Optimal Regression for Reasoning about Knowledge and Actions

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Abstract

We show how in the propositional case both Reiter's and Scherl & Levesque's solutions to the frame problem can be modelled in dynamic epistemic logic (DEL), and provide an optimal regression algorithm for the latter. Our method is as follows: we extend Reiter's framework by integrating observation actions and modal operators of knowledge, and encode the resulting formalism in DEL with announcement and assignment operators. By extending Lutz' recent satisfiability-preserving reduction to our logic, we establish optimal decision procedures for both Reiter's and Scherl & Levesque's approaches: satisfiability is NP-complete for one agent, PSPACE-complete for multiple agents and EXPTIME-complete when common knowledge is involved.

Introduction

Thielscher (1999) distinguishes two versions of the frame problem. The *representational version* is the problem of designing a logical language and a semantics such that domains can be described without expliciting the interaction between *every* action and fluent: basically, when there are n actions and m fluents, the domain description should be much smaller than $2 \times n \times m$. The *inferential version* of the frame problem is more demanding: given a solution of the representational version, the problem is to design an 'efficient' decision procedure, where 'efficient' roughly means that its computational complexity should not be too high.

Reiter (1991) solved the representational frame problem by means of successor state axioms (SSAs). In the propositional case fluents only have situation arguments, and SSAs take the form

$$\begin{aligned}&\forall x \forall s(p(do(x,s)) \leftrightarrow (\neg Poss(x,s) \vee \\ &(x = a_1 \wedge \gamma^+(a_1,p,s)) \vee \cdots \vee (x = a_n \wedge \gamma^+(a_n,p,s)) \vee \\ &(p(s) \wedge \neg(x = a'_1 \wedge \gamma^-(a'_1,p,s)) \wedge \cdots \wedge \\ &\neg(x = a'_m \wedge \gamma^-(a'_m,p,s)))))\end{aligned}$$

where $a_1, \dots, a_n$ are the actions potentially making p true, and $a'_1, \dots, a'_m$ are the actions potentially making p false. For a given action a_i, let us note $\mathit{Eff}^+(a_i)$ the set of those fluents which a_i may make true, and $\mathit{Eff}^-(a_i)$ the set of those fluents which a_i may make false (in (Reiter 1991) these sets are left implicit). Then for every fluent $p \in \mathit{Eff}^+(a_i)$, the formula $\gamma^+(a_i,p,s)$ characterizes the conditions under which a_i makes p true, and $\gamma^-(a_i,p,s)$ characterizes the conditions under which a_i makes p false. $\gamma^+(a_i,p,s)$ and $\gamma^-(a_i,p,s)$ must be *uniform in* s, which in particular means that they do not contain the do-function.[1]

Reiter's central idea is that due to inertia the sets $\mathit{Eff}^+(a_i)$ and $\mathit{Eff}^-(a_i)$ are 'small' subsets of the set of all fluents. For that reason the size of the set of all SSAs can be expected to be of the order of the number of actions, and thus much smaller than the product of the number of actions with the number of fluents. Hence SSAs count as a solution to the representational frame problem. Reiter's solution was extended in (Scherl & Levesque 2003) to sensing actions.

When SSAs are available for every fluent p, one can reduce ('regress') any formula φ to an equivalent formula $\mathrm{reg}(\varphi)$ not mentioning actions. This leads to a straightforward decision procedure in the propositional case, that has been implemented in the GOLOG language (Levesque *et al.* 1997). However, the reduced formula can be exponentially larger than the original formula, and therefore the inferential frame problem has to be considered unsolved in Reiter's and Scherl & Levesque's approaches.

In this paper we solve the inferential frame problem for the propositional case. For the extension to knowledge, among the epistemic actions we only consider *observations*: all agents observe *that* some proposition holds in the world, and update their epistemic state accordingly.[2] We give a satisfiability-preserving polynomial transformation eliminating action operators from formulas. This provides an optimal regression procedure for reasoning about actions: both in Reiter's case (without knowledge operators) and in the single-agent case the decision procedure works in nondeterministic polynomial time; in the multiagent case it works in PSPACE, and in the case of common knowledge in EXPTIME. All these results are optimal because they match the complexity of the underlying epistemic logic.

[1] In later work Reiter et col. generalized SSAs to equivalences $\forall x \forall s(p(do(x,s)) \leftrightarrow \psi(a,s)$. We do not consider this here.

[2] Note that observations are different from the sensing actions present in (Scherl & Levesque 2003). By performing the latter, the agents observe *whether* some proposition holds in the world or not.

Technically, our approach builds on recent progress in the field of *dynamic epistemic logics*. In this family of logics situation terms are left implicit, and there is no quantification over actions. Thus the central device in Reiter's solution is not available. We show that nevertheless one can do without it, and recast this framework in the dynamic epistemic logic DEL_N^C of (van Ditmarsch, van der Hoek, & Kooi 2005; Kooi 2007).[3] DEL_N^C being an extension of Plaza's public announcement logic, we extend Lutz' optimal decision procedure for the latter (Lutz 2006) to DEL_N^C, and show that we keep optimality: checking satisfiability of DEL_N^C-formulas is shown to have the same complexity as checking satisfiability in the underlying epistemic logic.

Background: Epistemic Logic EL_N^C

Let P be a countably infinite set of propositional letters, and let N be a finite set of agents. For convenience we slightly abuse notation and identify N with the set of integers $\{1, \ldots, |N|\}$. The *language* $\mathcal{L}_{\mathrm{EL}_N^C}$ *of epistemic logic with common knowledge* is defined by the BNF:

$$\varphi \ ::= \ p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathbf{K}_i\varphi \mid \mathbf{C}_G\varphi$$

where p ranges over P, i ranges over N, and G ranges over $\wp(N)$. The formula $\mathbf{K}_i\varphi$ reads 'agent i knows that φ', and $\mathbf{C}_G\varphi$ reads 'it is common knowledge in group G that φ'. We use the common abbreviations for $\varphi \vee \psi$, $\varphi \rightarrow \psi$, $\varphi \leftrightarrow \psi$, and $\mathbf{E}_G\varphi$. We recall that the latter is defined as: $\mathbf{E}_G\varphi = \bigwedge_{i \in G} \mathbf{K}_i\varphi$. The language $\mathcal{L}_{\mathrm{EL}_N}$ is obtained from $\mathcal{L}_{\mathrm{EL}_N^C}$ by dropping the operator of common knowledge.

An EL_N^C*-model* is a tuple $M = \langle W, K, V \rangle$, where W is a nonempty set of possible worlds; $K : N \rightarrow \wp(W \times W)$ associates an equivalence relation K_i to each $i \in N$; and $V : P \rightarrow \wp(W)$ associates an interpretation $V(p) \subseteq W$ to each $p \in P$.

For convenience, we define $K_i(w) = \{w' \mid (w, w') \in K_i\}$. The relation K_i models agent i's knowledge: $K_i(w)$ is the set of worlds that agent i considers to be possible at w.

The *satisfaction relation* '$\Vdash$' is defined as:

$M, w \Vdash p$	iff	$w \in V(p)$
$M, w \Vdash \neg\varphi$	iff	not $M, w \Vdash \varphi$
$M, w \Vdash \varphi \wedge \psi$	iff	$M, w \Vdash \varphi$ and $M, w \Vdash \psi$
$M, w \Vdash \mathbf{K}_i\varphi$	iff	$K_i(w) \subseteq [\![\varphi]\!]_M$
$M, w \Vdash \mathbf{C}_G\varphi$	iff	$(\bigcup_{i \in G} K_i)^*(w) \subseteq [\![\varphi]\!]_M$

where $[\![\varphi]\!]_M = \{w \in W \mid M, w \Vdash \varphi\}$ is the extension of φ in the model M, and the '$*$' in the last clause is reflexive and transitive closure.

A formula $\varphi \in \mathcal{L}_{\mathrm{EL}_N^C}$ is: *valid in a* EL_N^C*-model* M (notation: $M \Vdash \varphi$) iff $[\![\varphi]\!]_M = W$; EL_N^C*-valid* (notation: $\models_{\mathrm{EL}_N^C} \varphi$) iff $M \Vdash \varphi$ for all EL_N^C-models M; EL_N^C-*satisfiable* iff $\not\models_{\mathrm{EL}_N^C} \neg\varphi$. Similar notions are defined for the variant EL_N without common knowledge.

We recall that EL_N-satisfiability is NP-complete if $N = 1$, PSPACE-complete if $N \geq 2$, and EL_N^C-satisfiability is EXPTIME-complete (Fagin *et al.* 1995).

[3]A similar idea is outlined independently in (van Benthem 2007).

Reiter-Style Action Theories

In this section we extend the account of Reiter's solution in (Demolombe, Herzig, & Varzinczak 2003), where Reiter-style action theories are formulated in a propositional dynamic logic (PDL) framework.

Action Descriptions

In (Reiter 1991) and (Scherl & Levesque 2003) a number of simplifying assumptions are made. The most important are:

H1. All action laws are known by all agents.

H2. All action occurrences are public.

H3. All actions are deterministic.

H4. The set of fluents affected by an action is much smaller than the entire set P of fluents of the language.

H5. There is no action changing the truth value of an infinity of fluents.

The first two hypotheses say that the agents' knowledge about action types (H1) and about action instances (H2) is accurate. H3 is about the nature of the world. The last two hypotheses guarantee that Reiter's proposal indeed solves the representational frame problem, and are justified by the underlying hypothesis of inertia: (ontic) actions only change small parts of the world, leaving the rest unchanged. H4 says just this. Reiter does not explicit H5, but it is necessary when fluents are propositional. (One could argue that H5 is entailed by H4.)

Remark 1 Scherl & Levesque moreover suppose that there is only one agent. We do not make this restriction in this paper, and also consider the multiagent case.

Let A be a countably set of action letters (abstract atomic actions), and let a range over A.

Definition 2 We define an *action description* as the tuple $D = \langle Poss, Eff^+, Eff^-, \gamma^+, \gamma^- \rangle$ such that:

- $Poss : A \rightarrow \mathcal{L}_{\mathrm{EL}_N^C}$ assigns a formula to each action that describes its executability precondition;
- $Eff^+ : A \rightarrow \wp(P)$ assigns a finite set of possible positive effects to each action;
- $Eff^- : A \rightarrow \wp(P)$ assigns a finite set of possible negative effects to each action;
- γ^+ is a family of functions $\gamma^+(a) : Eff^+(a) \rightarrow \mathcal{L}_{\mathrm{EL}_N^C}$. It thus assigns a formula to each pair (a, p) that describes the precondition for the action a making p true; and
- γ^- is a family of functions $\gamma^-(a) : Eff^-(a) \rightarrow \mathcal{L}_{\mathrm{EL}_N^C}$. It thus assigns a formula to each pair (a, p) that describes the precondition for the action a making p false.

If $Eff^+(a) = Eff^-(a) = \emptyset$, then we call a an *epistemic action*. In the sequel, all epistemic actions are *observations*.

H1 and H2 make that the functions in D do not depend on agents. H3 makes that for any action a, its ontic effect can be characterized by $\gamma^+(a)$ and $\gamma^-(a)$. Finiteness of Eff^+ and Eff^- is due to H5. Finally, H4 allows to claim that the representational frame problem is solved by such action descriptions. In addition, Reiter (and we) assume:

H6. All $\gamma^+(a,p) \wedge \gamma^-(a,p)$ are inconsistent in EL^C_N.

Remark 3 Note that (Scherl & Levesque 2003) restrict the ranges of $Poss$, γ^+ and γ^- to boolean formulas. We extend them to the epistemic logical formulas in $\mathcal{L}_{\mathrm{EL}^C_N}$. This allows for actions such as 'make a phone call', whose precondition of execution is that the phone number is known.

Example 4 To illustrate the definition, suppose that a robot does not know whether the light is on or not. The available ontic action is toggling a switch, with $Poss(toggle) = \top$, $\mathit{Eff}^+(toggle) = \mathit{Eff}^-(toggle) = \{light\}$, $\gamma^+(toggle, light) = \neg light$, and $\gamma^-(toggle, light) = light$. The observations are $oDark$ and $oBright$, with $Poss(oDark) = \neg light$, and $Poss(oBright) = light$, and $\mathit{Eff}^+(oDark) = \mathit{Eff}^-(oDark) = \mathit{Eff}^+(oBright) = \mathit{Eff}^-(oBright) = \emptyset$.

Models for an Action Description

Let D be an action description for the action letters in A. Models for D are obtained by adding transition relations to the models of epistemic logic.

Definition 5 A *D-model* is a tuple $M = \langle W, K, T, V\rangle$, where $\langle W, K, V\rangle$ is an EL^C_N-model and $T : A \to \wp(W \times W)$ associates a relation T_a to each $a \in A$.

The relation T_a models the transition relation associated to the abstract action a: letting $T_a(w) = \{w' \mid (w, w') \in T_a\}$, $T_a(w)$ is the set of possible results of the execution a at w.

Moreover D-models satisfy the following constraints:

C1. *No-Forgetting:* $(T_a \circ K_i)(w) \subseteq (K_i \circ T_a)(w)$.

C2. *No-Learning:* if $T_a(w) \neq \emptyset$, then $(K_i \circ T_a)(w) \subseteq (T_a \circ K_i)(w)$.

C3. *Determinism:* if $v_1, v_2 \in T_a(w)$, then $v_1 = v_2$.

C4. *Executability:* $T_a(w) \neq \emptyset$ iff $\langle W, K, V\rangle, w \Vdash Poss(a)$.

C5. *Postcondition:* if $v \in T_a(w)$, then

- $p \notin \mathit{Eff}^+(a)$ and $w \notin V(p)$ implies $v \notin V(p)$;
- $p \notin \mathit{Eff}^-(a)$ and $w \in V(p)$ implies $v \in V(p)$;
- $p \in \mathit{Eff}^+(a)$ and $\langle W, K, V\rangle, w \Vdash \gamma^+(a,p)$ implies $v \in V(p)$;
- $p \in \mathit{Eff}^-(a)$ and $\langle W, K, V\rangle, w \Vdash \gamma^-(a,p)$ implies $v \notin V(p)$;
- $p \in \mathit{Eff}^+(a)$ and $\langle W, K, V\rangle, w \nVdash \gamma^+(a,p)$ and $w \notin V(p)$ implies $v \notin V(p)$;
- $p \in \mathit{Eff}^-(a)$ and $\langle W, K, V\rangle, w \nVdash \gamma^-(a,p)$ and $w \in V(p)$ implies $v \in V(p)$.

C1 implements H1 and H2. It guarantees that every world in $(T_a \circ K_i)(w)$ has an antecedent. This is also called *perfect recall* in (Fagin *et al.* 1995). In other words, there is no action able to make agents forget facts. C2 is motivated by H1–H3. For epistemic actions learning about the mere occurrence of an observation is sufficient for each agent to make his epistemic state evolve: the execution of an observation action a eliminates the possible worlds where $Poss(a)$ is false. C1 and C2 together correspond to Scherl & Levesque's SSA for knowledge in the case of an ontic action. C3 is motivated by H3. C4 defines the condition for an action be executable. C5 corresponds to Reiter's SSA for facts (as opposed to knowledge). Note that its consistency is guaranteed by H6.

Validity in D-models

We now introduce a combination of epistemic logic and PDL which will be interpreted in D-models. The language $\mathcal{L}_D$ extends $\mathcal{L}_{\mathrm{EL}^C_N}$ with dynamic operators, and is defined by the BNF:

$$\varphi \ ::= \ p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathbf{K}_i\varphi \mid \mathbf{C}_G\varphi \mid [a]\varphi$$

where p ranges over P, i ranges over N, G ranges over $\wp(N)$, and a ranges over A. The formula $[a]\varphi$ reads 'φ holds after all possible executions of a'. We use the common abbreviation $\langle a\rangle\varphi = \neg[a]\neg\varphi$. Thus $\langle a\rangle\top$ expresses that a is executable, and $[a]\bot$ expresses that a is inexecutable.

We define the *satisfaction relation* '$\Vdash$' as for EL^C_N, plus:

$$M, w \Vdash [a]\varphi \quad \text{iff} \quad T_a(w) \subseteq [\![\varphi]\!]_M$$

A formula $\varphi \in \mathcal{L}_D$ is: *valid in a D-model M* (notation: $M \Vdash \varphi$) iff $[\![\varphi]\!]_M = W$; *D-valid* (notation: $\models_D \varphi$) iff $M \Vdash \varphi$ for all D-models M; and *D-satisfiable* iff $\not\models_D \neg\varphi$.

For our running example we have

$$\not\models_D [toggle]\mathbf{K}_i light$$
$$\models_D [oDark][toggle]\mathbf{K}_i light$$
$$\models_D \neg\mathbf{K}_i\neg light \to [toggle]\neg\mathbf{K}_i light$$

Remark 6 Although epistemic actions do not change the world, note that $[a]Poss(a)$ is not D-valid, even if a is an epistemic action. To see this, consider a such that $Poss(a)$ is the so-called Moore-sentence: $p \wedge \neg\mathbf{K}_i p$. Then after learning that $p \wedge \neg\mathbf{K}_i p$ holds the agent will know that p, hence $\neg\mathbf{K}_i p$ does not hold any longer.

Regression

Let an action description D be given. Table 1 shows a number of valid D-equivalences. In each of those validities the complexity of the formula under the scope of the dynamic operator $[\cdot]$ decreases from the left to the right. For formulas without the common knowledge operator this allows for the definition of a procedure reg_D, called regression in (Reiter 2001), that repeatedly applies these equivalences until the resulting formula does not contain dynamic operators any more. It follows that for every domain description D and formula φ without $\mathbf{C}_G$ we have:

$$\models_D \varphi \quad \text{iff} \quad \models_{\mathrm{EL}^C_N} \mathrm{reg}_D(\varphi)$$

For example, $[toggle]\mathbf{K}_i light$ is first reduced to $Poss(toggle) \to \mathbf{K}_i[toggle]light$ (by 7) and then to $\mathbf{K}_i\neg light$ (by 4); and $[oDark]\mathbf{K}_i\neg light$ is first reduced to $Poss(oDark) \to \mathbf{K}_i[oDark]\neg light$ (by 7)

1. $[a]p \leftrightarrow (Poss(a) \rightarrow p)$
   if $p \notin \mathit{Eff}^+(a) \cup \mathit{Eff}^-(a)$
2. $[a]p \leftrightarrow (Poss(a) \rightarrow (\gamma^+(a,p) \vee p))$
   if $p \in \mathit{Eff}^+(a)$ and $p \notin \mathit{Eff}^-(a)$
3. $[a]p \leftrightarrow (Poss(a) \rightarrow (\neg\gamma^-(a,p) \wedge p))$
   if $p \notin \mathit{Eff}^+(a)$ and $p \in \mathit{Eff}^-(a)$
4. $[a]p \leftrightarrow (Poss(a) \rightarrow (\gamma^+(a,p) \vee (\neg\gamma^-(a,p) \wedge p)))$
   if $p \in \mathit{Eff}^+(a) \cap \mathit{Eff}^-(a)$
5. $[a]\neg\varphi \leftrightarrow (Poss(a) \rightarrow \neg[a]\varphi)$
6. $[a](\varphi_1 \wedge \varphi_2) \leftrightarrow ([a]\varphi_1 \wedge [a]\varphi_2)$
7. $[a]\mathbf{K}_i\varphi \leftrightarrow (Poss(a) \rightarrow \mathbf{K}_i[a]\varphi)$

Table 1: Relevant D-validities

and then to $\neg light \rightarrow \mathbf{K}_i(\neg light \rightarrow \neg light)$ (by 5 and then 1). The latter being EL_N-valid, it follows that $\models_D [oDark][toggle]\mathbf{K}_i light$.

Unfortunately, reg_D is a suboptimal decision procedure because there are formulas such that $\mathrm{reg}_D(\varphi)$ is exponentially larger than φ (Reiter 2001, Section 4.6).

Dynamic Epistemic Logic DEL_N^C

A different tradition in modelling knowledge and change has been followed in (Plaza 1989; Baltag, Moss, & Solecki 1998; van Benthem 2006). Logics in this tradition are, e.g., that of (van Ditmarsch, van der Hoek, & Kooi 2005; Kooi 2007), that are based on public announcements and public assignments.

Syntax

The *language of dynamic epistemic logic with common knowledge* $\mathcal{L}_{\mathrm{DEL}_N^C}$ is defined by the following BNF:

$$\begin{array}{lll} \varphi & ::= & p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathbf{K}_i\varphi \mid \mathbf{C}_G\varphi \mid [!\varphi]\varphi \mid [\sigma]\varphi \\ \sigma & ::= & \epsilon \mid p := \varphi, \sigma \end{array}$$

where p ranges over P, i ranges over N, G ranges over $\wp(N)$, and ϵ is an empty assignment. Again, the formula $[\alpha]\varphi$ is read 'φ holds after all possible executions of α'. The action $!\varphi$ is the public announcement of φ.[4] The action $p := \varphi$ is the public assignment of the truth value of φ to the atom p. For example, $p := \bot$ is a public assignment, and $\mathbf{K}_i[p := \bot]\neg p$ is a formula. When assignments are made in parallel, the same propositional letter can appear only once on the left hand side of the operator '$:=$'. For convenience, we identify $(p_1 := \varphi_1, \ldots, p_n := \varphi_n)$ with the set $\{p_1 := \varphi_1, \ldots, p_n := \varphi_n\}$, thus ϵ is identified with $\emptyset$.

The fragment of DEL_N^C without assignments is Plaza's public announcement logic with common knowledge (PAL_N^C) (Plaza 1989), whose fragment without common knowledge we note PAL_N.

Announcements model epistemic effects of actions, while assignments model ontic effects of actions. For example, the epistemic action *oDark* of Example 4 is modelled as $!\neg light$, and the ontic action *toggle* as the assignment $\sigma_{toggle} = (light := \neg light)$. In other words, the truth value of *light* is toggled.

[4]Note that announcement operators are different from the standard PDL test operator (usually noted $\varphi?$): the former have epistemic effects, but the latter has not.

1. $[!\varphi]p \leftrightarrow (\varphi \rightarrow p)$
2. $[!\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[!\varphi]\psi)$
3. $[!\varphi](\psi_1 \wedge \psi_2) \leftrightarrow ([!\varphi]\psi_1 \wedge [!\varphi]\psi_2)$
4. $[!\varphi]\mathbf{K}_i\psi \leftrightarrow (\varphi \rightarrow \mathbf{K}_i[!\varphi]\psi)$
5. $[\sigma]p \leftrightarrow \sigma(p)$
6. $[\sigma]\neg\varphi \leftrightarrow \neg[\sigma]\varphi$
7. $[\sigma](\varphi \wedge \psi) \leftrightarrow ([\sigma]\varphi \wedge [\sigma]\psi)$
8. $[\sigma]\mathbf{K}_i\varphi \leftrightarrow \mathbf{K}_i[\sigma]\varphi$

Table 2: Relevant DEL_N^C-validities.

Semantics

DEL_N^C-models are tuples $M = \langle W, K, V\rangle$ that are defined just as for epistemic logic EL_N^C. The *satisfaction relation* '$\Vdash$' is as there, plus:

$$\begin{array}{lll} M, w \Vdash [!\varphi]\psi & \text{iff} & M, w \Vdash \varphi \text{ implies } M^{!\varphi}, w \Vdash \psi \\ M, w \Vdash [\sigma]\varphi & \text{iff} & M^\sigma, w \Vdash \varphi \end{array}$$

where $M^{!\varphi}$ and M^σ are modifications of the epistemic model M that are defined as follows:

$$\begin{array}{lll} M^{!\varphi} & = & \langle W^{!\varphi}, K^{!\varphi}, V^{!\varphi}\rangle \\ W^{!\varphi} & = & W \cap [\![\varphi]\!]_M \\ K_i^{!\varphi} & = & K_i \cap ([\![\varphi]\!]_M \times [\![\varphi]\!]_M) \\ V^{!\varphi}(p) & = & V(p) \cap [\![\varphi]\!]_M \end{array}$$

and

$$\begin{array}{lll} M^\sigma & = & \langle W, K, V^\sigma\rangle \\ V^\sigma(p) & = & [\![\sigma(p)]\!]_M \end{array}$$

and where $\sigma(p)$ is the formula assigned to p in σ. If there is no such a formula, i.e., if there is no $p := \varphi$ in σ, then $\sigma(p) = p$. (In particular $\epsilon(p) = p$, for all p.)

As usual, a formula $\varphi \in \mathcal{L}_{\mathrm{DEL}_N^C}$ is: *valid in a* DEL_N^C*-model* M (notation: $M \Vdash \varphi$) iff $[\![\varphi]\!]_M = W$, DEL_N^C*-valid* (notation: $\models_{\mathrm{DEL}_N^C} \varphi$) iff $M \Vdash \varphi$ for all epistemic models M, and DEL_N^C*-satisfiable* iff $\not\models_{\mathrm{DEL}_N^C} \neg\varphi$. For example, $\mathbf{K}_i p \rightarrow [q := p]\mathbf{K}_i q$ is DEL_N^C-valid.

A number of relevant DEL_N^C-validities are listed in Table 2. When there are no $\mathbf{C}_G$ operators then the equivalences in Table 2 obviously allow the definition of a regression procedure $\mathrm{reg}_{\mathrm{DEL}_N}$, that eliminates dynamic operators from an expression (van Ditmarsch, van der Hoek, & Kooi 2005):

$$\models_{\mathrm{DEL}_N^C} \varphi \quad \text{iff} \quad \models_{\mathrm{EL}_N^C} \mathrm{reg}_{\mathrm{DEL}_N}(\varphi)$$

DEL_N-regression has the same problem of D-regression: the size of the resulting formula $\mathrm{reg}_{\mathrm{DEL}_N}(\varphi)$ can be exponentially larger than that of φ (Lutz 2006, Theorem 2). Moreover, no such equivalences exist for the $\mathbf{C}_G$ operator (Baltag, Moss, & Solecki 1998).

In the next sections we provide a better solution. The first step is to formally link Reiter-style action descriptions D with DEL_N^C.

Translating Reiter-Style Theories into DEL_N^C

The D-validities presented in Table 1 are similar to the DEL_N^C-validities presented in Table 2. We show in this section that: (1) the executability preconditions *Poss* in D can be modelled in DEL_N^C as public announcements, because once an action is executed, all the agents now know that it was executable at the previous instant; and (2) the changes brought about by actions can be modelled as public assignments.

Definition 7 Let an action description D be given. The *translation δ_D from $\mathcal{L}_D$ to $\mathcal{L}_{\mathrm{DEL}_N^C}$ formulas* is defined as follows:

$$\begin{array}{lcl}
\delta_D(p) & = & p \\
\delta_D(\neg\varphi) & = & \neg\delta_D(\varphi) \\
\delta_D(\varphi\wedge\psi) & = & \delta_D(\varphi)\wedge\delta_D(\psi) \\
\delta_D(\mathbf{K}_i\varphi) & = & \mathbf{K}_i(\delta_D(\varphi)) \\
\delta_D([a]\varphi) & = & [!Poss(a)][\sigma_a]\delta_D(\varphi)
\end{array}$$

where σ_a is the complex assignment:

$$\begin{array}{l}
\{p := \gamma^+(a,p)\vee p \mid p\in \mathit{Eff}^+(a) \text{ and } p\notin \mathit{Eff}^-(a)\} \quad \cup \\
\{p := \neg\gamma^-(a,p)\wedge p \mid p\notin \mathit{Eff}^+(a) \text{ and } p\in \mathit{Eff}^-(a)\} \quad \cup \\
\{p := \gamma^+(a,p)\vee(\neg\gamma^-(a,p)\wedge p) \mid p\in \mathit{Eff}^+(a)\cap \mathit{Eff}^-(a)\}
\end{array}$$

$\mathcal{L}_D$-formulas do not have the common knowledge operator, there is thus no clause for it. Also note that $\delta_D(a)$ is well-defined because $\mathit{Eff}^+(a)$ and $\mathit{Eff}^-(a)$ are finite by H5. For example, $\delta_D([oDark]\neg light) = [!\neg light][\epsilon]\neg light$, which is equivalent to $\top$ (remember that ϵ is the empty assignment); and $\delta_D([toggle]\neg light) = [!\top][light := \neg light \vee (\neg light\wedge light)]\neg light$, which is equivalent to $light$.

We now show that this translation is polynomial. We therefore define the function len() that returns the *length* of a given expression. In the case of sets and tuples, we count the length of each element and also the commas and delimiters, while for formulas each atom and each operator has length 1. For example, $\mathrm{len}(\langle t_1,\ldots,t_n\rangle) = (1+\mathrm{len}(t_1)) + \cdots + (1+\mathrm{len}(t_n))+1$, and $\mathrm{len}([\{p:=q, q:=p\wedge q\}]\mathbf{K}_i p) = 1+\mathrm{len}(\{p:=q,q:=p\wedge q\})+\mathrm{len}(\mathbf{K}_i p) = 12+2+1 = 15$;

Lemma 8 Let D be a finite Reiter-style action description and let $\varphi\in\mathcal{L}_D$. Then $\mathrm{len}(\delta_D(\varphi)) \leq \mathcal{O}(\mathrm{len}(D)\times\mathrm{len}(\varphi))$.

And also the following holds (cf. Table 1 and 2):

Proposition 9 Let D be a Reiter-style action description and let $\varphi\in\mathcal{L}_D$. Then φ is D-satisfiable if and only if $\delta_D(\varphi)$ is DEL_N^C-satisfiable.

Hence D-satisfiability check can be polynomially transformed into DEL_N^C-satisfiability check.[5]

Optimal Regression for DEL_N^C

We now give a polynomial satisfiability-preserving reduction from DEL_N^C to EL_N^C. The idea is first eliminate assignments, and then apply Lutz' reduction to eliminate announcements (Lutz 2006).

[5] The reader can find the proofs of Lemma 8 and Proposition 9 in (van Ditmarsch, Herzig, & de Lima 2007).

Eliminating Assignments

We apply a technique that is fairly standard in automated theorem proving (Nonnengart & Weidenbach 2001).

Proposition 10 Let $[p_1 := \varphi_1,\ldots,p_n := \varphi_n]\psi$ be a subformula of a $\mathcal{L}_{\mathrm{DEL}_N^C}$-formula χ. Let ψ' be obtained from ψ by substituting every occurrence of p_k by x_{p_k}, where x_{p_k} is a new propositional letter not occurring in χ. Let χ' be obtained from χ by replacing $[p_1 := \varphi_1,\ldots,p_n := \varphi_n]\psi$ by ψ'. Let B abbreviate the conjunction of equivalences (biimplications) $\bigwedge_{1\leq k\leq n}(x_{p_k}\leftrightarrow\varphi_k)$.

1. If $\chi\in\mathcal{L}_{\mathrm{DEL}_1}$ then χ is DEL_1-satisfiable iff
$$\chi'\wedge\mathbf{K}_i B$$
is DEL_1-satisfiable.
2. If $\chi\in\mathcal{L}_{\mathrm{DEL}_N}$, $N\geq 2$, then χ is DEL_N-satisfiable iff
$$\chi'\wedge\bigwedge_{\ell\leq \mathrm{md}(\varphi)}\mathbf{E}_N^{\ell}B$$
is DEL_N-satisfiable, where the modal depth $\mathrm{md}(\varphi)$ is the maximal number of nested modal operators of ψ, and $\mathbf{E}_G^{\ell}\varphi$ means '$\mathbf{E}_G\ldots\mathbf{E}_G\varphi$, $\ell\geq 0$ times'.
3. If $\chi\in\mathcal{L}_{\mathrm{DEL}_N^C}$ then χ is DEL_N^C-satisfiable iff
$$\chi'\wedge\mathbf{C}_N B$$
is DEL_N^C-satisfiable.

Proof. To simplify suppose the subformula of χ is $[p:=\varphi]\psi$. $\Rightarrow$: Suppose $M = \langle W,K,V\rangle$ is an EL_N^C-model such that $M,w\Vdash\chi$. We construct an EL_N^C-model $M_{x_p} = \langle W,K,V_{x_p}\rangle$, where $V_{x_p}(p) = V(p)$ for all $p\neq x_p$, and $V_{x_p}(x_p) = [\![\varphi]\!]_M$. We then prove that $M_{x_p}\Vdash[p:=\varphi]\psi\leftrightarrow\psi'$, from which all three cases follow.
$\Leftarrow$: We suppose w.l.o.g. that M is generated from w, and observe that $\mathbf{K}_i$ is a 'master modality' for single-agent EL_N, and $\mathbf{C}_G$ for EL_N^C, and the conjunction up to modal depth of $\mathbf{E}_G$-operators for multi-agent EL_N: for example, when $M,w\Vdash\mathbf{C}_G\chi$ then $M\Vdash\mathbf{C}_G\chi$. ∎

Renaming avoids exponential blow-up. This allows the definition of reduction operators $\mathrm{reg}_{\mathrm{DEL}_1}$, $\mathrm{reg}_{\mathrm{DEL}_N}$, and $\mathrm{reg}_{\mathrm{DEL}_N^C}$ that iteratively eliminate all assignments. For instance, consider the following DEL_N-unsatisfiable formula: $\neg[!\neg light][light := \neg light]\mathbf{K}_i light$. Its reduction is $\neg[!\neg light]\mathbf{K}_i x_{light}\wedge\mathbf{K}_i(x_{light}\leftrightarrow\neg light)$.

Proposition 11 $\mathrm{reg}_{\mathrm{DEL}_1}$, $\mathrm{reg}_{\mathrm{DEL}_N}$ and $\mathrm{reg}_{\mathrm{DEL}_N^C}$ are polynomial transformations, and preserve satisfiability in the respective logics.

Proof. Satisfiability-equivalence follows from Proposition 10. For the single-agent and the common-knowledge case we prove that the size of the reduction of χ is at most $\mathrm{len}(\chi)\times(\mathrm{len}(\chi)+6)$, and for the case of DEL_N we prove that the size of the reduction of χ is at most $\mathrm{len}(\chi)^2\times(\mathrm{len}(\chi)+6)$. Indeed, in Proposition 10 the size of χ' is at most $\mathrm{len}(\chi)$, the size of each equivalence in B is at most $\mathrm{len}(\chi)+4$, and the number of these equivalences is bound by the number of (atomic) assignments in χ, which is at most

$\mathrm{len}(\chi)$. In the case of the $\mathbf{E}$ operator the number of equivalences has to be multiplied by the modal depth of χ, which is at most $\mathrm{len}(\chi)$. ∎

Eliminating Announcements

Once assignments are eliminated, we can eliminate announcements by Lutz' procedure. We don't have space to go into the details, and just restate the relevant theorem.

Proposition 12 ((Lutz 2006)) PAL_N-satisfiability is NP-complete if $|N| \leq 1$, and PSPACE-complete if $|N| \geq 2$. PAL_N^C-satisfiability is EXPTIME-complete.

Via Proposition 9 one obtains:

Corollary 13 D-satisfiability without $\mathbf{C}_G$-operators is NP-complete if $|N| = 1$, and PSPACE-complete if $|N| \geq 2$. D-satisfiability with $\mathbf{C}_G$-operators is EXPTIME-complete.

Conclusions

We have modelled the frame problem in dynamic epistemic logic by providing correspondents for situation calculus style ontic and observation actions, and we have given complexity results using that translation. As far as we know, this is the first optimal decision procedure for a Reiter-style solution to the frame problem.

Scherl & Levesque's epistemic extension of Reiter's solution allows not only for observations, but also for sensing actions $?\varphi$, which test whether some boolean formula φ is true. Such sensing actions can be viewed as abbreviating the nondeterministic composition of two announcements: $?\varphi = !\varphi \cup !\neg\varphi$. Expansion of such abbreviations leads to exponential blow-up, which does not allow to extend our approach and integrate primitive sensing actions: it is not clear how the associated successor state axiom

$$\begin{aligned}[?\varphi]\mathbf{K}_i\psi \quad\leftrightarrow\quad & ((\varphi \rightarrow \mathbf{K}_i(\varphi \rightarrow [?\varphi]\psi))\wedge \\ & (\neg\varphi \rightarrow \mathbf{K}_i(\neg\varphi \rightarrow [?\varphi]\psi)))\end{aligned}$$

could be compiled into Lutz' polynomial transformation. Further evidence that the presence of sensing actions increases complexity is provided by the result in (Herzig *et al.* 2000) that plan verification in this case is Π_2^p-complete. We leave integration of sensing actions as future work.

We moreover intend to generalize our results to non-public actions, as in (Baltag, Moss, & Solecki 1998; Bacchus, Halpern, & Levesque 1999).

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