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OPTIMAL QUANTIZATION FOR THE CANTOR DISTRIBUTION GENERATED BY INFINITE SIMILUTUDES

MRINAL KANTI ROYCHOWDHURY

Abstract. Let P be a Borel probability measure on $\mathbb{R}$ generated by an infinite system of similarity mappings $\{S_j : j \in \mathbb{N}\}$ such that $P = \sum_{j=1}^{\infty} \frac{1}{2^j} P \circ S_j^{-1}$, where for each $j \in \mathbb{N}$ and $x \in \mathbb{R}$, $S_j(x) = \frac{1}{3^j}x + 1 - \frac{1}{3^{j-1}}$. Then, the support of P is the dyadic Cantor set C generated by the similarity mappings $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ such that $f_1(x) = \frac{1}{3}x$ and $f_2(x) = \frac{1}{3}x + \frac{2}{3}$ for all $x \in \mathbb{R}$. In this paper, using the infinite system of similarity mappings $\{S_j : j \in \mathbb{N}\}$ associated with the probability vector $(\frac{1}{2}, \frac{1}{2^2}, \cdots)$, for all $n \in \mathbb{N}$, we determine the optimal sets of n-means and the nth quantization errors for the infinite self-similar measure P. The technique obtained in this paper can be utilized to determine the optimal sets of n-means and the nth quantization errors for more general infinite self-similar measures.

1. Introduction

The history of the theory and practice of quantization dates back to 1948, although similar ideas had appeared in the literature in 1897 (see [S]). It is used in many applications such as signal processing and telecommunications, data compression, pattern recognitions and cluster analysis (for details see [GG, GN]). It is also closely connected with centroidal Voronoi tessellations. Let $\mathbb{R}^d$ denote the d-dimensional Euclidean space, $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^d$ for any $d \geq 1$, and P be a Borel probability measure on $\mathbb{R}^d$. For a finite set $\alpha \subset \mathbb{R}^d$ and $a \in \alpha$, by $M(a|\alpha)$ we denote the set of all elements in $\mathbb{R}^d$ which are nearest to a among all the elements in α, i.e.,

$$M(a|\alpha) = \{x \in \mathbb{R}^d : \|x - a\| = \min_{b \in \alpha} \|x - b\|\}.$$

$M(a|\alpha)$ is called the *Voronoi region* generated by $a \in \alpha$. On the other hand, the set $\{M(a|\alpha) : a \in \alpha\}$ is called the *Voronoi diagram* or *Voronoi tessellation* of $\mathbb{R}^d$ with respect to the set α.

Definition 1.1. *A set $\alpha \subset \mathbb{R}^d$ is called a centroidal Voronoi tessellation (CVT) with respect to a probability distribution P on $\mathbb{R}^d$, if it satisfies the following two conditions:*
(i) $P(M(a|\alpha) \cap M(b|\alpha)) = 0$ for $a, b \in \alpha$, and $a \neq b$;
(ii) $E(X : X \in M(a|\alpha)) = a$ for all $a \in \alpha$,
where X is a random variable with distribution P, and $E(X : X \in M(a|\alpha))$ represents the conditional expectation of the random variable X given that X takes values in $M(a|\alpha)$.

For details about CVT and its application one can see [DFG]. If α is a finite set, the error $\int \min_{a\in\alpha} \|x - a\|^2 dP(x)$ is often referred to as the *cost,* or *distortion error* for α with respect to the probability measure P, and is denoted by $V(\alpha) := V(P; \alpha)$. On the other hand, $\inf\{V(P; \alpha) : \alpha \subset \mathbb{R}^d, \text{ card}(\alpha) \leq n\}$ is called the *nth quantization error* for the probability measure P, and is denoted by $V_n := V_n(P)$. If $\int \|x\|^2 dP(x) < \infty$, then there is some set α for which the infimum is achieved (see [GKL, GL1, GL2]). Such a set α for which the infimum occurs and contains no more than n points is called an *optimal set of n-means.* Elements of an optimal set of n-means are called *optimal quantizers.* In some literature it is also refereed to as *principal points* (see [MKT], and the references therein). To see some work on optimal sets of n-means one is refereed to [DR, GL3, RR, R1, R2]. It is known that for a continuous probability measure an optimal set of n-means always has exactly n-elements (see [GL2]). For a Borel probability

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measure P on $\mathbb{R}^d$, an optimal set of n-means forms a CVT with n-means (n-generators) of $\mathbb{R}^d$; however, the converse is not true in general (see [DFG, R]). A CVT with n-means is called an *optimal CVT with n-means* if the generators of the CVT form an optimal set of n-means with respect to the probability distribution P. Let us now state the following proposition (see [GG, Chapter 6 and Chapter 11] and [GL2, Section 4.1]).

Proposition 1.2. *Let α be an optimal set of n-means for a continuous Borel probability measure P on $\mathbb{R}^d$. Let $a \in \alpha$ and $M(a|\alpha)$ be the Voronoi region generated by $a \in \alpha$. Then, for every $a \in \alpha$, (i) $P(M(a|\alpha)) > 0$, (ii) $P(\partial M(a|\alpha)) = 0$, (iii) $a = E(X : X \in M(a|\alpha))$, and (iv) P-almost surely the set $\{M(a|\alpha) : a \in \alpha\}$ forms a Voronoi partition of $\mathbb{R}^d$.*

Let C be the Cantor set generated by the contractive similarity mappings f_1 and f_2 given by $f_1(x) = \frac{1}{3}x$ and $f_2(x) = \frac{1}{3}x + \frac{2}{3}$ for all $x \in \mathbb{R}$. Then, C satisfies

$$C = \bigcup_{\omega \in \{1,2\}^\infty} \bigcap_{n=1}^{\infty} f_{\omega|_n}([0,1]),$$

where for $\omega := \omega_1\omega_2\cdots \in \{1,2\}^\infty$, $f_{\omega|_n} = f_{\omega_1} \circ f_{\omega_2} \circ \cdots \circ f_{\omega_n}$. Notice that for any $\omega := \omega_1\omega_2\cdots \in \{1,2\}^\infty$ and $n \in \mathbb{N}$, $\omega|_n := \omega_1\omega_2\cdots\omega_n \in \{1,2\}^*$, where $\{1,2\}^*$ denotes the set of all words over the alphabet $\{1,2\}$ including the empty word $\emptyset$. Define $\mu := \frac{1}{2}\mu \circ f_1^{-1} + \frac{1}{2}\mu \circ f_2^{-1}$. Then, μ is a self-similar measure on $\mathbb{R}$ such that $\mu(C) = 1$.

Definition 1.3. *For $n \in \mathbb{N}$ with $n \geq 2$ let $\ell(n)$ be the unique natural number with $2^{\ell(n)} \leq n < 2^{\ell(n)+1}$. For $I \subset \{1,2\}^{\ell(n)}$ with card$(I) = n - 2^{\ell(n)}$ let $\beta_n(I)$ be the set consisting of all midpoints a_σ of intervals $f_\sigma([0,1])$ with $\sigma \in \{1,2\}^{\ell(n)} \setminus I$ and all midpoints $a_{\sigma 1}$, $a_{\sigma 2}$ of the basic intervals of $f_\sigma([0,1])$ with $\sigma \in I$. Formally, $\beta_n(I) = \{a_\sigma : \sigma \in \{1,2\}^{\ell(n)} \setminus I\} \cup \{a_{\sigma 1} : \sigma \in I\} \cup \{a_{\sigma 2} : \sigma \in I\}$.*

In [GL3], Graf and Luschgy showed that $\beta_n(I)$ forms an optimal set of n-means for the probability distribution μ, and the nth quantization error is given by

$$V(\beta_n(I)) = \frac{1}{18^{\ell(n)}} \cdot \frac{1}{8}\Big(2^{\ell(n)+1} - n + \frac{1}{9}(n - 2^{\ell(n)})\Big).$$

Let $\{S_j\}_{j=1}^\infty$ be an infinite collection of contractive similitudes on $\mathbb{R}$ such that $S_j(x) = \frac{1}{3^j}x + 1 - \frac{1}{3^{j-1}}$ for all $x \in \mathbb{R}$ and all $j \in \mathbb{N}$. S_j has the similarity ratio s_j, where $s_j = \frac{1}{3^j}$, for each $j \in \mathbb{N}$. Let π be the coding map from the symbol or coding space $\mathbb{N}^\infty$ into $[0,1]$ such that

$$\{\pi(\omega)\} = \bigcap_{n=1}^{\infty} S_{\omega|_n}([0,1]),$$

where for $\omega := \omega_1\omega_2\cdots \in \mathbb{N}^\infty$, $\omega|_n := \omega_1\omega_2\cdots\omega_n$, and $S_{\omega|_n} = S_{\omega_1} \circ S_{\omega_2} \circ \cdots \circ S_{\omega_n}$. Then, π is a continuous map from the coding space onto the set J given by

$$J := \pi(\mathbb{N}^\infty) = \bigcup_{\omega \in \mathbb{N}^\infty} \bigcap_{n=1}^{\infty} S_{\omega|_n}([0,1]),$$

which is called the limit set of the infinite iterated function system (IFS) $\{S_j\}_{j=1}^\infty$. Observe that J satisfies the natural invariance equality: $J = \cup_{j=1}^\infty S_j(J)$. Due to an infinite iterated function system, the limit set J is not necessarily compact (see [HMU]). Let $(\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \cdots)$ be the probability vector associated with the infinite IFS $\{S_j\}_{j=1}^\infty$. Then, there exists a unique Borel probability measure P on $\mathbb{R}$, actually on J, such that $P = \sum_{j=1}^\infty \frac{1}{2^j} P \circ S_j^{-1}$ (see [RM]). For each $j \in \mathbb{N}$, we have $S_j(x) = (f_2^{(j-1)} \circ f_1)(x)$ for $x \in \mathbb{R}$, where f_1 and f_2 are the two similarity mappings generating the dyadic Cantor set C as defined before, and $f_2^{(j-1)}$ denotes the $j-1$ iteration of the mapping f_2. Therefore, by iterated application of some of the S_j we can generate the mappings $f_\omega := f_{\omega_1} \circ f_{\omega_2} \circ \cdots \circ f_{\omega_n}$ for all finite words $\omega := \omega_1\omega_2\cdots\omega_n \in \{1,2\}^*$ except of those where $\omega_n = 2$. Because of the contraction property, we have

$$\lim_{n\to\infty} f_{\omega_1\omega_2\cdots\omega_n}([0,1]) = \lim_{n\to\infty} f_{w_1\ldots w_{n-1}2}([0,1]),$$

i.e., the limit does not depend on the last letter ω_n. Therefore, the infinite coding mappings agree with those of the Cantor set and thus, we get $J = C$. Again, for any $j \in \mathbb{N}$,

$$P(S_j([0,1])) = \frac{1}{2^j}, \text{ and } \mu(f_2^{(j-1)} \circ f_1([0,1])) = \frac{1}{2^{j-1}}\frac{1}{2} = \frac{1}{2^j}.$$

Thus, we see that $P = \mu$. Hence, we can say that for any $n \in \mathbb{N}$, the optimal sets of n-means for P are same as the optimal sets of n-means for μ. In this paper, using the infinite system of similarity mappings $\{S_j : j \in \mathbb{N}\}$ associated with the probability vector $(\frac{1}{2}, \frac{1}{2^2}, \cdots)$, for all $n \in \mathbb{N}$, we determine the optimal sets of n-means and the nth quantization errors for the infinite self-similar measure P. Due to infinite number of mappings the technique used in this paper is completely different from the technique used by Graf-Luschgy in [GL3]. For $j \in \mathbb{N}$, let $r_j \in (0, \frac{1}{2})$. Let $\{S_j\}_{j=1}^\infty$ be a collection of contractive similarity mappings on $\mathbb{R}$ such that for $x \in \mathbb{R}$,

$$S_j(x) = \left\{\begin{array}{ll} r_1 x & \text{if } j = 1, \\ r_1 r_2 \cdots r_j x + 1 - r_1 r_2 \cdots r_{j-1} & \text{if } j \geq 2. \end{array}\right.$$

Let $(p_1, p_2, \cdots)$ be a probability vector with $p_j > 0$ for all $j \in \mathbb{N}$. Then, for the infinite self-similar measure P given by $P = \sum_{j=1}^\infty p_j P \circ S_j^{-1}$ the optimal sets of n-means and the nth quantization errors are not known yet for all infinite probability vectors $(p_1, p_2, \cdots)$ and all $0 < r_j < \frac{1}{2}$. The technique developed in this paper can be used to investigate the optimal sets of n-means and the nth quantization errors for such a more general infinite self-similar measure P.

2. Basic definitions, lemmas and proposition

Let $\mathbb{N}$ denote the set all natural numbers, i.e., $\mathbb{N} = \{1, 2, \cdots\}$. By a *string* or a *word* ω over the alphabet $\mathbb{N}$, we mean a finite sequence $\omega := \omega_1\omega_2\cdots\omega_k$ of symbols from the alphabet, where $k \geq 1$, and k is called the length of the word ω. The length of a word ω is denoted by $|\omega|$. A word of length zero is called the *empty word*, and is denoted by $\emptyset$. We denote the set of all words of length k by $\mathbb{N}^k$. By $\mathbb{N}^*$ we denote the set of all words over the alphabet $\mathbb{N}$ of some finite length k including the empty word $\emptyset$. For any two words $\omega := \omega_1\omega_2\cdots\omega_k$ and $\tau := \tau_1\tau_2\cdots\tau_\ell$ in $\mathbb{N}^*$, $\omega\tau := \omega_1\cdots\omega_k\tau_1\cdots\tau_\ell$ is the concatenation of the words ω and τ. For $n \geq 1$ and $\omega = \omega_1\omega_2\cdots\omega_n \in \mathbb{N}^*$ we define $\omega^- := \omega_1\omega_2\cdots\omega_{n-1}$, i.e., ω^- is the word obtained from the word ω by deleting the last letter of ω. Notice that ω^- is the empty word if the length of ω is one. For $\omega \in \mathbb{N}^*$, by (ω, ∞) it is meant the set of all words $\omega^-(\omega_{|\omega|} + j)$, obtained by concatenation of the word ω^- with the word $\omega_{|\omega|} + j$ for $j \in \mathbb{N}$, i.e.,

$$(\omega, \infty) = \{\omega^-(\omega_{|\omega|} + j) : j \in \mathbb{N}\}.$$

Write $P := \sum_{j=1}^\infty p_j P \circ S_j^{-1}$, where $p_j = \frac{1}{2^j}$ for all $j \in \mathbb{N}$ and $\{S_j\}_{j=1}^\infty$ is the infinite collection of similitudes with similarity ratios $s_j := \frac{1}{3^j}$ for $j \in \mathbb{N}$ as defined in the previous section. Then, P has support lying in the closed interval $[0,1]$. This paper deals with this probability measure P. For $\omega = \omega_1\omega_2\cdots\omega_n \in \mathbb{N}^n$, write

$$S_\omega := S_{\omega_1} \circ \cdots \circ S_{\omega_n}, \quad J_\omega := S_\omega(J), \quad s_\omega := s_{\omega_1}\cdots s_{\omega_n}, \quad p_\omega := p_{\omega_1}\cdots p_{\omega_n},$$

where $J := J_\emptyset = [0,1]$. If ω is the empty word $\emptyset$, then S_ω represents the identity mapping on $\mathbb{R}$, and $p_\omega = 1$. Then, for any $\omega \in \mathbb{N}^*$, we write

$$J_{(\omega,\infty)} := \bigcup_{j=1}^\infty J_{\omega^-(\omega_{|\omega|}+j)} \text{ and } p_{(\omega,\infty)} := P(J_{(\omega,\infty)}) = \sum_{j=1}^\infty P(J_{\omega^-(\omega_{|\omega|}+j)}) = \sum_{j=1}^\infty p_{\omega^-(\omega_{|\omega|}+j)}.$$

Notice that for any $k \in \mathbb{N}$, $p_{(k,\infty)} = 1 - \sum_{j=1}^k p_j$, and for any word $\omega \in \mathbb{N}^*$, $p_{(\omega,\infty)} = p_{\omega^-} - p_\omega$.

Lemma 2.1. *Let $f : \mathbb{R} \to \mathbb{R}^+$ be Borel measurable and $k \in \mathbb{N}$. Then,*
$$\int f(x)dP(x) = \sum_{\omega \in \mathbb{N}^k} p_\omega \int (f \circ S_\omega)(x)dP(x).$$

Proof. We know $P = \sum_{j=1}^\infty p_j P \circ S_j^{-1}$, and so by induction $P = \sum_{\omega \in \mathbb{N}^k} p_\omega P \circ S_\omega^{-1}$, and thus the lemma is established. □

Lemma 2.2. *Let X be a random variable with probability distribution P. Then, the expectation $E(X)$ and the variance $V := V(X)$ of the random variable X are given by*
$$E(X) = \frac{1}{2} \text{ and } V = \frac{1}{8}.$$

Proof. Using Lemma 2.1, we have
$$E(X) = \int x dP(x) = \sum_{j=1}^\infty \frac{1}{2^j} \int S_j(x)dP(x) = \sum_{j=1}^\infty \frac{1}{2^j} \int \Big(\frac{1}{3^j}x + 1 - \frac{1}{3^{j-1}}\Big) dP(x)$$
$$= \sum_{j=1}^\infty \Big(\frac{1}{6^j} E(X) + \frac{1}{2^j}(1 - \frac{1}{3^{j-1}})\Big) = \frac{1}{5}E(X) + 1 - \frac{3}{5},$$
which implies $E(X) = \frac{1}{2}$. Now,
$$E(X^2) = \int x^2 dP(x) = \sum_{j=1}^\infty \frac{1}{2^j} \int \Big(\frac{1}{3^j}x + 1 - \frac{1}{3^{j-1}}\Big)^2 dP(x)$$
$$= \sum_{j=1}^\infty \frac{1}{2^j} \int \Big(\frac{1}{9^j}x^2 + \frac{2}{3^j}(1 - \frac{1}{3^{j-1}})x + (1 - \frac{1}{3^{j-1}})^2\Big) dP(x).$$
Since,
$$\sum_{j=1}^\infty \frac{1}{18^j} = \frac{1}{17} \text{ and } \sum_{j=1}^\infty \frac{1}{2^j} \int \frac{2}{3^j}(1 - \frac{1}{3^{j-1}})x dP = \frac{2}{85}, \text{ and } \sum_{j=1}^\infty \frac{1}{2^j}(1 - \frac{1}{3^{j-1}})^2 = \frac{28}{85},$$
we have $E(X^2) = \frac{1}{17}E(X^2) + \frac{2}{85} + \frac{28}{85}$ which yields $E(X^2) = \frac{3}{8}$. Thus,
$$V = E(X^2) - (E(X))^2 = \frac{3}{8} - \frac{1}{4} = \frac{1}{8},$$
which is the lemma. □

Lemma 2.3. *For any $k \geq 1$, we have*
$$E(X | X \in J_k \cup J_{k+1} \cup \cdots) = 1 - \frac{1}{2}\frac{1}{3^{k-1}}.$$

Proof. By the definition of conditional expectation, we have
$$E(X | X \in J_k \cup J_{k+1} \cup \cdots) = \frac{1}{\sum_{j=k}^\infty \frac{1}{2^j}} \Big(\sum_{j=k}^\infty \frac{1}{2^j} S_j(\frac{1}{2})\Big) = 2^{k-1} \sum_{j=k}^\infty \frac{1}{2^j}(1 - \frac{5}{2}\frac{1}{3^j}) = 1 - \frac{1}{2}\frac{1}{3^{k-1}},$$
which is the lemma. □

Now, the following remarks are in order.

Remark 2.4. For $k \in \mathbb{N}$, we have $S_k(\frac{1}{2}) = \frac{1}{3^k}\frac{1}{2} + 1 - \frac{1}{3^{k-1}} = 1 - \frac{5}{2}\frac{1}{3^k}$. Thus, by Lemma 2.3, for $k \in \mathbb{N}$,
$$E(X | X \in J_k \cup J_{k+1} \cup \cdots) = S_k(\frac{1}{2}) + \frac{1}{3^k} = \frac{1}{2}(S_k(1) + S_{k+1}(0)).$$

For any $x_0 \in \mathbb{R}$, we have $\int (x-x_0)^2 dP(x) = V(X) + (x_0 - E(X))^2$, and so, the optimal set of one-mean is the expected value and the corresponding quantization error is the variance V of the random variable X. For $\omega \in \mathbb{N}^k$, $k \geq 1$, using Lemma 2.1, we have

$$E(X : X \in J_\omega) = \frac{1}{P(J_\omega)} \int_{J_\omega} x dP(x) = \int_{J_\omega} x dP \circ S_\omega^{-1}(x) = \int S_\omega(x) dP(x) = E(S_\omega(X)).$$

Since S_j are similitudes, we have $E(S_j(X)) = S_j(E(X))$ for $j \in \mathbb{N}$, and so by induction, $E(S_\omega(X)) = S_\omega(E(X))$ for $\omega \in \mathbb{N}^k$, $k \geq 1$.

Remark 2.5. For words $\beta, \gamma, \cdots, \delta$ in $\mathbb{N}^*$, by $a(\beta, \gamma, \cdots, \delta)$ we denote the conditional expectation of the random variable X given $J_\beta \cup J_\gamma \cup \cdots \cup J_\delta$, i.e.,

$$a(\beta, \gamma, \cdots, \delta) = E(X | X \in J_\beta \cup J_\gamma \cup \cdots \cup J_\delta) = \frac{1}{P(J_\beta \cup \cdots \cup J_\delta)} \int_{J_\beta \cup \cdots \cup J_\delta} x dP(x).$$

Thus by Remark 2.4, for $\omega \in \mathbb{N}^*$, we have

$$(1) \qquad \begin{cases} a(\omega) = S_\omega(E(X)) = S_\omega(\frac{1}{2}), \text{ and} \\ a(\omega, \infty) = E(X | X \in J_{\omega^-(\omega_{|\omega|}+1)} \cup J_{\omega^-(\omega_{|\omega|}+2)} \cup \cdots) = S_{\omega^-(\omega_{|\omega|}+1)}(\frac{1}{2}) + s_{\omega^-(\omega_{|\omega|}+1)}. \end{cases}$$

Moreover, for any $\omega \in \mathbb{N}^*$ and $j \geq 1$, since $p_{\omega^-(\omega_{|\omega|}+j)} = p_{\omega^-} p_{\omega_{|\omega|}+j} = p_{\omega^-} p_{\omega_{|\omega|}} p_j = p_\omega p_j = p_{\omega j}$, and similarly $s_{\omega^-(\omega_{|\omega|}+j)} = s_\omega s_j = s_{\omega j}$, for any $x_0 \in \mathbb{R}$, we have

$$(2) \qquad \begin{cases} \int_{J_\omega} (x-x_0)^2 dP(x) = p_\omega \int (x-x_0)^2 dP \circ S_\omega^{-1}(x) = p_\omega \Big(s_\omega^2 V + (S_\omega(\frac{1}{2}) - x_0)^2 \Big), \text{ and} \\ \int_{J_{(\omega,\infty)}} (x-x_0)^2 dP(x) = \sum_{j=1}^\infty p_{\omega j} \Big(s_{\omega j}^2 V + (S_{\omega^-(\omega_{|\omega|}+j)}(\frac{1}{2}) - x_0)^2 \Big). \end{cases}$$

In the sequel, for $\omega \in \mathbb{N}^k$, $k \geq 1$, by $E(a(\omega))$, we mean the error contributed by $a(\omega)$ in the region J_ω; and similarly, by $E(a(\omega, \infty))$, it is meant the error contributed by $a(\omega, \infty)$ in the region $J_{(\omega,\infty)}$. We apologize for any abuse in notation. Thus, we have

$$(3) \qquad E(a(\omega)) := \int_{J_\omega} (x - a(\omega))^2 dP(x) \text{ and } E(a(\omega, \infty)) := \int_{J_{(\omega,\infty)}} (x - a(\omega, \infty))^2 dP(x).$$

We now prove the following lemma.

Lemma 2.6. *Let $\omega \in \mathbb{N}^*$. Let $E(a(\omega))$ and $E(a(\omega, \infty))$ be defined by (3). Then,*

$$E(a(\omega)) = E(a(\omega, \infty)) = p_\omega s_\omega^2 V.$$

Proof. In the first equation of (2) put $x_0 = a(\omega)$, and then $E(a(\omega)) = p_\omega s_\omega^2 V$. In the second equation of (2), put $x_0 = a(\omega, \infty)$, and then

$$(4) \qquad E(a(\omega, \infty)) = \sum_{j=1}^\infty p_{\omega j} \Big(s_{\omega j}^2 V + (S_{\omega^-(\omega_{|\omega|}+j)}(\frac{1}{2}) - a(\omega, \infty))^2 \Big).$$

Putting the values of $a(\omega, \infty)$ from (1), we have

$$\begin{aligned} & S_{\omega^-(\omega_{|\omega|}+j)}(\frac{1}{2}) - a(\omega, \infty) = S_{\omega^-(\omega_{|\omega|}+j)}(\frac{1}{2}) - S_{\omega^-(\omega_{|\omega|}+1)}(\frac{1}{2}) - s_{\omega^-(\omega_{|\omega|}+1)} \\ & = s_{\omega^-} \Big(S_{\omega_{|\omega|}+j}(\frac{1}{2}) - S_{\omega_{|\omega|}+1}(\frac{1}{2}) - s_{\omega^-(\omega_{|\omega|}+1)} \Big) \\ & = s_{\omega^-} \Big(\frac{1}{3^{\omega_{|\omega|}+j}} \frac{1}{2} - \frac{1}{3^{\omega_{|\omega|}+j-1}} - \frac{1}{2^{\omega_{|\omega|}+1}} \frac{1}{2} + \frac{1}{2^{\omega_{|\omega|}}} - s_{\omega^-(\omega_{|\omega|}+1)} \Big) \\ & = s_\omega \Big(\frac{1}{3^j} \frac{1}{2} - \frac{3}{3^j} - \frac{1}{6} + 1 - \frac{1}{3} \Big) = s_\omega \Big(\frac{1}{2} - \frac{5}{2} \frac{1}{3^j} \Big). \end{aligned}$$

Hence, (4) implies $E(a(\omega, \infty)) = p_\omega s_\omega^2 \sum_{j=1}^\infty \frac{1}{2^j} \Big(\frac{1}{9^j} V + (\frac{1}{2} - \frac{5}{2} \frac{1}{3^j})^2 \Big) = p_\omega s_\omega^2 V$. Thus, the proof of the lemma is complete. □

Remark 2.7. By (2) and Lemma 2.6, for any $x_0 \in \mathbb{R}$, we have

$$\left\{ \begin{array}{l} \int_{J_\omega} (x - x_0)^2 dP(x) = E(a(\omega)) + (x_0 - a(\omega))^2 p_\omega, \text{ and} \\ \int_{J_{(\omega,\infty)}} (x - x_0)^2 dP(x) = E(a(\omega)) + (x_0 - a(\omega, \infty))^2 (p_{\omega^-} - p_\omega). \end{array} \right. \tag{5}$$

Notice that by (1), we have $a(\omega, \infty) = a(\omega^-(\omega_{|\omega|} + 1)) + s_{\omega^-(\omega_{|\omega|}+1)}$. The expressions (1) and (5) are useful to obtain the optimal sets and the corresponding quantization errors with respect to the probability distribution P.

The following lemma is useful.

Lemma 2.8. *For any two words $\omega, \tau \in \mathbb{N}^*$, if $p_\omega = p_\tau$, then*

$$\int_{J_\omega} (x - a(\omega))^2 dP(x) = \int_{J_\tau} (x - a(\tau))^2 dP(x).$$

Proof. Let $\omega, \tau \in \mathbb{N}^*$. Let $\omega = \omega_1\omega_2 \cdots \omega_k$ and $\tau = \tau_1\tau_2 \cdots \tau_m$ for some $k, m \in \mathbb{N}$. Then, $p_\omega = p_\tau$ implies $\omega_1 + \omega_2 + \cdots + \omega_k = \tau_1 + \tau_2 + \cdots + \tau_m$, and so $s_\omega = s_\tau$. Thus,

$$\int_{J_\omega} (x - a(\omega))^2 dP(x) = p_\omega s_\omega^2 V = p_\tau s_\tau^2 V = \int_{J_\tau} (x - a(\tau))^2 dP(x),$$

which is the lemma. □

Definition 2.9. *For $n \in \mathbb{N}$ with $n \geq 2$ let $\ell(n)$ be the unique natural number with $2^{\ell(n)} \leq n < 2^{\ell(n)+1}$. Write*

$$\alpha(\ell(n)) := \{a(\omega) : \omega \in \mathbb{N}^* \text{ and } p_\omega = \frac{1}{2^{\ell(n)}}\} \cup \{a(\omega, \infty) : \omega \in \mathbb{N}^* \text{ and } p_\omega = \frac{1}{2^{\ell(n)}}\}.$$

For $I \subset \alpha(\ell(n))$ with card$(I) = n - 2^{\ell(n)}$, write

$$\begin{aligned} \alpha_n(I) :=\ & (\alpha(\ell(n)) \setminus I) \cup \{a(\omega 1) : a(\omega) \in I\} \cup \{a(\omega 1, \infty) : a(\omega) \in I\} \\ & \cup \{a(\omega^-(\omega_{|\omega|} + 1)) : a(\omega, \infty) \in I\} \cup \{a(\omega^-(\omega_{|\omega|} + 1), \infty) : a(\omega, \infty) \in I\}. \end{aligned}$$

Remark 2.10. In Definition 2.9, if $n = 2^{\ell(n)}$, then $I = \emptyset$, and so, $\alpha_n(I) = \alpha(\ell(n))$.

Using Definition 2.9, we now give a few examples.

Example 2.11. Let $n = 3$. Then, $\ell(n) = 1$, $\alpha(1) = \{a(1), a(1, \infty)\} = \{\frac{1}{6}, \frac{5}{6}\}$, card$(I) = 1$. If $I = \{a(1)\}$, then

$$\alpha_3(I) = \{a(11), a(11, \infty), a(1, \infty)\} = \{\frac{1}{18}, \frac{5}{18}, \frac{5}{6}\}.$$

If $I = \{a(1, \infty)\}$, then,

$$\alpha_3(I) = \{a(1), a(2), a(2, \infty)\} = \{\frac{1}{6}, \frac{13}{18}, \frac{17}{18}\}.$$

Example 2.12. Let $n = 4$. Then, $\ell(n) = 2$, $I = \emptyset$, and so

$$\alpha_4(I) = \alpha(2) = \{a(11), a(11, \infty), a(2), a(2, \infty)\} = \{\frac{1}{18}, \frac{5}{18}, \frac{13}{18}, \frac{17}{18}\}.$$

Example 2.13. Let $n = 5$. Then, $\ell(n) = 2$, $\alpha(2) = \{a(11), a(11, \infty), a(2), a(2, \infty)\}$, $I \subset \alpha(2)$ with card$(I) = 1$. If $I = \{a(11)\}$, then

$$\alpha_5(I) = \{a(111), a(111, \infty), a(11, \infty), a(2), a(2, \infty)\} = \{\frac{1}{54}, \frac{5}{54}, \frac{5}{18}, \frac{13}{18}, \frac{17}{18}\}.$$

If $I = \{a(2)\}$, then

$$\alpha_5(I) = \{a(11), a(11, \infty), a(21), a(21, \infty), a(2, \infty)\} = \{\frac{1}{18}, \frac{5}{18}, \frac{37}{54}, \frac{41}{54}, \frac{17}{18}\}.$$

If $I=\{a(11,\infty)\}$, then

$$\alpha_5(I)=\{a(11),a(12),a(12,\infty),a(2),a(2,\infty)\}=\{\frac{1}{18},\frac{13}{54},\frac{17}{54},\frac{13}{18},\frac{17}{18}\}.$$

If $I=\{a(2,\infty)\}$, then

$$\alpha_5(I)=\{a(11),a(11,\infty),a(2),a(3),a(3,\infty)\}=\{\frac{1}{18},\frac{5}{18},\frac{13}{18},\frac{49}{54},\frac{53}{54}\}.$$

Let us now prove the following proposition.

Proposition 2.14. *Let $\alpha_n(I)$ be the set as defined by Definition 2.9. Then*

$$\int \min_{a\in\alpha_n(I)}(x-a)^2dP(x)=\frac{1}{18^{\ell(n)}}\frac{1}{8}\Big(2^{\ell(n)+1}-n+\frac{1}{9}(n-2^{\ell(n)})\Big).$$

Proof. Using the definition of $\alpha_n(I)$, we have

$$\begin{aligned}&\int \min_{a\in\alpha_n(I)}(x-a)^2dP(x)\\ &=\sum_{a(\omega)\in\alpha(\ell(n))\setminus I}\int_{J_\omega}(x-a(\omega))^2dP(x)+\sum_{a(\omega,\infty)\in\alpha(\ell(n))\setminus I}\int_{J_{(\omega,\infty)}}(x-a(\omega,\infty))^2dP(x)\\ &+\sum_{a(\omega)\in I}\Big(\int_{J_{\omega1}}(x-a(\omega1))^2dP(x)+\int_{J_{(\omega1,\infty)}}(x-a(\omega1,\infty))^2dP(x)\Big)\\ &+\sum_{a(\omega,\infty)\in I}\Big(\int_{J_{\omega^-(\omega_{|\omega|}+1)}}(x-a(\omega^-(\omega_{|\omega|}+1)))^2dP(x)+\int_{J_{(\omega^-(\omega_{|\omega|}+1),\infty)}}(x-a(\omega^-(\omega_{|\omega|}+1),\infty))^2dP(x)\Big).\end{aligned}$$

Now, using Lemma 2.6, we have

$$\begin{aligned}&\sum_{a(\omega)\in\alpha(\ell(n))\setminus I}\int_{J_\omega}(x-a(\omega))^2dP(x)+\sum_{a(\omega,\infty)\in\alpha(\ell(n))\setminus I}\int_{J_{(\omega,\infty)}}(x-a(\omega,\infty))^2dP(x)\\ &=\sum_{a(\omega)\in\alpha(\ell(n))\setminus I}p_\omega s_\omega^2V+\sum_{a(\omega,\infty)\in\alpha(\ell(n))\setminus I}p_\omega s_\omega^2V\\ &=\frac{1}{18^{\ell(n)}}\frac{1}{8}\operatorname{card}(\alpha(\ell(n))\setminus I)=\frac{1}{18^{\ell(n)}}\frac{1}{8}(2^{\ell(n)+1}-n).\end{aligned}$$

Again, by Lemma 2.6, we have

$$\sum_{a(\omega)\in I}\Big(\int_{J_{\omega1}}(x-a(\omega1))^2dP(x)+\int_{J_{(\omega1,\infty)}}(x-a(\omega1,\infty))^2dP(x)\Big)=2p_1s_1^2V\sum_{a(\omega)\in I}p_\omega s_\omega^2,$$

and

$$\begin{aligned}&\sum_{a(\omega,\infty)\in I}\Big(\int_{J_{\omega^-(\omega_{|\omega|}+1)}}(x-a(\omega^-(\omega_{|\omega|}+1)))^2dP(x)+\int_{J_{(\omega^-(\omega_{|\omega|}+1),\infty)}}(x-a(\omega^-(\omega_{|\omega|}+1),\infty))^2dP(x)\Big)\\ &=2p_1s_1^2V\sum_{a(\omega,\infty)\in I}p_\omega s_\omega^2.\end{aligned}$$

Combining all these,

$$\begin{aligned}\int \min_{a\in\alpha_n(I)}(x-a)^2dP(x)&=\frac{1}{18^{\ell(n)}}\frac{1}{8}(2^{\ell(n)+1}-n)+2p_1s_1^2V\Big(\sum_{a(\omega)\in I}p_\omega s_\omega^2+\sum_{a(\omega,\infty)\in I}p_\omega s_\omega^2\Big)\\ &=\frac{1}{18^{\ell(n)}}\frac{1}{8}(2^{\ell(n)+1}-n)+\frac{1}{9}\frac{1}{8}\frac{1}{18^{\ell(n)}}\operatorname{card}(I)=\frac{1}{18^{\ell(n)}}\frac{1}{8}\Big(2^{\ell(n)+1}-n+\frac{1}{9}(n-2^{\ell(n)})\Big),\end{aligned}$$

which is the proposition. ∎

Corollary 2.15. Let V_n be the nth quantization error for every $n \geq 1$. Then,

$$V_n \leq \frac{1}{18^{\ell(n)}} \frac{1}{8}\Big(2^{\ell(n)+1} - n + \frac{1}{9}(n - 2^{\ell(n)})\Big).$$

In the next section, Theorem 3.15 gives the optimal sets of n-means and the nth quantization errors for all $n \geq 2$.

3. Optimal sets of n-means for all $n \geq 2$

In this section, first we give some basic lemmas and propositions that we need to state and prove Theorem 3.15 which gives the main result of the paper. To prove the lemmas and propositions, we will frequently use the formulas given by the expressions (1) and (5).

Lemma 3.1. *Let $\alpha := \{a_1, a_2\}$ be an optimal set of two-means, $a_1 < a_2$. Then, $a_1 = a(1) = \frac{1}{6}$, $a_2 = a(1, \infty) = \frac{5}{6}$ and the corresponding quantization error is $V_2 = \frac{1}{72} = 0.0138889$.*

Proof. by Corollary 2.15, $V_2 \leq \frac{1}{72} = 0.0138889$. Let $\alpha = \{a_1, a_2\}$ be an optimal set of two-means, $a_1 < a_2$. Since a_1 and a_2 are the centroids of their own Voronoi regions, we have $0 \leq a_1 < a_2 \leq 1$. If $a_1 \geq \frac{1}{3}$, then

$$\frac{1}{72} \geq V_2 \geq \int_{J_1} (x - \frac{1}{3})^2 dP = \frac{1}{48} > \frac{1}{72} > V_2,$$

which is a contradiction, and so $a_1 < \frac{1}{3}$. If $a_2 \leq \frac{2}{3}$, then

$$\frac{1}{72} \geq V_2 \geq \int_{J_{(1,\infty)}} (x - \frac{2}{3})^2 dP > \int_{J_2 \cup J_3 \cup J_4} (x - \frac{2}{3})^2 dP = \frac{3959}{279936} = 0.0141425 > V_2,$$

which leads to a contradiction. Thus, $\frac{2}{3} < a_2$. Since $0 \leq a_1 \leq \frac{1}{3} < \frac{2}{3} \leq a_2 \leq 1$, we have $\frac{1}{3} \leq \frac{a_1 + a_2}{2} \leq \frac{2}{3}$, and so $J_1 \subseteq M(a_1|\alpha)$ and $J_{(1,\infty)} \subseteq M(a_2|\alpha)$. Thus,

$$\int \min_{a \in \alpha} (x - a)^2 dP = \int_{J_1} (x - a_1)^2 dP + \int_{J_{(1,\infty)}} (x - a_2)^2 dP,$$

which is minimum when $a_1 = a(1) = S_1(\frac{1}{2}) = \frac{1}{6}$ and $a_2 = a(1, \infty) = S_2(\frac{1}{2}) + \frac{1}{3^2} = \frac{5}{6}$, and the corresponding quantization error is $V_2 = \frac{1}{72}$. Hence, the proof of the lemma is complete. □

Proposition 3.2. *Let α_n be an optimal set of n-means for $n \geq 2$. Then, $\alpha_n \cap J_1 \neq \emptyset$ and $\alpha_n \cap [\frac{2}{3}, 1] \neq \emptyset$. Moreover, the Voronoi region of any point in $\alpha_n \cap J_1$ does not contain any point from $[\frac{2}{3}, 1]$ and the Voronoi region of any point in $\alpha_n \cap [\frac{2}{3}, 1]$ does not contain any point from J_1.*

Proof. By Lemma 3.1, the proposition is true for $n = 2$. We now show that the proposition is true for all $n \geq 3$. Consider the set of three points β given by $\beta := \{a(11), a(11, \infty), a(1, \infty)\}$. Then, the distortion error is

$$\int \min_{a \in \beta} (x - a)^2 dP = \int_{J_{11}} (x - a(11))^2 dP + \int_{J_{(11,\infty)}} (x - a(11, \infty))^2 dP + \int_{J_{(1,\infty)}} (x - a(1, \infty))^2 dP = \frac{5}{648}.$$

Since V_n is the quantization error for n-means for all $n \geq 3$, we have $V_n \leq V_3 \leq \frac{5}{648} = 0.00771605$. Let $\alpha_n := \{a_1 < a_2 < \cdots < a_n\}$ be an optimal set of n-means for $n \geq 3$. Since the optimal quantizers are the centroids of their own Voronoi regions, we have $0 \leq a_1 < a_2 \cdots < a_n \leq 1$. Proceeding in the similar way as Lemma 3.1, it can be shown that $a_1 < \frac{1}{3}$ and $\frac{2}{3} < a_n$ yielding the fact that $\alpha_n \cap J_1 \neq \emptyset$ and $\alpha_n \cap [\frac{2}{3}, 1] \neq \emptyset$. Let $j = \max\{i : a_i \leq \frac{1}{3}\}$. Then, $a_j \leq \frac{1}{3}$. Suppose that the Voronoi region of a_j contains points from $[\frac{2}{3}, 1]$. Then, we must have $\frac{1}{2}(a_j + a_{j+1}) > \frac{2}{3}$ implying $a_{j+1} > \frac{4}{3} - a_j \geq \frac{4}{3} - \frac{1}{3} = 1$, which gives a contradiction. Hence, the Voronoi region of any point in $\alpha_n \cap J_1$ does not contain any point from $J_{(1,\infty)}$. Similarly, we can show that the Voronoi region of any point in $\alpha_n \cap [\frac{2}{3}, 1]$ does not contain any point from J_1. Thus, the proof of the proposition is complete. □

We need the following two lemmas to prove Lemma 3.5.

Lemma 3.3. *Let $V(P, J_1, \{a, b\})$ be the quantization error due to the points a and b on the set J_1, where $0 \leq a < b$ and $b = \frac{1}{3}$. Then, $a = a(11)$ and*

$$V(P, J_1, \{a, b\}) = \int_{J_{11}} (x - a(11))^2 dP + \int_{J_{(11,\infty)}} (x - \frac{1}{3})^2 dP = \frac{1}{648}.$$

Proof. Consider the set $\{a(11), \frac{1}{3}\}$. Then, as $S_{11}(1) = \frac{1}{9} < \frac{1}{2}(a(11) + \frac{1}{3}) = \frac{7}{36} < S_{12}(0) = \frac{2}{9}$, and $V(P, J_1, \{a, b\})$ is the quantization error due to the points a and b on the set J_1, we have

$$V(P, J_1, \{a, b\}) \leq \int_{J_{11}} (x - a(11))^2 dP + \int_{J_{(11,\infty)}} (x - \frac{1}{3})^2 dP = \frac{1}{2592} + \frac{1}{864} = \frac{1}{648} = 0.00154321.$$

If $\frac{1}{8} \leq a$, then

$$V(P, J_1, \{a, b\}) \geq \int_{J_{11}} (x - \frac{1}{8})^2 dP = \frac{11}{6912} = 0.00159144 > V(P, J_1, \{a, b\}),$$

which is a contradiction, and so we can assume that $a < \frac{1}{8}$. If the Voronoi region of b contains points from J_1, we must have $\frac{1}{2}(a + b) < \frac{1}{9}$ implying $a < \frac{2}{9} - b = \frac{2}{9} - \frac{1}{3} = -\frac{1}{9}$, which leads to a contradiction. So, we can assume that the Voronoi region of b does not contain any point from J_{11} yielding $a \geq a(11) = \frac{1}{18}$. If the Voronoi region of a contains points from $[\frac{2}{9}, \frac{1}{3}]$, we must have $\frac{1}{2}(a + \frac{1}{3}) > \frac{2}{9}$ implying $a > \frac{4}{9} - \frac{1}{3} = \frac{1}{9}$, and so $\frac{1}{9} < a \leq \frac{1}{8}$. But, then $\frac{1}{2}(\frac{1}{8} + \frac{1}{3}) = \frac{11}{48} < S_{121}(1)$ yielding

$$V(P, J_1, \{a, b\}) > \int_{J_{11}} (x - \frac{1}{9})^2 dP + \int_{J_{122} \cup J_{123} \cup J_{124} \cup J_{13}} (x - \frac{1}{3})^2 = \frac{2577311}{1632586752} = 0.00157867,$$

and so, $V(P, J_1, \{a, b\}) > 0.00157867 > V(P, J_1, \{a, b\})$, which gives a contradiction. Hence, the Voronoi region of a does not contain any point from $[\frac{2}{9}, \frac{1}{3}]$ yielding $a \leq a(11)$. Again, we have seen $a \geq a(11)$. Thus, $a = a(11)$ and

$$V(P, J_1, \{a, b\}) = \int_{J_{11}} (x - a(11))^2 dP + \int_{J_{(11,\infty)}} (x - \frac{1}{3})^2 dP = \frac{1}{648},$$

which is the lemma. □

Proceeding in the similar way as Lemma 3.3, the following lemma can be proved.

Lemma 3.4. *Let $V(P, J_{(1,\infty)}, \{a, b\})$ be the quantization error due to the points a and b on the set $J_{(1,\infty)}$, where $a = \frac{2}{3}$ and $\frac{2}{3} < b \leq 1$. Then, $b = a(2, \infty)$ and*

$$V(P, J_{(1,\infty)}, \{a, b\}) = \int_{J_2} (x - \frac{2}{3})^2 dP + \int_{J_{(2,\infty)}} (x - a(2, \infty))^2 dP = \frac{1}{648}.$$

Lemma 3.5. *Let α be an optimal set of three-means. Then, $\alpha = \{a(11), a(11, \infty), a(1, \infty)\} = \{\frac{1}{18}, \frac{5}{18}, \frac{5}{6}\}$, or $\alpha = \{a(1), a(2), a(2, \infty)\} = \{\frac{1}{6}, \frac{13}{18}, \frac{17}{18}\}$ with quantization error $V_3 = \frac{5}{648} = 0.00771605$.*

Proof. As shown in the proof of Proposition 3.2, if V_3 is the quantization error for three-means, we have $V_3 \leq \frac{5}{648} = 0.00771605$. Let α be an optimal set of three-means with $\alpha = \{a_1, a_2, a_3\}$, where $a_1 < a_2 < a_3$. By Propositoin 3.2, we have $0 \leq a_1 < \frac{1}{3}$ and $\frac{2}{3} < a_3 \leq 1$. We now show that α_3 does not contain any point from the open interval $(\frac{1}{3}, \frac{2}{3})$. For the sake of contradiction, assume that $a_2 \in (\frac{1}{3}, \frac{2}{3})$. The following two cases can arise:

Case 1: $a_2 \in [\frac{1}{2}, \frac{2}{3})$.

Then, $\frac{1}{2}(a_1+a_2) < \frac{1}{3}$ implying $a_1 < \frac{2}{3} - a_2 \leq \frac{2}{3} - \frac{1}{2} = \frac{1}{6} = a(1)$, otherwise, the quantization error can be strictly reduced by moving the point a_2 to $\frac{2}{3}$. Thus, by Lemma 3.4, we have

$$V_3 \geq \int_{J_1} (x - \frac{1}{6})^2 dP + \frac{1}{648} = \frac{11}{1296} = 0.00848765 > V_3,$$

which leads to a contradiction.

Case 2: $a_2 \in (\frac{1}{3}, \frac{1}{2}]$.

Then, $\frac{1}{2}(a_2+a_3) > \frac{2}{3}$ implying $a_3 > \frac{4}{3} - a_2 \geq \frac{4}{3} - \frac{1}{2} = \frac{5}{6} = a(1,\infty)$. Then, by Lemma 3.3, we have

$$V_3 \geq \frac{1}{648} + \int_{J_{(1,\infty)}} (x - a(1,\infty))^2 dP = \frac{11}{1296} = 0.00848765 > V_3,$$

which gives a contradiction.

Thus, by Case 1 and Case 2, we have $a_2 \notin (\frac{1}{3}, \frac{2}{3})$, i.e., either $a_2 \in [0, \frac{1}{3}]$ or $a_2 \in [\frac{2}{3}, 1]$. Let us first assume $a_2 \in [0, \frac{1}{3}] = J_1$. Set $\alpha_1 := \{a_1, a_2\}$ and $\alpha_2 := \{a_3\}$. Since $\alpha = \alpha_1 \cup \alpha_2$, by Lemma 2.1, we deduce

$$V_3 = \int_{J_1} \min_{a\in\alpha_1}(x-a)^2 dP + \int_{J_{(1,\infty)}} (x-a_3)^2 dP = \frac{1}{18} \int \min_{a\in 3\alpha_1}(x-a)^2 dP + \int_{J_{(1,\infty)}} (x-a_3)^2 dP.$$

We now show that $S_1^{-1}(\alpha_1)$ is an optimal set of two-means. If $S_1^{-1}(\alpha_1) := 3\alpha_1$ is not an optimal set of two-means, then we can find a set $\beta \subset \mathbb{R}$ with $\text{card}(\beta) = 2$ such that $\int \min_{b\in\beta}(x-b)^2 dP < \int \min_{a\in\alpha_1}(x-3a)^2 dP$. But, then $(\frac{1}{3}\beta)\cup\alpha_2$ is a set of cardinality three with $\int \min_{a\in\frac{1}{3}\beta\cup\alpha_2}(x-a)^2 dP < \int \min_{a\in\alpha}(x-a)^2 dP$, which contradicts the optimality of α. Thus, $S_1^{-1}(\alpha_1)$ is an optimal set of two-means, i.e., $S_1^{-1}(\alpha_1) = \{a(1), a(1,\infty)\}$ which gives $\alpha_1 = \{a(11), a(11,\infty)\}$. Again, V_3 being the quantization error, we must have $a_3 = a(1,\infty)$. Thus, under the assumption $a_2 \in [0, \frac{1}{3}] = J_1$, we have $\alpha = \{a(11), a(11,\infty), a(1,\infty)\}$, and then using (5), we have $V_3 = \frac{5}{648}$. Let us now assume $\frac{2}{3} \leq a_2$. Set $\beta := \{a_2, a_3\}$. Then,

$$V_3 = \int_{J_1} (x-a(1))^2 dP + \int_{J_{(1,\infty)}} \min_{b\in\beta}(x-b)^2 dP = \frac{1}{144} + \int_{J_{(1,\infty)}} \min_{b\in\beta}(x-b)^2 dP.$$

We show that $a_2 < S_2(1) = \frac{7}{9}$ and $S_3(0) = \frac{8}{9} < a_3$. If $a_2 \geq \frac{7}{9}$, then

$$V_3 \geq \frac{1}{144} + \int_{J_2} (x - \frac{7}{9})^2 dP = \frac{7}{864} = 0.00810185 > V_3,$$

which leads to a contradiction. If $a_3 \leq \frac{8}{9} = S_3(0)$, then,

$$V_3 \geq \frac{1}{144} + \int_{J_3\cup J_4\cup J_5} (x - \frac{8}{9})^2 dP = \frac{38951}{5038848} = 0.00773014 > V_3,$$

which give a contradiction. Thus, $a_2 < S_2(1) = \frac{7}{9}$ and $S_3(0) = \frac{8}{9} < a_3$ yielding

$$\int_{J_{(1,\infty)}} \min_{b\in\beta}(x-b)^2 dP = \int_{J_2} (x-a_2)^2 dP + \int_{J_{(2,\infty)}} (x-a_3)^2 dP,$$

which is minimum when $a_2 = a(2)$ and $a_3 = a(2,\infty)$. Hence, under the assumption $a_2 \in [\frac{2}{3}, 1]$, we obtain $\alpha = \{a(1), a(2), a(2,\infty)\}$ and $V_3 = \frac{5}{648}$. Thus, the proof of the lemma is complete. □

Proposition 3.6. *Let $n \geq 2$ and α_n be an optimal set of n-means. Then, α_n does not contain any point from the open interval $(\frac{1}{3}, \frac{2}{3})$.*

Proof. By Lemma 3.1 and Lemma 3.5, the proposition is true for $n = 2$ and $n = 3$. Let us now prove that the proposition is true for all $n \geq 4$. Consider the set of four points $\beta := \{a(11), a(11, \infty), a(2), a(2, \infty)\}$. Then, by Lemma 2.6, we have the distortion error as

$$\int \min_{a\in\beta}(x-a)^2 dP = 2\Big(E(11)+E(2)\Big) = \frac{1}{648}.$$

Since V_n is the quantization error for n-means for $n \geq 4$, we have $V_n \leq V_4 \leq \frac{1}{648} = 0.00154321$. Let $\alpha_n := \{a_1 < a_2 < \cdots < a_n\}$ be an optimal set of n-means for $n \geq 4$. Since the optimal quantizers are the centroids of their own Voronoi regions, we have $0 \leq a_1 < a_2 \cdots < a_n \leq 1$. Let $j := \max\{i : a_i \leq \frac{1}{3}\}$. Then, $a_j \leq \frac{1}{3}$. Proposition 3.2 implies that $2 \leq j \leq n-1$. We need to show that $\frac{2}{3} \leq a_{j+1}$. Suppose that $a_{j+1} \in (\frac{1}{3}, \frac{2}{3})$. Then, either $a_{j+1} \in [\frac{1}{2}, \frac{2}{3})$, or $a_j \in (\frac{1}{3}, \frac{1}{2}]$. First, assume that $a_{j+1} \in [\frac{1}{2}, \frac{2}{3})$. Then, $\frac{1}{2}(a_j + a_{j+1}) < \frac{1}{3}$ implying $a_j < \frac{2}{3} - a_{j+1} \leq \frac{2}{3} - \frac{1}{2} = \frac{1}{6} < \frac{2}{9} = S_{12}(0)$, and so

$$V_n \geq \int_{J_{12}\cup J_{13}} (x-\frac{1}{6})^2 dP = \frac{521}{279936} = 0.00186114 > V_n,$$

which leads to a contradiction. Next, assume that $a_{j+1} \in (\frac{1}{3}, \frac{1}{2}]$. Then, $\frac{1}{2}(a_{j+1} + a_{j+2}) > \frac{2}{3}$ implying $a_{j+2} > \frac{4}{3} - a_{j+1} = \frac{4}{3} - \frac{1}{2} = \frac{5}{6} > S_2(1)$, and so,

$$V_n \geq \int_{J_2} (x-\frac{5}{6})^2 dP = \frac{1}{288} = 0.00347222 > V_n,$$

which gives another contradiction. Hence, $\frac{2}{3} \leq a_{j+1}$, which completes the proof of the proposition. □

Lemma 3.7. *Let α_n be an optimal set of n-means for $n \geq 4$. Then, card$(\alpha_n \cap J_1) \geq 2$ and card$(\alpha_n \cap [S_2(0), 1]) \geq 2$.*

Proof. As shown in the proof of Proposition 3.6, since V_n is the quantization error for n-means for $n \geq 4$, we have $V_n \leq V_4 \leq \frac{1}{648} = 0.00154321$. By Proposition 3.2, we have card$(\alpha_n \cap J_1) \geq 1$ and card$(\alpha_n \cap [S_2(0), 1]) \geq 1$. First, we show that card$(\alpha_n \cap [S_2(0), 1]) \geq 2$. Suppose that card$(\alpha_n \cap [S_2(0), 1]) = 1$. Then,

$$V_n \geq \int_{J_{(1,\infty)}} (x-a(1,\infty))^2 dP = \frac{1}{144} = 0.00694444 > V_n,$$

which leads to a contradiction. So, we can assume that card$(\alpha_n \cap [S_2(0), 1]) \geq 2$ for $n \geq 4$. Next, suppose that card$(\alpha_n \cap J_1) = 1$. Then,

$$V_n \geq \int_{J_1} (x-a(1))^2 dP = \frac{1}{144} = 0.00694444 > V_n,$$

which leads to another contradiction. Thus, the lemma is established. □

Proposition 3.8. *Let α_n be an optimal set of n-means for P such that card$(\alpha_n \cap [S_{k+1}(0), 1]) \geq 2$ for some $k \in \mathbb{N}$ and $n \in \mathbb{N}$. Then, $\alpha_n \cap J_{k+1} \neq \emptyset$, $\alpha_n \cap [S_{k+2}(0), 1] \neq \emptyset$, and α_n does not contain any point from the open interval $(S_{k+1}(1), S_{k+2}(0))$. Moreover, the Voronoi region of any point in $\alpha_n \cap J_{k+1}$ does not contain any point from $[S_{k+2}(0), 1]$ and the Voronoi region of any point in $\alpha_n \cap [S_{k+2}(0), 1]$ does not contain any point from J_{k+1}.*

Proof. To prove the proposition it is enough to prove it for $k = 1$, and then inductively the proposition will follow for all $k \geq 2$. Fix $k = 1$. Suppose that card$(\alpha_n \cap [S_2(0), 1]) \geq 2$. By Lemma 3.5, it is clear that the proposition is true for $n = 3$. We now prove that the proposition is true for $n = 4$. Let $\alpha_4 := \{a_1, a_2, a_3, a_4\}$ be an optimal set of four-means, such that $0 < a_1 < a_2 < a_3 < a_4 < 1$. By Lemma 3.7, we have card$(\alpha_4 \cap J_1) = 2$ and card$(\alpha_4 \cap [S_2(0), 1]) = 2$. Let $V(P, \alpha_4 \cap [S_2(0), 1])$ be the quantization error contributed by the set $\alpha_4 \cap [S_2(0), 1]$. Let $\beta := \{a(11), a(11,\infty), a(2), a(2,\infty)\}$. The distortion error due to the set $\beta \cap [S_2(0), 1] := \{a(2), a(2,\infty)\}$ is given by

$$\int_{[S_2(0),1]} \min_{a\in\beta\cap[S_2(0),1]} (x-a)^2 dP = 2\int_{J_2} (x-a(2))^2 dP = \frac{1}{1296},$$

and so $V(P, \alpha_4 \cap [S_2(0), 1]) \leq \frac{1}{1296} = 0.000771605$. Suppose that $\alpha_4 \cap J_2 = \emptyset$, i.e., $S_2(1) < a_3$. Then,

$$V(P, \alpha_4 \cap [S_2(0), 1]) \geq \int_{J_2} (x - S_2(1))^2 dP = \frac{1}{864} = 0.00115741 > V(P, \alpha_4 \cap [S_2(0), 1]),$$

which is a contradiction. So, $a_3 \leq S_2(1)$. We now show that $\alpha_4 \cap [S_3(0), 1] \neq \emptyset$. Suppose that $\alpha_4 \cap [S_3(0), 1] = \emptyset$. Then, $a_4 < S_3(0)$, and so

$$V(P, \alpha_4 \cap [S_2(0), 1]) \geq \int_{J_3\cup J_4\cup J_5} (x - S_3(0))^2 dP = \frac{3959}{5038848} = 0.000785695 > V(P, \alpha_4 \cap [S_2(0), 1]),$$

which leads to a contradiction. Therefore, $S_3(0) \leq a_4$. Since, $a_3 \leq S_2(1)$ and $S_3(0) \leq a_4$, we can assume that α_4 does not contain any point from the open interval $(S_2(1), S_3(0))$. Since $\frac{1}{2}(a_3 + a_4) \geq \frac{1}{2}(\frac{2}{3} + \frac{8}{9}) = \frac{7}{9} = S_2(1)$, the Voronoi region of any point in $\alpha_4 \cap [S_3(0), 1]$ does not contain any point from J_2. If the Voronoi region of any point in $\alpha_4 \cap J_2$ contains points from $[S_3(0), 1]$, we must have $\frac{1}{2}(a_3 + a_4) > \frac{8}{9}$ implying $a_4 > \frac{16}{9} - a_3 \geq \frac{16}{9} - \frac{7}{9} = 1$, which leads to a contradiction. Hence, the Voronoi region of any point in $\alpha_4 \cap J_2$ does not contain any point from $[S_3(0), 1]$. Thus, the proposition is true for $n = 4$. Similarly, we can prove that the proposition is true for $n = 5, 6, 7$. We now prove that the proposition is true for all $n \geq 8$. Let $\alpha_n := \{a_1, a_2, \cdots, a_n\}$ be an optimal set of n-means for any $n \geq 8$ such that $0 < a_1 < a_2 < \cdots < a_n < 1$. Let $V(P, \alpha_n \cap [S_2(0), 1])$ be the quantization error contributed by the set $\alpha_n \cap [S_2(0), 1]$. Set $\beta := \{a(111), a(111,\infty), a(12), a(12,\infty), a(21), a(21,\infty), a(3), a(3,\infty)\}$. The distortion error due to the set $\beta \cap [S_2(0), 1] := \{a(21), a(21,\infty), a(3), a(3,\infty)\}$ is given by

$$\int_{[S_2(0),1]} \min_{a\in\beta\cap[S_2(0),1]} (x-a)^2 dP = 2\Big(E(a(21)) + E(a(3))\Big) = \frac{1}{11664},$$

and so $V(P, \alpha_n \cap [S_2(0), 1]) \leq \frac{1}{11664} = 0.0000857339$. Suppose that α_n does not contain any point from J_2. Since by Proposition 3.2, the Voronoi region of any point from $\alpha_n \cap J_1$ does not contain any point from $[S_2(0), 1]$, we have

$$V(P, \alpha_n \cap [S_2(0), 1]) \geq \int_{J_2} (x - \frac{7}{9})^2 dP = \frac{1}{864} = 0.00115741 > V(P, \alpha_n \cap [S_2(0), 1]),$$

which leads to a contradiction. So, we can assume that $\alpha_n \cap J_2 \neq \emptyset$. Suppose that $\alpha_n \cap [S_3(0), 1] = \emptyset$. Then, $a_n < S_3(0)$, and so

$$V(P, \alpha_n \cap [S_2(0), 1]) \geq \int_{J_3\cup J_4} (x - S_3(0))^2 dP = \frac{131}{279936} = 0.000467964 > V(P, \alpha_n \cap [S_2(0), 1]),$$

which gives another contradiction. Therefore, $\alpha_n \cap [S_3(0), 1] \neq \emptyset$. We now show that $\alpha_n \cap (S_2(1), S_3(0)) = \emptyset$. Let $j := \max\{i : a_i \leq S_2(1) \text{ for all } 1 \leq i \leq n\}$, and so $a_j \leq \frac{7}{9} = S_2(1)$. Suppose that $\frac{7}{9} < a_{j+1} < \frac{8}{9}$. Then, the following two cases can arise:

Case 1. $\frac{7}{9} < a_{j+1} \leq \frac{5}{6}$.

Then, $\frac{1}{2}(a_{j+1} + a_{j+2}) > \frac{8}{9}$ implying $a_{j+2} > \frac{16}{9} - a_{j+1} \geq \frac{16}{9} - \frac{5}{6} = \frac{17}{18}$, and so

$$V(P, \alpha_n \cap [S_2(0), 1]) \geq \int_{J_3} (x - \frac{17}{18})^2 dP = \frac{1}{5184} = 0.000192901 > V(P, \alpha_n \cap J_{(1,\infty)}),$$

which is contradiction.

Case 2. $\frac{5}{6} \leq a_{j+1} < \frac{8}{9}$.

Then, $\frac{1}{2}(a_j+a_{j+1})<\frac{7}{9}$ implying $a_j<\frac{14}{9}-a_{j+1}\leq\frac{14}{9}-\frac{5}{6}=\frac{13}{18}<S_{22}(0)$, and so

$$V(P,\alpha_n\cap J_{(1,\infty)})\geq\int_{J_{22}\cup J_{23}}(x-\frac{13}{18})^2dP=\frac{521}{5038848}=0.000103397>V(P,\alpha_n\cap J_{(1,\infty)}),$$

which gives a contradiction. Therefore, $\alpha_n\cap(S_2(1),S_3(0))\neq\emptyset$. Proceeding similarly, as shown for $n=4$, in this case we can also show that the Voronoi region of any point in $\alpha_n\cap J_2$ does not contain any point from $[S_3(0),1]$ and the Voronoi region of any point in $\alpha_n\cap[S_3(0),1]$ does not contain any point from J_2. Thus, the proof of the proposition is complete. □

Proposition 3.9. *Let α_n be an optimal set of n-means for $n\geq 2$. Then, there exists a positive integer k such that $\alpha_n\cap J_j\neq\emptyset$ for all $1\leq j\leq k$, and card$(\alpha_n\cap[S_{k+1}(0),1])=1$. Moreover, if $n_j:=$ card(α_j), where $\alpha_j:=\alpha_n\cap J_j$, then $n=\sum_{j=1}^k n_j+1$, with*

$$V_n=\sum_{j=1}^k p_js_j^2V_{n_j}+p_ks_k^2V.$$

Proof. By Proposition 3.2 and Proposition 3.6, we see that if α_n is an optimal set of n-means for $n\geq 2$, then $\alpha_n\cap J_1\neq\emptyset$, $\alpha_n\cap[S_2(0),1]\neq\emptyset$, and α_n does not contain any point from the open interval $(S_1(1),S_2(0))$. Proposition 3.8 says that if card$(\alpha_n\cap[S_{k+1}(0),1])\geq 2$ for some $k\in\mathbb{N}$, then $\alpha_n\cap J_{k+1}\neq\emptyset$ and $\alpha_n\cap[S_{k+1}(0),1])\neq\emptyset$. Moreover, α_n does not take any point from the open interval $(S_{k+1}(1),S_{k+2}(0))$. Thus, by Induction Principle, we can say that if α_n is an optimal set of n-means for $n\geq 2$, then there exists a positive integer k such that $\alpha_n\cap J_j\neq\emptyset$ for all $1\leq j\leq k$ and card$(\alpha_n\cap[S_{k+1}(0),1])=1$.

For a given $n\geq 2$, write $\alpha_j:=\alpha_n\cap J_j$ and $n_j:=$ card(α_j). Since α_j are disjoints for $1\leq j\leq k$, and α_n does not contain any point from the open intervals $(S_\ell(1),S_{\ell+1}(0))$ for $1\leq\ell\leq k$, we have $\alpha_n=\bigcup_{j=1}^k\alpha_j\cup\{a(k,\infty)\}$ and $n=n_1+n_2+\cdots+n_k+1$. Then, using Lemma 2.1, we deduce

$$V_n=\int\min_{a\in\alpha_n}(x-a)^2dP=\sum_{j=1}^k\int_{J_j}\min_{a\in\alpha_j}(x-a)^2dP+\int_{J_{(k,\infty)}}(x-a(k,\infty))^2dP$$
$$=\sum_{j=1}^k p_j\int\min_{a\in\alpha_j}(x-a)^2dP\circ S_j^{-1}(x)+\int_{J_{(k,\infty)}}(x-a(k,\infty))^2dP,$$

which yields

$$V_n=\sum_{j=1}^k p_js_j^2\int\min_{a\in S_j^{-1}(\alpha_j)}(x-a)^2dP+p_ks_k^2V. \tag{6}$$

We now show that $S_j^{-1}(\alpha_j)$ is an optimal set of n_j-means, where $1\leq j\leq k$. If $S_j^{-1}(\alpha_j)$ is not an optimal set of n_j-means, then we can find a set $\beta\subset\mathbb{R}$ with card$(\beta)=n_j$ such that $\int\min_{b\in\beta}(x-b)^2dP<\int\min_{a\in S_j^{-1}(\alpha_j)}(x-a)^2dP$. But, then $S_j(\beta)\cup(\alpha_n\setminus\alpha_j)$ is a set of cardinality n such that

$$\int\min_{a\in S_j(\beta)\cup(\alpha_n\setminus\alpha_j)}(x-a)^2dP<\int\min_{a\in\alpha_n}(x-a)^2dP,$$

which contradicts the optimality of α_n. Thus, $S_j^{-1}(\alpha_j)$ is an optimal set of n_j-means for $1\leq j\leq k$. Hence, by (6), we have

$$V_n=\sum_{j=1}^k p_js_j^2V_{n_j}+p_ks_k^2V.$$

Thus, the proof of the proposition is complete. □

Proposition 3.10. *Let α_n be an optimal set of n-means for $n\geq 2$. Then, for $c\in\alpha_n$, we have $c=a(\omega)$, or $c=a(\omega,\infty)$ for some $\omega\in\mathbb{N}^*$.*

Proof. Let α_n be an optimal set of n-means for $n \geq 2$ such that $c \in \alpha_n$. By Proposition 3.9, there exists a positive integer k_1 such that $\alpha_n \cap J_{j_1} \neq \emptyset$ for $1 \leq j_1 \leq k_1$, and $\text{card}(\alpha_n \cap [S_{k_1+1}(0), 1]) = 1$, and α_n does not contain any point from the open intervals $(S_\ell(1), S_{\ell+1}(0))$ for $1 \leq \ell \leq k_1$. If $c \in \alpha_n \cap [S_{k_1+1}(0), 1]$, then $c = a(k_1, \infty)$. If $c \in \alpha_n \cap J_{j_1}$ for some $1 \leq j_1 \leq k_1$ with $\text{card}(\alpha_n \cap J_{j_1}) = 1$, then $c = a(j_1)$. Suppose that $c \in \alpha_n \cap J_{j_1}$ for some $1 \leq j_1 \leq k_1$ and $\text{card}(\alpha_n \cap J_{j_1}) \geq 2$. Then, as similarity mappings preserve the ratio of the distances of a point from any other two points, using Proposition 3.9 again, there exists a positive integer k_2 such that $\alpha_n \cap J_{j_1 j_2} \neq \emptyset$ for $1 \leq j_2 \leq k_2$, and $\text{card}(\alpha_n \cap [S_{j_1(k_2+1)}(0), 1]) = 1$, and α_n does not contain any point from the open intervals $(S_{j_1\ell}(1), S_{j_1(\ell+1)}(0))$ for $1 \leq \ell \leq k_2$. If $c \in \alpha_n \cap [S_{j_1(k_2+1)}(0), 1]$ then $c = a(j_1 k_2, \infty)$. Suppose that $c \in \alpha_n \cap J_{j_1 j_2}$ for some $1 \leq j_2 \leq k_2$. If $\text{card}(\alpha_n \cap J_{j_1 j_2}) = 1$, then $c = a(j_1 j_2)$. If $\text{card}(\alpha_n \cap J_{j_1 j_2}) \geq 2$, proceeding inductively as before, we can find a word $\omega \in \mathbb{N}^*$, such that either $c \in \alpha_n \cap J_\omega$ with $\text{card}(\alpha_n \cap J_\omega) = 1$ implying $c = a(\omega)$, or $c \in \alpha_n \cap [S_{\omega^-(\omega_{|\omega|}+1)}, 1]$ with $\text{card}(\alpha_n \cap [S_{\omega^-(\omega_{|\omega|}+1)}, 1]) = 1$ implying $c = a(\omega, \infty)$. Thus, the proof of the proposition is complete. □

Proposition 3.11. *For any $n \geq 2$, let α_n be an optimal set of n-means with respect to the probability distribution P. Write*

$$W(\alpha_n) := \{\omega \in \mathbb{N}^* : a(\omega) \text{ or } a(\omega, \infty) \in \alpha_n\}, \text{ and}$$
$$\tilde{W}(\alpha_n) := \{\tau \in W(\alpha_n) : p_\tau s_\tau^2 \geq p_\omega s_\omega^2 \text{ for all } \omega \in W(\alpha_n)\}.$$

Then, for any $\tau \in \tilde{W}(\alpha_n)$ the set $\alpha_{n+1} := \alpha_{n+1}(\tau)$, where

$$\alpha_{n+1}(\tau) = \begin{cases} (\alpha_n \setminus \{a(\tau)\}) \cup \{a(\tau 1), a(\tau 1, \infty)\} \text{ if } a(\tau) \in \alpha_n, \\ (\alpha_n \setminus \{a(\tau, \infty)\}) \cup \{a(\tau^-(\tau_{|\tau|}+1)), a(\tau^-(\tau_{|\tau|}+1), \infty)\} \text{ if } a(\tau, \infty) \in \alpha_n, \end{cases}$$

is an optimal set of $(n+1)$-means.

Proof. Let us first claim that for any $\omega, \tau \in \mathbb{N}^*$, $p_\tau s_\tau^2 \geq p_\omega s_\omega^2$ if and only if

$$E(a(\tau 1)) + E(a(\tau 1, \infty)) + E(a(\omega)) \leq E(a(\tau)) + E(a(\omega 1)) + E(a(\omega 1, \infty)).$$

By Lemma 2.6, we have

$$LHS = 2p_{\tau 1} s_{\tau 1}^2 V + p_\omega s_\omega^2 V = \frac{1}{9} p_\tau s_\tau^2 V + p_\omega s_\omega^2 V,$$
$$RHS = p_\tau s_\tau^2 V + 2p_{\omega 1} s_{\omega 1}^2 V = p_\tau s_\tau^2 V + \frac{1}{9} p_\omega s_\omega^2 V.$$

Thus, $LHS \leq RHS$ if and only if $p_\tau s_\tau^2 \geq p_\omega s_\omega^2$, which is the claim.

We now prove the proposition by induction. By Lemma 3.1, we know that the optimal set of two-means is $\alpha_2 = \{a(1), a(1, \infty)\}$. Here $\tilde{W}(\alpha_2) = W(\alpha_2) = \{1\}$. Since $a(1) \in \alpha_2$, we have $\alpha_3 = \{a(11), a(11, \infty), a(1, \infty)\}$. Again, as $a(1, \infty) \in \alpha_2$, we have $\alpha_3 = \{a(1), a(2), a(2, \infty)\}$. Clearly by Lemma 3.5, the sets α_3 are optimal sets of three-means. Thus, the proposition is true for $n = 2$. Let us now assume that α_m is an optimal set of m-means for some $m \geq 2$. Write

$$W(\alpha_m) := \{\omega \in \mathbb{N}^* : a(\omega) \text{ or } a(\omega, \infty) \in \alpha_m\}, \text{ and}$$
$$\tilde{W}(\alpha_m) := \{\tau \in W(\alpha_m) : p_\tau s_\tau^2 \geq p_\omega s_\omega^2 \text{ for all } \omega \in W(\alpha_m)\}.$$

If $\tau \notin \tilde{W}(\alpha_m)$, i.e., if $\tau \in W(\alpha_m) \setminus \tilde{W}(\alpha_m)$, then by the claim, if $a(\tau) \in \alpha_m$ the error

$$\int \min\{(x-a)^2 : a \in (\alpha_m \setminus \{a(\tau)\}) \cup \{a(\tau 1), a(\tau 1, \infty)\}\} dP,$$

or, if $a(\tau, \infty) \in \alpha_m$ the error

$$\int \min\{(x-a)^2 : a \in (\alpha_m \setminus \{a(\tau, \infty)\}) \cup \{a(\tau^-(\tau_{|\tau|}+1)), a(\tau^-(\tau_{|\tau|}+1), \infty)\}\} dP$$

is either equal or larger, in fact strictly larger if n is not of the form 2^k for any positive integer k, than the corresponding error obtained in the case where $\tau \in \tilde W(\alpha_m)$. Hence, for any $\tau \in \tilde W(\alpha_n)$ the set $\alpha_{m+1} := \alpha_{m+1}(\tau)$, where

$$\alpha_{m+1}(\tau) = \left\{ \begin{array}{l} (\alpha_m \setminus \{a(\tau)\}) \cup \{a(\tau 1), a(\tau 1, \infty)\} \text{ if } a(\tau) \in \alpha_m, \\ (\alpha_m \setminus \{a(\tau, \infty)\}) \cup \{a(\tau^-(\tau_{|\tau|}+1)), a(\tau^-(\tau_{|\tau|}+1), \infty)\} \text{ if } a(\tau, \infty) \in \alpha_m, \end{array} \right.$$

is an optimal set of $(m+1)$-means. Thus, by the principle of mathematical induction, the proposition is true for all positive integers $n \geq 2$. $\square$

Lemma 3.12. *Let $n \in \mathbb{N}$ be such that $n = 2^k$ for some $k \geq 1$. Then,*

$$\alpha(k) := \{a(\omega) : p_\omega = \frac{1}{2^k}\} \cup \{a(\omega, \infty) : p_\omega = \frac{1}{2^k}\}$$

is an optimal set of n-means. Set $\alpha_j(k) := \alpha(k) \cap J_j$ for $1 \leq j \leq k$. Then, $S_j^{-1}(\alpha_j(k))$ is an optimal set of 2^{k-j}-means for $1 \leq j \leq k$. Moreover, $n = \sum_{j=1}^k 2^{k-j} + 1$ and

$$V_n = \sum_{j=1}^k \frac{1}{18^j} V_{2^{k-j}} + \frac{1}{18^k} V_1.$$

Proof. Let us prove the lemma by induction. If $n = 2$, i.e., when $k = 1$, by Lemma 3.1, we have $\alpha(1) = \{a(1), a(1, \infty)\} = \{a(\omega) : p_\omega = \frac{1}{2}\} \cup \{a(\omega, \infty) : p_\omega = \frac{1}{2}\}$ which is an optimal set of two-means. Here $\alpha_1(1) = \alpha(1) \cap J_1 = \{a(1)\}$. Notice that $\text{card}(\alpha_1(1)) = 1$, and the set $S_1^{-1}(\alpha_1) = \{\frac{1}{2}\}$ is an optimal set of one-mean. Moreover, $V_2 = \frac{1}{18} V_1 + \frac{1}{18} V_1$. Thus, the lemma is true for $n = 2$. Let the lemma be true if $n = 2^k$ for some $k = m$, where $m \in \mathbb{N}$ and $m \geq 2$. We will show that it is also true for $k = m+1$. We have

$$\alpha(m) = \{a(\omega) : p_\omega = \frac{1}{2^m}\} \cup \{a(\omega, \infty) : p_\omega = \frac{1}{2^m}\}.$$

List the elements of $\alpha(m)$ as $a_1, a_2, \cdots, a_{2^m}$, i.e., $\alpha(m) = \{a_j : 1 \leq j \leq 2^m\}$. Construct the sets A_j for $1 \leq j \leq 2^m$ as follows:

$$A_j := \left\{ \begin{array}{l} \{a(\omega 1), a(\omega 1, \infty)\} \text{ if } a_j = a(\omega) \text{ for some } \omega \in \mathbb{N}^*, \\ \{a(\omega^-(\omega_{|\omega|}+1)), a(\omega^-(\omega_{|\omega|}+1), \infty)\} \text{ if } a_j = a(\omega, \infty) \text{ for some } \omega \in \mathbb{N}^*. \end{array} \right.$$

For $1 \leq j \leq 2^m$, set $\alpha_{2^m+j} = (\alpha(m) \setminus \bigcup_{k=1}^{j} \{a_k\}) \cup A_1 \cup A_2 \cup \cdots \cup A_j$. Since α_{2^m} is an optimal set of 2^m-means, by Proposition 3.11, α_{2^m+1} is an optimal set of (2^m+1)-means, which implies α_{2^m+2} is an optimal set of (2^m+2)-means, and thus proceeding inductively, we can say that the set

$$\alpha_{2^{m+1}} := \alpha_{2^m+2^m} = (\alpha(m) \setminus \cup_{k=1}^{2^m} \{a_k\}) \cup A_1 \cup A_2 \cup \cdots \cup A_{2^m} = A_1 \cup A_2 \cup \cdots \cup A_{2^m}$$

is an optimal set of 2^{m+1}-means. Notice that for any $\omega \in \mathbb{N}^*$ if $a(\omega)$ or $a(\omega, \infty) \in A_j$, then $p_\omega = \frac{1}{2^{m+1}}$, and so

$$\alpha_{2^{m+1}} = \alpha(m+1) = \{a(\omega) : p_\omega = \frac{1}{2^{m+1}}\} \cup \{a(\omega, \infty) : p_\omega = \frac{1}{2^{m+1}}\}.$$

Therefore, by using the principle of mathematical induction, we can say that the set $\alpha(k)$ is an optimal set of n-means if $n \in \mathbb{N}$ and $n = 2^k$ for some $k \geq 1$. To complete the rest of the proof, we proceed as follows: For any $\omega = \omega_1\omega_2 \cdots \omega_{|\omega|} \in \mathbb{N}^*$, we have $a(\omega) := S_\omega(\frac{1}{2}) \in J_{\omega_1}$. Again, from the definitions of $a(\omega)$, $a(\omega, \infty)$, if $a(\omega) \in J_{\omega_1}$ and $|\omega| > 1$, then $a(\omega, \infty) \in J_{\omega_1}$. Keeping ω_1 fixed, if $\omega_1 < k$, we see that there are $2^{k-\omega_1-1}$ different $\tau \in \mathbb{N}^*$ such that $p_{\omega_1\tau} = \frac{1}{2^k}$. Thus, for any $\omega = \omega_1\omega_2 \cdots \omega_{|\omega|} \in \mathbb{N}^*$ with $|\omega| > 1$ and $p_\omega = \frac{1}{2^k}$, the optimal set $\alpha(k)$ contains $2^{k-\omega_1}$ elements from J_{ω_1}; in other words, $\text{card}(\alpha(k) \cap J_{\omega_1}) = 2^{k-\omega_1}$. If $|\omega| = 1$ and $p_\omega = \frac{1}{2^k}$, i.e., when $\omega = k$, then $a(k) \in J_k$, i.e., $\alpha(k)$ contains only one element from J_k. Besides, $\alpha(k)$ contains the element $a(k, \infty)$. Write $\alpha_j(k) := \alpha(k) \cap J_j$. Then, $\text{card}(\alpha_j(k)) = 2^{k-j}$ for $1 \leq j \leq k$. For any $1 \leq j \leq k-1$, by the definition of the mappings, we have

$$S_j^{-1}(\alpha_j(k)) = \{a(\omega_{j+1} \cdots \omega_{|\omega|}) : p_{\omega_{j+1}\cdots\omega_{|\omega|}} = \frac{1}{2^{k-j}}\} \cup \{a(\omega_{j+1} \cdots \omega_{|\omega|}, \infty) : p_{\omega_{j+1}\cdots\omega_{|\omega|}} = \frac{1}{2^{k-j}}\},$$

and $S_k^{-1}(\alpha_k(k)) = \{\frac{1}{2}\}$. Thus, for all $1 \leq j \leq k$, we can see that $S_j^{-1}(\alpha_j(k)) = \alpha(k-j)$. Hence, by the first part of the lemma, for each $1 \leq j \leq k$, the set $S_j^{-1}(\alpha_j(k))$ is an optimal set of 2^{k-j}-means. Now,

$$\begin{aligned} V_n &= \int \min_{a \in \alpha(k)} \|x-a\|^2 dP = \sum_{j=1}^k \int_{J_j} \min_{a \in \alpha_j(k)} (x-a)^2 dP + \int_{J_{(k,\infty)}} (x - a(k,\infty))^2 dP \\ &= \sum_{j=1}^k p_j \int \min_{a \in \alpha_j(k)} (x-a)^2 dP \circ S_j^{-1}(x) + \int_{J_k} (x-a(k))^2 dP, \end{aligned}$$

which yields

$$V_n = \sum_{j=1}^k \frac{1}{18^j} \int \min_{a \in S_j^{-1}(\alpha_j(k))} (x-a)^2 dP + \frac{1}{18^k} V_1 = \sum_{j=1}^k \frac{1}{18^j} V_{2^{k-j}} + \frac{1}{18^k} V_1.$$

Thus, the proof of the lemma is complete. □

Remark 3.13. The set $\alpha(k))$ given by Lemma 3.12 is a unique optimal set of n-means where $n = 2^k$ for some $k \in \mathbb{N}$.

In regard to Lemma 3.12 let us give the following example.

Example 3.14. Take $n = 16 = 2^4$. Then,

$$\begin{aligned} \alpha(4) = \{&a(1111), a(1111,\infty), a(112), a(112,\infty), a(121), a(121,\infty), a(13), \\ &a(13,\infty), a(211), a(211,\infty), a(22), a(22,\infty), a(31), a(31,\infty), a(4), a(5,\infty)\}. \end{aligned}$$

Since, $\alpha_j(4) = \alpha(4) \cap J_j$ for $1 \leq j \leq 4$, we have

$$\begin{aligned} &\alpha_1(4) = \{a(1111), a(1111,\infty), a(112), a(112,\infty), a(121), a(121,\infty), a(13), a(13,\infty)\}, \\ &\alpha_2(4) = \{a(211), a(211,\infty), a(22), a(22,\infty)\}, \\ &\alpha_3(4) = \{(31), a(31,\infty)\}, \\ &\alpha_4(4) = \{a(4)\}. \end{aligned}$$

Here, $S_1^{-1}(\alpha_1(4)) = \{a(111), a(111,\infty), a(12), a(12,\infty), a(21), a(21,\infty), a(3), a(3,\infty)\}$ is an optimal set of 2^3-means, $S_2^{-1}(\alpha_2(4)) = \{a(11), a(11,\infty), a(2), a(2,\infty)\}$ is an optimal set of 2^2-means, $S_3^{-1}(\alpha_3(4)) = \{a(1), a(1,\infty)\}$ is an optimal set of 2-means, and $S_4^{-1}(\alpha_4(4)) = \{\frac{1}{2}\}$ is an optimal set of one-mean. Moreover, we can see that

$$V_{16} = \frac{1}{18} V_8 + \frac{1}{18^2} V_4 + \frac{1}{18^3} V_2 + \frac{1}{18^4} V_1 + \frac{1}{18^4} V_1.$$

Let us now state and prove the main theorem of the paper.

Theorem 3.15. *For $n \in \mathbb{N}$ with $n \geq 2$ let $\ell(n) \in \mathbb{N}$ satisfy $2^{\ell(n)} \leq n < 2^{\ell(n)+1}$. Let $\alpha(\ell(n))$ and $\alpha_n(I)$ be the sets as defined by Definition 2.9. Then, $\alpha_n(I)$ is an optimal set of n-means with quantization error*

$$V_n = \frac{1}{18^{\ell(n)}} \frac{1}{8} \Big(2^{\ell(n)+1} - n + \frac{1}{9}(n - 2^{\ell(n)})\Big).$$

The number of such sets is ${}^{2^{\ell(n)}}C_{n-2^{\ell(n)}}$, where ${}^uC_v = \binom{u}{v}$ is a binomial coefficient.

Proof. By Lemma 3.12, $\alpha(\ell(n))$ is an optimal set of $2^{\ell(n)}$-means. Choose $I \subset \alpha(\ell(n))$ such that $\text{card}(I) = n - 2^{\ell(n)}$. List the elements of I as $a_1, a_2, \cdots, a_{n-2^{\ell(n)}}$, i.e., $I = \{a_j : 1 \leq j \leq n - 2^{\ell(n)}\}$. Construct the sets A_j for $1 \leq j \leq n - 2^{\ell(n)}$ as follows:

$$A_j := \left\{ \begin{array}{l} \{a(\omega 1), a(\omega 1, \infty)\} \text{ if } a_j = a(\omega) \text{ for some } \omega \in \mathbb{N}^*, \\ \{a(\omega^-(\omega_{|\omega|} + 1)), a(\omega^-(\omega_{|\omega|} + 1), \infty)\} \text{ if } a_j = a(\omega, \infty) \text{ for some } \omega \in \mathbb{N}^*. \end{array} \right.$$

For $1 \leq j \leq n - 2^{\ell(n)}$, set

$$\alpha_{2^{\ell(n)}+j} = (\alpha(\ell(n)) \setminus \bigcup_{k=1}^{j} \{a_k\}) \cup A_1 \cup A_2 \cup \cdots \cup A_j.$$

As shown in Lemma 3.12, proceeding inductively, we see that the set $\alpha_n(I) := \alpha_{2^{\ell(n)}+(n-2^{\ell(n)})} = (\alpha(\ell(n)) \setminus I) \cup A_1 \cup A_2 \cup \cdots \cup A_{n-2^{\ell(n)}}$ forms an optimal set of n-means. Then, using Proposition 2.14, we obtain the quantization error as

$$V_n = \int \min_{a \in \alpha_n} (x-a)^2 dP = \frac{1}{18^{\ell(n)}} \frac{1}{8} \Big(2^{\ell(n)+1} - n + \frac{1}{9}(n - 2^{\ell(n)}) \Big).$$

Since the subset I from the set $\alpha(\ell(n))$ can be chosen in ${}^{2^{\ell(n)}}C_{n-2^{\ell(n)}}$ different ways, the number of $\alpha_n(I)$ is ${}^{2^{\ell(n)}}C_{n-2^{\ell(n)}}$. Thus, the proof of the theorem is complete. ☐

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