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Optimal Quantum Trajectories for Continuous Measurement

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We define an ideal optimal quantum measurement as that measurement on the apparatus for which the average algorithmic information in the measurement record is minimized. We apply the definition to a chaotic system subject to continuous (Markov) quantum nondemolition measurements. For optimized measurements the average information in the record is much closer to the von Neumann entropy than in the nonoptimized case, but increases more quickly in the chaotic region than in the regular region.

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A quantum trajectory describes the evolution of the state of a quantum system, conditioned on a time sequence of measurements [1–3]. It connects a classical stochastic process, the measurement record, to a quantum stochastic process, the evolution of the conditional quantum state. A continuous measurement requires the system to be constantly coupled to an apparatus, or environment. There may be many ways of extracting information about the quantum source from this environment, and each will give rise to a different sort of quantum trajectory. For a given coupling to the environment, is there an optimal way of extracting information from the environment, enabling the most efficient characterization of the conditional state of the quantum source? In this Letter, we define such an optimal measurement to be that for which the average algorithmic information of the record is a minimum.

Consider first the case of a single generalized measurement, described in terms of effects and operations [4]. If a measurement is made upon a system in state ρ, the result a is a classical random variable distributed according to the probabilities $P(a) = \mathrm{tr}[\hat{F}(a)\rho]$, where $\hat{F}(a)$ is a positive operator (an effect [4]). The random variable a labels a particular (classical) apparatus state. The state of the system, conditioned on the result of the measurement, is given by $\rho^{(a)} = [P(a)]^{-1}\phi_a\rho$, where ϕ_a is an operation [4] which must satisfy $\mathrm{tr}(\phi_a\rho) = \mathrm{tr}[\hat{F}(a)\rho]$. Clearly, there are many possible operations which would satisfy this relation. We will only consider efficient measurements. An efficient measurement is that for which the statistics of the observed result depends only on the quantum state of the measured system. The state of the apparatus is completely known (i.e., it is a pure state). For an inefficient measurement the statistics of the result also depend on the statistics of unknown apparatus parameters. For efficient measurements all operations can be written in terms of operators by $\phi_a\rho = \hat{\Omega}_a\rho\hat{\Omega}_a^\dagger$. The unconditioned state of the system, that is the system state given at which the measurement has taken place but for which the outcome has not been determined, is given by

$$\rho' = \sum_a \phi_a\rho\,. \tag{1}$$

In general, this transforms pure states to mixed states. Throughout this paper we assume that the premeasurement state ρ of the system is known.

It is easy to verify that a unitary rearrangement of the measurement operators Ω_a will change the operations ϕ_a but will not change the unconditioned state (1) [5]. Physically, this arbitrariness corresponds to choosing to measure a different property of the apparatus, after it has interacted with the system. We wish to find the optimal measurement (that requiring the least information to keep track of the system) with the map between states, Eq. (1), held fixed. In reality this map is fixed by the interaction between the system and the measuring apparatus, and thus we are assuming this interaction is fixed. However, there are many different ways in which the apparatus may be read. We wish to define an optimal measurement by exploiting this flexibility. The objective is to extract sufficient information from the apparatus to enable us to track the conditional evolution of the known initial state in such a way that the average algorithmic information in the record is kept as small as possible.

The conditional algorithmic information of a particular result of the measurement is the length in bits of the shortest program returning a given the list $\{a, P(a)\}$. To a very good approximation the average conditional algorithmic information needed to describe the measurement results is given by [6]

$$I \approx -\sum_a P(a)\log_2 P(a)\,, \tag{2}$$

where $P(a) = \mathrm{tr}(\phi_a\rho)$. The quantity on the right hand side of Eq. (2) is the "ensemble entropy" $H(\mathcal{A})$ [5], for the particular ensemble $\mathcal{A}$ of operations ϕ_a, and represents a strict lower bound to the average conditional algorithmic information. The optimal measurement is that for which the average algorithmic information, needed to specify an outcome, is a minimum. How small can this be? The answer is provided by an inequality relating the

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ensemble entropy to the von Neumann entropy of the final state ρ' [5],

$$H(\mathcal{A}) \geq S(\rho'), \tag{3}$$

where $S(\rho) = -\mathrm{tr}(\rho \log_2 \rho)$. The von Neumann entropy is equal to the ensemble entropy if the conditional postmeasurement states $\rho^{(a)}$ are the eigenstates of the unconditional postmeasurement state ρ'. Thus, the optimal measurement ensemble $\mathcal{A}$ consists of operations yielding orthogonal conditional states, for which the algorithmic information of the record is a minimum and is given by the von Neumann entropy of ρ'. It is always possible (for efficient measurements) to choose a basis in which to observe the apparatus which will achieve this [5]. In reality it may be difficult, if not impossible, to do this, not least because it is necessary to know precisely the premeasurement state of the system ρ.

We now wish to consider the case of continuous (in time) measurements. This has been discussed by a number of authors [7–10]. We will follow the presentation of Wiseman [9] which is best suited to the quantum optical application discussed below. To proceed, we must specify the nature of the measurement interaction. We do this by specifying that the reduced system dynamics is given by a particular Markov master equation [11]. For simplicity consider the case of a single output channel so that

$$\dot{\rho} = -i[H, \rho] + (c\rho c^\dagger - \tfrac{1}{2} c^\dagger c\rho - \tfrac{1}{2} \rho c^\dagger c). \tag{4}$$

In an infinitesimal time interval dt there is a natural decomposition of the state $\rho(t + dt)$ into conditional states. For this purpose we define the sample space of measurement results, on this time interval, as $\{0, 1\}$. The probability of a result $\nu \in \{0, 1\}$ is given by

$$P(\nu) = \mathrm{tr}[\Omega_\nu(dt)\rho(t)\Omega_\nu^\dagger(dt)]. \tag{5}$$

The unconditioned density operator at time $t + dt$ is

$$\rho(t + dt) = \Omega_1(dt)\rho(t)\Omega_1^\dagger(dt) + \Omega_0(dt)\rho(t)\Omega_0^\dagger(dt). \tag{6}$$

In this context, the restriction to efficient measurements means that the environment to which the system is coupled must be in a pure state both before and after the measurement. See Ref. [9] for the generalization to inefficient measurements. The master equation, Eq. (4), is invariant under the transformation

$$c \longrightarrow c + \alpha, \quad H \longrightarrow H - i\,\tfrac{1}{2}(\alpha^* c - \alpha c^\dagger). \tag{7}$$

The measurement operators for a given α are now taken to be

$$\Omega_1(dt) = \sqrt{dt}\,(c + \alpha), \tag{8}$$

$$\Omega_0(dt) = 1 - (iH + \alpha^* c + \tfrac{1}{2} c^\dagger c + \tfrac{1}{2}|\alpha|^2)\,dt. \tag{9}$$

It is easily verified that with these definitions Eq. (6) is the solution to Eq. (4). The choice of α (which corresponds to a unitary rearrangement of the measurement operators) can be physically realized in the quantum optical context by the use of a coherent local oscillator [2].

The *standard unraveling*, assuming that there exists a natural separation of the reversible and irreversible parts of the evolution, is defined to be that for which $\alpha = 0$. In this case, the probability for the elementary positive result is

$$P(1) = \langle c^\dagger c\rangle dt, \tag{10}$$

where $\langle A\rangle = \mathrm{tr}[\rho(t)A]$. If the system starts in a pure state, the conditional state of the system at any time is also pure (as we are assuming perfect measurements). The unnormalized conditional state of the system given a particular result $\nu \in \{0, 1\}$ is $|\tilde{\psi}^{(\nu)}(t + dt)\rangle = \Omega_\nu(dt)|\psi_c(t)\rangle$ where the subscript c refers to the conditional state up to time t. In general, the measurement thus defined is not optimal as the conditional states $|\tilde{\psi}^{(0)}(t + dt)\rangle$, $|\tilde{\psi}^{(1)}(t + dt)\rangle$ need not be orthogonal.

In order to make the measurement optimal, it is necessary simply to choose α so that the inner product of these two conditional states equals zero. Assuming that we are following the conditional evolution so that at some time t the conditional state is $|\psi_c(t)\rangle$, then one easily sees that we must choose α at every infinitesimal time step to be equal to the negative of the conditional mean of c at that time, $\alpha = \alpha_c(t) = -\langle c\rangle_c(t)$. Then the probability for the elementary event is

$$\tilde{P}(1) = (\langle c^\dagger c\rangle_c - |\langle c\rangle_c|^2)dt. \tag{11}$$

This choice for the conditional evolution of a Markov open quantum system was first proposed by Diosi [12] in an unrelated context. Of course, we will need to know the initial state if we are to know the conditional means at every time. Comparing Eq. (11) with Eq. (10), we see that for optimal measurements the elementary event 1 is less likely, which explains why it keeps track of the system state more efficiently. If the master equation happens to have a pure state as its time dependent solution, then there will never be any detections and the measurement record will be empty. It is important to note that the choice of α depends on the past history of measurement results. Thus the ensemble of all possible realizations of this measurement scheme will not yield an ensemble of pure states at some final time t' which is the eigenensemble of the final unconditioned state $\rho(t')$. The optimality of the measurement is only true for each infinitesimal time increment, for each particular history of results. Nevertheless, one would expect that the average entropy of the measurement record would be closer to the von Neumann entropy of $\rho(t')$ for the case of the optimal unraveling than for the case of the standard unraveling.

We now illustrate the above discussion by considering the quantum dynamics of a parametrically kicked nonlinear optical oscillator subject to quantum nondemolition measurement of photon number. This model was first proposed by one of us [13] and in the classical limit exhibits a rich dynamical structure including regions of regular and chaotic behavior. Here we first describe the model with quantum counting nondemolition measurements. Recently, Barnum, Schack, and Caves [14] have

considered similar measurements on this system. We show first that the average algorithmic information needed to specify a measurement record, for a system initially localized in a chaotic region of phase space, is much greater than that needed for a system initially localized in the regular region, and very much greater than the von Neumann entropy of the unconditional quantum state. For optimized measurements the algorithmic information in the record is much smaller but for the chaotic case still rises more rapidly than the corresponding regular case.

Consider a single mode field of an optical cavity containing a medium with an intensity dependent refractive index (a Kerr medium). Such a system is modeled as a nonlinear oscillator in which the nonlinear frequency of oscillation is a linear function of the energy, with a Hamiltonian $\frac{\mu}{2}(a^\dagger)^2a^2$, where a is the annihilation operator for the cavity field. We suppose that periodically (at times separated by T) the oscillator is subject to a parametric amplification process which acts impulsively. We model the effect of the parametric kick by the fixed unitary transformations $\exp(\frac{r}{2}[(a^\dagger)^2 - a^2])$ [13]. In addition to the nonlinear oscillation between kicks, we now assume the system is subject to quantum nondemolition measurement of photon number, as in Ref. [15]. Unlike ordinary absorptive photon counting, quantum nondemolition counting does not alter the system's energy [16]. The effect of this continual nondemolition count process between kicks is modeled by a master equation for the unconditional state

$$\dot{\rho} = -i\frac{\mu}{2}[(a^\dagger)^2a^2,\rho] - \frac{\gamma}{2}[a^\dagger a,[a^\dagger a,\rho]], \tag{12}$$

where γ is a parameter that determines the signal-to-noise ratio for the measurement over a given bandwidth.

This master equation has a standard unraveling in terms of the operators

$$\Omega_1(dt) = \sqrt{\gamma dt}\,a^\dagger a\,, \tag{13}$$

$$\Omega_0(dt) = 1 - \left(i\frac{\mu}{2}(a^\dagger)^2a^2 + \frac{\gamma}{2}(a^\dagger a)^2\right)dt\,. \tag{14}$$

Because photon number is conserved, the measurement results are Poisson processes at the conditional rate $\gamma\langle(a^\dagger a)^2\rangle_c$. A course-grained history would be given in terms of the total count recorded in each of a series of equal time intervals on the interval $[0,T)$, between kicks. Indeed the conditional state before each kick only depends on the total count over this total interval. The probability to record m kicks on this interval is given by

$$P(m) = \langle\tilde{\psi}^{(m)}(t+T)|\tilde{\psi}^{(m)}(t+T)\rangle \tag{15}$$

$$= \sum_{n=0}^{\infty}\frac{(n^2\gamma T)^m}{m!}e^{-\gamma n^2T}|\langle n|\psi(t)\rangle|^2. \tag{16}$$

Equation (16) provides a particularly easy way to simulate a quantum trajectory corresponding to this course-grained history.

We first truncate the infinite oscillator Hilbert space to dimension R, chosen sufficiently large to ensure that states are normalized to within a specified tolerance for the number of kicks of interest. This depends to a large extent on the initial state [15]. Next, the initial state is chosen to be a coherent state localized in either the regular or the chaotic region of the classical phase portrait. A random positive integer m_1 is then chosen by sampling the distribution in Eq. (16). This number is the count recorded on the first interval, from which the conditional state is determined. If we are interested in computing ensemble average values, for example, of the average photon number, the conditional state can then be normalized and the first kick applied. We repeat this procedure for as many kicks as desired. The entire process is then repeated for another trial starting from the same initial state.

The case of optimized measurements, however, must be done quite differently. It is no longer the case that only the total count over the time between kicks determines the conditional state, due to the fact that the rate operator is continually adjusted as counts are recorded and the conditional mean changes. In this case we divide the interval between kicks into 100 equally spaced intervals. The duration $\Delta t = T/100$ of each interval is sufficiently small that it is most unlikely that more than one count will be recorded. A random number r, uniform on the unit interval, is then selected and compared with the probability $\bar{P}(1)\Delta t$, where $\bar{P}(1)$ is as in Eq. (11) with $c = \sqrt{\gamma}a^\dagger a$. If r is less than this probability, a one is recorded, otherwise the result is zero. Depending on the result of this record, the operator $\Omega_r(dt)$ is applied to the state at the start of the interval. The normalized state can then be computed before the next step. As a check on the method we can compute the average photon number for comparison with the results in [15]. In all cases the agreement is good.

To compute the average algorithmic information in the measurement record we proceed as follows. A quantum trajectory is generated by the methods described above and the resulting measurement records stored. While the probability of a measurement record could, in principle, be computed from the unnormalized conditional state up to the final kick, it is easier to simply count distinct measurement records and compute probabilities by dividing by the number of quantum trajectories simulated. This makes it easy to compute the entropy of the records and assign the average algorithmic information of the measurement records. The result depends on whether the initial state is localized in the regular or chaotic region. In Fig. 1 we plot the average algorithmic information and the von Neumann entropy for the regular case. We used three ensembles each with 2000 quantum trajectories to compute these quantities and calculate the error bars. Note that the algorithmic information of the optimized record is much closer to the von Neumann entropy than that of the standard record.

In Fig. 2 we plot the same quantities for a state initially localized in the chaotic region of phase space. Note first

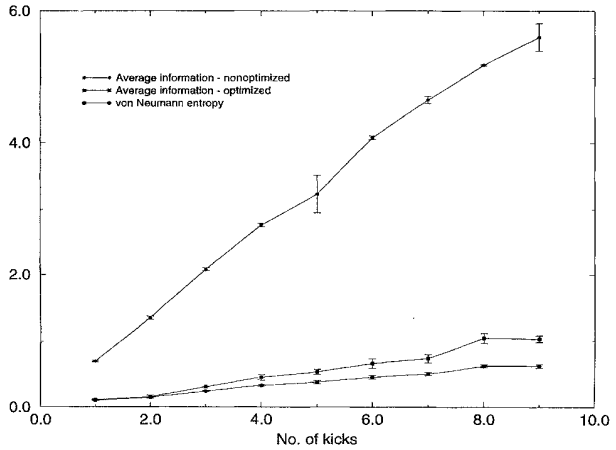

FIG. 1. The von Neumann entropy and optimized and nonoptimized average algorithmic information in the measurement record versus kick number for a quantum nondemolition measurement of photon number. The system is initially localized in the regular region. In all cases, $g = 1.5$, $\mu = 0.01\pi$, $\gamma = 0.001$, and the total number of trajectories is 6000.

that the rate of increase of the von Neumann entropy is higher than in the regular case. The rate of increase of the information in the optimal measurement record is higher also, and the difference is greater than for the von Neumann entropy. Finally, the rate of increase of the standard record's algorithmic information is also higher in the chaotic case, and the difference is by far the greatest out of the three quantities. The increase in information needed to track the state in the chaotic case does not, however, indicate the hypersensitivity to perturbation for chaotic systems discussed by Schack and Caves [17–19]. Such hypersensitivity can only occur for an environment in a mixed state. As we have treated only efficient measurements, the environment is effectively in a pure state, which is always known.

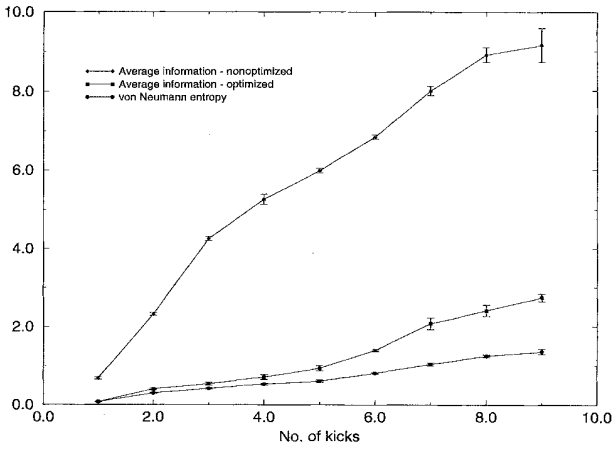

FIG. 2. As in Fig. 1, except with the system initially localized in the chaotic region.

The reason that the optimal measurement does not realize exactly the von Neumann entropy is that our optimal measurement is only optimal for each infinitesimal time interval for a particular trajectory, while the von Neumann entropy is defined by diagonalizing the density operator at every time. It would seem that it is not, in general, possible to have quantum trajectories corresponding to an average algorithmic information equal to the von Neumann entropy without modifying the master equation, that is by changing the coupling between the system and the apparatus at every time step. It is, however, possible that, for a given initial state and finite time interval, there may exist a different unraveling of the original master equation which will have a lower information content in its record, one closer to the von Neumann entropy. Indeed it is possible to think of simple systems in which there is an obvious measurement scheme which is more optimal, in this sense, than the optimal measurement which we have defined. However, we do not see how such a measurement scheme could be determined in general. Finally, we emphasize that we have only discussed the case of efficient measurements, for which the apparatus is in a pure state. The removal of this restriction requires a considerable increase in computational power due to the need to work with density matrices. We are currently attempting this calculation.

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