}$, where the function $\tau(\varphi)$ is given in equation (A-28).[35] Using the definition of φ and the law of motion for $\tilde{p}$ it follows that $\Phi(\tilde{\varphi},\varphi)-\varphi$ is a random variable with the standard normal distribution: $n\left(\Phi(\tilde{\varphi},\varphi)-\varphi\right)$.

By doing the change in variables from $\tilde{p}$ to $\tilde{\varphi}$ on the left-hand side of equation (A-30), and from $\tilde{p}$ to $\Phi(\tilde{\varphi},\varphi)$ on the right-hand side of equation (A-30), we obtain

$$
\begin{aligned}
q(\tilde{\varphi})\frac{d\tilde{\varphi}}{d\tilde{p}} &= \int_{-\bar{\varphi}}^{\bar{\varphi}} n\left(\Phi(\tilde{\varphi},\varphi)-\varphi\right)\frac{d\Phi(\tilde{\varphi},\varphi)}{d\tilde{\varphi}}\frac{d\tilde{\varphi}}{d\tilde{p}}\ q(\varphi)\ d\varphi + \left[1-\int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi)d\varphi\right] n\left(\tilde{\varphi}\right)\frac{d\Phi(\tilde{\varphi},\tilde{\varphi})}{d\tilde{\varphi}}\frac{d\tilde{\varphi}}{d\tilde{p}},\\
q(\tilde{\varphi}) &= \int_{-\bar{\varphi}}^{\bar{\varphi}} n\left(\Phi(\tilde{\varphi},\varphi)-\varphi\right)\frac{d\Phi(\tilde{\varphi},\varphi)}{d\tilde{\varphi}}\ q(\varphi)\ d\varphi + \left[1-\int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi)d\varphi\right] n\left(\tilde{\varphi}\right) \quad .
\end{aligned}
$$

We notice that $q(\cdot)$ attains its maximum at $\tilde{\varphi}=0$, and that it is symmetric, so that $q'(0)=q'''(0)=0$ and $q''(0)<0$. Furthermore, $q(\bar{\varphi})>0$. Then, for small $\bar{\varphi}=\frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}$, this function can be approximated by a quadratic function with

$$
q(\varphi) = q(0) + \frac{1}{2}q''(0)\ \ \varphi^2\ .
$$

The value of $q(\cdot)$ and its first and second derivatives with respect to $\tilde{\varphi}$, evaluated at $\tilde{\varphi}=0$,

[35] The monotonicity holds since Φ is increasing in $\tilde{\varphi}$.

are given by

$$
\begin{aligned}
q(0) &= \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi) n\left(-\varphi\right) \frac{d\Phi(0,\varphi)}{d\tilde{\varphi}} d\varphi + \left[1 - \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi) d\varphi\right] n(0), \\
q'(0) &= \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi) n'\left(-\varphi\right) \left(\frac{d\Phi(0,\varphi)}{d\tilde{\varphi}}\right)^2 d\varphi + \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi) n\left(-\varphi\right) \frac{d^2\Phi(0,\varphi)}{(d\tilde{\varphi})^2} d\varphi + \left[1 - \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi) d\varphi\right] n'(0), \\
q''(0) &= \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi) n''\left(-\varphi\right) \left(\frac{d\Phi(0,\varphi)}{d\tilde{\varphi}}\right)^3 d\varphi + \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi) n'\left(-\varphi\right) 3 \frac{d\Phi(0,\varphi)}{d\tilde{\varphi}} \frac{d^2\Phi(0,\varphi)}{(d\tilde{\varphi})^2} d\varphi \\
&\quad + \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi) n\left(-\varphi\right) \frac{d^3\Phi(0,\varphi)}{(d\tilde{\varphi})^3} d\varphi + \left[1 - \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi) d\varphi\right] n''(0).
\end{aligned}
$$

Using equation (A-28) gives

$$
\frac{d\Phi(0,\varphi)}{d\tilde{\varphi}} = \sqrt{1+\varphi^2} \quad , \quad \frac{d^2\Phi(0,\varphi)}{(d\tilde{\varphi})^2} = 0 \quad , \quad \frac{d^3\Phi(0,\varphi)}{(d\tilde{\varphi})^3} = -3\sqrt{1+\varphi^2}.
$$

Using that $n''(-\varphi) = n(-\varphi)(\varphi^2 - 1)$, we rewrite $q(0)$ and $q''(0)$ as

$$
\begin{aligned}
q(0) &= 2\int_0^{\bar{\varphi}} q(\varphi) n(\varphi) \sqrt{1+\varphi^2} d\varphi + \left(1 - 2\int_0^{\bar{\varphi}} q(\varphi) d\varphi\right) n(0), \\
q''(0) &= 2\int_0^{\bar{\varphi}} q(\varphi) n(\varphi) \sqrt{1+\varphi^2} (\varphi^4 - 4) d\varphi - \left(1 - 2\int_0^{\bar{\varphi}} q(\varphi) d\varphi\right) n(0).
\end{aligned}
$$

These two equations and the quadratic approximation for $q(\cdot)$ give a system of 2 equations in 2 unknowns: $q(0)$ and $q''(0)$:

$$
\begin{aligned}
q(0) + q''(0) &= 2\int_0^{\bar{\varphi}} \left(q(0) + \frac{1}{2} q''(0)\ \varphi^2\right) n(\varphi) \sqrt{1+\varphi^2} (\varphi^4 - 3) d\varphi \\
q(0) &= 2\int_0^{\bar{\varphi}} \left(q(0) + \frac{1}{2} q''(0)\ \varphi^2\right) n(\varphi) \sqrt{1+\varphi^2} d\varphi + \left(1 - 2\int_0^{\bar{\varphi}} \left(q(0) + \frac{1}{2} q''(0)\ \varphi^2\right) d\varphi\right) n(0) \quad .
\end{aligned}
$$

The equations above imply

$$
\begin{aligned}
\frac{q''(0)}{q(0)} &= -\frac{1 - 2\int_0^{\bar{\varphi}}\ n\left(\varphi\right) \sqrt{1+\varphi^2} (\varphi^4 - 3) d\varphi}{1 - \int_0^{\bar{\varphi}}\ n\left(\varphi\right) \varphi^2 \sqrt{1+\varphi^2} (\varphi^4 - 3) d\varphi} \equiv \theta(\bar{\varphi}) \\
q(0) &= \frac{n(0)}{1 + 2\int_0^{\bar{\varphi}} \left(n(0) - n(\varphi)\sqrt{1+\varphi^2}\right) d\varphi + \theta(\bar{\varphi}) \int_0^{\bar{\varphi}} \left(n(0) - n(\varphi)\sqrt{1+\varphi^2}\right) \varphi^2 d\varphi} \quad .
\end{aligned}
$$

Using these results into equation (34) to obtain

$$\mathcal{R}(\hat{\varphi}) = 2\int_0^{\bar{\varphi}} \tau(\varphi)q(\varphi)d\varphi + \left[1 - 2\int_0^{\bar{\varphi}} q(\varphi)d\varphi\right]\hat{\tau} \tag{A-31}$$

$$\approx \hat{\tau} - 2\hat{\tau}\left[-\int_0^{\bar{\varphi}} (1+\varphi^2)^{-1}\left(q(0) + \frac{1}{2}q''(0)\varphi^2\right)d\varphi + \int_0^{\bar{\varphi}}\left(q(0) + \frac{1}{2}q''(0)\varphi^2\right)d\varphi\right] \tag{A-32}$$

$$\approx \hat{\tau} - 2\hat{\tau}\left[\int_0^{\bar{\varphi}}\left(q(0) + \frac{1}{2}q''(0)\varphi^2\right)\frac{\varphi^2}{(1+\varphi^2)}d\varphi\right] \tag{A-33}$$

where we use the quadratic approximation of $q(\cdot)$, the definition of $\varphi(p)$, and the quadratic approximation of $\tau(p)$.

Notice that, given equation (30), the average frequency of price review can be always written as $n_r = 1/\mathcal{T}_r(0)$ where $\mathcal{T}_r(0) = \hat{\tau}\mathcal{R}(\hat{\varphi})$. This result follows directly from substituting equation (30) into equation (A-32). ■

C.12 Proof of Proposition 11.

In this section we report the numerical solution of the model to the following experiments: (i) a change in ψ holding ϕ fixed; (ii) a change in ϕ holding ψ fixed. These experiments are meant to capture the elasticities of n_a and n_r with respect to ψ and ϕ.

Figure A-6: Numerical and approximated $\bar{p}$ as a function of $\alpha \equiv \frac{\psi}{\phi}$

Note: fixed parameter values are $\rho = 0.02$, $\sigma = 0.2$; $\psi = 0.03$ when ϕ is allowed to change; $\phi = 0.06$ when ψ is allowed to change.

We show results for different parameterizations of $B = 5, 20, 50$. The first row in Figure A-6 display results for $\log(n_a), \log(n_r)$ to changes in $\log(\phi)$ holding $\psi = 0.03$ as in our benchmark calibration. The larger ϕ, the closer the value of α to zero. As we can see, the elasticity of n_r with respect to ϕ is roughly equal to - 1/2, independently of the level of B and the level of α. Similarly, the elasticity of n_a with respect to ϕ is not changing much to changes in the level of B, however it is sensitive to the level of α, being roughly equal to - 1/2 for large values of ϕ, i.e. for α closer to zero, and smaller at smaller values of ϕ.

The second row in Figure A-6 display results for $\log(n_a), \log(n_r)$ to changes in $\log(\psi)$ holding $\phi = 0.06$ as in our benchmark calibration. The smaller ψ, the closer the value of α to zero. The elasticities of n_a and n_r with respect to ψ are smaller at smaller values of ψ.

C.13 Proof of Proposition 12.

Proof. Rewrite equation (40) as

$$w(\Delta p) = \frac{1}{\sigma\sqrt{\hat{\tau}}} \frac{\int_{-\bar{\varphi}}^{\bar{\varphi}} \sqrt{1+\varphi^2}\; n\left(\frac{\Delta p \; \sqrt{1+\varphi^2}}{\sigma\sqrt{\hat{\tau}}} - \varphi\right) q(\varphi) d\varphi}{1 - \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi) d\varphi} + \frac{1}{\sigma\sqrt{\hat{\tau}}}\; n\left(\frac{\Delta p}{\sigma\sqrt{\hat{\tau}}}\right),$$

usign the change of variable $\varphi = \tilde{p}/(\sigma\sqrt{\tau(\tilde{p})})$, using the approximation for the optimal policty: $\tau(p) = \hat{\tau} - (p/\sigma)^2 = \frac{\hat{\tau}}{1+\varphi^2}$, which implies that $p(\varphi) = \sigma\sqrt{\hat{\tau}}\;\varphi/\sqrt{1+\varphi^2}$, and hence this change of variables gives the density of the normalized prices: $q(\varphi) = g(p(\varphi))dp(\varphi)/d\varphi$. Thus letting the normalized price changes be:

$$x \equiv \frac{\Delta p}{\sigma\sqrt{\hat{\tau}}} \;, \tag{A-34}$$

we define the density of the normalized price changes x as $v(\cdot)$, satisfying $v(x)/(\sigma\sqrt{\hat{\tau}})$ and then the distribution of normalized price adjustment have density given by the change of variable formula: $v(x) = w\left(\frac{\Delta p}{\sigma\;\sqrt{\hat{\tau}}}\right)\;\sigma\sqrt{\hat{\tau}}$

$$v(x) \;=\; \frac{\int_{-\bar{\varphi}}^{\bar{\varphi}} \sqrt{1+\varphi^2}\; n\left(x\;\sqrt{1+\varphi^2} - \varphi\right) q(\varphi) d\varphi}{1 - \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi) d\varphi} + \; n\left(x\right) \quad \text{for } |x| > \bar{\varphi}\;. \tag{A-35}$$

Finally we can use the approximation for $q(\varphi) \approx q(0) + \frac{1}{2}q''(0)\varphi^2$ and the formulas for $q(0)$ and $q(0)''$ developed in the proof of Proposition 10. The expressions obtained there for $q(0)$ and $q''(0)$ are a function of $\bar{\varphi}$. Thus we can write:

$$\begin{aligned} v(x;\bar{\varphi}) \;&=\; \frac{\int_{-\bar{\varphi}}^{\bar{\varphi}} \left[q(0) + \frac{1}{2}q''(0)\varphi^2\right] \sqrt{1+\varphi^2}\; n\left(x\;\sqrt{1+\varphi^2} - \varphi\right) d\varphi}{1 - \int_{-\bar{\varphi}}^{\bar{\varphi}} \left[q(0) + \frac{1}{2}q''(0)\varphi^2\right] d\varphi} \\ &+ \quad n\left(x\right) \quad \text{for } |x| > \bar{\varphi}\;. \end{aligned} \tag{A-36}$$

where we include $\bar{\varphi}$ as an argument to emphasize that this density does not depend on any other parameter. This is the expression in the proof. Finally, we notice that the mode and minimum are equal to $\bar{\varphi}$. ∎

C.14 Proof of Proposition 14.

Proof. Consider

$$G(\hat{p}\ ;\bar{T},\pi) \equiv \mathbb{E}\left(\int_0^{\bar{T}} e^{-\rho t}\left[(\pi t-\hat{p})+\sigma W(t)\right] dt\right) \tag{A-37}$$

where $W(t)$ is a standard brownian motion. If $\hat{p}$ and the $\bar{T}$ are optimal, then $G(\cdot;\bar{T},\pi)$ should be maximized at $\hat{p}$. We will show that, provided that the stopping time is positive and finite,

$$\frac{\partial G(0\ ;\bar{T},\pi)}{\partial \hat{p}} < 0 \text{ if } \pi > 0\ . \tag{A-38}$$

We write equation (A-37) as

$$G(\hat{p}\ ;\bar{T},\pi) \equiv \mathbb{E}\left(\int_0^{\bar{T}} e^{-\rho t}\left[(\pi t-\hat{p})^2+2\sigma W(t)\,(\pi t-\hat{p})+\sigma^2 W(t)^2\right] dt\right)$$

and thus

$$\begin{aligned}
\frac{\partial G(\hat{p}\ ;\bar{T},\pi)}{\partial \hat{p}} &= \mathbb{E}\left(\int_0^{\bar{T}} e^{-\rho t}\left[-2\,(\pi t-\hat{p})-2\sigma W(t)\right] dt\right) \\
&= -\mathbb{E}\left(\int_0^{\bar{T}} e^{-\rho t} 2\,(\pi t-\hat{p})\, dt\right) - \int_0^{\infty} \mathbb{E}\left(1_{t\le\bar{T}}\right) e^{-\rho t} 2\sigma \mathbb{E}\left(W(t)|t\le\bar{T}\right) dt
\end{aligned}$$

Since $W(t)$ is Martingale, by the optional sampling theorem, $\mathbb{E}\left(W(t)|t\le\bar{T}\right)=0$, and hence

$$\frac{\partial G(\hat{p}\ ;\bar{T},\pi)}{\partial \hat{p}} = -\mathbb{E}\left(\int_0^{\bar{T}} e^{-\rho t}\ 2\ (\pi\ t-\hat{p})\ dt\right)\ .$$

so if $\pi > 0$ then the optimal value for $\hat{p} > 0$. Equating this first order condition to zero and rearranging:

$$\hat{p} = \pi\, \frac{\mathbb{E}\left(\int_0^{\bar{T}} e^{-\rho t}\ t\ dt\right)}{\mathbb{E}\left(\int_0^{\bar{T}} e^{-\rho t}\ dt\right)}$$

Taking ρ to zero:

$$\hat{p} = \pi\, \frac{\mathbb{E}\left(\frac{\bar{T}^2}{2}\right)}{\mathbb{E}\left(\bar{T}\right)} = \pi\ \mathbb{E}\left[\left(\frac{\bar{T}}{\mathbb{E}\left(\bar{T}\right)}\right)\frac{\bar{T}}{2}\right]$$

C.15 Proof of Proposition 15.

Proof. It is mathematically simpler to solve the menu cost model as concentrating in the steady state case, i.e. when $\rho = 0$. This problem corresponds to the the limit as $\rho \downarrow 0$. Thus when $\pi > 0$ and $\sigma = 0$ we can write the objective function as:

$$\begin{aligned}
& \min_{\hat{p},\tau} \frac{1}{\tau}\left[\int_0^\tau B\left(\pi t - \hat{p}\right)^2 \; dt + (\psi+\phi)\right] = \min_{\hat{p},\tau} \frac{1}{\tau}\left[B\int_0^\tau \left(\pi^2 t^2 - 2\hat{p}\pi t + \hat{p}^2\right) \; dt + (\psi+\phi)\right] \\
= \; & \min_{\hat{p},\tau} \frac{1}{\tau}\left[B\left(\frac{\pi^2\tau^3}{3} - \frac{2\hat{p}\pi\tau^2}{2} + \hat{p}^2\tau\right) + (\psi+\phi)\right] = \min_{\tau} \frac{1}{\tau}\left[B\left(\frac{\pi^2\tau^3}{3} - \frac{\pi^2\tau^3}{2} + \frac{\pi^2\tau^3}{4}\right) + (\psi+\phi)\right] \\
= \; & \min_{\tau} \frac{1}{\tau}\left[B\frac{\pi^2\tau^3}{12} + (\psi+\phi)\right] = \min_{\tau}\left[B\frac{\pi^2\tau^2}{12}\right] + \frac{(\psi+\phi)}{\tau}
\end{aligned}$$

where we use that $\frac{\tau\pi}{2} = \arg\min_{\hat{p}} \frac{2\hat{p}\tau^2}{2} + \hat{p}^2\tau$. The first order condition for τ gives

$$\frac{2 \; B \; \pi^2 \; \tau}{12} = \frac{(\psi+\phi)}{\tau^2} \; .$$

This implies the following optimal rules:

$$\tau^* = \left(\frac{6 \; (\psi+\phi)}{B \; \pi^2}\right)^{1/3}, \; \hat{p} = \left(\frac{3 \; \pi \; (\psi+\phi)}{4 \; B}\right)^{1/3}, \text{and } \hat{p} - \underline{p} = \pi\tau = \left(\frac{6 \; \pi \; (\psi+\phi)}{B}\right)^{1/3} \tag{A-39}$$

The value of n_a is obtained as $n_a = 1/\tau^*$. The values for $\tau(\tilde{p})$ are obtained by requiring that the review happens exactly at the time of an adjustment: $\tau(\tilde{p})\pi = \tilde{p} - \underline{p}$ in the range of inaction. The optimal policy has this form because, due to the deterministic evolution of $\tilde{p}(t)$, if $\phi > 0$ it is optimal to review only at the time of an adjustment.

Now consider the menu cost version of this model for $\rho > 0$, $\sigma = 0$ and $\pi > 0$. We use this model to show that for all $\rho > 0$, the optimal return point and boundaries satisfy $\bar{p} - \hat{p} < \hat{p} - \underline{p}$. In the range of inaction $\tilde{p} \in (\underline{p}, \bar{p})$ the value function satisfy the ODE:

$$\rho V(\tilde{p}) = B\tilde{p}^2 - \pi V'(\tilde{p})$$

with value matching, optimality of $\hat{p}$ and smooth pasting conditions:

$$V(\underline{p}) = V(\hat{p}) + \psi + \phi, \; V'(\hat{p}) = 0 \text{ and } V'(\underline{p}) = 0 \; .$$

The solution of the ODE in the range of inaction is:

$$V(\tilde{p}) = \frac{B}{\rho}\tilde{p}^2 - \frac{\pi}{\rho}\frac{2 \; B}{\rho}\tilde{p} + \left(\frac{\pi}{\rho}\right)^2 \frac{2 \; B}{\rho} + A \; e^{-\rho/\pi \; \tilde{p}}$$

for some constant A to be determined. Let $a_0\tilde{p} + a_1\tilde{p} + a_2\tilde{p}^2$ be the particular solution of the ODE. Its coefficients must solve:

$$\rho\left(a_0 + a_1\tilde{p} + a_2\tilde{p}^2\right) = B\tilde{p}^2 - \pi\left(a_1 + 2a_2\tilde{p}\right)$$

and thus

$$a_2 = \frac{B}{\rho}\ ,\ a_1 = -\frac{\pi}{\rho}\frac{2\ B}{\rho}\ ,\ a_0 = \left(\frac{\pi}{\rho}\right)^2 \frac{2\ B}{\rho}$$

Notice that the quadratic function showing the particular solution of the ODE is the value of a policy where the fixed cost is never paid. Hence in the range of inaction, where the solution to this ODE must hold, the value function has to be smaller. This implies that the constant A has to be negative. We now use that $A < 0$ implies that $\bar{p} - \hat{p} < \hat{p} - \underline{p}$. To see this, denote the quadratic particular solution of the ODE as $Q(\tilde{p})$, so we have $V'(\tilde{p}) = Q'(\tilde{p}) - A(\rho/\pi)^{-\rho/\pi\tilde{p}}$. This derivative is zero at $\hat{p}$ where the function attains its minimum, and it is positive for higher values and negative for lower ones. From $V'(\hat{p}) = 0$ we obtain:

$$2\ \frac{B}{\rho}\ \hat{p} - \frac{\pi}{\rho}\frac{B}{\rho}\ 2 = \frac{\rho}{\pi}\ A\ e^{-\frac{\rho}{\pi}\ \hat{p}}$$

so that

$$A = \left(\hat{p} - \frac{\pi}{\rho}\right)\ 2\ \frac{B}{\rho}\ \frac{\pi}{\rho}\ e^{\frac{\rho}{\pi}\ \hat{p}}$$

From $V(\bar{p}) - V(\hat{p}) = \phi + \psi$ we obtain:

$$\frac{B}{\rho}(\bar{p}^2 - \hat{p}^2) - \frac{\pi}{\rho}\frac{B}{\rho}2\ (\bar{p} - \hat{p}) + A\ \left(e^{-\frac{\rho}{\pi}\bar{p}} - e^{-\frac{\rho}{\pi}\ \hat{p}}\right) = \phi + \psi$$

and substituting A into this expression we get:

$$\frac{B}{\rho}(\bar{p}^2 - \hat{p}^2) - \frac{\pi}{\rho}\frac{B}{\rho}2\ (\bar{p} - \hat{p}) + \left(\hat{p} - \frac{\pi}{\rho}\right) 2\ \frac{B}{\rho}\frac{\pi}{\rho}\ \left(e^{\frac{\rho}{\pi}(\hat{p}-\bar{p})} - 1\right) = \phi + \psi$$

We now obtain an expression for $\bar{p}$ as a function of $\hat{p}$ and the parameters $\pi(\phi + \psi)/B$ by letting ρ to go to zero. To do so, we use that $e^x - 1 = x + (1/2)x^2 + o(x^2)$ and apply it to $e^{\frac{\rho}{\pi}(\hat{p}-\bar{p})} - 1$ obtaining:

$$\bar{p}^2 - \hat{p}^2 - \frac{\pi}{\rho}2\ (\bar{p} - \hat{p}) + \left(\hat{p} - \frac{\pi}{\rho}\right) 2\frac{\pi}{\rho}\ \left(\frac{\rho}{\pi}(\hat{p} - \bar{p}) + \frac{1}{2}(\frac{\rho}{\pi})^2(\hat{p} - \bar{p})^2 + o(\rho^2)\right) = \frac{\phi + \psi}{B}\rho$$

In this expression we can simplify the terms that are multiplied by $1/\rho$. To see this we can developed the third term into:

$$\begin{aligned}\frac{\phi + \psi}{B}\rho \ &= \ \bar{p}^2 - \hat{p}^2 - \frac{\pi}{\rho}2\ (\bar{p} - \hat{p}) + \left(\hat{p} - \frac{\pi}{\rho}\right) 2\ \ (\hat{p} - \bar{p}) \\ &+ \ \left(\hat{p} - \frac{\pi}{\rho}\right) 2\frac{\pi}{\rho} \left(\frac{1}{2}(\frac{\rho}{\pi})^2(\hat{p} - \bar{p})^2 + o(\rho^2)\right)\end{aligned}$$

or canceling the terms multiplied by $1/\rho$

$$\frac{\phi + \psi}{B}\rho \ = \ \bar{p}^2 - \hat{p}^2 + \hat{p}\ 2\ \ (\hat{p} - \bar{p}) + \left(\hat{p} - \frac{\pi}{\rho}\right) \left(\frac{\rho}{\pi}(\hat{p} - \bar{p})^2 + o(\rho)\right)$$

developing the parenthesis

$$\frac{\phi+\psi}{B}\rho \;\;=\;\; \bar{p}^2-\hat{p}^2+\hat{p}\;2\;\;(\hat{p}-\bar{p})+\hat{p}\frac{\rho}{\pi}(\hat{p}-\bar{p})^2-(\hat{p}-\bar{p})^2+o(\rho)$$

developing the square

$$\frac{\phi+\psi}{B}\rho \;\;=\;\; \bar{p}^2-\hat{p}^2+2\hat{p}^2-2\hat{p}\bar{p}+\hat{p}\frac{\rho}{\pi}(\hat{p}-\bar{p})^2-\hat{p}^2-\bar{p}^2+2\bar{p}\hat{p}+o(\rho)$$

canceling the common terms

$$\frac{\phi+\psi}{B}\rho \;\;=\;\; \hat{p}\frac{\rho}{\pi}(\hat{p}-\bar{p})^2+o(\rho)$$

multiplying π/ρ:

$$\frac{\phi+\psi}{B}\pi \;\;=\;\; \hat{p}\;(\hat{p}-\bar{p})^2+\frac{o(\rho)}{\rho}$$

and taking ρ to zero:

$$\frac{\phi+\psi}{B}\pi \;\;=\;\; \hat{p}\;(\hat{p}-\bar{p})^2$$

We can write $\bar{p}=a\;\hat{p}$ for a constant $a>1$ to be determined:

$$\frac{\phi+\psi}{B}\pi \;\;=\;\; \hat{p}^3\;(1-a)^2$$

and replacing the expression for $\hat{p}^3$:

$$\frac{\phi+\psi}{B}\pi \;\;=\;\; \left(\frac{3}{4}\frac{\phi+\psi}{B}\pi\right)(1-a)^2$$

or solving for the positive value of a:

$$\sqrt{\frac{4}{3}}+1 \;\;=\;\; a$$

or

$$\bar{p} \;\;=\;\; \left(\sqrt{\frac{4}{3}}+1\right)\hat{p}$$

Now we compare $\bar{p}-\hat{p}$ with $\hat{p}-\underline{p}$:

$$\frac{\bar{p}-\hat{p}}{\hat{p}-\underline{p}} \;\;=\;\; \frac{\sqrt{\frac{4}{3}}}{8^{\frac{1}{3}}}=\frac{1}{\sqrt{3}}$$

Online Appendices

Price setting with observation and menu costs

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D More detailed information on country surveys

Figure OA-1 plots the CDF for the frequencies of review and adjustment. The source of the data are Stahl (2005) for Germany, Loupias and Ricart (2004) for France, Fabiani et al. (2004) for Italy and Greenslade and Parker (2008) for UK. We tossed those observation that were either missing or reporting an irregular frequency of review or adjustment.

Figure OA-1: Cumulative distribution of frequency of price adjustment and review

Note: Frequencies are measured on a per-year basis.

D.1 Table 1

Table OA-1: Price-reviews and price-changes per year: medians and means

	AT	BE	FR	GE	IT	NL	PT	SP	EURO	CAN	UK	US
						Medians						
Review	4	1	4	3	1 2	4	2	1	2.7	12	4	2
Change	1	1	1	1	1 1	1	1	1	1	4	2	1.4
						Means						
Review	12.3	1.2	23.2	4.9	27.7	52.4	3.6	1.89	16.9	99.7	39.2	30
Change	2.6	0.9	3.5	2.1	5.1	2.3	1.9	1.85	3.3	61.3	33.5	27

Number of changes and reviews per year. The sources for the medians are Fabiani et al. (2007) 2003 Euro area survey, Amirault et al. (2006) 2003 Canadian survey, Greenslade and Parker (2008) 2008 UK survey. The sources for the means are discussed below.

D.1.1 Computation of the means

Austria: The source of the data is Table 2 and Table 3 in Kwapil et al. (2005). In order to compute the means, we assigned a yearly frequency of 0.5 to frequency smaller than a year, and took the midpoint of all intervals; we assigned a value of 75 to the group of firms reporting a frequency of price adjustment higher than 50. Then we averaged across the different frequencies, using the fraction of firms at each frequency as a weight.

Belgium: The source of the data is Section IV in Aucremanne and Druant (2005). From Aucremanne and Druant (2005), we have information on the average time between consecutive price reviews to be 13 months and the average number of consecutive price changes to be about 10 months. From section IV.1.2 "Overall, the average duration between two consecutive price reviews is 10 months." From section IV.2, "...This implies that the average duration between two consecutive price changes is almost 13 months..."The following table is from section IV.3. counts the number of firms in the sample that review and adjust prices in a given pair of durations. Below we copy Table 17 - Duration of prices from these authors:

Table OA-2: Belgium: Duration of prices (number of firms in each bin)

	Price change			
	$<= 1$	> 1 and < 12	12	> 12
Price review				
$<= 1$	**31**	12	8	1
> 1 and < 12	1	**197**	72	21
12	2	15	**436**	37
> 12	0	1	5	**51**

Source: NBB, Aucremanne and Druant (2005). duration $<= 1$: price is changed/reviewed monthly or more frequently.duration > 1 and < 12 : price is changed/reviewed with a frequency from one month up to one year.duration $= 12$: price is changed /reviewed once a year.duration > 12 : price is changed/reviewed less than once a year.

France and Italy: The source is the raw data from Loupias and Ricart (2004) and Fabiani et al. (2004). We removed missing observations from both series of the frequencies of adjustment and review and averaged across the remaining observations. Notice that we are keeping those firms for which we only have observation of one of the two frequencies.

Germany: The source is the raw data from Stahl (2005). We removed missing observations from both series of the frequencies of adjustment and review. Then we averaged across the different frequencies, using the fraction of firms at each frequency as a weight. Notice that we are keeping those firms for which we only have observation of one of the two frequencies. In addition, the highest frequency of observation for price adjustment is monthly, while the highest frequency of price review is daily. In order to make the data comparable, we assigned a monthly frequency to all observations at a frequency higher than monthly.

Netherlands: The source of the data is Tables 4A-B in Hoeberichts and Stokman (2006). In order to compute the means, we assigned a yearly frequency of 0.5 to firms reporting of adjusting occasionally. Then we averaged across the different frequencies, using the fraction of firms at each frequency as a weight.

Portugal: The source of the data is Char 15-16 in Martins (2005). In order to compute the means, we assigned a yearly frequency of 0.5 to firms reporting of adjusting/reviewing less than once a year, and a yearly frequency of 18 to firm reporting of adjusting/reviewing more than twelve times a year. Then we averaged across the different frequencies, using the fraction of firms at each frequency as a weight.

Spain: I order to compute the mean, we used data in Tables A10-A11 in Alvarez and Hernando (2005). We assigned a frequency of 6, 2.5 and 0.5 to firms reviewing/adjusting more than four times, between two and three times and less than once a year respectively. Then we averaged across the different frequencies, using the fraction of firms at each frequency as a weight.

Moreover, Alvarez and Hernando (2005) show that for Spain, quoting from their section 4.4:
When we compare the frequencies of price reviews and of changes, restricting the comparison to those firms that responded to both questions we observe that price changes occur only slightly less frequently than price reviews. The correlation between both frequencies is very high. For instance, among those firms reviewing their prices four or more times a year, 89% declare changing their prices at least four times a year, 4% change them two or three times a year, 6% once a year and 1% less than once a year.

The following table constructed from the first row of tables A10 and A11 in Alvarez and Hernando (2005)

Table OA-3: Spain: Frequency of price reviews and price changes (% of firms in each bin)

	At least 4 times per year	2 or 3 times per year	once a year	< once a year
Price change	13.9	15.1	56.8	14.3
Price review	14.0	15.6	63.1	7.4

Source: Alvarez and Hernando (2005)

Euro: We used the 2003 nominal GDP to compute the weights and averaged across the countries.

Canada: The source of the data is Figure 1 and Table 14 in Amirault et al. (2006). We assigned a frequency of 0.5 to firms reporting to review sporadically. We took the midpoint in each closed interval for the frequency of price changes (e.g. 3 for firms reporting between 2 and 4), and assigned 547.5=365*1.5 frequency of price changes to firms reporting to adjust prices more than 365 times a year. Then we averaged across the different frequencies, using the fraction of firms at each frequency as a weight.

UK: The source of the data is Table C on page 406 and chart 5 on page 407 in Greenslade and Parker (2008). In computing the mean, we excluded firms reporting "irregularly" and "other". Then we averaged across the different frequencies, using the fraction of firms at each frequency as a weight.

US: Source data from Blinder et al. (1998) 1992 US survey. To compute the means we use Table 4.1 and Table 4.7, interpolating the bins. Both means and medians are based on a small number of responses (186 and 121), and both are sensitive to details used for the interpolation.

D.2 Figure ??

This figure includes data from Germany, Italy and France. The source of the data is Stahl (2005) for Germany, Loupias and Ricart (2004) for France, Fabiani et al. (2004) for Italy and Greenslade and Parker (2008) for UK. We removed from the raw data all missing information, i.e. mostly firms do not reporting any regular frequency of price review, or not reporting a frequency of price adjustment. Given that the highest frequency of observation for adjustment on the German data is monthly, we assigned a monthly frequency of review to those firms reviewing more than 12 times a year. We included in the analysis those secotors for which we have observations for at least 3 firms.

Section 3 clearly show that, in advanced economies, the median firm reviews its price more often than it adjusts.

D.3 Evidence about the distribution of price changes

D.4 Evidence about the hazard rate of price changes

Figure OA-2: Frequency of review vs. frequency of adjustment: across sectors

Note: Variables are intended as log-averages across firms in each sector. Sources: Stahl (2005), Loupias and Ricart (2004), Fabiani et al. (2004) and Greenslade and Parker (2008). Each point in the scatter plot refers to a NACE 2 digits sector; 01: crop and animal production, hunting; 10: food products; 11: beverages; 13: textiles; 15: leather products; 16: wood products; 17: paper products; 18: recorded media; 19: coke and petroleum products; 20: chemical products; 21: pharmaceutical products; 22: plastic products; 23: non-metallic mineral products; 24: basic metals; 25: fabricated metal products, except machinery and equipment; 26: computer, electronic and optical products; 27: electric motors, generators, transformers; 28: machinery and equipment; 29: motor vehicles; 30: other transport equipment; 31: furniture; 32: other manufacturing; 33: installation of machinery and equipment; 45: wholesale and retail trade and repair of motor vehicles; 46: wholesale trade, except of motor vehicles and motorcycles; 55: accommodation; 81: services to buildings and landscape activities; 98: undifferentiated goods- and services-producing activities of private households.

Figure OA-3: Distribution of price changes in Cavallo (2009)

Figure OA-4: Distribution of price changes in Alvarez et al. (2006)

Figure OA-5: Distribution of price changes in Klenow and Kryvtsov (2008)

Figure OA-6: Hazard rate in Cavallo (2009)

Figure OA-7: Hazard rate in Nakamura and Steinsson (2008)

Figure OA-8: Hazard rate in Klenow and Kryvtsov (2008)

Figure OA-9: Hazard rate in Klenow and Kryvtsov (2008)

Figure OA-10: Hazard rate in Alvarez et al. (2005)