Optimal Rejection of Persistent Bounded Disturbances

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Abstract—**In this paper, we formulate the problem of optimal disturbance rejection in the case where the disturbance is generated as the output of a stable system in response to an input which is assumed to be of unit amplitude, but is otherwise arbitrary. The objective is to choose a controller that minimizes the maximum amplitude of the plant output in response to such a disturbance. Mathematically, this corresponds to requiring uniformly good disturbance rejection over all time. Since the problem of optimal tracking is equivalent to that of optimal disturbance rejection if a feedback controller is used (see [7, sect. 5.6]), the theory presented here can also be used to design optimal controllers that achieve uniformly good tracking over all time rather than a tracking error whose L_2-norm is small, as is the case with the currently popular H_∞ theory. The present theory is a natural counterpart to the existing theory of optimal disturbance rejection (the so-called H_∞ theory) which is based on the assumption that the disturbance to be rejected is generated by a stable system whose input is square-integrable and has unit energy. It is shown that the problem studied here has quite different features from its predecessor. Complete solutions to the problem are given in several important cases, including those where the plant is minimum phase or when it has only a single unstable zero. In other cases, procedures are given for obtaining bounds on the solution and for obtaining suboptimal controllers.**

I. Introduction

In this paper, we study the following problem. Suppose one is given a (possibly unstable) plant P, which is being subjected to a disturbance d at its output.[1] Suppose, in addition, that the disturbance d can be thought of as the output of a system W, which is in turn driven by an input v that is bounded in time by 1, but is otherwise arbitrary. The objective is to design a controller C that stabilizes the plant P and at the same time optimally rejects the disturbance; in other words, C stabilizes P, and results in the smallest possible maximum output amplitude in response to the disturbance.

The problem under study here differs in important respects from those previously investigated in the literature. The classes of problems previously explored can be placed into two categories. In the first, it is assumed that d is a *known* disturbance, e.g., a step, a sinusoid, white noise, etc. This problem is one of regulation or filtering, and has been treated by a large number of researchers over the years. In the second, it is assumed that v is a square-integrable signal of unit energy but is otherwise arbitrary, and that W is a stable transfer matrix. The objective is to minimize the maximum energy of the resulting output signal y. Conceptually, this problem represents an important advance beyond that of regulation mentioned above, since one is attempting to minimize the worst possible adverse impact of a *class* of disturbances, rather than just a single fixed disturbance. Mathematically, the resulting problem is the so-called H_∞-norm minimization. Results from the theory of functions analytic on the unit disk of the complex plane (the theory of Hardy spaces) can be used to good advantage in solving this problem; see [1]–[6], [7, ch. 6] for a discussion of these results.

As mentioned above, the idea of minimax optimization, i.e., of minimizing the worst possible impact of a class of disturbances, represents an important conceptual advance. A mathematical framework for studying such problems is given in [1], in terms of multiplicative seminorms. In addition, in [1] explicit solutions are given for the case where the disturbance input is square-integrable (i.e., is an L_2-function), and the plant is scalar and has a single simple zero in the open right half-plane. Using the results of [2], [3] on minimax interpolation, it is possible to obtain explicit solutions for the case of L_2-disturbances and scalar plants with multiple RHP zeros. The studies in [4]–[6] enable one to tackle the case of multivariable plants and L_2-disturbances. In each case, the type of cost function that can be minimized using the methods of [1]–[6] is

$$J = \max_{d \in L_2, \|d\|_2 \le 1} \|z\|_2. \tag{1.1}$$

If H_{zd} denotes the transfer matrix from d to z, then, as is well known [9, ch. 3],

$$\max_{\|d\|_2 \le 1} \|z\|_2 = \|H_{zd}\|_\infty. \tag{1.2}$$

Suppose, in contrast, that one is interested in minimizing the L_∞-norm of z, i.e.,

$$J = \max_d \|z\|_\infty. \tag{1.3}$$

This means that one is interested in uniformly good disturbance rejection at all instants of time. If d varies over the unit ball of L_2, then

$$\max_{\|d\|_2 \le 1} \|z\|_\infty = \|H_{zd}\|_2. \tag{1.4}$$

Thus, the problem of minimizing the maximum L_∞-norm of z in response to a set of L_2-norm bounded disturbances can be solved using Wiener–Hopf methods [11], [17]. If, on the other hand, one assumes that the disturbances themselves are L_∞-norm bounded, then the cost function to be minimized becomes

$$\max_{\|d\|_\infty \le 1} \|z\|_\infty. \tag{1.5}$$

While this problem can be considered to belong to the general class of problems posed in [1], the contents of [1]–[6] are of no help in obtaining the minimum of such a cost function.

The present paper is devoted precisely to the minimization of cost functions of the type above. Complete solutions are given to the problems of optimal controller synthesis in several important situations, including those where the plant is minimum phase or has a single unstable zero. In the general case, methods are given

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[1] It is really not necessary to assume that the disturbance enters additively at the output; this is simply for ease of exposition.

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for estimating the optimal performance. It turns out that the L_∞-optimal controller is in general not the same as the L_2-optimal controller resulting from the methods of [1]–[7]. Even in cases where the optimal achievable performances are the same, the methods used to arrive at the end results are quite different. The result is a theory that complements the H_∞-optimization theory and draws on it in many ways, but is fundamentally different.

II. Preliminaries and Problem Statement

In this section, we introduce the various norms that are used in the paper, and define precisely the problem at hand. As there are several norms that arise naturally in connection with the problem studied here, it is worthwhile to study these in some detail. For further details concerning the various norm properties given below, good references are [9], [10].

Let $\mathbb{R}$ denote the field of real numbers, and suppose $f: \mathbb{R} \to C$ is Lebesgue measurable. Two norms that can be defined are

$$\|f\|_2 = \left\{ \int_{-\infty}^{\infty} |f(t)|^2 \right\}^{1/2}, \tag{2.1}$$

$$\|f\|_\infty = \operatorname*{ess.\ sup}_{t\in(-\infty,\infty)} |f(t)|. \tag{2.2}$$

Let L_2 (respectively, L_∞) denote the set of all $f(\cdot)$ such that $\|f\|_2$ is finite (respectively, such that $\|f\|_\infty$ is finite).

Now suppose h is a distribution with support in the inverval $[0, \infty)$ of the form

$$h(t) = \sum_{i=0}^{\infty} h_i \delta(t - t_i) + h_a(t) \tag{2.3}$$

where $0 \le t_0 < t_1 < \cdots$, δ denotes the unit impulse distribution, and $h_a(\cdot)$ is Lebesgue measurable. The set A consists of all distributions h of the form (2.3) such that

$$\|h\|_A = \sum_{i=0}^{\infty} |h_i| + \int_0^\infty |h_a(t)|\ dt \tag{2.4}$$

is finite. Note that A is precisely the set of impulse responses of BIBO stable systems.

Every $h \in A$ is Laplace transformable, and the region of convergence of the Laplace transform includes the closed right half-plane $C_+ = \{s: \text{Re } s \ge 0\}$. Thus, if $h \in A$, then the quantity

$$\hat{h}(s) = \int_0^\infty h(t)e^{-st}\ dt = \sum_{i=0}^{\infty} h_i e^{-st_i} + \int_0^\infty h_a(t)e^{-st}\ dt \tag{2.5}$$

is well-defined whenever Re $s \ge 0$. Moreover, it is easy to see that

$$|\hat{h}(s)| \le \|h\|_A \qquad \text{for all } s \in C_+. \tag{2.6}$$

Let $\hat{A}$ denote the set of Laplace transforms of distributions in A. Then clearly $\hat{A}$ is a linear space. We can define two distinct norms on $\hat{A}$. Suppose $F \in \hat{A}$ and let f denote the inverse Laplace transform of F. Then we define

$$\|F\|_{\hat{A}} = \|f\|_A. \tag{2.7}$$

$$\|F\|_\infty = \sup_{\omega\in(-\infty,\infty)} |F(j\omega)|. \tag{2.8}$$

Note that $\|F\|_\infty \le \|F\|_{\hat{A}}$ for all $F \in \hat{A}$. Also, note that the usage $\|F\|_\infty$ to denote the norm of a function of "frequency" is quite consistent with the definition (2.2) of the norm of a function of "time."

Suppose $H \in \hat{A}$, so that H is the transfer function of a distribution h in A. We can associate with H an operator from L_∞ into L_∞ (which is also denoted by H), defined by $Hf = h*f$. Then

$$\sup_{f\in L_\infty - 0} \frac{\|Hf\|_\infty}{\|f\|_\infty} = \|H\|_{\hat{A}}. \tag{2.9}$$

On the other hand, H also maps L_2 into L_2 by the association $Hf = h*f$. The gain of this map is given by

$$\sup_{f\in L_2 - 0} \frac{\|Hf\|_2}{\|f\|_2} = \|H\|_\infty. \tag{2.10}$$

To summarize, given a transfer function $H \in \hat{A}$, corresponding to a stable system, one can associate with it two distinct "gains." The quantity $\|H\|_{\hat{A}}$ is the gain of the system viewed as a map between bounded input–output pairs, while $\|H\|_\infty$ is the gain of the system viewed as a map between finite energy input–output pairs.

Since the interest in this paper is purely in lumped systems, it is understood that hereafter all elements of $\hat{A}$ are rational unless explicitly stated to the contrary.

In other work, e.g., [7] the symbol S is used to denote the set of all proper stable rational functions, equipped with the norm (2.9). With the convention above that $\hat{A}$ consists only of rational functions unless specified otherwise, we see that $\hat{A}$ and S are both normed spaces whose underlying linear vector spaces are the same, but whose norms are different.

Next, consider the discrete-time analogs of the various norms described above. Given a sequence $\{f_i\}$, we can define three norms on it, as follows:

$$\|f\|_1 = \sum_{i=0}^{\infty} |f_i|, \tag{2.11}$$

$$\|f\|_2 = \left\{ \sum_{i=0}^{\infty} |f_i|^2 \right\}^{1/2}, \tag{2.12}$$

$$\|f\|_\infty = \sup_i |f_i|. \tag{2.13}$$

We define the sequence spaces l_1, l_2, l_∞, respectively, to consist of those sequences $\{f_i\}$ such that $\|f\|_1$, $\|f\|_2$, $\|f\|_\infty$ is finite. It is easily shown that l_1 is a commutative Banach algebra with identity.

It is known [9] that a linear time-invariant discrete-time system with unit pulse response $\{h_i\}$ is BIBO-stable if and only if $h \in l_1$. Given such a sequence, we can associate with it its z-transform

$$F(z) = \sum_{i=0}^{\infty} f_i z^i. \tag{2.14}$$

Note that we use z^i instead of z^{-i} as is customary. The effect of this is that a z-transform represents a stable system if all of its poles are *outside* the unit disk rather than inside it. Let $\hat{A}_d$ denote the set of z-transforms of l_1 sequences. Then $\hat{A}_d$ is precisely the set of digital transfer functions of BIBO-stable discrete-time systems.

As before, it is possible to define two distinct norms on $\hat{A}_d$. Given $H \in \hat{A}_d$, let $\{h_i\}$ denote its inverse z-transform, and define

$$\|H\|_{\hat{A}_d} = \|h\|_1 = \sum_{i=0}^{\infty} |h_i|, \tag{2.15}$$

$$\|H\|_\infty = \max_{\theta\in[0,2\pi]} |H(e^{j\theta})|. \tag{2.16}$$

Note that $\|H\|_\infty \le \|H\|_{\hat{A}_d}$ for all $H \in \hat{A}_d$. The interpretation of these two norms is as follows. Given $H \in \hat{A}_d$, one can associate with it an operator, which maps a sequence $\{f_i\}$ into its convolution with $\{h_i\}$. Then

$$\|H\|_{\hat{A}_d} = \sup_{f \in l_\infty - 0} \frac{\|Hf\|_\infty}{\|f\|_\infty}, \tag{2.17}$$

$$\|H\|_\infty = \sup_{f \in l_2 - 0} \frac{\|Hf\|_2}{\|f\|_2}. \tag{2.18}$$

As is the case with continuous-time systems, it is understood hereafter that all elements of $\hat{A}_d$ are rational unless explicitly stated to the contrary.

In some papers, e.g., [4], [5], the symbol RH_∞ is used to denote the set of rational functions that are analytic on the closed unit disk, equipped with the norm (2.18). In analogy with the continuous-time case, we see that $\hat{A}_d$ and RH_∞ are distinct normed spaces whose underlying linear vector spaces are the same.

We are now in a position to state precisely the problem studied in this paper. Suppose a plant P is given, together with two stable transfer matrices T and W. Let $S(P)$ denote the set of all controllers that stabilize P; then the objective is to find a controller in $S(P)$ that minimizes the cost function

$$J = \|T(I+PC)^{-1}W\|_{\hat{A}} \text{ or } J = \|T(I+PC)^{-1}W\|_{\hat{A}_d} \tag{2.19}$$

where the first functional pertains to continuous-time systems while the second is for discrete-time systems.

The problem to be solved can be restated in a more convenient form using the results of [11], [12] that give a simple parametrization of all controllers that stabilize a given plant, together with an expression for all the resulting stable transfer matrices. In fact, the problem of minimizing J of (2.19) with respect to $C \in S(P)$ is equivalent to that of minimizing a functional of the form

$$J = \|F - GRH\|_{\hat{A}} \text{ or } \|F - GRH\|_{\hat{A}_d} \tag{2.20}$$

by a suitable choice of a matrix R with elements in $\hat{A}$ or $\hat{A}_d$. The interpretation of the cost function (2.19), as well as the reformulation of the problem in the form (2.20), is discussed in [7, sect. 6.1, 6.2].

This section is concluded with some observations. First, note that there is no *norm preserving* map between $\hat{A}$ and $\hat{A}_d$. Thus, it is necessary to treat continuous-time and discrete-time systems separately. This is in contrast to the case of H_∞-norm optimization, where the two cases can be treated in a common framework by employing a bilinear transform (see [7, sect. 6.4]). Another casualty of switching from the H_∞-norm criterion to the $\hat{A}$-norm criterion is the ability to discard inner factors; the reason is that multiplication by an inner function *does not in general preserve $\hat{A}$-norms.* In fact, the only inner functions in $\hat{A}$ with unit norm are ± 1; the only inner functions in $\hat{A}_d$ with unit norm are $\pm z^m$, where m is a nonnegative integer. One can easily compute that

$$\left\| \frac{s-a}{s+a} \right\|_{\hat{A}} = 3 \quad \text{if } a > 0, \tag{2.21}$$

$$\left\| \frac{z-a}{1-az} \right\|_{\hat{A}_d} = 1 + 2|a| \quad \text{if } |a| < 1. \tag{2.22}$$

Thus, in computing the minimum of the function $\|f - rg\|_{\hat{A}}$, it is *not* possible to discard any common inner factors of f and g. This too is in contrast with H_∞-norm minimization, where multiplication by inner functions preserves norms, so that if $f = f_1 h$, $g = g_1 h$, and h is inner, then the problem of minimizing $\|f - rg\|_\infty$ is equivalent to that of minimizing $\|f_1 - rg_1\|_\infty$.

III. Simple Cases

In this section, we first consider the minimization of functions of the type $\|f - rg\|_{\hat{A}}$ in the special case where g has no zeros in the open right half-plane and has possibly some zeros on the $j\omega$-axis. It is shown that the infimum of the above norm as r varies equals zero. Since these facts are well known in the case where the norm in question is the H_∞-norm, these results are perhaps not surprising. However, the path towards the solution in the current situation is quite different from that in H_∞-norm minimization. Specifically, in the latter theory one uses the notion of an outer function to show that various H_∞-norms can be made arbitrarily small. In contrast, in the case of the norm $\|\cdot\|_{\hat{A}}$, one is required to estimate the *time domain* norms of various quantities, which necessitates rather different reasoning. Next, it is shown that, if g has only a single simple zero in the extended RHP,[2] then $\|f - rg\|_\infty$ and $\|f - rg\|_{\hat{A}}$ have the same minima, and that in fact the same choice of r achieves each minimum. Finally, it is shown that a simplifying argument used in H_∞-norm minimization does not work in the case of $\hat{A}$-norm minimization.

Theorem 3.1: Let $f, g \in \hat{A}$, and suppose that two conditions are satisfied: i) the only C_+-zeros of g are on the $j\omega$-axis, and ii) whenever $g(j\omega) = 0$, we have that $f(j\omega) = 0$; if in addition $g(\infty) = 0$, then $f(\infty) = 0$. Under these conditions, we have

$$\inf_{r \in \hat{A}} \|f - rg\|_{\hat{A}} = 0. \tag{3.1}$$

The proof of this theorem is based on several lemmas.

Lemma 3.1: Suppose $a \in \hat{A}$ and that $a(0) = 0$. Then

$$\left\| \left[\frac{\epsilon}{s+\epsilon} \right]^k a(s) \right\|_{\hat{A}} \to 0 \quad \text{as } \epsilon \to 0 \tag{3.2}$$

for each integer $k \ge 1$.

In the above equation, we have been slightly sloppy in writing, e.g., $\|\epsilon/s + \epsilon a(s)\|_{\hat{A}}$ for the norm of the *function* $s \mapsto \frac{\epsilon}{s+\epsilon} a(s)$. This sloppiness should prove harmless and saves us from pedantry.

Proof: The result is first established for $k = 1$. By partial fraction expansion,

$$\frac{\epsilon}{s+\epsilon} a(s) = \frac{\epsilon a(-\epsilon)}{s+\epsilon} + \sum_{i=1}^{k} \sum_{j=1}^{m_i} \frac{c_{ij}(\epsilon)}{(s+p_i)^j} \tag{3.3}$$

where $-p_1, \cdots, -p_k$ are the distinct poles of $a(s)$, of multiplicities $m_1, \cdots, m_k$ (and none of these is zero). Now the constants $c_{ij}(\epsilon)$ are easily shown to be $O(\epsilon)$, so the norm of these terms goes to zero as $\epsilon \to 0$. Finally, noting that the norm of $(\epsilon/s + \epsilon)$ is one for all ϵ, we get

$$\left\| \frac{\epsilon}{s+\epsilon} a(-\epsilon) \right\|_{\hat{A}} = |a(-\epsilon)| \to 0 \quad \text{as } \epsilon \to 0 \tag{3.4}$$

since $a(0) = 0$.

Next, if $k \ge 1$, then we have that

$$\left\| \left[\frac{\epsilon}{s+\epsilon} \right]^k a(s) \right\|_{\hat{A}} \le \left\| \frac{\epsilon}{s+\epsilon} \right\|_{\hat{A}}^{k-1} \cdot \left\| \frac{\epsilon}{s+\epsilon} a(s) \right\|_{\hat{A}}$$

$$= \left\| \frac{\epsilon}{s+\epsilon} a(s) \right\|_{\hat{A}} \tag{3.5}$$

which goes to zero as $\epsilon \to 0$. This concludes the proof.

[2] The extended RHP consists of the closed RHP plus the point at infinity.

Lemma 3.2: Suppose $a(j\omega_i) = 0$ for some ω_i. Then

$$\left\| \left[\frac{\epsilon s}{s^2+\epsilon s+\omega_i^2} \right]^k a(s) \right\|_{\hat{A}} \to 0 \quad \text{as } \epsilon \to 0 \tag{3.6}$$

for each integer k.

The proof is omitted as it is entirely similar to that of Lemma 3.1 above.

Lemma 3.3: Suppose $a(\infty) = 0$. Then

$$\left\| \left[\frac{\epsilon s}{\epsilon s+1} \right]^k a(s) \right\|_{\hat{A}} \to 0 \quad \text{as } \epsilon \to 0 \tag{3.7}$$

for each integer k.

Proof: As before, by partial fraction expansion, we have

$$\frac{\epsilon s a(s)}{\epsilon s+1} = \frac{-a(-1/\epsilon)}{\epsilon s+1} + \sum_i \sum_j \frac{c_{ij}(\epsilon)}{(s+p_i)^j} \tag{3.8}$$

where p_i are the distinct poles of $a(s)$. It can once again be verified that the constants c_{ij} are $O(\epsilon)$; the reason is that the pole at $-1/\epsilon$ contributes a term that becomes smaller and smaller as $\epsilon \to 0$. Hence, the norm of all terms except the first is $O(\epsilon)$. Now the norm of the first term is $|a(-1/\epsilon)|$, which is also $O(\epsilon)$. Therefore,

$$\left\| \frac{\epsilon s}{\epsilon s+1} a(s) \right\|_{\hat{A}} \to 0 \quad \text{as } \epsilon \to 0. \tag{3.9}$$

For $k \geq 1$, we note that

$$\frac{\epsilon s}{\epsilon s+1} = 1 - \frac{1}{\epsilon s+1}. \tag{3.10}$$

Hence, the norm of this function in $\hat{A}$ equals two. Thus, for $k \geq 2$,

$$\left\| \left[\frac{\epsilon s}{\epsilon s+1} \right]^k a(s) \right\|_{\hat{A}} \leq \left\| \frac{\epsilon s}{\epsilon s+1} \right\|_{\hat{A}}^{k-1} \cdot \left\| \frac{\epsilon s}{\epsilon s+1} a(s) \right\|_{\hat{A}}$$

$$= 2^{k-1} \left\| \frac{\epsilon s}{\epsilon s+1} a(s) \right\|_{\hat{A}} \to 0 \quad \text{as } \epsilon \to 0. \tag{3.11}$$

Proof of Theorem 3.1: Let $\pm j\omega_i$, $i = 1, \cdots, k$ denote the distinct $j\omega$-axis zeros of g, other than at the origin, and let m_i denote the multiplicity of $j\omega_i$ as a zero of g. Let m_0, m_∞ denote the multiplicities of zero and infinity as zeros of g; if g does not vanish at either point, simply set the corresponding multiplicity to zero. Then g can be expressed in the form

$$g(s) = u(s) \cdot \left[\frac{s}{s+1} \right]^{m_0} \cdot \left[\frac{1}{s+1} \right]^{m_\infty} \cdot \prod_{i=1}^{k} \left[\frac{s^i+\omega_i^2}{(s+1)^2} \right]^{m_i} \tag{3.12}$$

where u is a unit of $\hat{A}$. Now define

$$v_\epsilon = u(s) \cdot \left[\frac{s+\epsilon}{s+1} \right]^{m_0} \cdot \left[\frac{\epsilon s+1}{s+1} \right]^{m_\infty} \cdot \prod_{i=1}^{k} \left[\frac{s^2+\epsilon s+\omega_i^2}{(s+1)^2} \right]^{m_i}. \tag{3.13}$$

Then v_ϵ is a unit of $\hat{A}$ for each $\epsilon > 0$, i.e., $1/v_\epsilon \in \hat{A}$ for all $\epsilon > 0$. Now define $r_\epsilon = f/v_\epsilon$. Then $r_\epsilon \in \hat{A}$ for all $\epsilon > 0$. The claim is that $\|f - r_\epsilon g\|_{\hat{A}} \to 0$ as $\epsilon \to 0$.

To establish the claim, note that $f - r_\epsilon g = f(1 - g/v_\epsilon)$. Note also that

$$\frac{s}{s+\epsilon} = 1 - \frac{\epsilon}{s+\epsilon}, \tag{3.14a}$$

$$\frac{1}{\epsilon s+1} = 1 - \frac{\epsilon s}{\epsilon s+1}, \tag{3.14b}$$

$$\frac{s^2+\omega_i^2}{s^2+\epsilon s+\omega_i^2} = 1 - \frac{\epsilon s}{s^2+\epsilon s+\omega_i^2}. \tag{3.14c}$$

If we substitute the above into (3.13) and expand the various powers using the binomial expansion, we get

$$1 - \frac{g}{v_\epsilon} = \sum_{i,j,l} c_{ijl} \left[\frac{\epsilon}{s+\epsilon} \right]^i \left[\frac{\epsilon s}{\epsilon s+1} \right]^j \left[\frac{\epsilon s}{s^2+\epsilon s+\omega_i^2} \right]^l. \tag{3.15}$$

Now by Lemmas 3.1–3.3 and the hypotheses on f, it follows that $f(1 - g/v_\epsilon) \to 0$ as $\epsilon \to 0$.

Theorem 3.2: Suppose $f, g \in \hat{A}$, and suppose g has only one simple zero in the extended closed RHP, at $s = \sigma$. Then

$$\min_{r \in \hat{A}} \|f - rg\|_{\hat{A}} = \min_{r \in \hat{A}} \|f - rg\|_\infty = |f(\sigma)| \tag{3.16}$$

and the unique choice of r that attains each minimum is

$$r(s) = \frac{f(s) - f(\sigma)}{g(s)}. \tag{3.17}$$

Proof: Since $\|f - rg\|_\infty \leq \|f - rg\|_{\hat{A}}$ for each $r \in \hat{A}$, it follows that

$$\inf_{r \in \hat{A}} \|f - rg\|_\infty \leq \inf_{r \in \hat{A}} \|f - rg\|_{\hat{A}}. \tag{3.18}$$

Now it is known [2], [3] that the choice r in (3.17) achieves the minimum on the left side of (3.18), and that, with this choice of r, $f - rg$ equals the constant function $f(\sigma)$, whose $\hat{A}$-norm and H_∞-norm are each equal to $|f(\sigma)|$.

The discrete-time analogs of Theorems 3.1 and 3.2 are easy and left to the reader.

Next it is shown that a simplifying argument that is very useful in H_∞-norm minimization is not applicable in $\hat{A}$-norm minimization. For purposes of discussion, we recall the following known result. It should be noted that Lemma 3.4 below is not the most general result of its kind; but it is adequate for the purposes of illustrating the point we wish to make here.

Lemma 3.4 [1], [7, Lemma (6.4.10)]: Suppose $f, g \in \hat{A}$, and factor g as a product uvb, where u is a unit of $\hat{A}$, the zeros of v are all on the $j\omega$-axis or at infinity, and the zeros of b are all in the open right half-plane. Finally, suppose $f(j\omega) = 0$ whenever $v(j\omega) = 0$, and that $f(\infty) = 0$ if $v(\infty) = 0$. Under these conditions,

$$\inf_{r \in \hat{A}} \|f - rg\|_\infty = \inf_{r \in \hat{A}} \|f - rb\|_\infty. \tag{3.19}$$

Thus, Lemma 3.4 allows one to replace the problem of minimizing $\|f - rg\|_\infty$ by the simpler one of minimizing $\|f - rb\|_\infty$; this minimization problem is easily solved using interpolation theory [2], [3]. It is therefore worthwhile to ask whether an analogous result holds for $\hat{A}$-norm minimization. The discussion below, while it does not settle the question, does show that the method of proof used in the case of H_∞ breaks down in the case of $\hat{A}$.

Suppose r_0 achieves the minimum on the right side of (3.19). In proving Lemma 3.4, one modifies v to v_ϵ as in (3.13), and sets $r_\epsilon = r_0/uv_\epsilon$. Then

$$f - r_\epsilon g = f - r_\epsilon uvb = f - r_0 b \frac{v}{v_\epsilon}. \tag{3.20}$$

In the case of H_∞, the argument is that v/v_ϵ approaches 1 uniformly on all closed subsets that do not contain any zeros of v, while f is zero at the zeros of v (see [7, pp. 176–178] for more details). Thus,

$$\|f-r_\epsilon g\|_\infty \to \|f-r_0 b\|_\infty = \min_{r\in\hat{A}} \|f-rb\|_\infty. \tag{3.21}$$

We now show by example that such an approach fails in general in the case of $\hat{A}$.

Example 3.1: Let

$$f(s)=\frac{s}{s+1},\ g(s)=\frac{s(s-1)}{(s+1)^2},$$

$$u(s)=1,\ v(s)=\frac{s}{s+1},\ b(s)=\frac{s-1}{s+1},\ v_\epsilon(s)=\frac{s+\epsilon}{s+1}. \tag{3.22}$$

Then, since b has only a simple RHP zero, the minimum of $\|f - rb\|_{\hat{A}}$ is readily computed using the results of [2], [3] to be 0.5, achieved by the choice $r_0(s) = 0.5$. Now let

$$r_\epsilon(s)=\frac{r_0(s)v(s)}{v_\epsilon(s)}=0.5\frac{s}{s+\epsilon}. \tag{3.23}$$

Then

$$\begin{aligned} f(s)-r_\epsilon(s)g(s)&=0.5+0.5\epsilon\frac{s-1}{(s+1)(s+\epsilon)} \\ &=0.5+\frac{\epsilon/(1-\epsilon)}{s+1}-\frac{\epsilon(1+\epsilon)/2(1-\epsilon)}{s+\epsilon}. \end{aligned} \tag{3.24}$$

As $\epsilon \to 0$, the middle term looks like $\epsilon/(s+1)$, which is $O(\epsilon)$, while the last term looks like $-\epsilon/2(s+\epsilon)$. Hence, asymptotically

$$f(s)-r_\epsilon(s)g(s)\approx 0.5-\frac{\epsilon}{2(s+\epsilon)}, \tag{3.25}$$

$$\lim_{\epsilon\to 0} \|f-r_\epsilon g\|_{\hat{A}}=0.5+0.5=1, \tag{3.26}$$

which is larger than $0.5 = \|f - r_0 b\|_{\hat{A}}$.

IV. General Case

In this section, the problem of minimizing $\|f - rg\|_{\hat{A}}$ is studied without any simplifying assumptions on g. It turns out that an exact solution to the optimization problem is available only in one particular case, namely in the discrete-time case where g has one or more zeros at the origin and possibly one other simple zero inside the closed unit disk. In all other cases, a technique is presented for obtaining bounds on the optimal performance. A general conclusion that emerges is that the choice of r that minimizes the function $\|f - rg\|_\infty$ does not in general minimize $\|f - rg\|_{\hat{A}}$, in contrast to the situation in Theorem 3.2.

Theorem 4.1 below represents the only complete solution available in the general case.

Theorem 4.1: Suppose $f \in \hat{A}_d$, and suppose $g \in \hat{A}_d$ is of the form

$$g(z)=z^m\frac{z-a}{1-az},\ |a|<1. \tag{4.1}$$

Let $\Sigma_{i=0}^\infty f_i z^i$ denote the power series of f, and define

$$h(z)=\left[f(z)-\sum_{i=0}^{m-1} f_i z^i\right]\Big/z^m. \tag{4.2}$$

Then

$$\inf_{r\in\hat{A}_d} \|f-gr\|_{\hat{A}_d}=\sum_{i=0}^{m-1}|f_i|+|h(a)|. \tag{4.3}$$

Moreover, the unique choice of r that attains the infimum in (4.3) is

$$r(z)=\frac{h(z)-h(a)}{z-a}\cdot(1-az). \tag{4.4}$$

Proof: Let $r \in \hat{A}_d$ be arbitrary. Then $f - gr$ has the form

$$f-gr=\sum_{i=0}^{m-1} f_i z^i+z^m\left[h-r\frac{z-a}{1-az}\right]. \tag{4.5}$$

The inverse z-transform of the first term vanishes at all time instants beyond the mth, while the inverse z-transform of the second term vanishes at all time instants before the mth. Hence, from the definition (2.15), it follows that

$$\begin{aligned} \|f-gr\|_{\hat{A}_d}&=\sum_{i=0}^{m-1}|f_i|+\left\|z^m\left[h-r\frac{z-a}{1-az}\right]\right\|_{\hat{A}_d} \\ &=\sum_{i=0}^{m-1}|f_i|+\left\|\left[h-r\frac{z-a}{1-az}\right]\right\|_{\hat{A}_d}. \end{aligned} \tag{4.6}$$

Now from the discrete-time analog of Theorem 3.2, we know that the unique choice of $r \in \hat{A}_d$ that minimizes the second term is given by (4.4), and that the minimum value is $|h(a)|$. The result follows.

Example 4.1: Consider the problem of minimizing the cost functional

$$J=\|f-rg\|_{\hat{A}_d}, \tag{4.6}$$

where

$$f=\frac{z-0.5}{1-0.5z},\ g=z^2\frac{z+0.5}{1+0.5z}. \tag{4.7}$$

Since f is already an inner function, it follows from [15, pp. 136–139] that the optimal choice of r if one wishes to minimize $\|f - gr\|_\infty$ is $r = 0$, which gives an H_∞-optimal value for J of $\|f\| = 1$. On the other hand, applying the results of Theorem 4.1, we first observe that

$$f(z)=-0.5+0.75z+\cdots \tag{4.8}$$

$$h(z)=\frac{f(z)+0.5-0.75z}{z^2}=\frac{0.375}{1-0.5z}. \tag{4.9}$$

Hence, the optimal value of J as defined above is $0.5 + 0.75 + |h(-0.5)| = 1.54$. Moreover, the optimizing choice of r is given by

$$r(z)=\frac{h(z)-h(-0.5)}{1+0.5z}(z+0.5)=0.15\frac{1+0.5z}{1-0.5z}. \tag{4.10}$$

Hence, in this case the minimum value of $\|f - rg\|_{\hat{A}}$ is 55 percent larger than the minimum value of $\|f - rg\|_\infty$, and is achieved for a different choice of r.

Since there exist sequences $\{h_i(z)\}$ of functions such that $\|h_i\|_\infty = 1$ but $\|h_i\|_{\hat{A}_d} \to \infty$, it is easy enough to construct examples where the minimum value of $\|f - rg\|_{\hat{A}}$ is arbitrarily larger than the minimum value of $\|f - rg\|_\infty$. This is left to the reader.

Now we study the general case, without assuming anything about the form of g. In this case, no closed form solution is available, but a method is presented for obtaining an upper bound on the minimum value of $f - rg\|_{\hat{A}}$. Since the minimum of $\|f - rg\|_\infty$ is a lower bound for the minimum value of $\|f - rg\|_{\hat{A}}$, it is in fact possible to bracket the latter minimum.

To provide the motivation for the discussion below, we first briefly review the philosophy behind the H_∞-optimal solution, as exemplified by the discussion in [16] of the problem of Nevanlinna–Pick interpolation. To keep matters simple, we focus on the discrete-time case, and restrict attention to the case where the function g has two distinct simple zeros in the closed unit disk, each lying in the open unit disk. That is, we assume that $g(z)$ has the form

$$g(z)=\frac{(z-\alpha)(z-\beta)}{(1-\alpha z)(1-\beta z)}, \tag{4.11}$$

where $|\alpha|$, $|\beta| < 1$. Let a, b denote the values of f at α and β, respectively. Thus, the H_∞-optimization problem is to find a function of minimum H_∞ norm such that its value at α is a and its value at β is b. Now, if $\alpha = 0$ and $a = 0$, then by the Schwarz lemma, the optimal interpolating function is easily shown to be bz/β. Suppose it is not the case that $a = 0$ and $\alpha = 0$. Then the idea is to map the closed unit disk into itself by means of a bilinear transformation of the form

$$z\leftarrow\frac{z-\alpha}{1-\alpha z}, \quad x\leftarrow\frac{x-a}{1-ax}. \tag{4.12}$$

These transformations map a into 0, b into $(b - a)/(1 - ab)$, α into 0, and β into $(\beta - \alpha)/(1 - \alpha\beta)$. This transforms the original interpolation problem into another one where one of the values of z lies at the origin, and where the value at the origin of the function to be interpolated is also 0. The optimal interpolating function is now (in the new coordinates)

$$h(z)=\frac{(b-a)/(1-ba)}{(\beta-\alpha)/(1-\beta\alpha)}z. \tag{4.13}$$

By carrying out the inverse of the transformations (4.12), it is now possible to recover the optimal interpolating function in the original coordinates.

The main fact used repeatedly in the above argument is the fact that the bilinear transformation (4.12) maps the closed unit disk into itself, so that the H_∞-optimal value of the original interpolation problem is less than or equal to one if and only if the optimal value of the modified problem is less than or equal to one, i.e., if and only if

$$\left|\frac{b-a}{1-ba}\right|\le\left|\frac{\beta-\alpha}{1-\beta\alpha}\right|. \tag{4.14}$$

This is indeed the same as the result obtained by testing the nonnegative definiteness of the so-called Pick matrix. Now, when one wishes to minimize the $\hat{A}_d$ norm, it is quite feasible to carry out a transformation of the *independent* variable (i.e., z), but it is not possible to transform the *dependent* variable without affecting norms. Rather than attempting to give a general theory, we illustrate the technique by means of an example.

Example 4.2: Consider the problem of minimizing

$$J=\|f-gr\|_{\hat{A}_d}, \tag{4.15}$$

where

$$f=\frac{z-0.8}{1-0.8z}, \quad g=\frac{(z-0.4)(z+0.8)}{(1-0.4z)(1+0.8z)}. \tag{4.16}$$

Since f is already inner and has one fewer zero than g, it follows by Lemma 4.1 that the choice of r that achieves the H_∞-minimum is $r = 0$, which corresponds to $\|f - gr\|_\infty = 1$. Hence, this is also a lower bound for J, i.e., $J \geq 1$ irrespective of how r is chosen. If the "naive" choice of $r = 0$ is applied to the optimization problem at hand, the resulting value of J is $\|f\|_{\hat{A}_d} = 2.6$. On the other hand, it is possible to do better by employing the transformation of variables described in the preceding paragraph.

Let

$$\lambda=\frac{z-0.4}{1-0.4z}, \quad z=\frac{\lambda+0.4}{1+0.4\lambda}. \tag{4.17}$$

Then, since $f(0.4) = -10/17$, we can write

$$f=-\frac{10}{17}+\lambda\cdot\frac{9}{17}\frac{1-0.4z}{1-0.8z} \tag{4.18}$$

Let

$$h=\frac{9}{17}\frac{1-0.4z}{1-0.8z}, \quad g_1=\frac{z+0.8}{1+0.8z}. \tag{4.19}$$

Then

$$f-gr=-\frac{10}{17}+\lambda(h-rg_1). \tag{4.20}$$

Now, by Theorem 3.2, we know that we can make $\|h - rg_1\|_{\hat{A}_d}$ equal $|h(-0.8)|$ by an appropriate choice of r, namely

$$r(z)=\frac{h(z)-h(-0.8)}{g_1(z)}=0.1291\,\frac{1+0.8z}{1-0.8z}. \tag{4.21}$$

With choice of r, we get

$$f-gr=-\frac{10}{17}+h(-0.8)\lambda=-0.5882+0.4261\lambda. \tag{4.22}$$

$$\|f-gr\|_{\hat{A}_d}\le 0.5882+0.4261\|\lambda\|_{\hat{A}_d}=1.3552. \tag{4.23}$$

It is of course possible to compute $\|f - gr\|_{\hat{A}_d}$ precisely by taking inverse z-transforms; in this case it turns out that the upper bound in (4.23) is exact. Compare the number 1.3552 obtained above to 2.6, which corresponds to using the H_∞-optimal choice of r. At this stage, we still do not know the optimal value of J, but we do know that it is somewhere between 1 and 1.3552.

It should be clear that there is nothing special in the fact that the function g in each of the above examples has only two unstable zeros; in fact, the iterative procedure is applicable to functions having an arbitrary number of unstable zeros. Moreover, it has an obvious discrete-time analog.

V. Disturbance Rejection

In this section, the various minimization results presented in earlier sections are interpreted in terms of disturbance rejection.

Theorem 5.1: Suppose a plant p is of the form gp_1, where $g \in \hat{A}$ has zeros only on the $j\omega$-axis or at infinity, and p_1 has no zeros in the extended RHP (but it can have poles there). Finally, suppose that w is a multiple of g in $\hat{A}$. Under these conditions,

$$\inf_{c\in S(p)}\|(1+pc)^{-1}w\|_{\hat{A}}=0. \tag{5.1}$$

Proof: The theorem is first proved under the assumption that p is stable in addition to satisfying the above hypotheses. In this case, the assumptions imply that p_1 is a unit of $\hat{A}$. Moreover, as shown in [7, sect. 6.4], the problem can be reformulated as one of minimizing

$$J=\|w(1-rp)\|_{\hat{A}} \tag{5.2}$$

with respect to $r \in \hat{A}$. Now let $q = rp_1$ denote another free parameter. Then

$$J=\|w(1-gq)\|_{\hat{A}}. \tag{5.3}$$

Since w is a multiple of g, it follows from Theorem 5.1 that

$$\inf_q J=0. \tag{5.4}$$

This establishes the theorem in the case where p is stable.

To establish the theorem for the case where p is unstable, note first of all that the plant p is strongly stabilizable, i.e., there exists a *stable* controller c_0 that stabilizes p; the reason is that p satisfies the parity interlacing property of [13]. Accordingly, choose a stable c_0 that stabilizes p, and let p_0 denote $p/(1+c_0p)$. Then, as shown in [7, sect. 5.3], the RHP zeros of p_0 are precisely the same as those of p, since c_0 is stable; in other words, $p_0 = gu$, where u is a unit of $\hat{A}$. Next, observe [1, p. 307], [7, Theorem (5.3.10)], [14, Theorem (8.4.8)] that, if c is any controller that stabilizes p_0, then the controller $c + c_0$ stabilizes the original plant p. Also,

$$\frac{1}{1+p(c+c_0)}=\frac{1}{1+p_0c}\cdot\frac{1}{1+pc_0}. \tag{5.5}$$

Hence, if the controller $c + c_0$ is applied to the original plant p, the resulting cost functional of (5.1) becomes

$$J=\left\|\frac{w}{1+p(c+c_0)}\right\|_{\hat{A}}=\left\|\frac{w}{1+p_0c}\cdot\frac{1}{1+pc_0}\right\|_{\hat{A}}$$

$$\leq\left\|\frac{w}{1+p_0c}\right\|_{\hat{A}}\cdot\left\|\frac{1}{1+pc_0}\right\|_{\hat{A}}. \tag{5.6}$$

However, by the earlier discussion of the stable plant case, it follows that the first quantity on the right side can be made arbitrarily small by a suitable choice of $c \in S(p_0)$. This concludes the proof.

Next, we interpret Theorem 3.2 in terms of optimal disturbance rejection.

Theorem 5.2: Suppose p is a stable scalar plant, and suppose p has only one RHP zero, namely a simple one at $s = \sigma$. Suppose also that w is a unit of $\hat{A}$. Under these conditions,

$$\min_{c\in S(p)}\left\|\frac{w}{1+pc}\right\|_{\hat{A}}=\min_{c\in S(p)}\left\|\frac{w}{1+pc}\right\|_{\infty}=|w(\sigma)|. \tag{5.7}$$

Moreover, the unique controller that achieves each minimum is $c = r/(1-pr)$, where

$$r(s)=\frac{w(s)-w(\sigma)}{w(s)p(s)}. \tag{5.8}$$

Note that if p is an *unstable* scalar plant with only one RHP zero, then it is no longer true in general that

$$\min_{c\in S(p)}\left\|\frac{w}{1+pc}\right\|_{\hat{A}}=\min_{c\in S(p)}\left\|\frac{w}{1+pc}\right\|_{\infty}. \tag{5.9}$$

The reason is that, if p is unstable, then the cost function on the left side of (5.7) becomes

$$\|w(y-nr)d\|_{\hat{A}} \tag{5.10}$$

where $p = n/d$ is a coprime factorization and $xn + yd = 1$. However, in this case, it is *not* in general possible to discard the inner factor of wd, as in the case of H_∞-norm minimization. This is illustrated by example.

Example 5.1: Let

$$p(z)=\frac{z}{z-0.5},\quad w(z)=1. \tag{5.11}$$

Then a coprime factorization n/d of p and a particular solution x, y of the Aryabhatta identity $xn + yd = 1$ are given by

$$n(z)=z,\ d(z)=z-0.5,\ x(z)=2,\ y(z)=-2. \tag{5.12}$$

Hence,

$$\min_{c\in S(p)}\|(1+pc)^{-1}\|_{\hat{A}_d}=\min_{r\in\hat{A}_d}\|(y-nr)d\|_{\hat{A}_d}, \tag{5.13}$$

and of course the same is true of H_∞-norms as well. Let us first examine the H_∞ optimum. It is routine to show using the methods of [2], [3] that the optimal choice of r is

$$r_0(z)=\frac{1}{1-0.5z} \tag{5.14}$$

which results in $\|(y - nr_0)d\|_\infty = 2$. The corresponding optimal controller is

$$c=\frac{x+dr_0}{y-nr_0}=-0.75. \tag{5.15}$$

When one attempts to minimize $\|(y - nr)d\|_{\hat{A}_d}$, it is not possible to discard the inner factor of d. In fact, the choice of r_0 in (5.14) results in

$$y-nr_0=-\frac{2(z-0.5)}{1-0.5z},$$

$$\|y-nr_0\|_{\hat{A}_d}=4. \tag{5.15}$$

Using the results of Theorem 4.1, it can be shown that

$$\min_{r\in\hat{A}_d}\|-2z+1-r(z)z(z-0.5)\|_{\hat{A}_d}=3 \tag{5.16}$$

corresponding to the choice $r = 0$, and the $\hat{A}_d$-optimal controller is $c = x/y = -1$.

We present one last result.

Theorem 5.2: Consider the discrete-time case, and suppose the scalar plant p has two properties: i) the only poles of p inside the closed unit disk are at the origin; and ii) p has only one zero inside the closed unit disk, namely a simple zero at $z = a$. Finally, suppose w is a unit of $\hat{A}_d$. Under these conditions,

$$\min_{c\in S(p)}\left\|\frac{w}{1+pc}\right\|_{\hat{A}_d}=\min_{c\in S(p)}\left\|\frac{w}{1+pc}\right\|_{\infty}. \tag{4.17}$$

Moreover, both minima are attained by the same controller.

The proof is easy and is based on the fact that multiplication by the function z^n where n is some integer preserves $\hat{A}_d$-norms.

VI. Multivariable Systems

In this section, we present a simple extension of Theorem 5.1 to multivariable minimum phase systems. At present this is the only result that is available. A preliminary lemma which facilitates the proof is presented first. Note that, in this section, $M(\hat{A})$ denotes the set of matrices, of whatever dimensions, with elements in $\hat{A}$.

Lemma 6.1: Suppose $F, G, H \in M(\hat{A})$, and suppose G, H have the following forms: $G = D_gU_g$, $H = U_hD_h$, where U_g, U_h are unit matrices in $M(\hat{A})$, and D_g, D_h are diagonal matrices. Further, if

$$D_g=\text{diag }\{d_{g1},\cdots,D_{gn}\},\ D_h=\text{diag }\{d_{h1},\cdots,D_{hm}\}, \tag{6.1}$$

then each d_{gi}, d_{hj} has its zeros only on the $j\omega$-axis or at infinity. Finally, suppose $F(j\omega) = 0$ whenever $d_{gi}(j\omega)$ or $d_{hj}(j\omega)$ equals zero. Under these conditions,

$$\inf_{R\in M(\hat{A})}\|F-GRH\|_{\hat{A}}=0. \tag{6.2}$$

Theorem 6.1: Suppose a plant P is of the form gP_1, where $g \in \hat{A}$ has zeros only on the $j\omega$-axis or at infinity, and P_1 has full row rank at all s in the extended RHP. Suppose $T, W \in M(\hat{A})$, and either T or W is of the form gM for some $M \in M(\hat{A})$. Under these conditions,

$$\inf_{c\in S(P)}\|T(I+PC)^{-1}W\|_{\hat{A}}=0. \tag{6.3}$$

The proof of Lemma 6.1 and Theorem 6.1 are very similar to their scalar counterparts and are therefore omitted.

VII. Conclusions

In this paper, we have formulated the problem of optimal disturbance rejection in the case where the disturbance is generated as the output of a stable system in response to an input which is assumed to be of unit amplitude, but is otherwise arbitrary. The objective is to choose a controller that minimizes the maximum amplitude of the plant output in response to such a disturbance. Mathematically, this corresponds to requiring uniformly good disturbance rejection over all time. Since the problem of optimal tracking is equivalent to that of optimal disturbance rejection if a feedback controller is used (see [7, sect. 5.6]), the theory presented here can also be used to design optimal controllers that achieve uniformly good tracking over all time rather than a tracking error whose L_2-norm is small, as is the case with the currently popular H_∞ theory.

It has been shown that some results from the H_∞ theory carry over to the present setting, but a great many do not. Specifically, it has been shown that arbitrarily good disturbance rejection is possible in the case of minimum phase plants, subject to certain technical assumptions; this result is analogous to that in the H_∞ theory. Similarly, it has been shown that in the case of stable scalar plants with exactly one unstable zero, the optimal achievable performance in the case of bounded disturbances is exactly the same as that achievable with square-integrable disturbances, and is in fact achieved with the same controller. In other situations, it has been shown by example that the optimal performance achievable in the case of bounded disturbances can be worse than in the case of disturbances with finite energy, and can require a different choice of optimal controller. Closed form optimal solutions have been obtained for some special cases of scalar plants, and a method has been presented for estimating the optimum in the general case as well as for generating suboptimal controllers.

To the best of the author's knowledge, this is the first paper on this subject, which stands as a complement to the theory of H_∞-norm minimization. It is hoped that further research will shed light on most of the questions left unanswered here.

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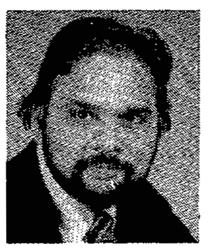

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