Optimal Reinsurance with Regulatory Initial Capital and Default Risk

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1. Introduction

- What is a one-period reinsurance stochastic model?

In a one-period reinsurance model, we assume that the underlying (aggregate) loss faced by an insurer in a fixed time period is a non-negative random variable X with survival function $S_X(x) = \Pr\{X > x\} = 1 - F_X(x)$.

In a reinsurance contract, a reinsurer agrees to pay the part of the loss X, denoted by $I(X)$, to the insurer at the end of the contract term, while the insurer will pay a reinsurance premium, denoted by P_I, to the reinsurer when the contract is signed, where the function $I(x)$ is called ceded loss function. Thus, under the reinsurance contract I, the retained loss for the insurer is $R(X) = X - I(X)$, where the function $R(x) = x - I(x)$ is called retained loss function.

- What are the reasonable assumptions on a reinsurance contract I?

In the reinsurance study, we often assume that a feasible reinsurance contract I should satisfy the following two conditions:

(1) $I : [0, \infty) \to [0, \infty)$ such that $I(0) = 0$ and I is non-decreasing;

(2) $I(y) - I(x) \leq y - x$, for any $0 \leq x \leq y$.

The first condition means that the larger is the incurred loss by an insurer, the larger is the covered loss by a reinsurer. The second condition implies that the growth rate of the covered loss by a reinsurer should not be faster than the growth rate of the underlying loss faced by an insurer.

Mathematically, these two conditions imply that both $I(x)$ and $R(x) = x - I(x)$ are continuous and non-decreasing on $[0, \infty)$. We denote the set of all feasible reinsurance contracts satisfying conditions (1) and (2) by $\mathcal{I}$.

- What are the common reinsurance contracts?

(1) Proportional or quota-share reinsurance with $I(X) = aX$, $0 \leq a \leq 1$.

(2) Stop-loss reinsurance with $I(X) = (X - d)_+$, $d \geq 0$.

(3) Limited stop-loss reinsurance with $I(X) = (X - d)_+ \wedge m,\ d \geq 0,\ m \geq 0$.

(4) The combination of the quota-share and limited stop-loss reinsurance with $I(X) = a(X - d)_+ \wedge m,\ 0 \leq a \leq 1,\ d \geq 0,\ m \geq 0$.

Here $(a)_+ = \max\{a, 0\},\ a \wedge b = \min\{a, b\}$. In addition, we denote $a \vee b = \max\{a, b\}$ and interpret "increasing" as "non-decreasing", while "decreasing" as "non-increasing".

- What is an optimal reinsurance contract?

An optimal reinsurance contract is a ceded loss functions I^* that is the best one in some sense or under a certain optimization criterion.

- What are commonly used optimization criteria in the optimal reinsurance study?

  Two of the commonly used optimization criteria are

  (1) maximize the expected utility of an insurer's terminal wealth,

  (2) minimize the risk measure of an insurer's total retained risk.

- What is a risk measure?
  A risk measure is a mapping from the set of risks or loss random variables to real numbers.

- What risk measures are good or reasonable?

  It is a controversial issue. In insurance and finance, many people think a good risk measure should be coherent.

  A risk measure is said to be coherent if it satisfies the following four conditions:

(1) **Translation Invariance**

(2) **Subadditivity**

(3) **Positive Homogeneity**

(4) **Monotonicity**

- What are good risk measures in the sense of statistics or estimation?

In statistics, many people think that a good risk measure should be able to be estimated easily from historical data such that the verification and comparison competing estimation procedures are possible. Such risk measures are called elicitable.

- Important examples of coherent and/or elicitable risk measures:

1. **(Value-at-Risk)** The Value-at-Risk (VaR) of a risk X at risk level $\alpha \in (0, 1)$, denoted by $VaR_\alpha(X)$, is

$$VaR_\alpha(X) = \inf\{x \in R : \Pr(X > x) \leq \alpha\}.$$

2. **(Conditional Tail Expectation)** For a risk X with $E(|X|) < \infty$, the conditional tail expectation (CTE) of a loss X at risk level $0 < \alpha < 1$, denoted by $CTE_\alpha(X)$, is defined as

$$CTE_\alpha(X) = E(X|X > VaR_\alpha(X))$$

provided that $P(X > VaR_\alpha(X)) > 0$.

3. **(TVaR or ES)** For a risk X with $E(|X|) < \infty$. The Tail Value-at-Risk (TVaR) or the expected shortfall (ES) at risk level $0 < \alpha < 1$ is defined by

$$ES_\alpha(X) = TVaR_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha VaR_q(X)dq.$$

4. **Expectile:** For a risk X with $E(X^2) < \infty$, the expectile of X at risk level $\alpha \in (0,1)$ is defines as

$$e_\alpha(X) = \text{argmin}_{x \in (-\infty,\infty)} E[\alpha((X-x)_+)^2 + (1-\alpha)((x-X)_+)^2]$$

or equivalently $e_\alpha(X)$ is the unique solution to the equation of

$$\alpha\, E[(X-x)_+] = (1-\alpha)E[(x-X)_+].$$

It is known that

– VaR is elicitable but not coherent.

– TVaR is coherent but not elicitable.

– Expectile is elicitable and coherent.

- Some recent references on optimal reinsurance under different risk measures include
  - Cai and Tan (2007),
  - Cai et al. (2008),
  - Balbás et al. (2009),
  - Cheung (2010),
  - Chi and Tan (2011),
  - Chi (2012),
  - Asimit, Badescu and Verdoncj (2013),

- Cai and Weng (2014).

- In addition, the one-period reinsurance model with one loss variable has been extended to models with more insurance lines of business or to models that discuss the interests of both insurers and reinsurers. Recent references on these issues can be found in
  - Hürlimann (2011),
  - Cai and Wei (2012),
  - Cai et al. (2013),
  - Cheung et al. (2013),
  - and references therein.

- In most studies on optimal reinsurance, one assumes that a reinsurer will pay the promised loss $I(X)$ regardless of its solvency or equivalently, one ignores the potential default by a reinsurer.

- However, in a reinsurance contract I, a reinsurer may fail to pay the promised amount $I(X)$ or a reinsurer may default due to different reasons.

- One of the main reasons could be that the promised amount $I(X)$ exceeds the reinsurer's solvency.

- The larger is the initial reserve of a reinsurer, the smaller is the likelihood that default will occur.

- This is why the initial capital of a seller (reinsurer) of a reinsurance contract should meet some requirements by regulations to reduce default risk.

- Recently, default risks in reinsurance designs or other related studies have been discussed in
  - Asimit, Badescu, and Cheung (2013),
  - Bernard and Ludkovski (2012),
  - Burren (2013),
  - Cummins et al. (2002),
  - Dana and Scarsini (2007),
  - Menegatti (2009),
  - and references therein.

- In the recent references for reinsurance designs with default risks, a constant initial capital or reserve for a reinsurer is assumed regardless of how large a reinsurer's promised amount $I(X)$ is.

- A reasonable requirement on a reinsurer could be that the larger is the promised indemnity of a reinsurer, the larger the initial reserve of a reinsurer should be.

- In this paper, we assume that the initial capital or reserve of a seller (reinsurer) of a reinsurance contract I is determined through regulation by the value-at-risk (VaR) of its promised indemnity $I(X)$, and denote the initial capital of the reinsurer by $\omega_I = \mathrm{VaR}_\alpha(I(X))$, where $0 < \alpha < 1$ is called the risk level.

- We assume that the reinsurer charges a reinsurance premium P_I based on the promised indemnity $I(X)$. The insurer is aware of the potential default by the reinsurer but the worst case for the insurer is that the reinsurer only pays $\omega_I + P_I$ if $I(X) > \omega_I + P_I$.

- When the insurer is seeking for optimal reinsurance strategies and taking account of the potential default by the reinsurer, the insurer assumes the worst indemnity $I(X) \wedge (\omega_I + P_I)$ from the reinsurer.

- Indeed, when $\omega_I = \mathrm{VaR}_\alpha(I(X))$, we know $\Pr\{I(X) > \omega_I + P_I\} \leq \alpha$ or the probability of default by the reinsurer is not greater than the value α, which could be an acceptable risk level for the insurer.

- Hence, under the proposed reinsurance model, the total retained risk or cost of the insurer is $X - I(X) \wedge (\omega_I + P_I) + P_I$ and the insurer's terminal wealth is $w_0 - X + I(X) \wedge (\omega_I + P_I) - P_I$, where w_0 is the initial capital of the insurer.

- In our proposed model, we emphasize that
  - the initial reserve of the reinsurer is determined by the VaR of the reinsurer's promised indemnity due to regulatory requirements,
  - the insurer believes that the guaranteed or minimum available capital of the reinsurer at the end of the contract is the initial reserve plus the reinsurance premium,
  - and the probability of default by the reinsurer is not greater than the risk level of the VaR.

- In the first part of the paper, we assume that the insurer wants to determine an optimal reinsurance strategy I^* that maximizes the expected utility of its terminal wealth of $w_0 - X + I(X) \wedge (\omega_I + P_I) - P_I$ under an increasing concave utility function v. That is, we study the following optimization problem:

$$\max_{I \in \mathcal{I}} \mathbb{E}\left[v\left(w_0 - X + I(X) \wedge (\omega_I + P_I) - P_I\right)\right] \tag{1}$$
$$\text{such that } P_I = (1+\theta)\mathbb{E}[I(X)] = p,$$

where $0 < p \leq (1+\theta)\mathbb{E}(X)$ is a given reinsurance premium budget for the insurer.

- This optimal reinsurance problem can be viewed as the extension of the classical optimal reinsurance problem without default risk, which was first studied by Arrow (1963) and Borch (1960). As illustrated later in the paper, as $\alpha \to 0$, Problem (1) is reduced to the classical optimal reinsurance problem without default risk studied by Arrow (1963) and Borch (1960). We can also recover the solutions of Arrow (1963) and Borch (1960) from our solution to Problem (1).

- In the second part of the paper, we assume that the insurer wants to use VaR at a risk level $0 < \beta < 1$ to control its total retained risk of $X - I(X) \wedge (\omega_I + P_I) + P_I$ and then seeks an optimal reinsurance strategy I^* that minimizes this VaR. That is, we consider the following optimization problem:

$$\min_{I \in \mathcal{I}} \mathrm{VaR}_\beta \left(X - I(X) \wedge (\omega_I + P_I) + P_I \right). \tag{2}$$

- This problem is an extension of recent studies on optimal reinsurance under risk measures without default risk such as Balbás et al. (2009), Asimit, Badescu and Verdoncj (2013), Cai et al. (2008), Cheung (2010), Chi (2012), Chi and Tan (2011), and references therein. In particular, and as will be shown later, when $\alpha \leq \beta$, Problem (2) reduces to the problem without default risk, which was studied by Cheung et al. (2014).

- As illustrated in the paper, the solutions to Problems (1) and (2) are more complicated than those without default risk. Furthermore, the optimal reinsurance strategies in the presence of regulatory initial capital and the counterparty default risk are different both from the optimal reinsurance strategies in the absence of the counterparty default risk and from the optimal reinsurance strategies in the presence of the counterparty default risk but without the regulatory initial capital.

- To avoid tedious discussions and arguments, in this paper, we simply assume that the survival function $S_X(x)$ of the underlying loss random variable X is continuous and strictly decreasing on $(0, \infty)$ with $0 < S_X(0) \leq 1$ or equivalently $0 \leq F_X(0) < 1$.

- Furthermore, we assume that $P_I = (1+\theta)\mathbb{E}[I(X)]$, i.e., the reinsurance premium is determined by the expected value principle, where $\theta > 0$.

2 Optimal reinsurance maximizing the expected utility of an insurer's terminal wealth

In this section, we study Problem (1). First, we point out that by taking $u(x) = -v(w_0 - p - x)$, Problem (1) is equivalent to the following minimization problem:

$$\min_{I\in\mathcal{I}} \mathbb{E}\Big[u\Big(X - I(X) \wedge (\omega_I + P_I)\Big)\Big] \tag{3}$$
$$\text{such that } P_I = (1+\theta)\mathbb{E}[I(X)] = p,$$

where u is an increasing convex function.

Throughout this section, we assume $\mathbb{E}|u^{(k)}(X)| < \infty$ for $k = 0, 1, 2$ and denote the density function of $F_X(x)$ on $(0, \infty)$ by $f_X(x)$ or $f_X(x) = F'_X(x) = -S'_X(x)$ for $x \in (0, \infty)$.

For any fixed premium budget $0 < p \leq (1+\theta)\mathbb{E}[X]$, we denote the set of all feasible contracts with the given reinsurance premium p by

$$\mathcal{I}_p = \{I \in \mathcal{I} : P_I = (1+\theta)\mathbb{E}[I(X)] = p\}.$$

Note that if $p = (1+\theta)\mathbb{E}[X]$, then $\mathcal{I}_p = \{I(x) \equiv x\}$, which contains only one reinsurance contract $I(x) \equiv x$, and thus Problem (3) reduces to the trivial case. Hence, throughout this section, we assume $p \in \big(0,\ (1+\theta)\mathbb{E}[X]\big)$.

Then Problem (3) can be written as

$$\min_{I\in\mathcal{I}_p}\mathbb{E}\Big[u\Big(X-I(X)\wedge(\omega_I+P_I)\Big)\Big]=\min_{I\in\mathcal{I}_p}H(I), \tag{4}$$

where

$$H(I)=\mathbb{E}\Big[u\Big(X-I(X)\wedge(\omega_I+P_I)\Big)\Big].$$

To solve the infinite-dimensional optimization Problem (4), we first show that for any given reinsurance contract $I\in\mathcal{I}_p$, there exists a contract $k_I\in\mathcal{I}_p$ such that $H(k_I)\leq H(I)$ and k_I is determined by four variables. Thus, we can reduce the infinite-dimensional optimization Problem (4) to a finite-dimensional optimization problem.

The following theorem shows that for any given reinsurance contract $I\in\mathcal{I}_p$, there exists a contract $k_I\in\mathcal{I}_p$ such that $H(k_I)\leq H(I)$.

Theorem 1. *Denote* $a = \mathrm{VaR}_\alpha(X)$*. For any* $I \in \mathcal{I}_p$*, there exists* $k_I \in \mathcal{I}_p$ *such that* $H(k_I) \leq H(I)$ *and* k_I *is defined as*

$$\begin{aligned} k_I(x) &= (x - d_1)^+ - (x - (d_1 + I(a)))^+ + (x - d_2)^+ \\ &\quad -(x - (d_2 + p))^+ + (x - d_3)^+ \\ &= \begin{cases} 0, & \textit{for} \quad 0 \leq x < d_1, \\ x - d_1, & \textit{for} \quad d_1 \leq x < d_1 + I(a), \\ I(a), & \textit{for} \quad d_1 + I(a) \leq x < d_2, \\ I(a) + x - d_2, & \textit{for} \quad d_2 \leq x < d_2 + p, \\ I(a) + p, & \textit{for} \quad d_2 + p \leq x < d_3, \\ I(a) + p + x - d_3, & \textit{for} \quad d_3 \leq x, \end{cases} \end{aligned} \tag{5}$$

for some (d_1, d_2, d_3) *satisfies* $0 \leq d_1 \leq d_1 + I(a) \leq a \leq d_2 < d_2 + p \leq d_3 \leq \infty$*.*

Remark 1. *We point out that for any $I \in \mathcal{I}$, I is continuous and non-decreasing. Thus, $\omega_I = \mathrm{VaR}_\alpha(I(X)) = I(\mathrm{VaR}_\alpha(X)) = I(a)$. As $\alpha \to 0$ or $a \to \infty$, we have $I(a) \to I(\infty)$ and hence $I \wedge (\omega_I + P_I) = I$. In other words, as $\alpha \to 0$ or $a \to \infty$, Problem (3) is reduced to the following classical problem without the default risk:*

$$\min_{I \in \mathcal{I}} \mathbb{E}\left[u\left(X - I(X)\right)\right] \tag{6}$$
$$\textit{such that } P_I = (1+\theta)\mathbb{E}[I(X)] = p.$$

The optimal solution to Problem 6, given in Arrow (1963) or Borch (1960), is a stop-loss reinsurance $I_0^*(x) = (x - d^*)^+$, where d^* is uniquely determined by the premium condition. This classical result can also be recovered from our Theorem 1.

In the rest of this paper, we assume $0 < \alpha < S_X(0)$ and thus $0 < a < \infty$. Otherwise, all the VaRs considered in the paper at the risk level

$\alpha \in [S_X(0), 1)$ are equal to zero, which are trivial cases.

Theorem 1 reduces the infinite-dimensional optimization Problem (4) to a finite-dimensional optimization problem. To see that, we denote

$$\mathcal{I}_{p,0} = \{I \in \mathcal{I}_p \text{ and } I \text{ has the expression } (5)\}\,.$$

Then, thanks to Theorem 1, we see that Problem (4) is equivalent to the following minimization problem

$$\min_{I \in \mathcal{I}_{p,0}} \mathbb{E}\Big[u\Big(X - I(X) \wedge (\omega_I + P_I)\Big)\Big] = \min_{I \in \mathcal{I}_{p,0}} H(I). \tag{7}$$

It is still not easy to solve Problem (7) since it involves four variables of d_1, d_2, d_3 and $I(a)$. To solve Problem (7), we first need to discuss the properties of the set $\mathcal{I}_{p,0}$.

For any given $\xi \in [0, a]$, define contract $I_{0,\xi} \in \mathcal{I}$ as

$$
\begin{aligned}
I_{0,\xi}(x) &= (x-a+\xi)^+ - (x-a)^+ \\
&= \begin{cases} 0, & \text{for} \quad 0 \le x < a-\xi, \\ x-a+\xi, & \text{for} \quad a-\xi \le x < a, \\ \xi, & \text{for} \quad a \le x < \infty, \end{cases}
\end{aligned} \tag{8}
$$

and denote the reinsurance premium based on $I_{0,\xi}$ by $p_{0,\xi}$ or

$$
p_{0,\xi} = (1+\theta)\mathbb{E}\left[I_{0,\xi}(X)\right] = (1+\theta)\int_{a-\xi}^{a} S_X(x)\underset{\cdot}{\mathsf{x}}. \tag{9}
$$

Furthermore, define contract $I_{M,\xi} \in \mathcal{I}$ as

$$\begin{aligned} I_{M,\xi}(x) &= x - (x-\xi)^+ + (x-a)^+ \\ &= \begin{cases} x, & \text{for} \quad 0 \le x < \xi, \\ \xi, & \text{for} \quad \xi \le x < a, \\ \xi + x - a, & \text{for} \quad a \le x < a+p, \end{cases} \end{aligned} \tag{10}$$

and denote the reinsurance premium based on $I_{M,\xi}$ by $p_{M,\xi}$ or

$$p_{M,\xi} = (1+\theta)\mathbb{E}\left[I_{M,\xi}(X)\right] = (1+\theta)\left(\int_0^{\xi} + \int_a^{\infty}\right) S_X(x)\underset{.}{x}. \tag{11}$$

We denote

$$\xi_0 = \inf\left\{\xi \in [0,a] \text{ such that } p_{M,\xi} \ge p\right\} \tag{12}$$

and

$$\xi_M = \sup\left\{\xi \in [0, a] \text{ such that } p_{0,\xi} \le p\right\}. \tag{13}$$

Thus, we can write $\mathcal{I}_{p,0}$ as the union of disjoint non-empty sets, namely

$$\mathcal{I}_{p,0} = \bigcup_{\xi_0 \le \xi \le \xi_M} \left\{I \in \mathcal{I}_{p,0} : I(a) = \xi\right\}.$$

It will be proved in Theorem 2 that Problem (7) is equivalent to the following two-step minimization problem:

$$\min_{0 \le \xi \le a} \min_{I \in \mathcal{I}_{p,0},\ I(a)=\xi} H(I) = \min_{\xi_0 \le \xi \le \xi_M} \min_{I \in \mathcal{I}_{p,0},\ I(a)=\xi} H(I) = \min_{\xi_0 \le \xi \le \xi_M} H(I^*_\xi), \tag{14}$$

where $H(I^*_\xi) = \min_{I \in \mathcal{I}_{p,0},\ I(a)=\xi} H(I)$ and I^*_ξ is the minimizer that solves the inner minimization problem $\min_{I \in \mathcal{I}_{p,0},\ I(a)=\xi} H(I)$ for any given $\xi \in [\xi_0,\ \xi_M]$.

To derive the expression of the minimizer I_ξ^* of (14), we define contract $I_{1,\xi}(x) \in \mathcal{I}$ as

$$
\begin{aligned}
I_{1,\xi}(x) &= (x-a+\xi)^+ - (x-a-p)^+ \\
&= \begin{cases} 0, & \text{for} \quad 0 \le x < a-\xi, \\ x-a+\xi, & \text{for} \quad a-\xi \le x < a+p, \\ \xi+p, & \text{for} \quad a+p \le x < \infty, \end{cases}
\end{aligned}
\tag{15}
$$

and denote the reinsurance premium based on $I_{1,\xi}$ by $p_{1,\xi}$ or

$$
p_{1,\xi} = (1+\theta)\mathbb{E}\left[I_{1,\xi}(X)\right] = (1+\theta)\int_{a-\xi}^{a+p} S_X(x)\underset{\cdot}{\mathsf{x}}. \tag{16}
$$

Furthermore, we define contract $I_{2,\xi}(x) \in \mathcal{I}$ as

$$
\begin{aligned}
I_{2,\xi}(x) &= x - (x-\xi)^+ + (x-a)^+ - (x-a-p)^+ \\
&= \begin{cases} x, & \text{for} \quad 0 \le x < \xi, \\ \xi, & \text{for} \quad \xi \le x < a, \\ \xi + x - a, & \text{for} \quad a \le x < a+p, \\ \xi + p, & \text{for} \quad a+p \le x < \infty, \end{cases}
\end{aligned} \tag{17}
$$

and denote the reinsurance premium based on $I_{2,\xi}$ by $p_{2,\xi}$ or

$$
p_{2,\xi} = (1+\theta)\mathbb{E}\left[I_{2,\xi}(X)\right] = (1+\theta)\left(\int_0^{\xi} + \int_a^{a+p}\right) S_X(x)\underset{\cdot}{\mathsf{x}}. \tag{18}
$$

In addition, throughout this paper, we denote

$$
\xi_1 = \begin{cases} \sup\left\{\xi \in [\xi_0, \xi_M] :\, p_{2,\xi} < p\right\}, & \text{if} \quad p_{2,\xi_0} < p, \\ \xi_0, & \text{if} \quad p_{2,\xi_0} \ge p, \end{cases} \tag{19}
$$

and

$$\xi_2 = \begin{cases} \inf\left\{\xi \in [\xi_0, \xi_M] : \, p_{1,\xi} > p\right\}, & \text{if} \quad p_{1,\xi_M} > p, \\ \xi_M, & \text{if} \quad p_{1,\xi_M} \le p. \end{cases} \tag{20}$$

It is not hard to check by the definitions of ξ_1 and ξ_2 that

the set $[\xi_0,\, \xi_M]$ has only the following three possible partitions:

1. $[\xi_0,\, \xi_M] = [0,\, \xi_M]$ if $\xi_0 = \xi_1$;
2. $[\xi_0,\, \xi_M] = [\xi_0,\, a]$ if $\xi_2 = \xi_M$;
3. $[\xi_0,\, \xi_M] = [\xi_0,\, \xi_1] \cup [\xi_1,\, \xi_2] \cup [\xi_2,\, \xi_M]$ if $\xi_0 < \xi_1 < \xi_2 < \xi_M$.

Now, in the following lemma, for any given $\xi \in [\xi_0,\, \xi_M]$, we solve the inner minimization problem $\min\limits_{I \in \mathcal{I}_{p,0},\; I(a)=\xi} H(I)$ of (14).

Lemma 1. *For a given $\xi \in [\xi_0, \xi_M]$, let I_ξ^* be the optimal solution to the minimization problem $\min_{I \in \mathcal{I}_{p,0},\ I(a)=\xi} H(I)$. Then, I_ξ^* can be summarized as follows.*

(1) If $\xi_0 \leq \xi \leq \xi_1$ and $\xi_0 < \xi_1$, then

$$I_\xi^*(x) = x - (x-\xi)^+ + (x-a)^+ - (x-a-p)^+ + (x-d_{3,\xi})^+$$
$$= \begin{cases} x, & \text{for} \quad 0 \leq x < \xi, \\ \xi, & \text{for} \quad \xi \leq x < a, \\ \xi + x - a, & \text{for} \quad a \leq x < a+p, \\ \xi + p, & \text{for} \quad a+p \leq x < d_{3,\xi}, \\ \xi + p + x - d_{3,\xi}, & \text{for} \quad d_{3,\xi} \leq x, \end{cases}$$

where $d_{3,\xi}$ is determined by $(1+\theta)\mathbb{E}\left[I_\xi^*(X)\right] = p$.

(2) If $\xi_1 \le \xi \le \xi_2$ and $\xi_1 < \xi_2$, then

$$
\begin{aligned}
I_\xi^*(x) &= (x-d_{1,\xi})^+ - (x-d_{1,\xi}-\xi)^+ + (x-a)^+ - (x-a-p)^+ \\
&= \begin{cases} 0, & \textit{for} \quad 0 \le x < d_{1,\xi}, \\ x-d_{1,\xi}, & \textit{for} \quad d_{1,\xi} \le x < d_{1,\xi}+\xi, \\ \xi, & \textit{for} \quad d_{1,\xi}+\xi \le x < a, \\ \xi+x-a, & \textit{for} \quad a \le x < a+p, \\ \xi+p, & \textit{for} \quad a+p \le x < \infty, \end{cases}
\end{aligned}
$$

where $d_{1,\xi}$ is determined by $(1+\theta)\mathbb{E}\left[I_\xi^*(X)\right] = p$.

(3) If $\xi_2 \le \xi \le \xi_M$ and $\xi_2 < \xi_M$, then

$$
\begin{aligned}
I_\xi^*(x) &= (x-a+\xi)^+ - (x-a)^+ + (x-d_{2,\xi})^+ - (x-d_{2,\xi}-p)^+ \\
&= \begin{cases} 0, & \text{for} \quad 0 \le x < a-\xi, \\ x-a+\xi, & \text{for} \quad a-\xi \le x < a, \\ \xi, & \text{for} \quad a \le x < d_{2,\xi}, \\ \xi + x - d_{2,\xi}, & \text{for} \quad d_{2,\xi} \le x < d_{2,\xi}+p, \\ \xi + p, & \text{for} \quad d_{2,\xi}+p \le x < \infty, \end{cases}
\end{aligned}
$$

where $d_{2,\xi}$ is determined by $(1+\theta)\mathbb{E}\left[I_\xi^*(X)\right] = p$.

For any given $\xi \in [\xi_0, \xi_M]$ and the corresponding optimal ceded loss function I_ξ^* given in Lemma 1, we define the function h of ξ as

$$h(\xi) = H(I_\xi^*). \tag{21}$$

Thus, Lemma 1 implies that $\min_{I \in \mathcal{I}_{p,0},\ I(a)=\xi} H(I) = H(I^*_\xi) = h(\xi)$.

Now, using Lemma 1, we obtain the optimal solution to Problem (4) in the following theorem.

Theorem 2. *Assume* $0 < p < (1+\theta)\mathbb{E}[X]$. *Then Problem (4) is equivalent to Problem (14) and the optimal solution to Problem (4), denoted by* I^*, *is summarized as follows.*

(1) If $\xi_1 = a$, *the optimal solution is*

$$
\begin{aligned}
I^*(x) &= x - (x-a-p)^+ + (x - d_{3,a})^+ \\
&= \begin{cases} x, & \text{for} \quad 0 \le x < a+p, \\ a+p, & \text{for} \quad a+p \le x < d_{3,a}, \\ a+p+x-d_{3,a}, & \text{for} \quad d_{3,a} \le x, \end{cases}
\end{aligned}
$$

where $d_{3,a}$ *is determined by* $(1+\theta)\mathbb{E}\left[I^*(X)\right] = p$.

(2) If $\xi_1 < a$ and $h'(\xi_M) \le 0$, then $\xi_M = a$ and the optimal solution is

$$
\begin{aligned}
I^*(x) &= x - (x-a)^+ + (x - d_{2,a})^+ - (x - d_{2,a} - p)^+ \\
&= \begin{cases} x, & \textit{if} \quad 0 \le x < a, \\ a, & \textit{if} \quad a \le x < d_{2,a}, \\ a + x - d_{2,a}, & \textit{if} \quad d_{2,a} \le x < d_{2,a} + p, \\ a + p, & \textit{if} \quad d_{2,a} + p \le x < \infty, \end{cases}
\end{aligned}
$$

where $d_{2,a}$ is determined by $(1+\theta)\mathbb{E}\left[I^*(X)\right] = p$.

(3) If $\xi_1 < a$ and $h'(\xi_M) > 0$, then there exists $\xi^* \in [\xi_2,\, \xi_M]$ such that $h'(\xi^*) = 0$ and the optimal solution is

$$
\begin{aligned}
I^*(x) &= (x-a+\xi^*)^+ - (x-a)^+ + (x-d_{2,\xi^*})^+ - (x-d_{2,\xi^*}-p)^+ \\
&= \begin{cases}
0, & \text{for} \quad 0 \le x < a-\xi^*, \\
x-a+\xi^*, & \text{for} \quad a-\xi^* \le x < a, \\
\xi^*, & \text{for} \quad a \le x < d_{2,\xi^*}, \\
\xi^* + x - d_{2,\xi^*}, & \text{for} \quad d_{2,\xi^*} \le x < d_{2,\xi^*} + p, \\
\xi^* + p, & \text{for} \quad d_{2,\xi^*} + p \le x < \infty,
\end{cases}
\end{aligned}
$$

where d_{2,ξ^*} is determined by $(1+\theta)\mathbb{E}\left[I^*(X)\right] = p$.

Remark 2. *It is easy to see that the optimal solution I^* in all three cases of Theorem 2 can be expressed using a unified formula as*

$$
\begin{aligned}
I^*(x) &= (x - d_1^*) - (x-a)^+ + (x - d_2^*)^+ - (x - d_2^* - p)^+ + (x - d_3^*)^+ \\
&= \begin{cases}
0, & \textit{for} \quad 0 \le x < d_1^*, \\
x - d_1^*, & \textit{for} \quad d_1^* \le x < a, \\
a - d_1^*, & \textit{for} \quad a \le x < d_2^*, \\
a - d_1^* + x - d_2^*, & \textit{for} \quad d_2^* \le x < d_2^* + p, \\
a - d_1^* + p, & \textit{for} \quad d_2^* + p \le x < d_3^*, \\
a - d_1^* + p + x - d_3^*, & \textit{for} \quad d_3^* \le x,
\end{cases}
\end{aligned}
\tag{22}
$$

where

$$
(d_1^*,\, d_2^*,\, d_3^*) = \begin{cases}
(0,\, a,\, d_{3,a}), & \textit{if} \quad \xi_1 = a, \\
(0,\, d_{2,a},\, +\infty), & \textit{if} \quad \xi_1 \le a \textit{ and } h'(\xi_M) \le 0, \\
(a - \xi^*,\, d_{2,\xi^*},\, +\infty), & \textit{if} \quad \xi_1 < a \textit{ and } h'(\xi_M) > 0.
\end{cases}
$$

Remark 3. *We point out that for a feasible contract* $I \in \mathcal{I}_p$*, if* $I(x) \leq \omega_I + P_I = I(a) + p$ *for all* $x \geq 0$*, then the contract is a default risk-free contact, i.e., the insurer will not face default risk with this contract.*

In (1) of Theorem 2, which corresponds to the case where $\xi_1 = a$*, if* $p_{2,a} = p$*, where* $p_{2,\xi}$ *is defined in (18), then* $d_{3,a} = \infty$ *and the optimal contract* I^* *is reduced to* $I^*(x) = x - (x - a - p)^+ = I^*(x) \wedge (I^*(a) + p) \leq I^*(a) + p$*, namely the optimal contract is a default risk-free contract. However, if* $p_{2,a} < p$*, then there does not exist a default risk-free contract in* $\mathcal{I}_p$*.*

In (2) and (3) of Theorem 2, which correspond to the case where $\xi_1 < a$*, it is obvious that the optimal solution* I^* *in both cases satisfy* $I^*(x) \leq I^*(a) + p$*, namely the insurer will not face default risk with the two optimal contracts.*

In summary, Theorem 2 suggests that, in order to lower default risk, an *insurer should choose a contract without default risk as long as this kind of contract is available. This leads to limits for indemnities on the tails of the optimal contracts.*

In addition, it has been mentioned that $I_0^* = (x - d^*)^+$ *is the optimal solution to the classical Problem (6) in the absence of default risk. Note that* $I_0^* \in \mathcal{I}_p$. *It is easy to check that in all three cases of Theorem 2, the optimal contract* I^* *of Theorem 2 satisfies* $\omega_{I^*} = I^*(a) > \omega_{I_0^*} = I_0^*(a) = (a - d^*)^+$, *which means that the reinsurer will have to set up a higher initial reserve if the insurer chooses the optimal contract* I^* *of Theorem 2 than if the insurer chooses* I_0^*. *In this way, the insurer can reduce the default risk.*

3 Optimal reinsurance minimizing the VaR of an insurer's total retained risk

In this section, we study Problem (2). To do so, for any $I \in \mathcal{I}$, we denote

$$V(I) = \mathrm{VaR}_\beta\Big(X - I(X) \wedge (\omega_I + P_I) + P_I\Big),$$

where and throughout this section $0 < \beta < S_X(0)$.

Thus Problem (2) is reformulated as

$$\min_{I \in \mathcal{I}} V(I). \tag{23}$$

Throughout this section, we denote $a = \mathrm{VaR}_\alpha(X)$ and $b = \mathrm{VaR}_\beta(X)$. For any $I \in \mathcal{I}$, we see that the function $x - I(x) \wedge (\omega_I + P_I)$ is continuous and non-decreasing on $[0, \infty)$. Thus, due to the translation invariance and preservation of VaR under continuous and non-decreasing functions, the objective function $V(I)$ can be expressed as

$$V(I) = b - I(b) \wedge (I(a) + P_I) + P_I.$$

Again, we can reduce the infinite-dimensional optimization Problem (23) to a finite-dimensional optimization problem. In doing so, we first give the following lemma.

Lemma 2. *For any $I_1, I_2 \in \mathcal{I}$, if $I_1(a) = I_2(a)$, $I_1(b) = I_2(b)$, and $P_{I_1} \leq P_{I_2}$, then $V(I_1) \leq V(I_2)$.*

Using this lemma, we can show in the following theorem that for any $I \in \mathcal{I}$, there exists $m_I \in \mathcal{I}$ such that $V(m_I) \leq V(I)$. The proof of the theorem is given in the appendix.

Theorem 3. *For any $I \in \mathcal{I}$, there exists $m_I \in \mathcal{I}$ satisfying $V(m_I) \le V(I)$ and m_I is defined as*

$$
\begin{aligned}
m_I(x) &= (x-d_1)^+ - (x - a\wedge b)^+ + (x-d_2)^+ - (x - a\vee b)^+ \\
&= \begin{cases} 0, & \text{for} \quad 0 \le x < d_1, \\ x - d_1, & \text{for} \quad d_1 \le x < a\wedge b, \\ a\wedge b - d_1, & \text{for} \quad a\wedge b \le x < d_2, \\ a\wedge b - d_1 + x - d_2, & \text{for} \quad d_2 \le x < a\vee b, \\ a + b - d_1 - d_2, & \text{for} \quad a\vee b \le x < \infty, \end{cases}
\end{aligned} \tag{24}
$$

where $d_1 = a\wedge b - I(a\wedge b)$ and $d_2 = a\vee b - (I(a\vee b) - I(a\wedge b))$ satisfying $0 \le d_1 \le a\wedge b \le d_2 \le a\vee b$.

For any $(d_1, d_2) \in [0,\, a\wedge b] \times [a\wedge b,\, a\vee b]$, we define

$$\mathcal{I}_{d_1,d_2} = \{I \in \mathcal{I} :\, I \text{ has the expression (24)}\}.$$

Then, for $I \in \mathcal{I}_{d_1,d_2}$, define the function

$$v(d_1, d_2) = V(I) = b - I(b) \wedge (I(a) + P_I) + P_I.$$

Thus, by Theorem 3, we see that Problem (23) is equivalent to the following optimization problem:

$$\min_{I \in \mathcal{I}_{d_1,\, d_2}} V(I) = \min_{(d_1,\, d_2) \in [0,\, a\wedge b] \times [a\wedge b,\, a\vee b]} v(d_1,\, d_2), \tag{25}$$

which is a finite-dimensional optimization problem.

By solving Problem (25), we obtain the optimal solution to Problem (23) in the following theorem. The proof of the theorem is given in the appendix.

Theorem 4. *Let I^* be the optimal solution to Problem (23).*

(1) If $\alpha \leq \beta$, then

$$
\begin{aligned}
I^*(x) &= (x - b \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X))^+ - (x-b)^+ \\
&= \begin{cases} 0, & \textit{for} \quad 0 \leq x < b \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X), \\ x - b \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X), & \textit{for} \quad b \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X) \leq x < b, \\ b - b \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X), & \textit{for} \quad b \leq x < \infty. \end{cases}
\end{aligned}
$$

(2) If $\alpha > \beta$ and $\alpha \leq \frac{1}{1+\theta}$, there exists $d_0 \in \mathbb{R}$ satisfying

$$
(1+\theta) \int_{d_0}^{b} S_X(x) x = b - a. \tag{26}
$$

Then

$$I^*(x) = (x - \max\{0,\, d_0 \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X)\})^+ - (x-b)^+$$
$$= \begin{cases} 0, & 0 \le x < \max\{0,\, d_0 \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X)\}, \\ x - \max\{0,\, d_0 \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X)\}, & \max\{0,\, d_0 \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X)\} \le x < b, \\ b - \max\{0,\, d_0 \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X)\}, & b \le x < \infty, \end{cases}$$

Remark 4. *The optimal solution I^* in the two cases of Theorem 4 can be expressed in a unified formula as*

$$I^*(x) = (x - d^*)^+ - (x-b)^+,$$

where $d^* = b \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X)$ if $\alpha \le \beta$ and $d^* = \max\{0,\, d_0 \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X)\}$ if $\alpha > \beta$ and $\alpha \le \frac{1}{1+\theta}$.

When $\alpha \leq \beta$, which means the reinsurer is more conservative than the insurer, we have $I(b) \leq I(a)$, thus Problem (2) is reduced to the following model without default risk:

$$\min_{I \in \mathcal{I}} \mathrm{VaR}_{\beta}(X - I(X) + P_I), \tag{27}$$

which was studied by Cheung et al. (2014).

Since the insurer measures its risk, based on VaR, at a higher risk level $\beta \geq \alpha$, the initial reserve $\omega_I = \mathrm{VaR}_{\alpha}(I(X))$ set by the reinsurer at a lower risk level α is high enough to ensure that default will not occur.

On the other hand, if the insurer is more conservative than the reinsurer or $\alpha > \beta$, in order to reduce the default risk, the insurer should require a lower deductible or ask the reinsurer to cover more loss, to force the reinsurer to set up a higher initial reserve. For the case where $\alpha > 1/(1+\theta)$, the *optimal solution I^* has no closed form and the case is not interesting since in practice, α is a small value and usually $\alpha < 1/(1+\theta)$ holds.*

4 Numerical Examples

In this section, we use numerical examples to illustrate the optimal solutions derived in Sections 2 and 3. We assume the underlying loss faced by the insurer has an exponential distribution or a Pareto distribution. Thus, in one case, the loss is light-tailed, and in the other case, the loss is heavy-tailed. We will calculate the optimal forms under the two loss distributions and consider the influence of the distribution, the risk level α, and the reinsurance premium budget p on the optimal reinsurance contracts.

Suppose random variables X and Y have exponential and Pareto distributions, respectively. Assume their survival functions are $S_X(x) = e^{-x/\mu}$ and $S_Y(x) = \left(\frac{\lambda}{x+\lambda}\right)^{\gamma}$ for any $x \geq 0$, respectively. All numerical results are given under the setting of $\theta = 0.1$, $\mu = 100$, $\lambda = 200$ and $\gamma = 3$. Under this setting, X and Y have the same mean 100, and the fixed premium budget $p \in (0, 110)$.

Example 1. (Numerical results for the optimal reinsurance maximizing the expected utility) *In this case, we know that the optimal contract can be expressed as the unified formula given by (22) as*

$$I^*(x) = (x - d_1^*) - (x - a)^+ + (x - d_2^*)^+ - (x - d_2^* - p)^+ + (x - d_3^*)^+.$$

We take the convex function $u(x) = x^2$, which means that the insurer would like to minimize the variance of its retained loss. We first consider the exponential underlying loss X with survival distribution $S_X(x) = e^{-x/100}$ for any $x \geq 0$ for the insurer. In this case, $a = \mathrm{VaR}_\alpha(X) = -\frac{\ln(1-\alpha)}{100}$. We obtain the optimal contract under $\alpha = 0.01$, $\alpha = 0.05$, and different premium budget values of p. The numerical results are given in Tables 1 and 2.

Table 1: Exponential Risk X with $\alpha = 0.01$

p	d_1^*	d_2^*	d_3^*
80.000	31.225	461.168	∞
99.200	9.921	460.940	∞
105.880	3.456	460.806	∞
108.100	1.396	460.809	∞
109.631	0	460.811	∞
109.800	0	460.517	649.089

Table 2: Exponential Risk X with $\alpha = 0.05$

p	d_1^*	d_2^*	d_3^*
80.000	28.691	302.681	∞
99.200	8.229	301.653	∞
105.880	1.971	301.407	∞
108.100	0	301.332	∞
109.631	0	299.573	431.620
109.800	0	299.573	420.917

Next, we consider the Pareto underlying loss Y with survival distribution $S_Y(y) = \left(\frac{200}{y+200}\right)^3$ for $y \geq 0$ for the insurer. In this case, $\mathrm{VaR}_\alpha(Y) = 200(\alpha^{-1/3} - 1)$. The numerical results for (d_1^*, d_2^*, d_3^*) when $\alpha = 0.01$, $\alpha = 0.05$, and different premium budget values of p are summarized in Tables 3 and 4.

Table 3: Pareto Risk Y with $\alpha = 0.01$

p	d_1^*	d_2^*	d_3^*
80.000	28.405	734.196	∞
99.200	6.305	732.488	∞
105.880	0	732.107	∞
108.100	0	728.318	1215.400
109.631	0	728.300	888.275
109.800	0	728.300	864.518

Table 4: Pareto Risk Y with $\alpha = 0.05$

p	d_1^*	d_2^*	d_3^*
80.000	19.200	356.748	∞
99.200	0	352.764	∞
105.880	0	342.900	633.469
108.100	0	342.884	520.208
109.631	0	342.884	464.486
109.800	0	342.884	459.096

It is easy to see from Tables 1-4 that when the risk level α is fixed, the optimal ceded loss function $I^*(x)$ associated with a higher reinsurance premium is larger than the optimal ceded loss function $I^*(x)$ with a lower reinsurance premium, which means that the larger is the premium charged by the reinsurer, the larger is the loss that the reinsurer should cover. Furthermore, Tables 1-4 suggest that when the reinsurance premium p is *fixed, the higher is the risk level α, the lower are the deductible levels d_i^* for $i = 1, 2, 3$, which means that the smaller is the reinsurer's initial reserve, the larger is the loss that the insurer should cover. Moreover, Tables 1-4 imply that when both the reinsurance premium p and risk level α are fixed, the optimal ceded loss function $I^*(x)$ for Pareto loss Y is larger than the optimal ceded loss function $I^*(x)$ for exponential loss X, which means that the insurer should cede more loss to the reinsurer for a heavy tailed loss or a riskier loss.* □

Example 2. (Numerical results for the optimal reinsurance minimizing the VaR of its retained risk) *In this case, denote the optimal reinsurance contract for X and Y by I_X^* and I_Y^*, respectively. Denote $d_{0,X}$ and $d_{0,Y}$ be the solutions to the equation (26) for X and Y, respectively. According to Theorem 4, we have*

$$I_X^*(x) = (x - d_X^*)^+ - (x - \mathrm{VaR}_\beta(X))^+$$

and

$$I_Y^*(y) = (y - d_Y^*)^+ - (y - \mathrm{VaR}_\beta(Y))^+,$$

where $d_X^* = \mathrm{VaR}_\beta(X) \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X)$ *if* $\alpha \leq \beta$ *and* $d_X^* = \max\{0, d_{0,X} \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X)\}$ *if* $\alpha > \beta$ *and* $\alpha \leq \frac{1}{1+\theta}$*, and* d_Y^* *has a similar expression as* d_X^**.*

Using Theorem 4, we obtain the optimal values of d_X^* *and* d_Y^* *for different risk levels of* (α, β) *in Table 5. Table 5 suggests that a Pareto loss* Y *will result in lower deductible levels than an exponential loss* X*. These numerical results are consistent with those in Example 4.1* □

Table 5: Deductible Values

(α, β)	d_X^*	$VaR_\beta(X)$	d_Y^*	$VaR_\beta(Y)$
(0.0100, 0.0500)	9.531	299.573	6.456	342.884
(0.0100, 0.0280)	9.531	357.555	6.456	458.634
(0.0185, 0.0150)	9.531	419.971	6.456	610.960
(0.0500, 0.0100)	0	460.517	0	728.318
(0.0280, 0.0100)	5.549	460.517	0	728.318
(0.0280, 0.0185)	9.531	398.999	4.448	556.205
(0.0150, 0.0185)	9.531	398.999	6.456	556.205

5. Concluding remarks

- We propose a reinsurance risk model that incorporates the regulatory requirements on the initial reserve of a reinsurance contract seller (reinsurer) and the possible default by the seller.

- Mathematically, the proposed model can be reduced to existing reinsurance risk models that do not consider the possible default by a reinsurer. Practically, the proposed model allows more realistic settings.

- We derive the optimal reinsurance strategy from the insurer's point of view under the proposed model. The results show that the regulatory initial reserve and the default risk have a significant impact on the optimal reinsurance strategy.

- The optimal reinsurance strategies under the proposed model are more complicated than those in the existing default risk-free reinsurance risk models.

- The model proposed in the paper can be further explored in different ways as well, something we plan to do in future work.

References:

1. Balbás, A., Balbás, B., and Heras, A. (2009) Optimal reinsurance with general risk measures. *Insurance: Mathematics and Economics* **44**(3): 374-384.
2. Asimit, A.V., Badescu, A.M. and Cheung, K.C., (2013) Optimal reinsurance in the presence of counterparty default risk. *Insurance: Mathematics and Economics* **53**(3): 590-697.
3. Asimit, A.V., Badescu, A.M. and Verdoncj, T. (2013) Optimal risk transfer under quantile-based risk measures. *Insurance: Mathematics and Economics* **53**(1): 252-265.
4. Arrow, K.J., (1963) Uncertainty and the welfare economic of medical care. *American Economic Review* **53**(5): 941-973.
5. Bernard, C. and Ludkovski, M., (2012) Impact of counterparty risk on the reinsurance market. *North American Actuarial Journal* **16**(1): 87-111.

6. Borch, K., (1960) An attempt to determine the optimal amount of stop loss reinsurance. *Transactions of the 16th International Congress of Actuaries*, vol **I**: 597-610.
7. Burren, D. (2013) Insurance demand and welfare-maximizing risk capital - some hints for the regulator in the case of exponential preferences and exponential claims. *Insurance: Mathematics and Economics* **53**(3): 551-568.
8. Cai, J., Fang, Y., Li, Z. and Willmot, G.E. (2013) Optimal reciprocal reinsurance treaties under the joint survival probability and the joint profitable probability. *Journal of risk and insurance* **80**(1): 145-168.
9. Cai, J., Tan, K.S., Weng, C. and Zhang, Y., 2008. Optimal reinsurance under VaR and CTE risk measures. *Insurance: Mathematics and Economics* **43**(1): 185-196.
10. Cai, J. and Wei, W. (2012). Optimal reinsurance with positively dependent risks. *Insurance: Mathematics and Economics* **50**(1): 57-63.

11. Cheung, K.C. (2010) Optimal reinsurer revisited - a geometric approach. *ASTIN Bulletin* **40**(1): 221-239.

12. Cheung, K.C., Sung, K. C. J., Yam, S. C. P. (2014) Risk-minimizing reinsurance protection for multivariate risks. *Journal of Risk and Insurance*. **81**(1):219-236.

13. Cheung, K.C., Sung, K. C. J., Yam, S. C. P. and Yung, S.P. (2014) Optimal reinsurance under general law-invariant risk measures. *Scandinavian Actuarial Journal* **1**: 72-91.

14. Chi, Y. (2012) Optimal reinsurance under variance related premium principles. *Insurance: Mathematics and Economics* **51**(2): 310-321.

15. Chi, Y. and Tan, K.S. (2011) Optimal reinsurance under VaR and CVaR risk measures: a simplified approach. *ASTIN Bulletin* **41**(2): 487-509.

16. Cummins, J., Doherty, N. and Lo, A., (2002) Can insurers pay for the big one? Measuring the capacity of the insurance market to respond to catastrophic losses. *Journal of Ranking and Finance* **26**: 557-583.

17. Dana, R.-A. and Scarsini, M. (2007) Optimal risk sharing with background risk. *Journal of Economic Theory* **133** (1): 152-176.

18. Hürlimann, W. (2011) Optimal reinsurance revisited - point of view of cedent and reinsurer. *ASTIN Bulletin* **41**(2): 547-574.

19. Menegatti, M. (2009) Optimal saving in the presence of two risks. *Journal of Economics* **96**(3): 277-288.

20. Ohlin, J., (1969) On a class of measures of dispersion with application to optimal reinsurance. *ASTIN Bulletin* **5**(2): 249-266.