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Jay M. Jarvis for the degree of Master of Science in Industrial Engineering presented on May 30, 1984.

Title: Optimal Product Layout in an Order Picking Warehouse

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Edward McDowell

The purpose of this paper is to provide, at a basic level, product layouts for an order picking warehouse that will minimize average order picking time. A statistical model is developed to ensure that optimal results rather than just 'good' results are obtained. It is shown that simple but general assignment algorithms can be used to optimally allocate products in an order picking warehouse. A simulation shows a 40 percent decrease in average order picking distance over an order picking warehouse with a random product layout.

Optimal Product Layout in an Order Picking Warehouse

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Jay M. Jarvis

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Professor of Industrial Engineering in charge of major

Redacted for Privacy

Head of department of Industrial Engineering

Redacted for Privacy

Dean of Graduate School

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INTRODUCTION

Order Picking Warehouses are a special class of warehouse that generally deal with a large number of smaller, fast moving products. The orders they fill usually contain many different products which individually do not constitute a full unit load. So, rather than traveling to a single location in the warehouse to pull an order, it is necessary to travel to a sequence of several locations before returning to the dock with a completed order. Order pickers are quite often people, who walk or drive a vehicle through the warehouse on order picking tours. Pick lists, which are generated either manually or by computer, are used to determine which products are on an order and their respective locations. In most cases these lists are generated with respect to the location of the products on them, so that the first item on the list is in the nearest location, the second item is in the second nearest location, and so on. In this way unnecessary backtracking while on a picking tour is eliminated. A shipping dock is usually the beginning and end of all order picking tours.

Many variations on this basic concept have been successfully implemented. Partial to complete automation has been accomplished by using a variety of 'new technology' ideas. Some of these include using automatic guided vehicles so that narrow aisle technology can be used, or using conveyor systems to transport products from their locations to the dock area. Different warehouse designs and layouts have been used to speed up material flow and use space more efficiently. One only need visit a few warehouses to see the vast array of ideas in use to make operations run more efficiently. Many of these ideas, though, are not cost effective for all order picking warehouses, whether it be because of their size, the nature of their business, or the economy.

A large number of order picking warehouses depend upon a high volume of product movement to keep their profits up. This is because profit margins are often very low in the warehousing business. Efficiency, then, can mean the difference between doing well or going out of business. A great deal of work has been done to improve the efficiency of order picking warehouses. Automation and warehouse design are two possible ways to increase overall efficiency, though they can be capital cost intensive. Another approach, is reducing picking time by optimizing product layout in the warehouse. The research done in product layout for the general warehouse problem has been extensive. This research usually falls under the heading of facilities layout. Unfortunately, because of the special nature of order picking warehouses, specifically that the order picking tours are designed to pick many products from many different locations rather than just a single location, the results from facilities layout research do not help very much. The problem of locating products optimally in an order picking warehouse has not received much attention. This then, is the subject of this paper.

Since very little work has been done in optimizing product layout in an order picking warehouse, this research will start out on the 'ground floor'. The results of this work are intended to be building blocks. So, in order that further research is not limited, optimal solutions rather than solutions that approximate optimality, are the goal. In order to meet this goal, a statistical approach is used. An order picking warehouse is described with several limiting assumptions. Then, using statistical methods, hypothesized product layouts are proven to be optimal for a number of different dock locations.

Concrete results in the area of product layout in order picking warehouses would not only benefit those warehouses that cannot afford higher cost alternatives, but might be a less expensive solution for those that can. These other alternatives might also be improved by optimizing the layout of their products. This research should prove to be fundamental to all work done to make order picking warehouses run more efficiently.

LITERATURE REVIEW

The standard warehouse layout, or facilities design problem has been studied for many years. Product layout in these warehouses is an important issue as it is in order picking warehouses, as they cannot afford to have their pickers spending more time than necessary traveling to and from product locations. Since an order picker usually travels to a single location to fill an order or to get a full pallet load, optimizing picking routes is generally a simple problem. The picker just takes the shortest route to the product location. Other than redesigning the warehouse, or adding expensive automation, product location is about the only area left to improve. Francis and White (1) have shown several solutions to this problem. For a warehouse with a single dock, a least cost layout is basically one in which the highest demand products are located closest to the dock and the lowest demand products are located farthest away from the dock. These results, however, can not be applied directly to order picking warehouses because of the difference in picking tours. They do however, lead to some important hypotheses for product layout in order picking warehouses.

Many private companies have tried different solutions to make their order picking warehouses more efficient. The Barret-Cravens Company (2) developed a system that implements two of their own ideas to improve efficiency: Random storage of products, and activity zones. Activity zones are divisions in the warehouse that divide the product locations into zones for different product demand groups. The activity zones at Barret-Cravens are set up so that the highest demand items are in the first zone, closest to the dock, and so on so that the last zone has the lowest demand items and is farthest away from the dock. The activity zones reduce overall travel time on order picking tours by making the more frequent trips shorter. The items are then assigned an activity zone and located randomly in the different aisles with the restriction that they are located in their assigned activity zone within those aisles. This feature allows Barret-Cravens to eliminate a back-up stock holding area. A product that would normally be in the holding area is placed in any available location in any aisle as long as it is within that product's activity zone. A computer keeps track of the changing product locations. Barret-Cravens also uses narrow aisle vehicles to optimize space, and a conveyor system to transport the picked orders to the loading dock in order to further reduce picking time.

The Western Electric Company (3) developed a computer simulation of an order picking warehouse. With this capability, they could simulate several picking tours for given product layouts, and then analyze such things as picker and row utilization, and average order picking time. They could then experiment with different product layouts until they found one that best utilized their resources and reduced picking time.

Both of these approaches, though, are non-optimizing for product layout in the area of reducing order picking time. One company implements some good ideas, but they never strive for an optimal product layout. The other company's procedure uses a computer simulation to test differing product layouts. Although this may produce a good product layout, it will never optimize.

Georgia Institute of Technology has done research into making order picking warehouses more efficient. Most notably, work has been done on optimizing order picking tour routes, in terms of reducing picking time. H. Donald Ratliff (4) has presented an algorithm that finds the picking route for an order that minimizes picking time. His solution though, works only for certain types of order picking warehouses. Most of the work at Georgia Institute of Technology has focused on order picking routes rather than product layout. The problem of finding a product layout in an order picking warehouse that minimizes order picking time, it seems, has not been approached.

PROBLEM STATEMENT

The problem is to find a product layout, given certain assumptions, that will minimize the expected picking time for any given order. Assuming that the time to actually pick a product off of the shelf will not vary for different locations, this problem can be equivalently solved by finding the product layout that minimizes expected distance traveled to pick any order. It should be obvious that if the distance traveled to pick an order is shortened, then the total picking time for that order will subsequently be shortened.

The warehouse considered in this paper is assumed rectangular with the picking aisles running perpendicular to the dock with cross-over aisles only at the ends. All picking aisles are of equal length (ℓ) and width (w). All aisles also have an equal number of product locations (bays). These bays may be of varying widths, but only one product type may be stored per bay. If necessary, a product type may be allocated to more than one bay. For example, if two bays are required, then that product type will be counted as two product types, both with the same demand. If there are (n) product types, (m) aisles, and (b) bays per aisle, then there are exactly the same number of product locations as product types $(n = m \times b)$. If there are less product types than locations, dummy products with no demand are created so that $(n = m \times b)$. All bays have equal depth. Given this and the aisle width (w), the distance between aisle (γ) and $(\gamma+1)$ is the same as the distance between aisle (δ) and $(\delta+1)$. This aisle to aisle width is (Δ). It is assumed that once an aisle is entered, it must be completely traversed. The aisles are assumed to be sufficiently wide to accommodate two-way traffic, so congestion should be negligible in a multiple picker situation. Each order is picked on a single tour and by a single order picker. It is also assumed that there is no demand dependence between product types. That is, because one product type is on an order, the probability of some other product type being on the order is not affected. Product demand is assumed constant. A diagram of this order picking warehouse is given in Figure 3 on page 21.

MODEL DEVELOPMENT

The following definitions will be used throughout this paper. Let the dimension perpendicular to the dock be called 'In-Aisle' and the dimension parallel to the dock be called 'Cross-Aisle'.

p_i = probability that product i will be picked on any given order

P_j = probability that aisle j will be traveled on any given order

d_j = distance from the dock to the center line of aisle j and back to the dock in the Cross-Aisle direction

$q_i = 1 - p_i$

$Q_j = 1 - P_j$

D_T = total distance traveled for an order

The terms 'probability that a product will be picked' (p_i) and 'product demand' will be used interchangeably throughout this paper. The following theorems will also be useful throughout this paper:

THEOREM 1

Given two groups of k numbers ordered so that

$$0 \le a_1^1 \le a_2^1 \le \cdots \le a_k^1 \le a_1^2 \le a_2^2 \le \cdots \le a_k^2$$

then

$$a_1^1 a_2^1 \cdots a_{k-1}^1 a_k^1 \le a_1^2 a_2^2 \cdots a_{k-1}^2 a_k^2$$

PROOF in Appendix.

THEOREM 2

For any $k \geq 2$ and if $0 \leq a_1^j \leq a_2^j \leq \ldots \leq a_n^j$, and if $j = 1,\ldots,k$ then,

$$Z2 = \sum_{i=1}^{n} \prod_{j=1}^{k} a_i^j \geq \sum_{i=1}^{n} \prod_{j=1}^{k} a^j_{\psi_j(i)}$$

where $\psi_1(i) = i$ $(i=1,\ldots,n)$ and $\psi_2,\psi_3,\ldots,\psi_k$ are any k permutations of $1,2,\ldots,n$.

In other words, any other arrangement of the numbers than is shown by Z2, is not a maximum.

PROOF by Hardy, Littlewood, and Polya (4).

For each of the n numbers in the k groups, take the largest item out of each group and multiply them together, the next largest out of each group and multiply them together,...,and the smallest out of each group and multiply them together. The sum of these products is a maximum over any other permutation of these numbers.

THEOREM 3

Given m groups of k numbers ordered so that

$$0 \leq a_1^1 \leq a_2^1 \leq \ldots \leq a_k^1 \leq a_1^2 \leq a_2^2 \leq \ldots \leq a_k^2 \leq a_1^3 \leq a_2^3$$
$$\leq \ldots \leq a_1^m \leq a_2^m \leq \ldots \leq a_k^m$$

then

$$Z3 = \sum_{j=1}^{m} \prod_{i=1}^{k} a_i^j \geq \sum_{j=1}^{m} \prod_{i=1}^{k} a^{\phi_i(j)}_{\psi_j(i)}$$

where $\phi_1(j) = j$ $(j=1,\ldots,m)$ and $\phi_2,\phi_3,\ldots,\phi_k$ are any k permutations of $1,2,\ldots,m$ and $\psi_1(i) = i$ $(i=1,\ldots,k)$ and $\psi_2,\psi_3,\ldots,\psi_m$ are any m permutations of $1,2,\ldots,k$.

In other words, any other arrangement of the numbers than is shown by Z3, is not a maximum.

PROOF in Appendix.

In other words, given these $(m \times k)$ numbers, in order to maximize the sum of the products, put the smallest k numbers in one group, the next smallest k numbers in another group,...,the largest k numbers in another group; take the product in each of these groups and then sum these products together.

SOLUTION

Considering that this warehouse has only the In-Aisle and Cross-Aisle dimensions (no height), the Total Distance Traveled for any order (D_T) can be split up into two separate parts: Travel in the In-Aisle direction (D_A) and travel in the Cross-Aisle direction (D_C), where

$$D_T = D_A + D_C$$

The Expected Total Distance Traveled can then be stated as

$$E(D_T) = E(D_A) + E(D_C) \ .$$

D_A and D_C are not completely independent. A product layout that gives a minimum Expected Distance Traveled for D_A does not necessarily give a minimum Expected Distance Traveled for D_C. Therefore, the procedure used here to minimize $E(D_T)$ is to first find the product layout that minimizes $E(D_A)$, and then to find the product layout that minimizes $E(D_C)$ and show that this product layout is the same as the layout that minimized $E(D_A)$. Therefore, this product layout minimizes Expected Total Distance Traveled, $E(D_T)$.

Because of the assumption that once an aisle is entered it must be completely traveled, the only way an aisle would not be traveled for a given order is if there were no products in that aisle on the order. Therefore, the following is the development of the probability that an aisle is traveled (P_j):

Let

$$A_j = \{i/\text{product } i \in \text{aisle } j\}$$

then

$$Q_j = \prod_{i \in A_j} (1-p_i)$$

$$Q_j = \prod_{i \in A_j} q_i = \text{probability that no products on the order are in aisle } j$$

and

$$P_j = 1 - \prod_{i \in A_j} (1-p_i)$$

$$P_j = 1 - \prod_{i \in A_j} q_i = \text{probability that at least one product in aisle } j \text{ is on the order.}$$

<u>Travel in the Picking Aisles (In-Aisle Direction)</u>

If one or more products on an order are located in a particular aisle, then it will be said that the order uses that aisle. Then

$$D_A = (\ell)(\text{number of aisles used on an order})$$

where ℓ is the length of an aisle, therefore,

$$E(D_A) = (\ell)E(\text{number of aisles used on an order})$$

because ℓ is a constant, minimizing $E(D_A)$ is equivalent to minimizing E(number of aisles used on an order).

In the case where an order uses an odd number of aisles, the order picker will end up at the back of the warehouse. Therefore, the order picker will need to travel an extra aisle in order to finally end up at the front of the warehouse (at the dock). This extra aisle can not be accounted for when using the expected number of aisles to be picked. However, the total number of aisles traveled on an order (including an extra aisle if necessary) is a monotonically increasing function against the number of aisles used on an order. Minimizing the expected number of aisles used on an order then, necessarily minimizes the expected total number of aisles traveled on an order. So, whether or not an extra aisle is needed to end up at the front of the warehouse, the result from minimizing the expected number of aisles will provide a minimum in either case. Continuing,

$$E(\text{number of aisles}) = \sum_{j=1}^{m} P_j$$

$$E(\text{number of aisles}) = \sum_{j=1}^{m} \left(1 - \prod_{i \in A_j} (1-p_i)\right)$$

$$E(\text{number of aisles}) = m - \sum_{j=1}^{m} \prod_{i \in A_j} (1-p_i)$$

Minimizing this is equivalent to

$$\text{Maximizing} \sum_{j=1}^{m} \prod_{i \in A_j} (1-p_i)$$

If the p_i's are placed in increasing order, or because p_i is between 0 and 1, if the $(1-p_i)$'s are placed in increasing order, then the maximum of $\sum_{j=1}^{m} \prod_{i \in A_j} (1-p_i)$ is given by Theorem 3. This proves the following theorem:

THEOREM 4

For the given warehouse, the product layout that minimizes Expected Distance Traveled in the In-Aisle direction is found by placing the products with the lowest demand in one aisle, the products with the next lowest demand in another aisle, and so on so that the products with the highest demand are grouped in another aisle. The placement of these aisles is not important as only the In-Aisle direction is being considered here.

Distance Traveled in the Cross-Aisle Direction

Since only the Cross-Aisle direction is considered in this section, the warehouse in Figure 3 can be diagrammed as shown below in Figure 1.

m ... α ... 4 2 1 3 ...α-1...m-1

0— —0— —0———0—□—0———0— —0— —0

Figure 1. Whole-Warehouse. □ = dock, 0 = aisle center line

For an order to be completely picked, all aisles used by that order must be traveled. Considering only the right side of the dock, an order picker will at least have an opportunity to travel those aisles used by an order, if he moves from the dock, in Figure 1, out to the farthest aisle away from the dock used by that order and then back to the dock. The same can be said for the left side of the dock. Together, these two distances, from both sides of the dock, represent the minimum possible distance an order picker must travel in the Cross-Aisle direction to complete a given order. The method used in this section then, is to find the product layout that will minimize the expected distance to the farthest aisle out used by an order on both sides of the dock. This layout minimizes the Expected Distance Traveled in the Cross-Aisle direction. The Expected Distance Traveled in the Cross-Aisle direction is identical to twice the expected distance to the farthest aisle out on an order for both sides of the dock. Therefore, $E(D_C)$ equals the sum over all the aisles of the probability that the aisle is the farthest aisle out on the order, times the distance from the dock to that aisle and back again. The probability that any aisle is the farthest aisle out on an order on its side of the dock is: The probability that that aisle is on the order (P_j) times the probabilities that all the aisles farther out on the same side of the dock are not on the order $(Q_{j+2}Q_{j+4}\ldots)$.

The solution procedure used is to show this for two cases. First, the dock is located such that there are aisles only to one side. This will be called the half-warehouse case. Second, the dock is located such that there are an equal number of aisles, symmetrically located on either side. This is the whole-warehouse case as shown in Figure 3.

Half-Warehouse

In the half warehouse case the aisles are all to the left or, as shown in Figure 2, to the right of the dock.

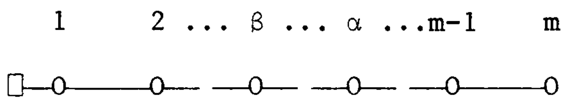

Figure 2. Half-Warehouse. □ = dock, O = aisle center line

THEOREM 5

For the warehouse with aisles only to one side of the dock, the product layout that minimizes Expected Distance Traveled in the Cross-Aisle direction is as follows: Place the products with the highest demand in aisle 1, closest to the dock; place the products with the next highest demand in aisle 2; and so on so that the products with the lowest demand are in aisle m, farthest away from the dock.

PROOF in Appendix.

Note that the product layout in Theorem 5 is just a specific case of the layout described by Theorem 4. Therefore, since the same product layout minimizes both Expected Distance Traveled in the In-Aisle direction and in the Cross-Aisle direction, this product layout minimizes the Expected Total Distance Traveled for any order, $E(D_T)$. Theorem 5 then, describes a product layout that will give a minimum expected picking time for a warehouse with aisles only to one side of the dock.

Whole-Warehouse

The whole-warehouse problem is just a generalization of the half-warehouse problem solved above. Even so, two more assumptions are necessary to make the solution procedure straight forward. First, the number of aisles (m), must be even. Second, the dock is located such that there are an equal number of aisles, symmetrically located, on either side, as shown in Figure 1.

This warehouse is basically two half-warehouses: One to the right of the dock, and one to the left of the dock. Because of this, the product layouts of both of these half-warehouses must conform to the results in Theorem 5 in order to give a minimum expected picking time. Any product layout for this warehouse that does not conform exactly to Theorem 5 can be improved upon. Moving the products around on their respective sides of the dock so that the highest demand products are in the aisle closest to the dock, and the lowest demand products are in the aisle farthest away from the dock, as in Theorem 5, will always give a smaller Expected Distance Traveled in the Cross-Aisle direction. In fact, it will give the minimum Expected Distance Traveled in the Cross-Aisle direction. This fact greatly reduces the complexity of the whole-warehouse problem.

THEOREM 6

For the warehouse with an equal number of aisles on either side of the dock, symmetrically located, the product layout that minimizes Expected Distance Traveled in the Cross-Aisle direction is as follows: Referring to Figure 1, place the products with the highest demand in aisle 1, closest to the dock; place the products with the next highest demand in aisle 2, closest to the dock but on the opposite side of aisle 1; place the next highest demand products in aisle 3; place the next highest demand products in aisle 4; and so on so that the next to lowest demand products are in aisle (m-1), farthest away from the dock on the same side as aisle 1; place the products with the lowest demand in aisle m, farthest away from the dock on the same side as aisle 2. Note that aisle 1 can be on either side of the dock as long as all other aisles are changed accordingly.

PROOF in Appendix.

Note that the product layout in Theorem 6 is just a specific case of the layout described in Theorem 4. Therefore, since the same product layout minimizes both Expected Distance Traveled in the In-Aisle direction and in the Cross-Aisle direction, this product layout minimizes Expected Total Distance Traveled for any order, $E(D_T)$. Theorem 6 then, gives a product layout that will give a minimum Expected Picking Time for a warehouse with an equal number of aisles, symmetrically located, on either side of the dock.

Example

A computer simulation was run to test the improvement of an order picking warehouse with a product layout as given by Theorem 6, over that of a warehouse with a random product layout. The order picking warehouse used in this example has ten aisles, 100 feet in length. The aisle to aisle distance (Δ), is 20 feet. The warehouse is filled to capacity with 500 different product types. The demand for each of the products is taken by generating five hundred exponential random variables with a mean of (0.2). The dock for the warehouse is located at the front of the warehouse and between aisles five and six.

Twenty random orders were generated using the probabilities assigned to the products. Therefore, the order size is a random variable. Each of the 20 orders was run through both the warehouse with the Theorem 6 product layout, and the warehouse with the random product layout, and the resulting distance traveled for each order was computed. The results showed a 40 percent decrease in mean distance traveled per order. The mean distance traveled in the Theorem 6 warehouse was 445 feet with a standard deviation of 135.6 feet, and the mean distance traveled in the random warehouse was 725 feet with a standard deviation of 107 feet. The average difference in distance traveled to pick an order, between the two warehouses, was 280 feet.

Extension: Variation in Warehouse Design

Another possible warehouse design is to turn the aisles 90 degrees and run them across the warehouse, as in Figure 4 on page 21, rather than running them perpendicular to the dock. Let the aisle closest to the dock be called aisle 1, and number the remaining aisles so that the aisle farthest away from the dock is aisle (m). A product layout that minimizes Expected Total Distance Traveled for this warehouse can be found quite easily, given the results in the previous sections.

Note that if a dock is located at one of the ends of the first row (see 'pseudo-docks' in Figure 4), the problem is identical to the half-warehouse problem already discussed. If the dock is located

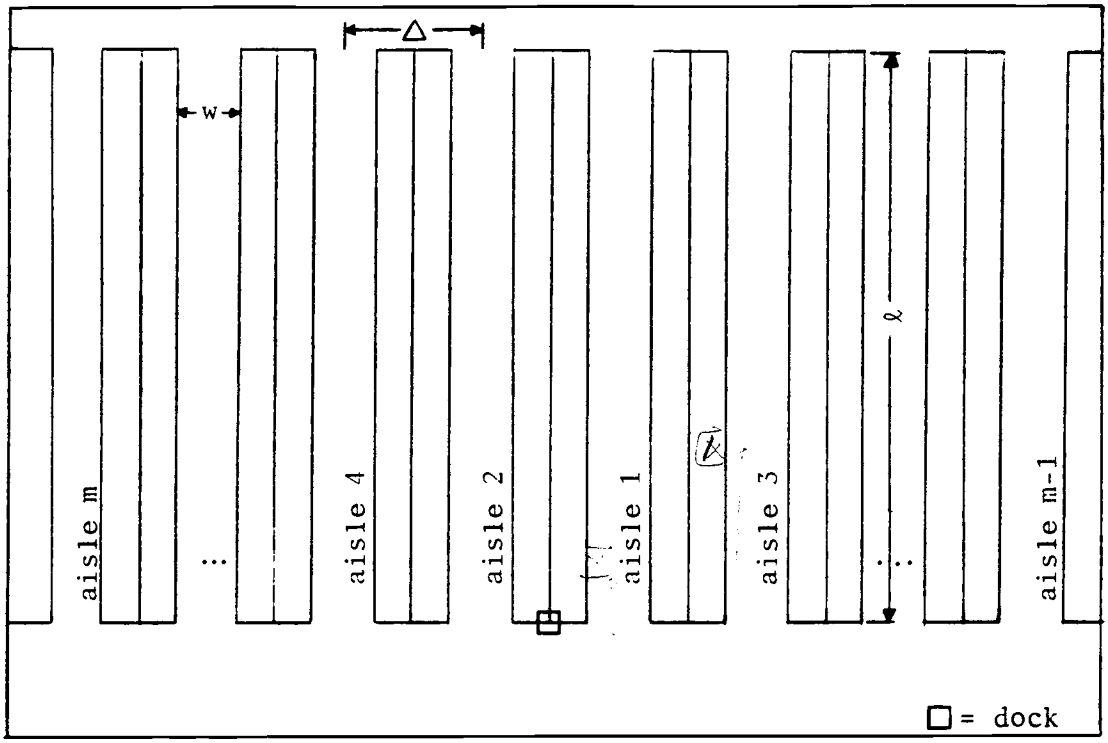

Figure 3. Whole-Warehouse with aisles perpendicular to dock

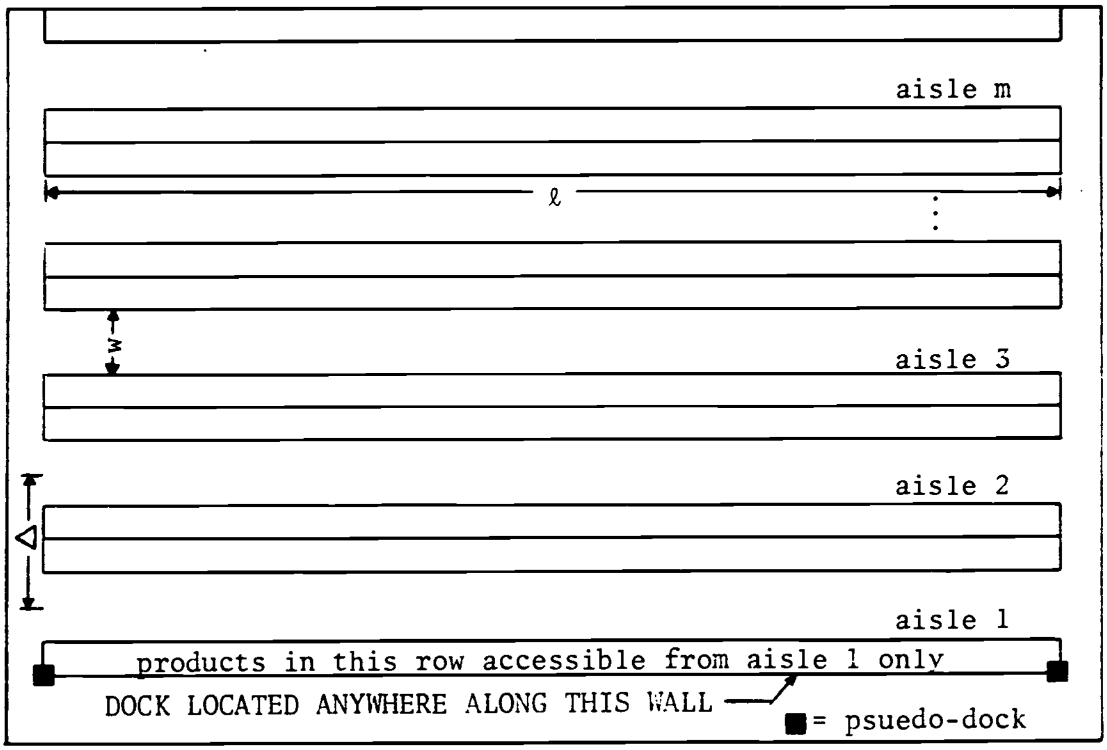

Figure 4. Whole-Warehouse with aisles parallel to dock

anywhere between these two positions, the problem changes slightly. In this case, locate the two 'pseudo-docks' as shown in Figure 4. The only difference between this warehouse and the half-warehouse discussed previously will show up in the case when the picker has to pick an even number of aisles. The picker travels from the dock to the closest 'pseudo-dock', performs the picking tour as if it were a half-warehouse problem, and then travels from the same 'pseudo-dock' back to the dock. This distance between the dock and the closest 'pseudo-dock', is an extra distance that has to be traveled twice. This distance, however, is fixed, and will not affect the product layout. In the case where there are an odd number of aisles to be picked, the distance traveled does not change from the half-warehouse problem. The picker travels from the dock to one of the 'pseudo-docks', performs the picking tour as if it were a half-warehouse problem, and then travels from the opposite 'pseudo-dock', back to the dock. The distance from the dock to the 'pseudo-dock' and from the opposite 'pseudo-dock' back to the dock is equal to traveling an extra aisle to get back to the dock, as is required in the half-warehouse case.

Therefore, no matter where the dock is located, this problem can be likened to the half-warehouse problem. The product layout that minimizes Expected Total Distance Traveled for this warehouse is the same as given by Theorem 5.

SUMMARY AND CONCLUSION

The final results in this paper are actually quite simple in themselves, and should seem fairly obvious to somebody with an acquaintance of warehousing. This paper has shown product layouts that minimize expected picking time for an order picking warehouse with a number of different dock locations, regardless of whether or not the warehouse has enough products to fill all its product locations. Because the order picking warehouse used in this paper was limited in many ways by certain assumptions, the results given are very basic. Once again, these results are only intended to be building blocks for later research.

Further research in order picking warehouses is unlimited. Areas that warrant further research include finding an optimal product layout for a warehouse with aisles that allow the order picker to turn around rather than having to completely traverse an aisle every time an aisle is entered; taking into consideration fluxuating demand on products, and products that have a demand that is dependent on other products' demand; considering an order picking warehouse with aisle cross-overs in the middle of the aisle rather than just at the end of the aisles; an order picking warehouse with aisles that have differing numbers of product locations; considering an order picking warehouse that has two or more shelves at each product location; considering the effect of product type grouping within the warehouse; and considering different aisle configurations within an order picking warehouse. These are just a few of the possible research areas.

There is basically only one advantage to reducing order picking time: reduced cost. Because many order picking warehouses depend upon high volume product movement, any overall reduction in picking times could mean significant cost savings. These savings can be seen in reductions in the number of order pickers, or in shorter total order picking hours. The increased efficiency could mean the difference between an order picking warehouse doing well in a delicate industry or going out of business.

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APPENDIX

APPENDIX

PROOF OF THEOREM 1

Show for

$$0 \le a_1^1 \le a_1^2$$

then

$$a_1^1 \le a_1^2 \quad \text{by definition}$$

Assume true for

$$0 \le a_1^1 \le a_2^1 \le \cdots \le a_{k-1}^1 \le a_1^2 \le a_2^2 \le \cdots \le a_{k-1}^2$$

then

$$(a_1^1 a_2^1 \ldots a_{k-1}^1) \le (a_1^2 a_2^2 \ldots a_{k-1}^2)$$

Now for

$$0 \le a_1^1 \le a_2^1 \le \cdots \le a_{k-1}^1 \le a_k^1 \le a_1^2 \le a_2^2 \le \cdots \le a_{k-1}^2 \le a_k^2$$

implies

$$(a_1^1 a_2^1 \ldots a_{k-1}^1) a_k^1 \le (a_1^2 a_2^2 \ldots a_{k-1}^2) a_k^1$$

and

$$(a_1^2 a_2^2 \ldots a_{k-1}^2) a_k^1 \le (a_1^2 a_2^2 \ldots a_{k-1}^2) a_k^2$$

therefore,

$$(a_1^1 a_2^1 \ldots a_{k-1}^1 a_k^1) \le (a_1^2 a_2^2 \ldots a_{k-1}^2 a_k^2)$$

PROOF OF THEOREM 3

Show for

$$0 \le a_1^1 \le a_1^2 \le \cdots \le a_1^m$$

then

$$(a_1^1 a_1^2 a_1^3 \ldots a_1^m)$$

is a maximum, and is equal to any other permutation (as in Theorem 3).

Assume true for

$$0 \leq a_1^1 \leq a_2^1 \leq \cdots \leq a_{k-1}^1 \leq a_1^2 \leq a_2^2 \leq \cdots \leq a_{k-1}^2 \leq a_1^3 \leq a_2^3$$
$$\leq \cdots \leq a_1^m \leq a_2^m \leq \cdots \leq a_{k-1}^m$$

then

$$(a_1^1 a_2^1 \ldots a_{k-1}^1) + (a_1^2 a_2^2 \ldots a_{k-1}^2) + \cdots + (a_1^m a_2^m \ldots a_{k-1}^m)$$

is a maximum over any other permutation (as in Theorem 3).

Then for

$$0 \leq a_1^1 \leq a_2^2 \leq \cdots \leq a_{k-1}^1 \leq a_k^1 \leq a_1^2 \leq a_2^2 \leq \cdots \leq a_1^m \leq a_2^m$$
$$\leq a_{k-1}^m \leq a_k^m$$

then by Theorem 1

$$(a_1^1 a_2^1 \ldots a_{k-1}^1) \leq (a_1^2 a_2^2 \ldots a_{k-1}^2) \leq \cdots \leq (a_1^m a_2^m \ldots a_{k-1}^m)$$

Also, by definition

$a_k^1 \leq a_k^2 \leq \cdots \leq a_k^m$, let these be one group.

Let

$$A^j = (a_1^j a_2^j \ldots a_{k-1}^j)$$

then

$A^1 \leq A^2 \leq A^3 \leq \cdots \leq A^m$, let these be another group.

Then by Theorem 2

$$(a_k^1 A^1) + (a_k^2 A^2) + \cdots + (a_k^m A^m)$$

is a maximum over any other like permutation from these two groups. Expanding the A^j terms gives

$$(a_1^1 a_2^1 \ldots a_k^1) + (a_1^2 a_2^2 \ldots a_k^2) + \ldots + (a_1^m a_2^m \ldots a_k^m)$$

which is a maximum over any other like permutation as given by Theorem 3.

PROOF OF THEOREM 5

Assume the contrary: Given any product layout such that product (a) is in aisle (α) with probability of being picked equal to p_a, and product (b) is in aisle (β) with probability of being picked equal to p_b, and $p_a \geq p_b$, and $d_\alpha > d_\beta$ as in Figure 2, assume that this layout minimizes Expected Distance Traveled in the Cross-Aisle direction. Using this product layout, which does not conform to Theorem 5, interchange the locations of products (a) and (b) and compute the difference in Expected Distance Traveled in the Cross-Aisle direction.

Before the interchange, the Expected Distance Traveled is

$$\begin{aligned} E(D_C) = {} & (d_1 P_1 Q_2 \ldots Q_\beta \ldots Q_\alpha \ldots Q_m) + (d_2 P_2 Q_3 \ldots Q_\beta \ldots Q_\alpha \ldots Q_m) \\ & + \ldots + (d_{\beta-1} P_{\beta-1} Q_\beta \ldots Q_\alpha \ldots Q_m) + (d_\beta P_\beta Q_{\beta+1} \ldots Q_\alpha \ldots Q_m) \\ & + \ldots + (d_\alpha P_\alpha Q_{\alpha+1} \ldots Q_m) + (d_{\alpha+1} P_{\alpha+1} Q_{\alpha+2} \ldots Q_m) \\ & + \ldots + (d_{m-1} P_{m-1} Q_m) + (d_m P_m) \end{aligned}$$

Only the terms with P_β, Q_β, P_α, or Q_α will change with the interchange of products (a) and (b), all others remain constant.

Also, the terms with both Q_α and Q_β will remain constant with the interchange, as these terms have both p_a and p_b. Therefore, these constant terms can be eliminated from consideration. Then let the remaining terms be represented by

$$E(D_C') = (d_\beta P_\beta Q_{\beta+1} \cdots Q_\alpha \cdots Q_m) + \ldots + (d_\alpha P_\alpha Q_{\alpha+1} \cdots Q_m)$$

The terms $(Q_{\alpha+1} Q_{\alpha+1} \cdots Q_m)$ can be factored out and eliminated, as they also will remain constant. Then we are left with

$$E(D_C'') = (d_\beta P_\beta Q_{\beta+1} \cdots Q_\alpha) + (d_{\beta+1} P_{\beta+1} Q_{\beta+2} \cdots Q_\alpha)$$
$$+ \ldots + (d_{\alpha-1} P_{\alpha-1} Q_\alpha) + (d_\alpha P_\alpha)$$

Substituting $(1-Q_j)$ for P_j, and expanding gives

$$E(D_C'') = (d_\beta Q_{\beta+1} \cdots Q_\alpha) - (d_\beta Q_\beta Q_{\beta+1} \cdots Q_\alpha) + (d_{\beta+1} Q_{\beta+2} \cdots Q_\alpha)$$
$$- (d_{\beta+1} Q_{\beta+1} Q_{\beta+2} \cdots Q_\alpha)$$
$$+ \ldots + (d_{\alpha-1} Q_\alpha) - (d_{\alpha-1} Q_{\alpha-1} Q_\alpha) + (d_\alpha) - (d_\alpha Q_\alpha)$$

Group the first two terms together, the second two terms, and so on so that the last two terms are grouped together. Call the first term in each group (X), and the second term in each group (Y). Pair the (X) term of the first group with the (Y) term of the second group. Note that except for the d_j's, the terms within this pairing are identical. Pair the (X) term of the second group with the (Y) term of the third group, and so on so that the (X) term of the second to last group is paired with the (Y) term of the last group.

The terms left out of this pairing are $-(d_\beta Q_\beta Q_{\beta+1} \cdots Q_\alpha)$ which has both Q_α and Q_β, so it will remain constant with the interchange, and (d_α) which is a constant. The remaining terms (those that were paired above), represented by R, are of the general form

$$R = (d_\gamma Q_{\gamma+1} Q_{\gamma+2} \cdots Q_\alpha) - (d_{\gamma+1} Q_{\gamma+1} Q_{\gamma+2} \cdots Q_\alpha)$$

$$R = (d_\gamma - d_{\gamma+1})(Q_{\gamma+1} Q_{\gamma+2} \cdots Q_\alpha)$$

where γ is a positive integer.

By definition, $(d_\gamma - d_{\gamma+1}) = -\Delta$.

Let $Q_\alpha = T(1-p_a)$, therefore

$$R = (-\Delta)(Q_{\gamma+1} Q_{\gamma+2} \cdots Q_{\alpha-1} T(1-p_a))$$

Since $p_a \geq p_b$, then $(1-p_a) \leq (1-p_b)$. Because $(1-p_b)$ is greater than $(1-p_a)$, and because of the negative distance $(-\Delta)$, interchanging products (a) and (b) results in a decrease in the general form of these pairings.

Since all the terms in $E(D_C)$ are either constant or were grouped in such a way that they were shown to decrease with an interchange of products (a) and (b), $E(D_C)$ decreases with an interchange of products (a) and (b). This contradicts the assumption that the original product layout minimizes the Expected Distance Traveled in the Cross-Aisle direction. Therefore, Theorem 5 must give the product layout that minimizes Expected Distance Traveled in the Cross-Aisle direction.

PROOF OF THEOREM 6

The idea used in the proof of Theorem 6 is to, one by one, locate products in the warehouse so that Expected Distance Traveled in the Cross-Aisle direction is minimized. The first product is located optimally, then given this, a second product is introduced into the warehouse. The new product is tried in all possible locations (aisles), with the location giving the minimum Expected Distance Traveled being optimal. Then a third product is introduced, and a fourth, and so on until the warehouse is filled. Each new product is located such that the overall layout of all the products gives a minimum Expected Distance Traveled in the y-direction. This is most easily shown by an inductive proof. If (n) is the number of products in the warehouse, then

Show for n=1

For a single product, it is trivial to show that the location that would give the minimum Expected Distance Traveled in the y-direction, is in either aisle 1 or 2, closest to the dock. Since Theorem 6 would place the single product in aisle 1, Theorem 6 gives the product layout that minimizes Expected Distance Traveled in the y-direction.

Assume true for n=k-1

Let (k) be less than or equal to the number of products that would fill the warehouse.

Show for n=k

Given that a product layout conforms to Theorem 6, first (before showing for n=k products) show the change in Expected Distance Traveled in the Cross-Aisle direction if a product on the left side of the dock in aisle (α), is interchanged with a product in aisle $(\alpha-1)$ on the right side of the dock, as in Figure 1. The Expected Distance Traveled in the Cross-Aisle direction before the location interchange is

$$
\begin{aligned}
E(D_C) = {} & (d_1P_1Q_3 \cdots Q_{\alpha-1} \cdots Q_{m-1}) + (d_3P_3Q_5 \cdots Q_{\alpha-1} \cdots Q_{m-1}) \\
& + \ldots + (d_{\alpha-3}P_{\alpha-3}Q_{\alpha-1} \cdots Q_{m-1}) + (d_{\alpha-1}P_{\alpha-1}Q_{\alpha+1} \cdots Q_{m-1}) \\
& + \ldots + (d_{m-3}P_{m-3}Q_{m-1}) + (d_{m-1}P_{m-1}) \\
& + (d_2P_2Q_4 \cdots Q_\alpha \cdots Q_m) + (d_4P_4Q_6 \cdots Q_\alpha \cdots Q_m) \\
& + \ldots + (d_{\alpha-2}P_{\alpha-2}Q_\alpha \cdots Q_m) + (d_\alpha P_\alpha Q_{\alpha+2} \cdots Q_m) \\
& + \ldots + (d_{m-2}P_{m-2}Q_m) + (d_mP_m)
\end{aligned}
$$

Only the terms with a P_α, Q_α, $P_{\alpha-1}$, or $Q_{\alpha-1}$ will change with the interchange of the two products. All other terms can be eliminated. Then let

$$
\begin{aligned}
E(D_C') = {} & (d_1P_1Q_3 \cdots Q_{\alpha-1} \cdots Q_{m-1}) + (d_3P_3Q_5 \cdots Q_{\alpha-1} \cdots Q_{m-1}) \\
& + \ldots + (d_{\alpha-3}P_{\alpha-3}Q_{\alpha-1} \cdots Q_{m-1}) + (d_{\alpha-1}P_{\alpha-1}Q_{\alpha+1} \cdots Q_{m-1}) \\
& + (d_2P_2Q_4 \cdots Q_\alpha \cdots Q_m) + (d_4P_4Q_6 \cdots Q_\alpha \cdots Q_m) \\
& + \ldots + (d_{\alpha-2}P_{\alpha-2}Q_\alpha \cdots Q_m) + (d_\alpha P_\alpha Q_{\alpha+2} \cdots Q_m)
\end{aligned}
$$

Substituting $(1-Q_j)$ for P_j and then expanding gives

$$
\begin{aligned}
E(D_C') = {} & (d_1 Q_3 \cdots Q_{\alpha-1} \cdots Q_{m-1}) - (d_1 Q_1 Q_3 \cdots Q_{\alpha-1} \cdots Q_{m-1}) \\
& + (d_3 Q_5 \cdots Q_{\alpha-1} \cdots Q_{m-1}) - (d_3 Q_3 Q_5 \cdots Q_{\alpha-1} \cdots Q_{m-1}) \\
& + \cdots + (d_{\alpha-3} Q_{\alpha-1} \cdots Q_{m-1}) - (d_{\alpha-3} Q_{\alpha-3} Q_{\alpha-1} \cdots Q_{m-1}) \\
& + (d_{\alpha-1} Q_{\alpha+1} \cdots Q_{m-1}) - (d_{\alpha-1} Q_{\alpha-1} Q_{\alpha+1} \cdots Q_{m-1}) \\
& + (d_2 Q_4 \cdots Q_\alpha \cdots Q_m) - (d_2 Q_2 Q_4 \cdots Q_\alpha \cdots Q_m) \\
& + (d_4 Q_6 \cdots Q_\alpha \cdots Q_m) - (d_4 Q_4 Q_6 \cdots Q_\alpha \cdots Q_m) \\
& + \cdots + (d_{\alpha-2} Q_\alpha \cdots Q_m) - (d_{\alpha-2} Q_{\alpha-2} Q_\alpha \cdots Q_m) \\
& + (d_\alpha Q_{\alpha+2} \cdots Q_m) - (d_\alpha Q_\alpha Q_{\alpha+2} \cdots Q_m)
\end{aligned}
$$

The top four lines are from the right side of the dock and the bottom four lines are from the left side of the dock. On each side of the dock, group the first two terms together, the second two terms, and so on so that the last two terms are grouped together. Each of these groups has the same $\pm\, d_j$ in each term. Call the first term in each group (X), and the second term in each group (Y). Take the (X) term from the first group and the (Y) term from the second group on each side of the dock and group the four terms together. Then do the same with the (X) term from the second group and the (Y) term from the third group on each side of the dock. Continue this until the last terms are grouped together. The terms left out of this grouping are

$$- (d_1Q_1Q_3 \ldots Q_{\alpha-1} \ldots Q_{m-1}) - (d_2Q_2Q_4 \ldots Q_{\alpha} \ldots Q_m)$$

$$+ (d_{\alpha-1}Q_{\alpha+1} \ldots Q_{m-1}) + (d_{\alpha}Q_{\alpha+2} \ldots Q_m)$$

The second two terms are constant as they do not contain P_{α}, Q_{α}, $P_{\alpha-1}$, or $Q_{\alpha-1}$.

Let L represent the first two terms, which change as follows

$$d_1 = d_2 \quad \text{by definition} \quad (\text{let } d = d_1 = d_2)$$

therefore,

$$L = -d(Q_1Q_3 \ldots Q_{\alpha-1} \ldots Q_{m-1} + Q_2Q_4 \ldots Q_{\alpha} \ldots Q_m).$$

Let

$$Q_{\alpha-1} = T(1-p_{\theta}) \quad \text{and} \quad Q_{\alpha} = S(1-p_{\phi})$$

where (θ) is a product in aisle $(\alpha-1)$ and (ϕ) is a product in aisle (α). Then

$$L = -d(Q_1Q_3 \ldots T(1-p_{\theta}) \ldots Q_{m-1} + Q_2Q_4 \ldots S(1-p_{\phi}) \ldots Q_m)$$

Because it was assumed that the warehouse product layout conformed to Theorem 6

$$Q_1 \leq Q_2, Q_3 \leq Q_4, \ldots, T \leq S, \ldots, Q_{m-1} \leq Q_m$$

Case 1

If $p_{\theta} \geq p_{\phi}$ or $(1-p_{\theta}) \leq (1-p_{\phi})$, then, not considering the $(-d)$, Theorem 2 says L is a maximum. Then considering the $(-d)$, L is a minimum. So interchanging the locations of products θ and ϕ will result in an increase in L.

Case 2

If $p_\theta \leq p_\phi$ or $(1-p_\theta) \geq (1-p_\phi)$, then, not considering the $(-d)$, Theorem 2 says L could be maximized by interchanging p_θ and p_ϕ. Considering the $(-d)$, L could be minimized (decreased) by interchanging the locations of products θ and ϕ.

The only terms left in $E(D_C')$ are the terms that were grouped above. These, represented by R, all have the general form

$$R = (d_\gamma Q_{\gamma+2}\cdots Q_{\alpha-1}\cdots Q_{m-1}) - (d_{\gamma+2}Q_{\gamma+2}\cdots Q_{\alpha-1}\cdots Q_{m-1})$$
$$+ (d_{\gamma+1}Q_{\gamma+3}\cdots Q_{\alpha}\cdots Q_m) - (d_{\gamma+3}Q_{\gamma+3}\cdots Q_{\alpha}\cdots Q_m)$$

where γ is an odd, positive integer.

$$R = (d_\gamma - d_{\gamma+2})(Q_{\gamma+2}\cdots Q_{\alpha-1}\cdots Q_{m-1})$$
$$+ (d_{\gamma+1} - d_{\gamma+3})(Q_{\gamma+3}\cdots Q_{\alpha}\cdots Q_m)$$

but by definition, $(d_\gamma - d_{\gamma+2}) = (d_{\gamma+1} - d_{\gamma+3}) = -\Delta$. Then

$$R = (-\Delta)(Q_{\gamma+2}\cdots Q_{\alpha-1}\cdots Q_{m-1} + Q_{\gamma+3}\cdots Q_{\alpha}\cdots Q_m)$$

Substitute as above

$$T(1-p_\theta) = Q_{\alpha-1} \quad \text{and} \quad S(1-p_\phi) = Q_\alpha \text{ , then}$$

$$R = (-\Delta)(Q_{\gamma+2}\cdots T(1-p_\theta)\cdots Q_{m-1} + Q_{\gamma+3}\cdots S(1-p_\phi)\cdots Q_m)$$

Again, by the assumption above

$$Q_{\gamma+2} \leq Q_{\gamma+3}, Q_{\gamma+4} \leq Q_{\gamma+5}, \ldots, T \leq S, \ldots, Q_{m-1} \leq Q_m$$

Case 1

If $p_\theta \geq p_\phi$ or $(1-p_\theta) \leq (1-p_\phi)$, then, not considering the (-d), Theorem 2 says R is a maximum. Then considering the (-d), R is a minimum. So interchanging the locations of products θ and ϕ will result in an increase in R.

Case 2

If $p_\theta \leq p_\phi$ or $(1-p_\theta) \geq (1-p_\phi)$, then, not considering the (-d), Theorem 2 says R could be maximized by interchanging p_θ and p_ϕ. Considering the (-d), R could be minimized (decreased) by interchanging the locations of products θ and ϕ.

In Case 1, all the terms were constant, or were grouped so that they showed an increase in $E(D_C)$ when products θ and ϕ were interchanged, and $p_\theta \geq P_\phi$.

In Case 2, all the terms were constant, or were grouped so that they showed a decrease in $E(D_C)$ when products θ and ϕ were interchanged, and $p_\theta \leq p_\phi$.

With these results, the proof is fairly simple. If (k-1) products are located optimally, the remaining empty locations are assumed to be filled with dummy products with $p_i = 0$. This gives

$$P_1 \geq P_2 \geq P_3 \geq \cdots \geq P_{m-1} \geq P_m$$

We wish to locate the k-th product so that $E(D_C)$ for the warehouse with all k products is minimized. It is given that the product in aisle (β) with the highest demand is product (a), and the product in aisle (β) with the lowest demand is product (b). It is also given that $p_a \geq p_k \geq p_b$. If Theorem 6 were true, aisle (β) would be the optimal location for product (k). There are two cases to consider.

Case I (Aisle β is on the left side of the dock)

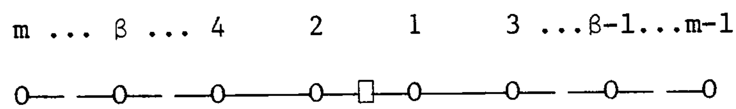

Figure 5. Whole-Warehouse with β on the left side of the dock. □ = dock, 0 = aisle center line

First, put product (k) in aisle m (the last aisle). Since aisles (β) and (m) are on the same side of the dock, Theorem 5 says $E(D_C)$ can be improved by moving product (k) up to aisle (β) and shifting all those products with $p_i \leq p_k$ down one location to make room for product (k) in aisle (β). Theorem 5 also says that if product (k) is moved to aisle (β-2) or closer to the dock on the same side, then $E(D_C)$ will show an increase over that when product (k) was in aisle (β). So aisle (β) is the best location on the left side of the dock for product (k).

Because $p_a \geq p_k \geq p_b$, and $P_1 \geq P_2 \geq \cdots \geq P_{m-1} \geq P_m$, product (k)'s optimal location on the right side of the dock, as given by Theorem 5, is in aisle (β-1). If product (k), which is in aisle (β), is interchanged with the product having the lowest demand in aisle (β-1), because p_k is smaller than p_i for this product, Case 1 above shows that $E(D_C)$ will increase. Now with product (k) on the right side of the dock, Theorem 5 says that any aisle closer to the dock, on this same side of the dock, than aisle (β-1) will show additional increases in $E(D_C)$, and moving it to any location farther away from the dock on that side will also cause an increase in $E(D_C)$.

Therefore, moving product (k) in any direction out of aisle (β) causes an increase in $E(D_C)$. Aisle (β) must then be the optimal location for product (k).

<u>Case II</u> (Aisle β is on the right side of the dock)

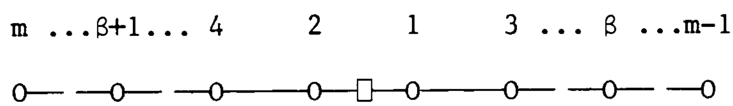

Figure 6. Whole-Warehouse with β on the right side of the dock. □ = dock, 0 = aisle center line

Once again, start by putting product (k) in aisle m. Because product (k) has demand such that $p_a \geq p_k \geq p_b$, and because all the other products are located optimally $(P_1 \geq P_2 \geq \ldots \geq P_m)$, aisle (β+1) is the best location on the left side of the dock, as given by Theorem 5. Therefore, moving product (k) to aisle (β+1), still on the left side of the dock, will cause decreases in $E(D_C)$, and moving it closer than aisle (β+1) will cause increases in $E(D_C)$.

If product (k) in aisle (β+1), is interchanged with product (b) which has the lowest demand in aisle (β), Case 2 above says the $E(D_C)$ will decrease. Because $p_k \leq p_i$ for all products in aisle (β-2) and closer to the dock, and $p_k \geq p_i$ for all the products in aisle (β+2) and farther away from the dock, Theorem 5 says the $E(D_C)$ will increase if product (k) is moved in either direction on that side of the dock.

Therefore, moving product (k) in any direction out of aisle (β) causes an increase in $E(D_C)$. Aisle (β) must then be the optimal location for product (k).

Both Case I and Case II show aisle (β) to be the location for product k that minimizes the Expected Distance Traveled in the In-Aisle direction. This concurs with the location that Theorem 6 would give.