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Optimal relaxation parameter for the Uzawa Method

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Summary. We consider the Uzawa method to solve the stationary Stokes equations discretized with stable finite elements. An iteration step consists of a velocity update $\mathbf{u}^{n+1}$ involving the (augmented Lagrangian) operator $-\nu\Delta - \rho\nabla\text{div}$ with $\rho \geq 0$, followed by the pressure update $p^{n+1} = p^n - \alpha\nu\text{div}\,\mathbf{u}^{n+1}$, the so-called Richardson update. We prove that the inf-sup constant β satisfies $\beta \leq 1$ and that, if $\sigma = 1 + \rho\nu^{-1}$, the iteration converges linearly with a contraction factor $\beta^2\alpha\sigma^{-1}\big(2\sigma - \alpha\big)$ provided $0 < \alpha < 2\sigma$. This yields the optimal value $\alpha = \sigma$ regardless of β.

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1 Introduction

Given an open bounded polygon Ω in $\mathbb{R}^d$, with $d \geq 2$, we consider the stationary Stokes equations, namely the simplest model for incompressible viscous flows:

$$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in } \Omega, \tag{1.1}$$

$$\text{div}\,\mathbf{u} = 0, \quad \text{in } \Omega, \tag{1.2}$$

with vanishing Dirichlet boundary condition $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$ and pressure mean-value $\int_\Omega p = 0$. Here the unknowns are the (vector) velocity field $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ and the (scalar) pressure $p \in L_0^2(\Omega)$; the forcing function satisfies $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\nu = Re^{-1}$ is the reciprocal of the Reynolds number.

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In view of the incompressibility constraint (1.2), the momentum equation (1.1) is equivalent to the *augmented Lagrangian* formulation with $\rho \geq 0$

$$-\nu\triangle\mathbf{u} - \rho\nabla\text{div }\mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in } \Omega.$$

This equivalence is no longer true at the discrete level, where the additional operator $-\rho\nabla\text{div}$ may improve the convergence of iterative methods [3]. We provide a quantitative measure of such improvement in this paper.

The following (infinite-dimensional) Uzawa algorithm to solve the Stokes system is known to converge for appropriate values of the *relaxation parameter* α [2–6].

Algorithm 1 (Uzawa Method) *Given a suitable relaxation parameter* $\alpha > 0$ *and initial guess* p^0*:*

Step 1: Find $\mathbf{u}^{n+1} \in \mathbf{H}_0^1(\Omega)$ *as the solution of*

$$-\nu\triangle\mathbf{u}^{n+1} - \rho\nabla\text{div }\mathbf{u}^{n+1} + \nabla p^n = \mathbf{f}, \qquad \textit{in } \Omega;$$

Step 2: Find $p^{n+1} \in L_0^2(\Omega)$ *from the Richardson update*

$$p^{n+1} = p^n - \alpha\nu\text{div }\mathbf{u}^{n+1}.$$

Convergence of Algorithm 1 for $\rho = 0$ is proved via boundedness and coercivity of the Schur complement operator $S = -\text{div }(-\triangle^{-1})\nabla$ with sufficiently small $\alpha < 1$ in [2,4]. In [6] Temam shows the convergence range $0 < \alpha < 2$ also for $\rho = 0$, but does not quite prove that Algorithm 1 is a contraction and thus cannot find the optimal value of α. For the case $\rho > 0$, Fortin and Glowinski prove convergence for $0 < \alpha \leq 2\rho/\nu$ using a spectral analysis [3].

The choice of relaxation parameter $\alpha > 0$ is crucial for the convergence of Uzawa method because a small value of α yields a large contraction factor whereas a large value may lead to divergence. It is the purpose of this note to show convergence for all $0 < \alpha < 2(1 + \rho/\nu)$ and that $\alpha = 1 + \rho/\nu$ is an optimal choice. This has been already instrumental in [1] for $\rho = 0$. Our analysis is in the spirit of that in [6] for $\rho = 0$, but it gives rise to more precise bounds.

We consider now a finite element discretization. Let $\mathfrak{T} = \{K\}$ be a shape-regular partition of Ω of local meshsize h into closed elements K; $\mathfrak{T}$ can be highly graded though. The finite element spaces to be used for approximating the velocity space $\mathbf{H}_0^1(\Omega)$ and pressure space $L_0^2(\Omega)$ are:

$$\mathbb{V}_h := \{\mathbf{v}_h \in \mathbf{H}_0^1(\Omega) : \mathbf{v}_h|_K \in \mathcal{P}(K), \text{ for all } K \in \mathfrak{T}\},$$
$$\mathbb{P}_h := \{\mathbf{v}_h \in L^2(\Omega) : p_h|_K \in \mathcal{Q}(K), \text{ for all } K \in \mathfrak{T}\},$$

where $\mathcal{P}(K)$ and $\mathcal{Q}(K)$ are spaces of polynomials with degree bounded uniformly with respect to $K \in \mathfrak{T}$ [2,4]. These spaces are compatible, namely they satisfy the following discrete *inf-sup condition: There exists a constant* $\beta > 0$ *such that* [2,4]

$$\inf_{p_h \in \mathbb{P}_h} \sup_{\mathbf{v}_h \in \mathbb{V}_h} \frac{\langle \operatorname{div} \mathbf{v}_h \,,\ p_h \rangle}{\|\nabla \mathbf{v}_h\| \|p_h\|} \geq \beta; \tag{1.3}$$

hereafter $\|\cdot\|$ indicates the L^2-norm in Ω. Hence, there is a unique solution $(\mathbf{u}_h, p_h) \in \mathbb{V}_h \times \mathbb{P}_h$ to the following discrete Stokes problem [2,4]:

$$\begin{aligned} \nu \langle \nabla \mathbf{u}_h \,,\ \nabla \mathbf{w}_h \rangle - \langle p_h \,,\ \operatorname{div} \mathbf{w}_h \rangle = \langle \mathbf{f} \,,\ \mathbf{w}_h \rangle \,, \qquad & \forall \mathbf{w}_h \in \mathbb{V}_h, \\ \langle \operatorname{div} \mathbf{u}_h \,,\ q_h \rangle = 0, \qquad & \forall q_h \in \mathbb{P}_h. \end{aligned} \tag{1.4}$$

Proposition 1 **(Inf-Sup Constant)** *Let* β *be the inf-sup constant of* (1.3). *Then we have*

$$\beta \leq 1. \tag{1.5}$$

The discrete Uzawa method, a discrete version of Algorithm 1, is known to be an effective iteration to compute $(\mathbf{u}_h, p_h)$, and reads as follows [2,4].

Algorithm 2 **(Discrete Uzawa Method)** *For a suitable* $\alpha > 0$ *and initial guess* $p_h^0 \in \mathbb{P}_h$*:*

Step 1: Find $\mathbf{u}_h^{n+1} \in \mathbb{V}_h$ *as the solution of*

$$\begin{aligned} \nu \left\langle \nabla \mathbf{u}_h^{n+1} \,,\ \nabla \mathbf{w}_h \right\rangle + \rho \left\langle \operatorname{div} \mathbf{u}_h^{n+1} \,,\ \operatorname{div} \mathbf{w}_h \right\rangle - \left\langle p_h^n \,,\ \operatorname{div} \mathbf{w}_h \right\rangle \\ = \langle \mathbf{f} \,,\ \mathbf{w}_h \rangle \,, \quad \forall \mathbf{w}_h \in \mathbb{V}_h; \end{aligned} \tag{1.6}$$

Step 2: Find $p_h^{n+1} \in \mathbb{P}_h$ *from the Richardson update*

$$\left\langle p_h^{n+1} \,,\ q_h \right\rangle = \left\langle p_h^n \,,\ q_h \right\rangle - \alpha \nu \left\langle \operatorname{div} \mathbf{u}_h^{n+1} \,,\ q_h \right\rangle. \tag{1.7}$$

In §§3 and 4, we prove the following sharp decay estimates.

Theorem 1 **(Convergence Rate for Pressure)** *If* $0 < \alpha < 2\sigma$*, then Algorithm 2 satisfies*

$$\left\| p_h - p_h^{n+1} \right\| \leq \left(1 - \alpha \beta^2 \sigma^{-2} \left(2\sigma - \alpha\right)\right)^{1/2} \|p_h - p_h^n\|, \tag{1.8}$$

where $\sigma := 1 + \frac{\rho}{\nu}$*. The same estimate is valid for Algorithm 1.*

Corollary 1 **(Convergence Rate for Velocity)** *Both Algorithms 1 and 2 satisfy*

$$\left\| \nabla (\mathbf{u}_h - \mathbf{u}_h^{n+1}) \right\| \leq \nu^{-1} \left(1 - \alpha \beta^2 \sigma^{-2} \left(2\sigma - \alpha\right)\right)^{n/2} \left\| p_h - p_h^0 \right\|. \tag{1.9}$$

Remark 1.1 (*Optimal relaxation Parameter*) Consider now the function $f(\alpha) = \big(1 - \frac{\alpha\beta^2}{\sigma^2}(2\sigma - \alpha)\big)$. We see that Algorithm 2 converges *linearly* with contraction factor $0 < f(\alpha) < 1$ provided $0 < \alpha < 2\sigma$. Since the minimum of $f(\alpha)$ is $1 - \beta^2$ at $\alpha = \sigma$, we conclude that the *optimal* value of α is

$$\alpha = 1 + \frac{\rho}{\nu}.$$

We observe that this result is independent of the domain Ω, and valid for both Algorithms 1 and 2, whereas the eigenvalues of the Schur complement operator, the discrete version of $S := \text{div}\,(\triangle + \rho\nu^{-1}\nabla\text{div}\,)^{-1}\nabla$, depend on Ω. It is plausible that for a given Ω and finite element pair $(\mathbb{V}_h, \mathbb{P}_h)$, a special analysis would yield a better value for α since Uzawa is simply a Richardson iteration for the Schur complement. It is also plausible that for a rectangular domain with high aspect ratio, $\alpha = \sigma$ is the only choice valid for all aspect ratios. This deserves further investigation.

We also point out that (1.9) improves upon [6], where $\mathbf{u}^n$ is shown to converge weakly in $\mathbf{H}_0^1(\Omega)$.

2 Proof of Proposition 1

In this section, we prove a couple of crucial properties of the divergence operator, in particular an upper bound for the inf-sup constant β of (1.3). Since the following known result plays a pivotal role in our subsequent discussion, we present its elementary proof; we refer to [6, p.140].

Lemma 2.1 (**Div-Grad Relation**) *For all* $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$, *we have*

$$\|\text{div}\,\mathbf{v}\| \le \|\nabla\mathbf{v}\|. \tag{2.1}$$

Proof Given $\mathbf{v} = (v_i)_{i=1}^d \in \mathbf{H}_0^1(\Omega)$, there exists a sequence $\{\mathbf{v}^n\} \in \mathbf{C}_0^\infty(\Omega)$ such that

$$\big\|\nabla(\mathbf{v}^n - \mathbf{v})\big\| \to 0 \qquad \text{as } n \to \infty. \tag{2.2}$$

Since $\mathbf{v}^n \in \mathbf{C}_0^\infty(\Omega)$, integration by parts implies

$$\begin{aligned}\big\|\text{div}\,\mathbf{v}^n\big\|^2 &= \int_\Omega \left(\sum_{i=1}^d \partial_{x_i} v_i^n\right)^2 d\mathbf{x}\\ &= \int_\Omega \sum_{i=1}^d \sum_{j=1}^d \partial_{x_i} v_i^n \partial_{x_j} v_j^n d\mathbf{x}\end{aligned}$$

$$= \int_\Omega \sum_{i=1}^d \sum_{j=1}^d \partial_{x_i} v_j^n \partial_{x_j} v_i^n \, d\mathbf{x}$$

$$\leq \int_\Omega \sum_{i=1}^d \sum_{j=1}^d (\partial_{x_i} v_j^n)^2 d\mathbf{x} = \left\| \nabla \mathbf{v}^n \right\|^2.$$

The assertion (2.1) follows from (2.2) upon passing to the limit $n \to \infty$. □

Applying Lemma 2.1, we can find an upper bound of the inf-sup constant β of (1.3).

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Proof of Proposition 1 Let $q_h \in \mathbb{P}_h$ be an arbitrary function. Then, the discrete inf-sup condition (1.3) is equivalent to the existence of a function $\mathbf{v}_h \in \mathbb{V}_h$ such that [2,4]

$$\langle \text{div } \mathbf{v}_h \, , \, q_h \rangle = \|q_h\|^2 \qquad \text{and} \qquad \|\nabla \mathbf{v}_h\| \leq \frac{1}{\beta} \|q_h\|. \tag{2.3}$$

Then, by virtue of (2.3) and Lemma 2.1, we obtain

$$\|q_h\|^2 = \langle \text{div } \mathbf{v}_h \, , \, q_h \rangle \leq \|\text{div } \mathbf{v}_h\| \; \|q_h\| \leq \|\nabla \mathbf{v}_h\| \; \|q_h\| \leq \frac{1}{\beta} \|q_h\|^2,$$

which implies the asserted estimate (1.5). □

3 Proof of Theorem 1: Case $\rho = 0$

In this section we prove Theorem 1 for $\rho = 0$. To this end, we use the following error functions:

$$\mathbf{E}_h^{n+1} = \mathbf{u}_h - \mathbf{u}_h^{n+1}, \qquad e_h^{n+1} = p_h - p_h^{n+1}.$$

We proceed in several steps. Upon subtracting (1.6) (with $\rho = 0$) from (1.4), we have

$$\nu \left\langle \nabla \mathbf{E}_h^{n+1} \, , \, \nabla \mathbf{w}_h \right\rangle - \left\langle e_h^n \, , \, \text{div } \mathbf{w}_h \right\rangle = 0, \qquad \forall \mathbf{w}_h \in \mathbb{V}_h. \tag{3.1}$$

Since $\mathbf{u}_h$ is a discrete divergence free function, (1.7) can be written by

$$\left\langle e_h^{n+1} \, , \, q_h \right\rangle = \left\langle e_h^n \, , \, q_h \right\rangle - \alpha \nu \left\langle \text{div } \mathbf{E}_h^{n+1} \, , \, q_h \right\rangle, \qquad \forall q_h \in \mathbb{P}_h. \tag{3.2}$$

If we choose $\mathbf{w}_h = \mathbf{E}_h^{n+1}$, then (3.1) becomes

$$\nu \left\| \nabla \mathbf{E}_h^{n+1} \right\|^2 - \left\langle e_h^n \, , \, \text{div } \mathbf{E}_h^{n+1} \right\rangle = 0. \tag{3.3}$$

In light of (3.2) and (3.3), we obtain

$$\begin{aligned}\nu\left\|\nabla\mathbf{E}_h^{n+1}\right\|^2 &= -\frac{1}{\alpha\nu}\left\langle e_h^n\,,\; e_h^{n+1}-e_h^n\right\rangle \\ &= -\frac{1}{2\alpha\nu}\left(\left\|e_h^{n+1}\right\|^2-\left\|e_h^n\right\|^2-\left\|e_h^{n+1}-e_h^n\right\|^2\right).\end{aligned} \tag{3.4}$$

Consequently, we arrive at

$$2\alpha\nu^2\left\|\nabla\mathbf{E}_h^{n+1}\right\|^2+\left\|e_h^{n+1}\right\|^2=\left\|e_h^n\right\|^2+\left\|e_h^{n+1}-e_h^n\right\|^2. \tag{3.5}$$

The next step is to estimate $\left\|e_h^{n+1}-e_h^n\right\|^2$. Choosing $q_h = e_h^{n+1}-e_h^n$ in (3.2) gives

$$\|e_h^{n+1}-e_h^n\|^2 = -\alpha\nu\left\langle\text{div }\mathbf{E}_h^{n+1}\,,\; e_h^{n+1}-e_h^n\right\rangle \le \alpha\nu\|\text{div }\mathbf{E}_h^{n+1}\|\;\|e_h^{n+1}-e_h^n\|,$$

whence

$$\left\|e_h^{n+1}-e_h^n\right\| \le \alpha\nu\left\|\text{div }\mathbf{E}_h^{n+1}\right\| \le \alpha\nu\left\|\nabla\mathbf{E}_h^{n+1}\right\|. \tag{3.6}$$

Replacing (3.6) into (3.5) immediately leads to

$$\nu^2\alpha(2-\alpha)\left\|\nabla\mathbf{E}_h^{n+1}\right\|^2+\left\|e_h^{n+1}\right\|^2 \le \left\|e_h^n\right\|^2. \tag{3.7}$$

We finally prove (1.8) for $\rho = 0$. Since $e_h^n \in \mathbb{P}_h$, there exists a function $\mathbf{v}_h \in \mathbb{V}_h$ such that

$$\left\langle\text{div }\mathbf{v}_h\,,\; e_h^n\right\rangle = \|e_h^n\|^2 \qquad \text{and} \qquad \|\nabla\mathbf{v}_h\| \le \frac{1}{\beta}\|e_h^n\|, \tag{3.8}$$

which is equivalent to (1.3) [2,4]. In view of (3.1) and (3.8), we get

$$\begin{aligned}\|e_h^n\|^2 &= \left\langle\text{div }\mathbf{v}_h\,,\; e_h^n\right\rangle \\ &= \nu\left\langle\nabla\mathbf{E}_h^{n+1}\,,\; \nabla\mathbf{v}_h\right\rangle \\ &\le \nu\|\nabla\mathbf{E}_h^{n+1}\|\|\nabla\mathbf{v}_h\| \\ &\le \frac{\nu}{\beta}\|\nabla\mathbf{E}_h^{n+1}\|\|e_h^n\|.\end{aligned}$$

Consequently,

$$\beta\left\|e_h^n\right\| \le \nu\left\|\nabla\mathbf{E}_h^{n+1}\right\|. \tag{3.9}$$

In light of (3.7) and (3.9), we easily obtain

$$\beta^2\alpha(2-\alpha)\left\|e_h^n\right\|^2+\left\|e_h^{n+1}\right\|^2 \le \left\|e_h^n\right\|^2,$$

which implies (1.8) for $\rho = 0$, and thus completes the proof of Theorem 1.

On the other hand, Corollary 1 follows from (3.3) together with Proposition 1 and Theorem 1.

4 Proof of Theorem 1: Case $\rho > 0$

In this section, we prove Theorem 1 for $\rho > 0$. We note that both (3.2) and (3.6) are still valid for $\rho > 0$, because they are solely based on the Richardson update (1.7). Upon subtracting (1.6) (with $\rho > 0$) from (1.4), we have

(4.1)
$$\nu \left\langle \nabla \mathbf{E}_h^{n+1}\,,\ \nabla \mathbf{w}_h \right\rangle + \rho \left\langle \mathrm{div}\ \mathbf{E}_h^{n+1}\,,\ \mathrm{div}\ \mathbf{w}_h \right\rangle - \left\langle e_h^n\,,\ \mathrm{div}\ \mathbf{w}_h \right\rangle = 0, \qquad \forall \mathbf{w}_h \in \mathbb{V}_h.$$

Choosing $\mathbf{w}_h = \mathbf{E}_h^{n+1}$ and applying (3.2), (4.1) becomes

$$\begin{aligned}
\nu \left\| \nabla \mathbf{E}_h^{n+1} \right\|^2 + \rho \left\| \mathrm{div}\ \mathbf{E}_h^{n+1} \right\|^2 &= \left\langle e_h^n\,,\ \mathrm{div}\ \mathbf{E}_h \right\rangle \\
&= -\frac{1}{\alpha\nu} \left\langle e_h^n\,,\ e_h^{n+1} - e_h^n \right\rangle \\
&= -\frac{1}{2\alpha\nu} \left(\left\| e_h^{n+1} \right\|^2 - \left\| e_h^n \right\|^2 - \left\| e_h^{n+1} - e_h^n \right\|^2 \right),
\end{aligned}$$

instead of (3.4). In light of (3.6), we thus obtain

(4.2)
$$2\alpha\nu^2 \left(\left\| \nabla \mathbf{E}_h^{n+1} \right\|^2 + \frac{\rho}{\nu} \left\| \mathrm{div}\ \mathbf{E}_h^{n+1} \right\|^2 - \frac{\alpha}{2} \left\| \mathrm{div}\ \mathbf{E}_h^{n+1} \right\|^2 \right) + \left\| e_h^{n+1} \right\|^2 \le \left\| e_h^n \right\|^2.$$

Replacing now

$$-\frac{\alpha}{2} \left\| \mathrm{div}\ \mathbf{E}_h^{n+1} \right\|^2 = -\frac{\alpha\nu}{2\rho} \left(\left\| \nabla \mathbf{E}_h^{n+1} \right\|^2 + \frac{\rho}{\nu} \left\| \mathrm{div}\ \mathbf{E}_h^{n+1} \right\|^2 \right) + \frac{\alpha\nu}{2\rho} \left\| \nabla \mathbf{E}_h^{n+1} \right\|^2,$$

into (4.2), we end up with

$$\begin{aligned}
&2\alpha\nu^2 \left(1 - \frac{\alpha\nu}{2\rho} \right) \left(\left\| \nabla \mathbf{E}_h^{n+1} \right\|^2 + \frac{\rho}{\nu} \left\| \mathrm{div}\ \mathbf{E}_h^{n+1} \right\|^2 \right) + \frac{\alpha^2\nu^3}{\rho} \left\| \nabla \mathbf{E}_h^{n+1} \right\|^2 \\
&\qquad + \left\| e_h^{n+1} \right\|^2 \le \left\| e_h^n \right\|^2. \qquad (4.3)
\end{aligned}$$

Using inf-sup property (3.8), we deduce

$$\begin{aligned}
\|e_h^n\|^2 &= \left\langle \mathrm{div}\ \mathbf{v}_h\,,\ e_h^n \right\rangle \\
&= \nu \left\langle \nabla \mathbf{E}_h^{n+1}\,,\ \nabla \mathbf{v}_h \right\rangle + \rho \left\langle \mathrm{div}\ \mathbf{E}_h^{n+1}\,,\ \mathrm{div}\ \mathbf{v}_h \right\rangle \\
&\le \nu \left\| \nabla \mathbf{E}_h^{n+1} \right\| \|\nabla \mathbf{v}_h\| + \rho \left\| \mathrm{div}\ \mathbf{E}_h^{n+1} \right\| \|\mathrm{div}\ \mathbf{v}_h\| \\
&\le \left(\nu \|\nabla \mathbf{v}_h\|^2 + \rho \|\mathrm{div}\ \mathbf{v}_h\|^2 \right)^{\frac{1}{2}} \left(\nu \left\| \nabla \mathbf{E}_h^{n+1} \right\|^2 + \rho \left\| \mathrm{div}\ \mathbf{E}_h^{n+1} \right\|^2 \right)^{\frac{1}{2}} \\
&\le \frac{\nu\sqrt{\sigma}}{\beta} \left\| \nabla e_h^n \right\| \left(\left\| \nabla \mathbf{E}_h^{n+1} \right\|^2 + \frac{\rho}{\nu} \left\| \mathrm{div}\ \mathbf{E}_h^{n+1} \right\|^2 \right)^{\frac{1}{2}},
\end{aligned}$$

where we have used (2.1) for $\mathbf{v} = \mathbf{v}_h$ and $\sigma = 1 + \frac{\rho}{\nu}$. Consequently, we infer that

$$\|e_h^n\|^2 \le \frac{\nu^2\sigma}{\beta^2}\left(\left\|\nabla \mathbf{E}_h^{n+1}\right\|^2 + \frac{\rho}{\nu}\left\|\mathrm{div}\ \mathbf{E}_h^{n+1}\right\|^2\right)$$

and, making use of (2.1) again for $\mathbf{v} = \mathbf{E}_h^{n+1}$, that

$$\|e_h^n\|^2 \le \frac{\nu^2\sigma^2}{\beta^2}\left\|\nabla \mathbf{E}_h^{n+1}\right\|^2.$$

Replacing these two inequalities into (4.3), we obtain

$$\left(\frac{2\alpha\beta^2}{\sigma}\left(1 - \frac{\alpha\nu}{2\rho}\right) + \frac{\alpha^2\beta^2\nu}{\rho\sigma^2}\right)\|e_h^n\|^2 + \left\|e_h^{n+1}\right\|^2 \le \left\|e_h^n\right\|^2,$$

or equivalently,

$$\begin{aligned}\left\|e_h^{n+1}\right\|^2 &\le \left(1 - \frac{\alpha\beta^2}{\sigma^2}\left(2\sigma - \frac{\alpha\nu}{\rho}(\sigma - 1)\right)\right)\|e_h^n\|^2 \\ &= \left(1 - \frac{\alpha\beta^2}{\sigma^2}(2\sigma - \alpha)\right)\|e_h^n\|^2.\end{aligned}$$

This is the asserted estimate (1.8). The proof of Theorem 1 for $\rho > 0$ is thus complete. Finally, the proof of Corollary 1 for $\rho > 0$ is identical to the case $\rho = 0$.

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