Optimal regulation of nonlinear dynamical systems on a finite interval

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Abstract

In this paper the optimal control of nonlinear dynamical systems on a finite time interval is considered. The free end-point problem as well as the fixed end-point problem is studied. The existence of a solution is proved and a power series solution of both the problems is constructed.

1. Introduction

We consider control processes in $\mathbb{R}^n$ of the form

$$(1.1) \qquad \dot{x} = F(x,u,t)$$

and investigate the problem of finding a bounded r dimensional feedback control u(x,t) which minimizes the integral

$$(1.2) \qquad J(\tau,b,u) = L(x(T)) + \int_{\tau}^{T} G(x,u,t)dt$$

for all initial states $x(\tau) = b$ in a neighborhood of the origin in $\mathbb{R}^n$.
In section 2 we treat the free end-point problem and in section 3 the fixed end-point problem. More specifically, in section 3 we require the final value x(T) of the state to be zero.
For the situation where F is linear and L and G are quadratic the solution of the optimal control problem is well known (e.g. see [2] section 3.21, [3] section 2.3, [4] section 9.7 for the free end-point problem and [2] section 3.22 for the fixed end-point problem).
Here we consider the situation where the states and controls remain in a neighborhood of a fixed point (for which we without loss of generality take the origin) where the functions F, G and L can be expanded in power series. An analogous problem has been considered by D.L. Lukes [1] (see also [5] section 4.3) for the infinite horizon case and our treatment will follow this paper to some extent, in particular as far as the free end-point case is concerned. The theory is more complete than the related Hamilton-Jacobi theory since existence and uniqueness proofs of optimal controls are given. For the solution of the fixed end-point problem we introduce a dual problem of (1.1) and (1.2) which we use to reduce the fixed end-point problem to a free end-point problem. Some examples are added to illustrate the theory.

Notation

The inner product of two vectors x and y we shall denote by x^Ty. The length of a vector x by $|x| = \sqrt{x^Tx}$ and the transposed of a matrix M by M^T. The notation $M>0$ and $M\geq 0$ means that M represents a (symmetric) positive definite and a non-negative definite matrix respectively. If f(x) denotes a vector function from $\mathbb{R}^n$ into $\mathbb{R}^m$, the following notation and definition of the functional matrix will be used:

$$f_x := \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_1} \\ & \ddots & \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

2. Free end-point problem

2.1. Assumptions

(i) $F(x,u,t) = A(t)x + B(t)u + f(x,u,t)$. Here $A(t)$ and $B(t)$ are continuous real matrix functions of dimension $n \times n$ and $n \times r$ respectively. The function $f(x,u,t)$ contains the higher order terms in x and u, and is continuous with respect to t. Furthermore $f(x,u,t)$ is given as a power series in (x,u) which starts with second order terms and converges about the origin, uniformly for $t \in [\tau,T]$.

(ii) $G(x,u,t) = x^T Q(t)x + u^T R(t)u + g(x,u,t)$. Here $Q(t)$ and $R(t)$ are continuous real matrix functions of dimension $n \times n$ and $r \times r$ respectively. The function $g(x,u,t)$ contains the higher order terms in x and u, and is continous with respect to t. Furthermore $g(x,u,t)$ is given as a power series in (x,u) which starts with third order terms and converges about the origin, uniformly for $t \in [\tau,T]$.

(iii) $L(x) = x^T Mx + \ell(x)$. Here M is a real matrix of dimension $n \times n$. The function $\ell(x)$ is given as a power series which starts with third order terms and converges about the origin.

(iv) $Q(t) \geq 0$ and $R(t) > 0$ for $t \in [\tau,T]$; $M \geq 0$.

We consider the class of feedback controls which are of the form

$$(2.1) \qquad u(x,t) = D(t)x + h(x,t)$$

Here $D(t)$ is a continuous matrix function of dimension $r \times n$. The function $h(x,t)$ contains the higher order terms in x and is continuous with respect to t. Furthermore $h(x,t)$ is given as a power series in x which starts with second order terms and converges about the origin, uniformly for $t \in [\tau,T]$. We shall denote the class of admissible feedback controls by Ω.

Definition of an optimal feedback control

A feedback control $u_* \in \Omega$ is called optimal if there exists an $\epsilon > 0$ and a neighborhood N_* of the origin in $\mathbb{R}^n$ such that for each $b \in N_*$ the response $x_*(t)$ satisfies $|x_*(t)| \leq \epsilon$ and $|u_*(x_*(t),t)| \leq \epsilon$ for $t \in [\tau,T]$, and furthermore $J(\tau,b,u_*) \leq J(\tau,b,u)$ among all feedback controls $u \in \Omega$ generating responses $x(t)$ with $|x(t)| \leq \epsilon$ and $|u(x(t),t)| \leq \epsilon$ for $t \in [\tau,T]$.

2.2. Statement of the main results

Theorem 2.1. (Main Theorem)
For the control process in $\mathbb{R}^n$

$$\dot{x} = F(x,u,t), \quad x(\tau) = b$$

with performance index

$$J(\tau,b,u) = L(x(T)) + \int_\tau^T G(x,u,t)dt$$

there exists a unique optimal feedback control $u_*(x,t)$. *This feedback control is the unique solution of the functional equation*

$$(*) \qquad F_u(x,u_*(x,t),t)J_x(t,x,u_*) + G_u(x,u_*(x,t),t) = 0$$

for small $|x|$ *and* $t \in [\tau,T]$. *Furthermore*

$$u_*(x,t) = D_*(t)x + h_*(x,t)$$

and

$$J(\tau,b,u_*) = b^T K_*(\tau)b + j_*(\tau,b),$$

where the matrix functions $D_*(t)$ *and* $K_*(t) \geq 0$ *depend only on the truncated problem.*

<u>Theorem 2.2.</u> (Truncated problem)
For the special case in which $f(x,u,t) = 0$, $g(x,u,t) = 0$ *and* $\ell(x) = 0$ *the optimal control is given by*

$$u_*(x,t) = D_*(t)x$$

where

$$D_*(t) = -R^{-1}(t)B^T(t)K_*(t).$$

Here $K_*(t) \geq 0$ *is a solution of the Riccati equation on* $[\tau,T]$:

$$\begin{cases} \dot{K}(t) + Q(t) + K(t)A(t) + A^T(t)K(t) - K(t)B(t)R^{-1}(t)B^T(t)K(t) = 0 \\ K(T) = M \end{cases}$$

Furthermore $D_*(t)x$ *is a global optimal control in the sense that we can take* $N_* = \mathbb{R}^n$ *and* $\epsilon = \infty$ *in the definition of optimal feedback control. Finally*

$$J(\tau,b,u_*) = b^T K_*(\tau)b.$$

<u>Remark.</u> Note that for $u \in \Omega$ the property $J(T,b,u) = L(b)$ holds.

2.3. <u>Construction of the optimal feedback control</u>

<u>Lemma 2.1.</u>
For each feedback control $u \in \Omega$, $u(x,t) = D(t)x + h(x,t)$, *there exists a neighborhood* N_u *of the origin in* $\mathbb{R}^n$ *in which*

$$J(\tau,b,u) = b^T\hat{K}(\tau)b + j(\tau,b).$$

Here $j(\tau,b)$ *contains the higher order terms in* b. *The matrix function* $\hat{K}(\tau) \geq 0$ *depends only on the truncated problem. Furthermore, the functional equation*

$$F(x,u(x,t),t)^T J_x(t,x,u) + J_t(t,x,u) + G(x,u(x,t),t) = 0$$

holds for each $x \in N_u$, $t \in [\tau,T]$.

Proof. The following differential equation holds:

$$\begin{cases} \dot{x} = (A(t) + B(t)D(t))x + B(t)h(x,t) + f(x,u(x,t),t) \\ x(\tau) = b \end{cases}$$

If we define $A_*(t) := A(t) + B(t)D(t)$ and $v(x,t) := B(t)h(x,t) + f(x,u(x,t),t)$ then this equation becomes

$$\begin{cases} \dot{x} = A_*(t)x + v(x,t) \\ x(\tau) = b \end{cases}$$

From the theory of ordinary differential equations it is known that there exists a neighborhood N_1 of the origin such that the solution exists for each $b \in N_1$, and furthermore

$$x(t) = \Phi(t)\Phi^{-1}(\tau)b + \mathcal{O}(|b|^2),$$

uniformly for $t \in [\tau,T]$. Here $\Phi(t)$ is a fundamental matrix of the linear equation $\dot{x} = A_*(t)x$ (i.e. a nonsingular matrix function of dimension n x n which satisfies $\dot{\Phi}(t) = A_*(t)\Phi(t)$). Hence

$$G(x(t),u(x(t),t),t) = x(t)^T Q(t)x(t) + x(t)^T D(t)^T R(t)D(t)x(t) + \mathcal{O}(|x|^3) =$$

$$= b^T\Phi^{-T}(\tau)\Phi^T(t)\{Q(t) + D(t)^T R(t)D(t)\}\Phi(t)\Phi^{-1}(\tau)b + \mathcal{O}(|b|^3),$$

uniformly for $t \in [\tau,T]$. Furthermore

$$L(x(T)) = x(T)^T M x(T) + \mathcal{O}(|x(T)|^3) =$$

$$= b^T \Phi^{-T}(\tau)\Phi^T(T) M \Phi(T)\Phi^{-1}(\tau) b + \mathcal{O}(|b|^3)$$

So

$$J(\tau,b,u) = b^T \hat{K}(\tau) b + \mathcal{O}(|b|^3),$$

where

$$\hat{K}(\tau) := \Phi^{-T}(\tau)\Phi^T(T) M \Phi(T)\Phi^{-1}(\tau) +$$

$$+ \int_\tau^T [\Phi^{-T}(\tau)\Phi^T(t)\{Q(t) + D(t)^T R(t) D(t)\}\Phi(t)\Phi^{-1}(\tau)]dt \tag{2.2}$$

It is easy to verify that $\hat{K}(\tau) \geq 0$ and $\hat{K}(T) = M$. It is known that there exists a neighborhood N_2 of the origin in $\mathbb{R}^n$ such that for each $s \in [\tau,T]$ and for each $b \in N_2$, the solution of $\dot{x} = F(x,u(x,t),t)$ with $x(s) = b$, exists on $[s,T]$. Now let $N_u := N_1 \cap N_2$, $s \in [\tau,T]$ and $b \in N_u$. If $x(t,s,b)$ denotes the solution of $\dot{x} = F(x,u(x,t),t)$ with $x(s) = b$ than we can write

$$J(t,x(t,s,b),u) = L(x(T,s,b)) + \int_\tau^T G(x(\xi,s,b),u(x(\xi,s,b),\xi),\xi)d\xi$$

for $t \in [s,T]$. One can verify that it is allowed to differentiate this equation with respect to t. Setting $t = s$ afterwards we get the equation

$$F(b,u(b,s),s)^T J_x(s,b,u) + J_t(s,b,u) + G(b,u(b,s),s) = 0.$$

If we finally replace b and s by x and t we get the desired result. □

Remark. From the proof it follows that we even have

$$J(t,x,u) = x^T \hat{K}(t) x + \mathcal{O}(|x|^3)$$

uniformly for $t \in [\tau,T]$ and for small $|x|$.

Lemma 2.2. *The equation*

$$F_u(x,u_*,t)p + G_u(x,u_*,t) = 0$$

has a solution $u_*(x,p,t)$ *near the origin in* $\mathbb{R}^{2n}$ *for which* $u_*(0,0,t) = 0$ *for* $t \in [\tau,T]$. *Furthermore*

$$u_*(x,p,t) = -\tfrac{1}{2}R^{-1}(t)B^T(t)p + h_*(x,p,t),$$

where $h_*(x,p,t)$ *contains the higher order terms in* (x,p).

Proof. For each $t \in [\tau,T]$ we can use the result in [1], lemma 2.2. □

Lemma 2.3. *There exists a unique solution* $K_*(t)$ *on* $[\tau,T]$ *to the matrix differential equation (Riccati equation)*

$$\begin{cases} \dot{K}(t) + Q(t) + K(t)A(t) + A^T(t)K(t) - K(t)B(t)R^{-1}(t)B^T(t)K(t) = 0 \\ K(T) = M \end{cases}$$

The property $K_*(t) \geq 0$ *holds on* $[\tau,T]$.

Proof. See [3] section 2.3. □

Lemma 2.4. *Suppose there exists a feedback control* $u_*(x,t) = D_*(t)x + h_*(x,t)$, *which satisfies the nonlinear functional equation*

$$(*) \qquad F_u(x,u_*(x,t),t)J_x(t,x,u_*) + G_u(x,u_*(x,t),t) = 0$$

for small $|x|$ *and* $t \in [\tau,T]$. *Then* u_* *is the unique optimal feedback control. Furthermore*

$$D_*(t) = -R^{-1}(t)B^T(t)K_*(t)$$

and

$$J(\tau,b,u_*) = b^TK_*(\tau)b + j_*(\tau,b),$$

where $K_*(t)$ *is defined in lemma 2.3. The function* $j_*(\tau,b)$ *contains the higher terms in* b.

Proof. Consider the following real valued function defined for $t \in [\tau,T]$ and for (x,u) near the origin in $\mathbb{R}^{n+r}$:

$$(2.3) \qquad Q(t,x,u) := F(x,u,t)^T J_x(t,x,u_*) + Jt(t,x,u_*) + G(x,u,t)$$

By lemma 2.1.

$$Q(t,x,u_*(x,t)) = 0 \text{ near } x = 0 \text{ and for } t \in [\tau,T].$$

We have assumed that

$$Q_u(t,x,u_*(x,t)) = 0 \text{ near } x = 0 \text{ and for } t \in [\tau,T].$$

Furthermore the Hessian

$$Q_{uu}(t,0,0) = 2R(t) \text{ is positive definite for } t \in [\tau,T].$$

It follows that

$$Q_{uu}(t,x,u) > 0 \text{ for } |x| \text{ small, } |u| \text{ small and } t \in [\tau,T]$$

because $Q(t,x,u)$ is a continuous function. Hence we conclude that there exists an $\epsilon > 0$ such that

$$0 = Q(t,x,u_*(x,t)) \leq Q(t,x,u_1)$$

for $t \in [\tau,T]$, $|x| \leq \epsilon$ and $|u_1| \leq \epsilon$, while strict inequality holds for $u_1 \neq u_*(x,t)$. So

$$(2.4) \qquad 0 \leq F(x,u_1,t)^T J_x(t,x,u_*) + J_t(t,x,u_*) + G(x,u_1,t)$$

Now let N_* be a neighborhood of the origin in $\mathbb{R}^n$ such that for each $b \in N_*$ the solution $x_*(t)$ of $\dot{x} = F(x,u_*(x,t),t)$, $x(\tau) = b$, exists for $t \in [\tau,T]$, $|x_*(t)| \leq \epsilon$ and $|u_*(x_*(t),t)| \leq \epsilon$.

Furthermore let $u_1 \in \Omega$ be an arbitrary feedback control such that the solution $x_1(t)$ of $\dot{x} = F(x,u_1(x,t),t), x(\tau) = b$ is defined on $[\tau,T]$, and satisfies $|x_1(t)| \leq \epsilon$ and $|u_1(x_1(t),t)| \leq \epsilon$, if $b \in N_*$. Then we can write:

$$0 < \int_{\tau}^{T} \{F(x_1(t),u_1(x_1(t),t),t)^T J_x(t,x_1(t),u_*) + $$
$$+ J_t(t,x_1(t),u_*) + G(x_1(t),u_1(x_1(t),t),t)\}dt,$$

and so

$$0 < \int_{\tau}^{T} \{\frac{d}{dt}J(t,x_1(t),u_*)\}dt + \int_{\tau}^{T} G(x_1(t),u_1(x_1(t),t),t)dt$$

This yields the result

$$0 < J(T,x_1(T),u_*) - J(\tau,b,u_*) + \int_{\tau}^{T} G(x_1(t),u_1(x_1(t),t),t)dt$$

and thus

$$J(\tau,b,u_*) < J(\tau,b,u_1).$$

So $u_*(x,t)$ is the unique optimal feedback control.
By lemma 2.2. we have

$$u_*(x,t) = -\tfrac{1}{2}R^{-1}(t)B^T(t)J_x(t,x,u_*) + \mathcal{O}(|x|^2),$$

uniformly for $t \in [\tau,T]$ and in lemma 2.1. we have

$$J_x(t,x,u_*) = 2\hat{K}(t)x + \mathcal{O}(|x|^2)$$

So

$$(2.5) \qquad u_*(x,t) = -R^{-1}(t)B^T(t)\hat{K}(t)x + \mathcal{O}(|x|^2),$$

uniformly for $t \in [\tau,T]$. By lemma 2.1. we have

$$(2.6) \qquad F(x,u_*(x,t),t)^T J_x(t,x,u_*) + J_t(t,x,u_*) + G(x,u_*(x,t),t) = 0$$

for $|x|$ small and $t \in [\tau,T]$. Using (2.5) collecting the quadratic terms in x we find that $\hat{K}(t)$ is a solution of the Riccati equation. We also know that $\hat{K}(T) = M$ and by the uniqueness of the solution we have $\hat{K}(t) = K_*(t)$ on $[\tau,T]$.

This yields the result

$$u_*(x,t) = -R^{-1}(t)B^T(t)K_*(t)x + \mathcal{O}(|x|^2)$$

and

$$J(\tau,b,u_*) = b^T K_*(\tau)b + \mathcal{O}(|b|^3) \qquad \square$$

Proof of theorem 2.2. Let $u_*(x,t) = D_*(t)x$, where $D_*(t) = -R^{-1}(t)B^T(t)K_*(t)$ and the matrix $K_*(t)$ satisfies the Riccati equation, hence

$$x^T\{\dot{K}_*(t) + Q(t) + K_*(t)A(t) + A^T(t)K_*(t) -$$

$$+ K_*(t)B(t)R^{-1}(t)B^T(t)K_*(t)\}x = 0$$

for all $x \in \mathbb{R}^n$. So we can write

$$[(A(t) - B(t)R^{-1}(t)B^T(t)K_*(t))x]^T 2K_*(t)x + x^T\dot{K}_*(t)x +$$

$$+ x^TQ(t)x + x^TK_*(t)B(t)R^{-1}(t)B^T(t)K_*(t)x = 0$$

It follows that

$$[(A(t) + B(t)D_*(t))x]^T 2K_*(t)x + x^T\dot{K}_*(t)x +$$

$$x^TQ(t)x + [D_*(t)x]^TR(t)D_*(t)x = 0$$

This yields

$$F(x,u_*(x,t),t)^T 2K_*(t)x + x^T\dot{K}_*(t)x + G(x,u_*(x,t),t) = 0$$

By integrating this equation along the trajectory $\dot{x} = F(x,u_*(x,t),t)$, $x(\tau) = b$, where b is arbitrary in $\mathbb{R}^n$, we obtain the equation

$$J(\tau,b,u_*) = b^TK_*(\tau)b \quad (b \in \mathbb{R}^n)$$

It is now easy to verify that $u_*(x,t)$ satisfies the functional equation (*) in lemma 2.4. The global character of $u_*(x,t)$ follows by examining the proof of lemma 2.4. $\square$

Before giving the proof of the main theorem, we consider the Hamiltonian system in $\mathbb{R}^{2n}$:

$$(2.7)\qquad \begin{cases} \dot{x} = F(x,u_*(x,p.t),t) \\ \dot{p} = -\{F_x(x,u_*(x,p,t),t)p + G_x(x,u_*(x,p,t),t)\} \end{cases}$$

with the boundary values

$$\begin{cases} x(\tau) = b \\ p(T) = L_x(x(T)) \end{cases}$$

Here $u_*(x,p,t)$ is defined in lemma 1.2.

Lemma 2.5. *For small* $|b|$ *system* (2.7) *has a solution* $(x_*(t),p_*(t))$ *on* $[\tau,T]$ *with the property*

$$p_*(t) = 2K_*(t)x_*(t) + \mathcal{O}(|x_*(t)|^2),$$

uniformly for $t \in [\tau,T]$.

Proof. The Hamiltonian system has the form

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} A(t) & -\frac{1}{2}B(t)R^{-1}(t)B^T(t) \\ -2Q(t) & -A^T(t) \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} + h(x,p,t),$$

where the function $h(x,p,t)$ contains the higher order terms. First of all we shall prove that the lemma holds for the case that $h(x,p,t) = 0$. The solvability of the linear system together with the implicit function theorem will be used to obtain a proof for the general case. So we shall first consider the linear Hamiltonian system

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} A(t) & -\frac{1}{2}B(t)R^{-1}(t)B^T(t) \\ -2Q(t) & -A^T(t) \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix},$$

with $x(\tau) = b$ and $p(T) = 2Mx(T)$. This system has a solution $(x_*(t),p_*(t))$

with the property $p_*(t) = 2K_*(t)x_*(t)$, which can easily be verified. Note that this solution exists for each $b \in \mathbb{R}^n$. If we now consider this linear system as a final value problem: $x(T) = x_T, p(T) = p_T$, then the solution is given by

$$\begin{pmatrix} x \\ p \end{pmatrix}(t) = \Phi(t)\Phi^{-1}(T) \begin{pmatrix} x_T \\ p_T \end{pmatrix} \tag{2.8}$$

Here $\Phi(t)$ is a fundamental matrix of the problem. If we partition

$$\Phi(t)\Phi^{-1}(T) = \begin{bmatrix} \Theta_{11}(t,T) & \Theta_{12}(t,T) \\ \Theta_{21}(t,T) & \Theta_{22}(t,T) \end{bmatrix},$$

then (2.8) can be written as

$$x(t,x_T,p_T) = \Theta_{11}(t,T)x_T + \Theta_{12}(t,T)p_T$$

$$p(t,x_T,p_T) = \Theta_{21}(t,T)x_T + \Theta_{22}(t,T)p_T$$

So

$$x(t,x_T,2Mx_T) = (\Theta_{11}(t,T) + 2\Theta_{12}(t,T)M)x_T$$

We saw that for each $b \in \mathbb{R}^n$ there exists a solution on $[\tau,T]$ with $p(T) = 2Mx(T)$. So

$$\forall_{b \in \mathbb{R}^n} \; \exists_{x_T \in \mathbb{R}^n} : (\Theta_{11}(\tau,T) + 2\,\Theta_{12}(\tau,T)M)x_T = b$$

Hence the matrix

$$\Theta_{11}(\tau,T) + 2\Theta_{12}(\tau,T)M \tag{2.9}$$

is regular. We shall need this result later. Now consider the nonlinear Hamiltonian system as a final value problem : $x(T) = x_T, p(T) = p_T$. The solution has the form

$$\begin{pmatrix} x \\ p \end{pmatrix}(t) = \Phi(t)\Phi^{-1}(T)\begin{pmatrix} x_T \\ p_T \end{pmatrix} + v(t,x_T,p_T) ,$$

where $v(t,x_T,p_T)$ contains the second and higher order terms in x_T and p_T. It follows that

$$x(t,x_T,p_T) = \Theta_{11}(t,T)x_T + \Theta_{12}(t,T)p_T + \mathcal{O}(|\begin{pmatrix} x_T \\ p_T \end{pmatrix}|^2)$$

$$p(t,x_T,p_T) = \Theta_{21}(t,T)x_T + \Theta_{22}(t,T)p_T + \mathcal{O}(|\begin{pmatrix} x_T \\ p_T \end{pmatrix}|^2) ,$$

uniformly for $t \in [\tau,T]$. The question is: does there exist for arbitrary $b \in \mathbb{R}^n$, $|b|$ small, a vector $x_T \in \mathbb{R}^n$ such that $x(\tau,x_T,L_x(x_T)) = b$? Here the implicit function theorem can help us. Define

$$F(b,x_T) := x(\tau,x_T,L_x(x_T)) - b$$

Then $F(0,0) = 0$ and $F_{x_T}(0,0) = \Theta_{11}(\tau,T) + 2\Theta_{12}(\tau,T)M$. By (2.9) we have that $F_{x_T}(0,0)$ is regular. Thus there exists a neighborhood Ω of the origin in $\mathbb{R}^n$ and a function $\tilde{x}_T: \Omega \to \mathbb{R}^n$ such that

(i) $\tilde{x}_T(0) = 0$

(II) $F(b,\tilde{x}_T(b)) = 0$ for $b \in \Omega$

So $x(\tau,\tilde{x}_T(b), L_x(\tilde{x}_T(b))) = b$. Hence the Hamiltonian system (2.7) has a solution on $[\tau,T]$ for small $|b|$. From the considerations of the linear system we have

$$p_*(t) = 2K_*(t)x_*(t) + \mathcal{O}(|x_*(t)|^2),$$

uniformly for $t \in [\tau,T]$ □

<u>Proof of the main theorem.</u> It is sufficient to establish the existence of a feedback control $u_* \in \Omega$ which satisfies the functional equation (*). Define

(2.10) $$u_*(x,t) := u_*(x,p_*(x,t),t),$$

where $p_*(x,t)$ represents the solution of (2.7) and $u_*(x,p,t)$ is defined as in lemma 2.2. Then

$$u_*(x,t) = -\tfrac{1}{2}R^{-1}(t)B^T(t)p_*(x,t) + \mathcal{O}(|x|^2) =$$

$$= -R^{-1}(t)B^T(t)K_*(t)x + \mathcal{O}(|x|^2)$$

uniformly for $t \in [\tau,T]$. Thus we can conclude that $u_* \in \Omega$. Now let $s \in [\tau,T]$ fixed and choose $y \in \mathbb{R}^n$ so small that the solution of $\dot{x} = F(x,u_*(x,t),t)$, with $x(s) = y$, exists on $[\tau,T]$, and $x(\tau) =: b$ is so small that the solution of (2.7) exists. By the continuity and analyticity of $G(x,u_*(x,t),t)$ the following differentiation of the integral is allowed:

$$\frac{\partial J(s,y,u_*)}{\partial y} = \int_s^T \frac{\partial}{\partial y} G(x,u_*(x,t),t)dt + \frac{\partial}{\partial y} L(x(T)) =$$

$$= \int_s^T \{\frac{\partial x}{\partial y} \frac{\partial G(x,u_*(x,t),t)}{\partial x} + \frac{\partial u_*}{\partial y} \frac{\partial G(x,u_*(x,t),t)}{\partial u_*}\}dt + \frac{\partial}{\partial y} L(x(T)) =$$

$$= \int_s^T \{\frac{\partial x}{\partial y} [-\dot{p}_*(x,t) - \frac{\partial F(x,u_*(x,t),t)}{\partial x} p_*(x,t)] +$$

$$+ \frac{\partial u_*}{\partial y} \frac{\partial G(x,u_*(x,t),t)}{\partial u_*}\}dt + \frac{\partial}{\partial y} L(x(T)) =$$

$$= -\int_s^T \{\frac{\partial x}{\partial y} \dot{p}_*(x,t)\}dt + \frac{\partial}{\partial y} L(x(T)) +$$

$$+ \int_s^T \{\frac{\partial u_*}{\partial y} [-\frac{\partial F(x,u_*(x,t),t)}{\partial u_*} p_*(x,t)] - \frac{\partial x}{\partial y} \frac{\partial F(x,u_*(x,t),t)}{\partial x} p_*(x,t)\}dt =$$

$$= -\frac{\partial x}{\partial y} p_*(x,t) \Big|_s^T + \int_s^T \{\frac{d}{dt} \frac{\partial x}{\partial y} p_*(x,t)\}dt + \frac{\partial}{\partial y} L(x(T)) +$$

$$- \int_s^T [\frac{\partial}{\partial y} F(x,u_*(x,t),t)]p_*(x,t)dt =$$

$$= \quad p_*(y,s) - \frac{\partial x(T)}{\partial y} L_x(x(T)) + \frac{\partial}{\partial y} L(x(T)) = p_*(y,s).$$

So $J_y(s,y,u_*) = p_*(y,s)$ for small $|y|$ and $s \in [\tau,T]$. If we now replace s by t and y by x, and if we use lemma 2.2., we obtain

$$F_u(x,u_*(x,t),t)J_x(t,x,u_*) + G_u(x,u_*(x,t),t) = 0$$

for $|x|$ small and $t \in [\tau,T]$. So $u_*(x,t)$ satisfies (*). □

2.4 A method for calculating $u_*(x,t)$ and $J(t,x,u_*)$

In this section we shall use the following notation: if t(x) is a power series in x then the $k^{\underline{th}}$ order term will be denoted by $t^{(k)}(x)$ or $[t(x)]^{(k)}$.

$u_*(x,t)$ and $J_*(x,t) := J(t,x,u_*)$ can be expanded in power series:

$$u_*(x,t) = u_*^{(1)}(x,t) + u_*^{(2)}(x,t) + \ldots$$

$$J_*(x,t) = J_*^{(2)}(x,t) + J_*^{(3)}(x,t) + \ldots$$

We have seen that the lowest order terms are given by

$$u_*^{(1)}(x,t) = D_*(t)x$$

and

$$J_*^{(2)}(x,t) = x^T K_*(t)x,$$

where

$$D_*(t) = - R^{-1}(t)B^T(t)K_*(t)$$

and $K_*(t)$ is the solution of the Riccati equation. We indicate a method for computing the higher order terms analogous to the method followed in [1].

This method is based on the fact that $u_*(x,t)$ is a solution of the following two functional equations

$$\begin{cases} F(x,u_*(x,t),t)^T J_x(t,x,u_*) + J_t(t,x,u_*) + G(x,u_*(x,t),t) = 0 \\ F_u(x,u_*(x,t),t) J_x(t,x,u_*) + G_u(x,u_*(x,t),t) = 0 \end{cases}$$

In contrast to [1] where one has to solve linear equations, the problem defined here reduces to solving successively a set linear differential equations. We shall now give the result in the form of two equations :

$$\left.\begin{aligned} &(A_*(t)x)^T [J_*^{(m)}(x,t)]_x + [J_*^{(m)}(x,t)]_x = \\ &= -\sum_{k=3}^{m-1} [B(t)u_*^{(m-k+1)}(x,t)]^T [J_*^{(k)}(x,t)]_x + \\ &-\sum_{k=2}^{m-1} f^{(m-k+1)}(x,u_*(x,t),t)^T [J_*^{(k)}(x,t)]_x + \\ &-2\sum_{k=2}^{[\frac{m-1}{2}]} u_*^{(k)}(x,t)^T R(t) u_*^{(m-k)}(x,t) + \\ &- u_*^{(\frac{1}{2}m)}(x,t)^T R(t) u_*^{(\frac{1}{2}m)}(x,t) - g^{(m)}(x,u_*(x,t),t) \\ &\qquad (m = 3,4, \ldots) \end{aligned}\right\} \quad (A)$$

$$\left.\begin{aligned} &u_*^{(k)}(x,t) = -\tfrac{1}{2}R^{-1}(t)\{B^T(t) [J_*^{(k+1)}(x,t)]_x + \\ &+ \sum_{j=1}^{k-1} [f_u(x,u_*(x,t),t)]^{(j)} [J_*^{(k-j+1)}(x,t)]_x + \\ &+ [g_u(x,u_*(x,t),t)]^{(k)}\} \\ &\qquad (k = 2,3, \ldots) \end{aligned}\right\} \quad (B)$$

Here $A_*(t) := A(t) + B(t)D_*(t)$; $[k]$ denotes the integer part of k. Furthermore the term with $u_*^{(\frac{1}{2}m)}$ is to be omitted for odd values of m.

With the values $J_*^{(2)}(x,t)$ and $u_*^{(1)}(x,t)$ to start with, the higher order terms can be calculated from (A) and (B) in the sequence

$$J_*^{(3)}(x,t), u_*^{(2)}(x,t), J_*^{(4)}(x,t), u_*^{(3)}(x,t), \ldots$$

The sequence of terms $\{u_*^{(1)}, \ldots, u_*^{(m-2)}; J_*^{(2)}, \ldots, J_*^{(m-1)}\}$ determines $J_*^{(m)}$ in equation (A) by solving a partial differential equation with boundary value $J_*^{(m)}(x,T) = L^{(m)}(x)$. The sequence of terms $\{u_*^{(1)}, \ldots, u_*^{(k-1)}; J_*^{(2)}, \ldots, J_*^{(k+1)}\}$ determines $u_*^{(k)}$ in equation (B).

<u>Example</u>. $$\begin{cases} \dot{x} = x^3 + u, x(0) = x_0 \\ \min \int_0^T (x^2 + u^2)dt \end{cases}$$

Here $A(t) = 0, B(t) = 1, Q(t) = 1$ and $R(t) = 1$. Furthermore $f(x,u,t) = x^3$, $g(x,u,t) = 0$ and $L(x) = 0$. We have the Riccati equation

$$\begin{cases} \dot{K} + 1 - K^2 = 0 \\ K(T) = 0 \end{cases}$$

and the solution is given by $K_*(t) = \tanh(T - t)$. Hence

$$J_*^{(2)}(x,t) = x^T K_*(t) x = x^2 \tanh(T-t)$$

and

$$u_*^{(1)}(x,t) = - R^{-1}(t) B^T(t) K_*(t) x = -x \tanh(T-t)$$

Furthermore

$$A_*(t) = A(t) - B(t)R^{-1}(t)B^T(t)K_*(t) = -\tanh(T-t)$$

For m = 3 equation (A) reads as follows:

$$(-x \tanh(T-t))[J_*^{(3)}(x,t)]_x + [J_*^{(3)}(x,t)]_t = 0$$

If we set

$$J_*^{(3)}(x,t) = \alpha(t)x^3$$

then this equation becomes

$$-3x^3\alpha(t)\tanh(T-t) + \dot{\alpha}(t)x^3 = 0$$

or

$$\dot{\alpha}(t) - 3\alpha(t)\tanh(T-t) = 0$$

with the boundary value $\alpha(T) = 0$. This yields the solution $\alpha(t) = 0$ on $[\tau,T]$. So $J_*^{(3)}(x,t) = 0$ and equation (B) gives for $k = 2$: $u_*^{(2)}(x,t) = 0$

For $m = 4$ equation (A) becomes

$$(-x\tanh(T-t))[J_*^{(4)}(x,t)]_x + [J_*^{(4)}(x,t)]_t =$$

$$= -f^{(3)}(x,u_*,t)[J_*^{(2)}(x,t)]_x$$

Setting $J_*^{(4)}(x,t) = \alpha(t)x^4$ we have

$$\{-4\alpha(t)\tanh(T-t) + \dot{\alpha}(t)\}x^4 = -2\tanh(T-t)x^4$$

or

$$\dot{\alpha}(t) - 4\alpha(t)\tanh(T-t) + 2\tanh(T-t) = 0$$

with the boundary value $\alpha(T) = 0$. The solution of this differential equation is

$$\alpha(t) = \frac{1}{2} - \frac{1}{2}(\cosh(T-t))^{-4}$$

Thus

$$J_*^{(4)}(x,t) = \{\frac{1}{2} - \frac{1}{2}(\cosh(T-t))^{-4}\}x^4$$

Formula (B) gives for k = 3:

$$u_*^{(3)}(x,t) = -\frac{1}{2}R^{-1}(t)B^T(t)[J_*^{(4)}(x,t)]_x \quad ,$$

so

$$u_*^{(3)}(x,t) = \{ -1 + (\cosh(T-t))^{-4}\}x^3.$$

The higher order terms can be computed in a simular manner.

3. Fixed end-point problem

3.1. Assumptions

In this section we consider a problem similar to the problem discussed in section 2. The difference being that now we require the final value of the state to be zero : $x(T) = 0$. As a matter of course we can take now $L(x) = 0$. The basic assumptions made in section 2, remain. A new assumption is the controllability to the zero state of the linear system $\dot{x} = A(t)x + B(t)u$. Furthermore we restrict ourselves to feedback controls $u(x,t)$ with the following properties:

1. $u(x,t) = D(t)x + h(x,t)$. Here $D(t)$ is a continuous matrix function for $t \in [\tau,T)$.The function $h(x,t)$ contains the higher order terms in x and is continuous with respect to $t \in [\tau,T)$.Furthermore $h(x,t)$ is given as a power series in x which starts with second order terms and converges about the origin.
2. There exists a neighborhood N_u of the origin in $\mathbb{R}^n$ such that for $b \in N_u$ the solution $x(t,\tau,b)$ of (1.1) is defined on $[\tau,T]$ and in addition $x(T,\tau,b) = 0$.
3. $u(x(t,\tau,b),t)$ is a bounded function on $[\tau,T]$.

   We shall denote again the class of admissible feedback controls by Ω. If $u \in \Omega$ then it is clear that $u(x,t)$ has a singularity in $t = T$. Furthermore there exists for given $u \in \Omega$, $s \in [\tau,T)$, a neighborhood $N_{u,s}$ of the origin in $\mathbb{R}^n$ with the property that, if $c \in N_{u,s}$, the solution of $\dot{x} = F(x,u(x,t),t), x(s) = c$, is defined on $[s,T]$ and $x(T) = 0$. It is evident that

$$(3.1) \qquad N_{u,s} := \{x(s,\tau,b) \mid b \in N_u\}$$

represents such a neighborhood !

3.2. Statement of the main results

Theorem 3.1. (Main Theorem)
For the control process in $\mathbb{R}^n$

$$\dot{x} = F(x,u,t), x(\tau) = b, x(T) = 0$$

there exists a unique optimal feedback control $u_* \in \Omega$ *which minimizes the integral*

$$J(\tau,b,u) = \int_{\tau}^{T} G(x,u,t)dt$$

for all initial states b *in a neighborhood of the origin in* $\mathbb{R}^n$. *This feedback control is the unique solution of the functional equation*

$$(*) \qquad F_u(x,u_*(x,t),t)J_x(t,x,u_*) + G_u(x,u_*(x,t),t) = 0$$

for $t \in [\tau,T)$ *and small* $|x|$. *Furthermore*

$$u_*(x,t) = D_*(t)x + h_*(x,t)$$

and

$$J(\tau,b,u_*) = b^T K_*(\tau)b + j_*(\tau,b),$$

where the matrix functions $D_*(t)$ *and* $K_*(t)$ *are defined on* $[\tau,T)$ *and depend only on the truncated problem.*

The truncated problem is the case that $f(x,u,t) = 0$ and $g(x,u,t) = 0$. R.W. Brockett has proved in [2]that under our hypothesis an optimal control exists. One can easily show that his results can be written in the following form:

(3.2) $$u_*(x,t) = D_*(t)x$$

where

(3.3) $$D_*(t) = -R^{-1}(t)B^T(t)K_*(t).$$

Here $K_*(t)$ satisfies the Riccati equation on $[\tau,T)$:

$$\dot{K}(t) + Q(t) + K(t)A(t) + A^T(t)K(t) - K(t)B(t)R^{-1}(t)B^T(t)K(t) = 0$$

If $W_*(t)$ satisfies the dual Riccati equation

$$\begin{cases} \dot{W}(t) + B(t)R^{-1}(t)B^T(t) - W(t)A^T(t) - A(t)W(t) - W(t)Q(t)W(t) = 0 \\ W(T) = 0 \end{cases}$$

on $[\tau,T]$, then we have $K_*^{-1}(t) = W_*(t)$ for $t \in [\tau,T)$. Finally

$$J(\tau,b,u_*) = b^TK_*(\tau)b$$

3.3. Construction of the optimal feedback control

Lemma 3.1. *For each feedback control* $u \in \Omega$, $u(x,t) = D(t)x + h(x,t)$, *we have the property*

$$J(\tau,b,u) = b^T\hat{K}(\tau)b + j(\tau,b)$$

for $b \in N_u$. *The matrix function* $\hat{K}(\tau)$ *depends only on the truncated problem. Furthermore the functional equation*

$$F(x,u(x,t),t)^TJ_x(t,x,u) + J_t(t,x,u) + G(x,u(x,t),t) = 0$$

holds for $t \in [\tau,T)$ *and* $x \in N_{u,t}$.

Proof. The proof is analogous to the proof of lemma 2.1. Here we have $\Phi(T) = 0$. One can show that the solution of the differential equation $\dot{x} = F(x,u(x,t),t)$ is of the form $x(t) = \Phi(t)\Phi^{-1}(\tau)b + \mathcal{O}(|b|^2)$, again uniformly for $t \in [\tau,T]$. Note that $\hat{K}(t)$ may have a singularity in $t = T$. □

Lemma 3.2. *The exists a unique solution* $W_*(t)$ *on* $[\tau,T]$ *to the matrix differential equation (dual Riccati equation)*

$$\begin{cases} \dot{W}(t) + B(t)R^{-1}(t)B^T(t) - W(t)A^T(t) - A(t)W(t) - W(t)Q(t)W(t) = 0 \\ W(T) = 0. \end{cases}$$

The property $W_*(t) > 0$ *holds on* $[\tau,T)$. *If* $K_*(t) := W_*^{-1}(t)$ *on* $[\tau,T]$ *then* $K_*(t)$ *satisfies the Riccati equation*

$$\dot{K}(t) + Q(t) + K(t)A(t) + A^T(t)K(t) - K(t)B(t)R^{-1}(t)B^T(t)K(t) = 0$$

on $[\tau,T)$.

Proof. This lemma is a consequence of the analysis of R.W. Brockett in [2] section 3.22. □

Lemma 3.3. *Suppose there exists a feedback control* $u_* \in \Omega$, $u_*(x,t) = D_*(t)x + h_*(x,t)$, *which satisfies the functional equation*

$$(*) \qquad F_u(x,u_*(x,t),t)J_x(t,x,u_*) + G_u(x,u_*(x,t),t) = 0$$

for $t \in [\tau,T)$ *and* $x \in N_{u_*,t}$. *Then* u_* *is the unique optimal feedback control. Furthermore*

$$D_*(t) = -R^{-1}(t)B^T(t)K_*(t)$$

and

$$J(\tau,b,u_*) = b^TK_*(\tau)b + j_*(\tau,b),$$

where $K_*(t)$ *is defined in lemma* 3.2. *The function* $j_*(\tau,b)$ *contains the higher order terms in* b.

Proof. The method to proof that u_* represents the unique optimal feedback control is analogous to the method followed in lemma 2.4. Now we can choose

$|u_*(x_*(t),t) \quad | \le \epsilon$ and $|u_1(x_1(t),t) \quad | \le \epsilon$ because we have assumed that $u_*(x_*(t),t)$ and $u_1(x_1(t),t)$ are bounded functions on $[\tau,T]$. By lemma 2.2. we have

$$u_*(x,t) = -\frac{1}{2}R^{-1}(t)B^T(t)J_x(t,x,u_*) + \mathcal{O}(|x|^2)$$

and in lemma 3.1. we have

$$J_x(t,x,u_*) = 2\hat{K}(t)x + \mathcal{O}|x|^2)$$

for $t \in [\tau,T]$ and $x \in N_{u_*,t}$. So

$$(3.4) \qquad u_*(x,t) = -R^{-1}(t)B^T(t)\hat{K}(t)x + \mathcal{O}(|x|^2)$$

In the truncated case the corresponding formula is:

$$u_*(x,t) = -R^{-1}(t)B^T(t)\hat{K}(t)x.$$

Comparing this result with (3.2) and (3.3) it follows that $\hat{K}(t) = K_*(t)$ on $[\tau,T)$, where $K_*(t)$ is defined in lemma 3.2.

Conclusion:

$$u_*(x,t) = -R^{-1}(t)B^T(t)K_*(t)x + \mathcal{O}(|x|^2)$$

and

$$J(\tau,b,u_*) = b^TK_*(\tau)b + \mathcal{O}(|b|^3)$$ □

Before proving the main theorem we consider again the Hamiltonian system in $\mathbb{R}^{2n}$:

$$(3.5) \qquad \begin{cases} \dot{x} = F(x,u_*(x,p,t),t) \\ \dot{p} = -\{F_x(x,u_*(x,p,t),t)p + G_x(x,u_*(x,p,t),t)\} \end{cases}$$

with the boundary values

$$\begin{cases} x(\tau) = b \\ x(T) = 0 \end{cases}$$

Here $u_*(x,p,t)$ is defined in lemma 2.2.

Lemma 3.4. *For small* $|b|$ *system* (3.5) *has a solution* $(x_*(t),\ p_*(t))$ *with the property*

$$x_*(t) = \frac{1}{2}W_*(t)p_*(t) + \mathcal{O}(|p_*(t)|^2)$$

for $t \in [\tau,T]$. *Furthermore* $p_*(t)$ *is a bounded function on* $[\tau,T]$.

Proof. The Hamiltonian system has the form

$$\begin{pmatrix}\dot{x}\\ \dot{p}\end{pmatrix} \begin{bmatrix} A(t) & -\frac{1}{2}B(t)R^{-1}(t)B^T(t) \\ -2Q(t) & -A^T(t) \end{bmatrix} \begin{pmatrix}x\\ p\end{pmatrix} + h(x,p,t)$$

It can easily be verified that the linear system (i.e. the case that $h(x,p,t) = 0$) has for each $b \in \mathbb{R}^n$ a solution of the form $x_*(t) = \frac{1}{2}W_*(t)p_*(t)$. Analogous to the proof of lemma 2.5. we shall use the implicit function theorem to proof that the nonlinear system has a solution of the desired form. We need again a property which we shall derive from the solvability of the linear system. So consider again the linear Hamiltonian system as a final value problem. The solution can be written as

$$x(t,x_T,p_T) = \Theta_{11}(t,T)x_T + \Theta_{12}(t,T)p_T$$
$$p(t,x_T,p_T) = \Theta_{21}(t,T)x_T + \Theta_{22}(t,T)p_T$$

We have seen that for each $b \in \mathbb{R}^n$ there exists a solution on $[\tau,T]$ with $x(\tau) = b$ and $x(T) = 0$. So

$$\forall_{b \in \mathbb{R}^n}\ \exists_{p_T \in \mathbb{R}^n} : \Theta_{12}(\tau,T)p_T = b$$

Hence the matrix $\Theta_{12}(\tau,T)$ is regular. Now consider the nonlinear system as a final value problem. The solution has the form

$$x(t,x_T,p_T) = \Theta_{11}(t,T)x_T + \Theta_{12}(t,T)p_T + \mathcal{O}(|\binom{x_T}{p_T}|^2)$$

$$p(t,x_T,p_T) = \Theta_{21}(t,T)x_T + \Theta_{22}(t,T)p_T + \mathcal{O}(|\binom{x_T}{p_T}|^2)$$

The question is : does there exist for arbitrary $b \in \mathbb{R}^n$, $|b|$ small, a vector $p_T \in \mathbb{R}^n$ such that $x(\tau,0,p_T) = b$? Again, the implicit function theorem can help us. Define

$$F(b,p_T) := x(\tau,0,p_T) - b$$

Then $F(0,0) = 0$ and $Fp_T(0,0) = \Theta_{12}(\tau,T)$. So $Fp_T(0,0)$ is regular, and there exists a neighborhood Ω of the origin in $\mathbb{R}^n$ and a function $p_T: \Omega \to \mathbb{R}^n$ such that

(i) $\tilde{p}_T(0) = 0$
(ii) $F(b,\tilde{p}_T(b)) = 0$ for $b \in \Omega$.

Hence $x(\tau,0,\tilde{p}_T(b)) = b$ for $b \in \Omega$. Thus the Hamiltonian system (3.5) has a solution on $[\tau,T]$ for small $|b|$. From the considerations of the linear system we have

$$x_*(t) = \frac{1}{2}W_*(t)p_*(t) + \mathcal{O}(|p_*(t)|^2)$$

for $t \in [\tau,T]$. The boundedness of $p_*(t)$ on $[\tau,T]$ is a consequence of the continuity of the right hand side of (3.5) on $[\tau,T]$. □

Proof of the main theorem. It is sufficient to establish the existence of a feedback control $u_* \in \Omega$ which satisfies the functional equation (*) for $t \in [\tau,T)$ and small $|x|$. Define

$$u_*(x,t) := u_*(x,p_*(x,t),t)$$

where $p_*(x,t)$ represents the solution of (3.5) and $u_*(x,p,t)$ such as defined in lemma 2.2. Hence

$$u_*(x,t) = -\frac{1}{2}R^{-1}(t)B^T(t)p_*(x,t) + \mathcal{O}(|x|^2) =$$

$$= -R^{-1}(t)B^T(t)K_*(t)x + \mathcal{O}(x^2)$$

for $t \in [\tau,T]$. In lemma 3.4. we have seen that the solution of $\dot{x} = F(x,u_*(x,t),t), x(\tau) = b$ exists on $[\tau,T]$ for small $|b|$ and furthermore $x(T) = 0$. Because $p_*(t)$ is bounded on $[\tau,T]$ it follows that $u_*(x_*(t),t)$ is bounded on $[\tau,T]$. Hence we can conclude that $u_* \in \Omega$. An analogous argument as in the previous section shows us that u_* satisfies the functional equation (*). □

3.4. A method for calculating $u_*(x,t)$.

In chapter 1 we used the fact that the optimal feedback control $u_*(x,t)$ is a solution of the following two equations:

$$\begin{cases} F(x,u_*(x,t),t)^T J_x(t,x,u_*) + J_t(t,x,u_*) + G(x,u_*(x,t),t) = 0 \\ F_u(x,u_*(x,t),t) J_x(t,x,u_*) + G_u(x,u_*(x,t),t) = 0 \end{cases}$$

It turned out to be possible to calculate $u_*(x,t)$ from these equations using the boundary value $J(T,x,u_*) = L(x)$ to solve the partial differential equation. This method fails here. It is true that the optimal feedback control is again a solution of the two functional equations but we cannot solve the partial differential equation because the only information we have about J is that $J(T,0,u_*) = 0$ and this is not sufficient. This is a reason for us to follow a different method here. Consider the following free end-point problem

$$\begin{cases} \dot{p} = \tilde{F}(p,y,t), p(\tau) = c \\ \min \int_\tau^T \tilde{G}(p,y,t)dt \end{cases}$$

Note that p plays the role of state vector and y plays the role of control vector. The functions $\tilde{F}$ and $\tilde{G}$ are defined as follows

$$\tilde{F}(p,y,t) := -\{F_x(y,u_*(y,p,t),t)p + G_x(y,u_*(y,p,t),t)\}$$

$$\tilde{G}(p,y,t) := [F_x(y,u_*(y,p,t),t)p + G_x(y,u_*(y,p,t),t)]^T x + $$

$$- \{F(y,u_*(y,p,t),t)^T p + G(y,u_*(y,p,t),t)\}$$

Here $u_*(x,p,t)$ is defined in lemma 2.2. We shall call this control system the <u>dual system</u>. It is easy to verify that

$$\tilde{F}(p,y,t) = -A^T(t)p - 2Q(t)y + \tilde{f}(p,y,t)$$

and

$$\tilde{G}(p,y,t) = \frac{1}{4}p^T B(t)R^{-1}(t)B^T(t)p + y^T Q(t)y + \tilde{g}(p,y,t).$$

Here the functions $\tilde{f}$ and $\tilde{g}$ contain the higher order terms in y and p. It is clear that the dual system can be solved by the method described in section 2, provided that $Q(t) > 0$ on $[\tau,T]$. However, what is the connection with the original system? The two systems have one important common property; namely they both generate the same Hamiltonian system:

$$\begin{cases} \dot{x} = F(x,u_*(x,p,t),t) \\ \dot{p} = -\{F_x(x,u_*(x,p,t),t)p + G_x(x,u_*(x,p,t),t)\} . \end{cases}$$

The boundary values however are different. In the original case we have $x(\tau) = b$, $x(T) = 0$ and in the dual case $p(\tau) = c$, $x(T) = 0$. Namely, if $y_*(p,x,t)$ here plays the role of $u_*(x,p,t)$ in lemma 2.2. then it is easy to verify that $y_*(p,x,t) = x$ and furthermore $-\{\tilde{F}_p(p,y_*(p,x,t),t)x + \tilde{G}_p(p,y_*(p,x,t),t)\} = F(x,u_*(x,p,t),t)$. This argument enables us to construct the solution of the original system from the solution of the dual system. If $y_*(p,t)$ denotes the optimal feedback control with respect to the dual problem then it follows that $x_*(p,t) = y_*(p,t)$ is the solution of the Hamiltonian system. From this we can calculate $p_*(x,t)$ by the regular transformation $p_*(x,t) = 2K_*(t)x_*(t) + \mathcal{O}(|x_*(t)|^2)$ (see lemma 3.4.) Finally we can calculate the optimal feedback control with respect to the original system by $u_*(x,t) = u_*(x,p_*(x,t),t)$. In the case that $Q(t)$ is not positive definite but only positive semi definite, it does not seem to be possible to introduce a dual system with the properties sketched above.

<u>Example</u>

$$\begin{cases} \dot{x} = x^3 + u, x(0) = x_0, x(T) = 0 \\ \min \int_0^T (x^2 + u^2)dt \end{cases}$$

Here $A(t) = 0, B(t) = 1$, $Q(t) = 1$ and $R(t) = 1$. Furthermore $f(x,u,t) = x^3$ and $g(x,u,t) = 0$. The linear system $\dot{x} = u$ is controllable and the condition $Q > 0$ holds. Hence we can use the method described above.
The equation $F_u(x,u,t)p + G_u(x,u,t) = 0$ gives $u_*(x,p,t) = -\frac{1}{2}p$, so the dual system has the following form

$$\begin{cases} \dot{p} = -2y - 3y^2 p, p(0) = p. \\ \min \int_0^T (\frac{1}{4}p^2 + y^2 + 2y^3 p)dt \end{cases}$$

The method of chapter 1 gives the result

$$y_*(p,t) = \frac{1}{2}p \tanh(T-t) - \frac{1}{8}p^3 \tanh^4(T-t) + \ldots$$

Hence

$$x_*(p,t) = \frac{1}{2}p \tanh(T-t) - \frac{1}{8}p^3 \tanh^4(T-t) + \ldots$$

and it follows that

$$p_*(x,t) = 2x \operatorname{cotanh}(T-t) + 2x^3 + \ldots$$

Finally we find

$$u_*(x,t) = -\frac{1}{2}p_*(x,t) = - x \operatorname{cotanh}(T-t) - x^3 + \ldots$$

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