IIE Transactions
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/uiie20

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Hong-Zhong Huang [a], Zhi-Jie Liu [b] & D. N. P. Murthy [c]

[a] School of Mechanical, Electronic, and Industrial Engineering, University of Electronic Science and Technology of China, Chengdu, 610054, China

[b] School of Mechanical Engineering, Dalian University of Technology, Dalian, 116023, China

[c] School of Engineering, The University of Queensland, Brisbane, 4072, Australia

Version of record first published: 30 May 2007.

To cite this article: Hong-Zhong Huang, Zhi-Jie Liu & D. N. P. Murthy (2007): Optimal reliability, warranty and price for new products, IIE Transactions, 39:8, 819-827

To link to this article: http://dx.doi.org/10.1080/07408170601091907

IIE Transactions (2007) **39**, 819–827

ISSN: 0740-817X print / 1545-8830 online
DOI: 10.1080/07408170601091907

Optimal reliability, warranty and price for new products

HONG-ZHONG HUANG[1,*], ZHI-JIE LIU[2] and D. N. P. MURTHY[3]

[1]*School of Mechanical, Electronic, and Industrial Engineering, University of Electronic Science and Technology of China, Chengdu 610054, China*
E-mail: hzhuang@uestc.edu.cn
[2]*School of Mechanical Engineering, Dalian University of Technology, Dalian 116023, China*
[3]*School of Engineering, The University of Queensland, Brisbane 4072, Australia*

Received December 2005 and accepted September 2006

The success of a new product depends on both engineering decisions (product reliability) and marketing decisions (price, warranty). A higher reliability results in a higher manufacturing cost and higher sale price. Consumers are willing to pay a higher price only if they can be assured about product reliability. Product warranty is one such tool to signal reliability with a longer warranty period indicating better reliability. Better warranty terms result in increased sales and also higher expected warranty servicing costs. Warranty costs are reduced by improvements in product reliability. Learning effects result in the unit manufacturing cost decreasing with total sales volume and this in turn impacts on the sale price. As such, reliability, price and warranty decisions need to be considered jointly. The paper develops a model to determine the optimal product reliability, price and warranty strategy that achieve the biggest total integrated profit for a general repairable product sold under a free replacement-repair warranty strategy in a market and looks at two scenarios for the pricing and warranty of the product. The model assumes that the sale rate increases as the warranty period increases and decreases as the price increases. The maximum principle method is used to obtain optimal solutions for dynamic price and warranty situations. Finally, numerical examples are given to illustrate the proposed model.

Keywords: Warranty cost, reliability, cumulative sales, warranty policy, maximum principle

1. Introduction

Products are becoming increasingly complex resulting in customers often being uncertain about the performance of these products. The willingness of a consumer to pay a high price depends on he/she being convinced about product quality. A warranty is seller's assurance to a buyer that a product is as has been represented. It may be condidered to be a contractual agreement between buyers and sellers that is entered into on the sale of the product (Blischke and Murthy, 1996). A warranty can be viewed as a signal that conveys information about product reliability and as such it serves as an important marketing tool.

Servicing a warranty involves additional costs to the manufacturer and this has an impact on the profit levels. The warranty cost is a function of the product reliability and can be reduced through the use of various techniques to improve product reliability (such as reliability improvement through development and redundancy processes). However, this improvement generally involves incurring additional costs and as a result the manufacturing cost increases, which may imply a higher selling price. The design phase must incorporate marketing variables and the design and marketing decisions must be made jointly.

In this paper, we develop a model to determine the optimal sale price, warranty period and product reliability to maximize the discounted profit for a repairable product sold with a free replace-repair warranty policy. We assume that the learning effects result in the unit manufacturing cost decreasing with total sales volume. We consider two scenarios for the pricing and warranty. In the first scenario the price and warranty period are constant whereas in the second scenario they change over the product life cycle. The outline of this paper is as follows. Section 2 presents a brief literature review on marketing and/or technical strategy decisions involving pricing and/or warranty. In Section 3, we give the mathematical details of the model formulation. Section 4 examines the optimal policies for the proposed model. Section 5 deals with illustrative numerical examples. Finally, in Section 6, we draw conclusions and highlight of some extensions worthy of future investigation.

2. Review of the literature

There are many different aspects to warranty decisions that have been studied by researchers from diverse disciplines because of their importance. Blischke and Murthy (1996)

*Corresponding author

have highlighted several of these issues and a review of the literature published in the last 10 years has been performed by Murthy and Djamaludin (2002). Many researchers have investigated optimal strategies that link engineering issues (such as reliability improvement through redundancy and development, maintenance, testing policies, burn-in, etc.) with warranty to either maximize the manufacturers' profit or minimize the total cost. In this context we highlight the papers of Murthy and Nguyen (1987), Mi (1997), Monga and Zuo (1998), Pohl and Dietrich (1999), Hussain and Murthy (2003) and Shue and Chien (2005).

Many researchers have used sales models that consider the determination of optimal marketing and/or technical strategies, such as advertising, price, quality, etc. We highlight the papers of Dockner and Gaunersdorfer (1996), Mendez and Narasimhan (1996), Teng and Thompson (1996), Zhao and Zheng (2000), Chen and Chu (2001) and Kamrad *et al*. (2005). Price and warranty are two major commercial variables that influence sales decisions and ultimately the profit levels. Glickman and Berger (1976) presented a model to determine the optimal price and warranty period that maximizes a manufacturer's profit for a failure-free warranty policy. Nguyen and Murthy (1988) developed a model for obtaining the optimal reliability allocation taking into account the manufacturing and warranty costs. Murthy (1990) developed a model to obtain the optimal price, warranty period and product reliability to maximize a manufacturer's profit. In these models, the manufacturing cost and the price are assumed to be constant over the product life cycle. Teng and Thompson (1996) considered the optimal price and quality policies for the introduction of a new product. They assumed that the unit cost declines along the learning curve and investigated the dynamics between price and quality for the new product. Lin and Shue (2005) and Wu *et al.* (2006) modified the Teng-Thompson price-quality model into a price-warranty decision model in which the warranty length replaces the quality level. Wu *et al.* (2006) derived a normal lifetime distributed product whereas Lin and Shue (2005) investigated numerous basic lifetime distributions. DeCroix (1999) presented a game-theoretic model that represents firms in an oligopoly that choose warranty, reliability and price levels for their goods and examines the Nash equilibria for this game.

3. Model formulation

3.1. *Nomenclature and Notation*

θ = reliability parameter (design variable);
$F(t,\theta)$ = cumulative distribution function for the first time to failure;
$f(t,\theta)$ = probability density function associated with $F(t,\theta)$;
$r(t,\theta)$ = hazard function associated with $F(t,\theta)$;
$P(t)$ = unit sale price at time t (marketing variable);
$W(t)$ = duration of warranty period for products sold at time t (marketing variable);
$\omega(W,\theta)$ = expected warranty cost per unit sold;
$c(t,\theta)$ = total manufacturing cost of unit product, which includes unit development cost and production cost – to indicate that this is a function of θ;
$c_m(t,\theta)$ = unit production cost – to indicate that this is a function of θ;
c_r = expected cost of servicing a warranty claim;
δ = discount rate;
L = product life cycle;
$q(t)$ = sales (production) rate at time t, $0 \le t < L$;
$Q(t)$ = accumulated sales in [0, t], $0 \le t < L$;
Q_0 = parameter characterizing past sales or production experience;
Q_M = maximum sales potential;
ψ = parameter to reflect the relative influence of innovators in the sales model;
$S(t,\theta)$ = expected number of failures for an item over the interval [0, t);
π_L = expected discounted integral profit during the interval [0, L).

3.2. *Product warranty policy*

Manufacturers provide many types of warranty policies for different products. In this paper, we consider the Free Replace-repair Warrenty (FRW) policy which was defined as follows by Blischke and Murthy (1996).

"Under the policy, the seller agrees to repair or provide replacements for failed items free of charge up to a time W from the time of the initial purchase. FRW is widely applied, which include consumer products (such as automobiles, refrigerators, TVs, electronic components) and industrial/commercial products (such as pumps, trucks, office equipment)."

3.3. *Product failure model*

Let $F(t,\theta)$ denote the cumulative failure distribution function for the first time to failure. Let $f(t,\theta)\{= \mathrm{d}F(t,\theta)/\mathrm{d}t\}$ and $r(t,\theta)\{= f(t,\theta)/[1-F(t,\theta)]\}$ denote the failure density and hazard functions associated with $F(t,\theta)$. The parameter θ is a design variable and an indicator of product reliability with a smaller value of θ implying better product quality. For an exponential failure distribution, θ is the failure rate, and for Weibull and gamma distributions it is the scale parameter. We assume that manufacturers can choose any θ in the interval $[\theta_{\min}, \theta_{\max}]$ with $\theta_{\min}$ and $\theta_{\max}$ representing the achievable limits. The design reliability θ is decided before the launch of a product, so it is constant over the product life cycle.

The possibility of a second failure occuring after an initial failures has occured depends on the action taken to rectify the problem under the warranty terms. The number of failures under a warranty is a random variable and Blichke and Murthy (1996) looks at the following three cases:

Case (1): Replace failed item with a new one.

If the replacement times are negligible in relation to the mean time between failures, then failures over time occur according to a renewal process associated with $F(t,\theta)$. The expected number of failures over $[0, t)$, $S(t,\theta)$, is given by $M(t,\theta)$, the renewal function associated with $F(t,\theta)$,

$$M(t,\theta) = F(t,\theta) + \int_0^t M(t-x,\theta) f(x,\theta)\mathrm{d}x. \quad (1)$$

Case (2): The repaired items have a distribution $G(t,\theta)$ which is different from $F(t,\theta)$.

In this case the failures over the warranty period occur according to a delayed renewal process, then the expected number of failures over $[0, t)$ (under the assumption that repair times are negligible), $S(t,\theta)$, is given by the delayed renewal function $M_{\mathrm{d}}(t,\theta)$ which is obtained by solving the following integral equation:

$$M_{\mathrm{d}}(t,\theta) = F(t,\theta) + \int_0^t M_G(t-x,\theta) f(x,\theta)\mathrm{d}x. \quad (2)$$

where $M_G(\bullet)$ is the ordinary renewal function associated with $G(t,\theta)$. Note that when $G(t,\theta)$ equals $F(t,\theta)$, the delayed renewal process reduces to an ordinary renewal process.

Case (3): The failed item is rectified through minimal repair.

In this case the hazard function after repair is the same as that just before item failure. This type of rectification model is appropriate for multicomponent products where item failure is due to the failure of a single or only a few components. Only the failed components are either repaired or replaced and they have a negligible effect on the hazard function of the item as a whole. Under the assumption that repair times are negligible failures over rime occur according to a nonhomogeneous Poisson process with an intensity function given by the failure rate function $r(t,\theta)$ so that expected number of failures over $[0, t)$,$S(t,\theta)$, is given by

$$S(t,\theta) = \int_0^t r(t,\theta)\mathrm{d}t. \quad (3)$$

3.4. *Sales model*

The product sales volume depends on many factors, such as marketing variables (e.g., price, advertising, product quality), the competitive environment, and so on. A number of mathematical models have been used to investigate product sales. Customers often use warranty information to assess the quality of a product and to decide if the price is appropriate. Thus, price and warranty are the two major factors that decide the sales volume of a product. In the Glickman–Berger model (Glickman and Berger, 1976), demand is represented by a displaced log-linear function of the form $q(p, W) = k_1 P^{-a}(W + k_2)^b$, where $k_1 > 0, k_2 > 0$, $a > 1$ and $0 < b < 1$. Product reliability cannot be directly observed, so it has no impact on the purchase decision. Bass' growth model (Bass, 1969; Robinson and Lakhani, 1975) is an epidemic model which can only be applied to initial purchases. The Bass model assumes that the purchasers are divided into innovators and imitators based upon the timing of adoption by the various groups. The innovators decide to adopt an innovation independently of the decisions of other individuals in a social system. The volume of sales to this group is simply proportional to the number of potential customers who do not already own the product. However, the imitators are influenced in the timing of adoption by the pressures of the social system, the pressure increasing for later adopters with the number of previous adopters. Sales to the second group are again proportional to the number of people who do not have the product, but it is also proportional to the number of people who do have the product.

In this paper, we refer to the sales models presented by Bass (1969), Robinson and Lakhani (1975) and Glickman and Berger (1976) and consider both price and warranty length as the decision variables that are to be determined dynamically to maximize the overall profit. Assume that the demand is decreasing in price and increasing in warranty so that a longer warranty signals a more reliable product. This assumption can be justified by a number of empirical studies. For example, Wiener (1985) states that in the markets for appliances and motor vehicles, warranties are accurate signals of product reliability. Douglas *et al.* (1993) found that a more intensive warranty is associated with a higher-quality product in the US automobile market. Of course other attributes such as style, size, performance, etc., will also influence consumer demand, but these factors are suppressed in our analysis to allow us to focus on the variables of interest. In modern society, many products evolve so rapidly that the useful life of the product is much longer than the product life cycle. As a result, repeat purchases for these products are insignificant. Thus, we confine our attention to first purchase sales and the sales rate, $q(t)$, is modeled as follows:

$$q(t) = \frac{\mathrm{d}Q(t)}{\mathrm{d}t} = k_1(W(t)+k_2)^{\alpha} P(t)^{-\beta}\left[1 - \frac{Q(t)}{Q_M}\right] \times \left[\psi + \frac{Q(t)}{Q_M}\right], \quad Q(0) = Q_0, \quad (4)$$

where $Q(t)$ is the accumulated sales volume over $[0, t]$.

The salient features of the model are as follows.

1. The square brackets reflect the concept of sales as a diffusion process involving innovators and imitators as in the Bass model (Bass, 1969; Robinson and Lakhani, 1975). The parameter ψ reflect the relative influence of innovators. For example, Bass' study indicates that the constant ψ is typically a few hundredths for consumer durables. This indicates that innovators are only a dominant factor in the marketplace during the short period required to achieve the first few percent of market penetration (Robinson and Lakhani, 1975).

2. The term before the square bracket captures the effect of price and warranty on the sales rate. The parameter α ($0 < \alpha < 1$) is the displaced warranty elasticity parameter and the parameter β (> 1) is the price elasticity. Note the warranty period has a positive effect (with the sale rate increasing as the warranty period increases) and the sale price has a negative effect (with the sales rate decreasing as the price increases).
3. The parameter k_1 is a scale factor, to reflect the competitor and other market influences, such as the number of potential consumers, the consumer purchasing power, etc. We assume that k_1 is constant. It is a valid approximation because the market factors frequently exert the same influence on both a new product and its existing competitors and if a competitor changes its selling strategy, then the new product will also change in response. k_2 is a constant of time displacement which allows for the possibility of nonzero demand when t is zero.

We consider two scenarios for the sale price and warranty period.

Scenario 1: Stable market conditions. In this case the price and warranty period are constant over the product life cycle so that $P(t) = P_0$, $W(t) = W_0$.

Scenario 2: $P(t)$ and $W(t)$ change over time. We assume that the changes occur at discrete points along the time axis so that it is a discontinuous function.

The accumulated sales volume, $Q(t)$, is given by

$$Q(t) = Q_0 + \int_0^t q(t)\mathrm{d}t, \qquad (5)$$

where Q_0 is a parameter which captures the past experience at $t = 0$, and is obtained from research and development and pilot plant operations, and $Q_L = Q(L)$ is the total sales volume over the life cycle.

3.5. *Manufacturing cost model*

Optimal product development strategies involving a warranty are typically developed and used under the assumption that the manufacturing unit costs are constant. Firms enjoy large cost reductions as they gain experience in the production of a product. Learning curves are often used to mathematically represent this concept but can only be applied to situations where the production rate has stabilized. The total manufacturing cost is the sum of the research and development cost (to improve reliability) and the production cost. The development cost is incurred before the launch of the product. It usually depends on the development time and work effort, etc. In general, the smaller the value of θ, the higher the manufacturing process costs.

In general, the development cost is a monotonically increasing function of product reliability. Generally the larger the development costs to achieve the same improvement, the better is the initial reliability. This means that it is easier to e.g., increase the reliability of a component from 70 to 75% than to increase its reliability from 90 to 95%. Thus, the derivative of the cost function (with respect to reliability) is a monotonically increasing function. In this paper, we refer to the model presented by Mettas (2000), which takes an exponential behavior and is modeled as follows:

$$c(\theta) = A_1 + B_1 \exp\left(k\frac{\theta_{\max} - \theta}{\theta - \theta_{\min}}\right), \quad A_1, B_1, k > 0, \qquad (6)$$

where θ is the product reliability parameter. The constant k represents the difficulty in increasing reliability, which depends on the design complexity, technological and resource limitations, etc. Clearly, the more difficult it is to improve the product reliability (the parameter k is bigger), the greater the cost is. As expected, when θ decreases (the product becomes more reliable), the development cost increases.

The production cost depends on the total number produced the product reliability, etc. The first reported use of a learning curve was by Wright in 1936 (Loerch, 1999), and since then numerous papers have reported its use in industrial applications (see, e.g., Mazzola and McCardle (1997), Loerch (1999) and Smunt (1999)). Many forms exist to mathematically depict learning effects. Probably the most popular are the "power" or "exponential" forms which relate the cost of an item to its place in the sequence of items produced. In this paper, the production cost is given by

$$c_{\mathrm{m}}(t, \theta) = K c_{\mathrm{m0}}(\theta)\left[\frac{Q_0}{Q(t)}\right]^{\mu}, \qquad (7)$$

where $c_{\mathrm{m}}(t, \theta)$ denotes the production cost per item at time t, $c_{\mathrm{m0}}(\theta)$ is the initial production cost per item, i.e., $c_{\mathrm{m}}(0, \theta)$, $Q(t)$ is the accumulated sales volume until time t, and the learning parameter μ ($0 < \mu < 1$) is a constant. The parameter K represents the influence of various factors such as inflation and production rate. For example, Lee (1997) incorporated the production rate into the computation of learning curve costs. We assume that the initial production cost $c_{\mathrm{m0}}(\theta)$ depends on product reliability, has the form:

$$c_{\mathrm{m0}}(\theta) = A_2 + B_2\theta^{-i}, \qquad (8)$$

where A_2, B_2 and i are positive-valued parameters. Note that this form matches the relationship that the production cost $c_{\mathrm{m0}}(\theta)$ is decreasing in the failure parameter θ, and satisfies the intuitive properties: $c_{m0}(\theta)$ is convex in θ, $c_{\mathrm{m0}}(\theta) \to \infty$ as $\theta \to 0$ and $c_{\mathrm{m0}}(\theta) \to A_2$ as $\theta \to \infty$.

As a result, the total manufacturing cost per item is given by

$$\begin{aligned} c(t, \theta) &= \frac{c(\theta)}{Q_L} + c_{\mathrm{m}}(t, \theta) \\ &= \frac{A_1 + B_1 \exp(k((\theta_{\max} - \theta)/(\theta - \theta_{\min})))}{Q_L} \\ &\quad + K(A_2 + B_2\theta^{-i})\left[\frac{Q_0}{Q(t)}\right]^{\mu}. \end{aligned} \qquad (9)$$

3.6. *Expected warranty cost model*

When an item is returned for repair under warranty, the manufacturer incurs many costs, such as transportation costs, administrative costs, material costs and labor costs, etc. We aggregate all of these costs into a single cost termed the "repair cost" for each claim. Because some of the costs are uncertain, this cost is a random variable (Blischke and Murthy, 1996). Let c_r denote the expected value of this cost. The expected warranty cost for unit product, $\omega(W, \theta)$ is given by

$$\omega(W, \theta) = c_r S(W, \theta), \tag{10}$$

where c_r is the cost of each replacement or repair.

As discussed in Section 3.3, the number of claims over the warranty period $S(W, \theta)$ depends on the type of repair action required and this in turn determines the warranty costs.

3.7. *Expected profit*

The expected discounted profit over the life cycle of the product, π depends on: (i) the sales volume; (ii) the unit manufacturing cost; (iii) the sale price; and (iv) the expected warranty cost. This is given by

$$\pi = \int_0^L [P(t) - c(t, \theta) - \omega(W, \theta)]q(t)e^{-\delta t}\mathrm{d}t. \tag{11}$$

The objective of the manufacturer is to select the reliability parameter θ, the warranty period $W(t)$ and the price $P(t)$ which maximize the expected discounted profit π given by Equation (11). Let P^*, W^* and θ^* be the optimal values of P, W and θ which maximize π. To improve clarity, we will in future omit the function arguments when this does not cause confusion.

4. Model optimization analysis

4.1. *Stable market (scenario 1)*

When the price and warranty period are constant over the product life cycle, a necessary condition for P^*, W^* and θ^* to be optimal is that they satisfy (a subscript on a variable denotes partial differential with respect to that variable):

$$\begin{aligned} \pi_P &= 0, \\ \pi_W &= 0, \\ \pi_\theta &= \tau; \end{aligned} \tag{12}$$

where τ is an arbitrary constant with sign given as follows:

$$\tau = \begin{cases} \leq 0 & \text{if } \theta^* = \theta_{\min} \\ 0 & \text{if } \theta_{\min} < \theta < \theta_{\max} \\ \geq 0 & \text{if } \theta^* = \theta_{\max} \end{cases}, \tag{13}$$

4.2. *Dynamic market (scenario 2)*

Under dynamic market conditions we face a dynamic optimization problem. To obtain the optimal solution, we apply the maximum principle (Sethi and Thompson, 1981). The following analysis procedures follows that in Teng and Thompson (1996). The current value Hamiltonian function is given as follows:

$$H = [P(t) - c(t, \theta) - \omega(W, \theta) + \lambda(t)]q(t), \tag{14}$$

where $\lambda(t)$ is the current value adjoint variable which satisfies the following differential equation (for convenience, a dot above a variable denotes the first derivative with respect to time):

$$\begin{aligned} \dot{\lambda}(t) &= \delta\lambda(t) - H_Q \\ &= \delta\lambda + c_Q q - [P - c - \omega + \lambda]q_Q, \end{aligned} \tag{15}$$

with the transversality condition at $t = L$, $\lambda(t) = 0$.

The following necessary conditions hold for an optimal solution:

$$H_P = 0 \Rightarrow P - c - \omega + \lambda = -q/q_P, \tag{16}$$

$$H_W = 0 \Rightarrow P - c - \omega + \lambda = \omega_W q/q_W, \tag{17}$$

and the second-order conditions for H-maximization are:

$$H_{PP} < 0 \Rightarrow 2q_P - \left(\frac{q}{q_P}\right) q_{PP} < 0, \tag{18}$$

$$H_{WW} < 0 \Rightarrow -\omega_{WW} q - 2\omega_W q_W + \frac{\omega_W q}{q_W} q_{WW} < 0, \tag{19}$$

$$H_{PP}H_{WW} - (H_{PW})^2 > 0, \tag{20}$$

where

$$H_{PW} = H_{WP} = q_W - \omega_W q_P - \frac{q}{q_P} q_{PW}. \tag{21}$$

We can obtain the following result from Equations (16) and (17):

$$\omega_W = -\frac{q_W}{q_P}. \tag{22}$$

According to Sethi and Thompson (1981), the economic interpretations of the Hamiltonian function in Equation (14) can be derived. Multiplying Equation (14) by $\mathrm{d}t$ gives:

$$H\mathrm{d}t = [P - c - \omega]q\mathrm{d}t + \lambda\mathrm{d}Q. \tag{23}$$

The first term $[P - c - \omega]q\mathrm{d}t$ represents the current profit from time t to $t + \mathrm{d}t$. The second term $\lambda(t)$ represents the future benefit (at time t) of having one more unit produced. Therefore, the second term $\lambda\mathrm{d}Q$ represents the future benefit of the incremental sales $\mathrm{d}Q$. Thus, $H\mathrm{d}t$ can be interpreted as the total profit of the manufacturer from the interval t to $t + \mathrm{d}t$.

We followed the proof procedures of Lemmas 1, 2 and 3 of Teng and Thompson (1996) to derive the following lemmas:

Lemma 1. *If* $H_{WW} + \omega_W H_{PW} > 0$, *then* $H_{PW} > 0$ *and* $H_{PW} + \omega_W H_{PP} < 0$.

Lemma 2. *If* $H_{PW} + \omega_W H_{PP} > 0$, *then* $H_{WP} > 0$ *and* $H_{WW} + \omega_W H_{PW} < 0$.

Lemma 3. *If* $H_{PW} = H_{WP} < 0$, *then* $H_{PW} + \omega_W H_{PP} > 0$ *and* $H_{WW} + \omega_W H_{PW} < 0$.

The condition $H_{PW} > 0$ indicates that the total profit will increase following the simultaneous increase (or decrease) of both price and warranty. Therefore, the best combined policy for both price and warranty should be to increase or decrease both at the same time. On the contrary, when $H_{PW} = H_{WP} < 0$, the best policy is to change price and warranty in opposite directions; i.e., when one is increased the other one must be decreased.

When the sales function is given by Equation (4), which is separable in cumulative sales, we simplify as follows:

$$q = F(W, P)G(Q). \qquad (24)$$

In order to analyze the relationship between optimal price and warranty policy, we take the time derivative of the optimal price given by Equation (16), substitute Equation (15) for $\dot{\lambda}$ and Equation (22) for ω_W and rearrange terms. Thus, the system of equations that characterizes the optimal solution is given as follows:

$$H_{PP}\dot{P} + H_{PW}\dot{W} = -\delta\lambda F_P G - F^2 G_Q G, \qquad (25)$$

$$(H_{PW} + \omega_W H_{PP})\dot{P} + (H_{WW} + \omega_W H_{PW})\dot{W} = 0. \qquad (26)$$

Thus, we can obtain:

$$\dot{P} = \frac{H_{WW} + \omega_W H_{PW}}{H_{PP}H_{WW} - H_{PW}^2}(-\delta\lambda F_P - F^2 G_Q)G,$$

$$\dot{W} = \frac{H_{PW} + \omega_W H_{PP}}{H_{PW}^2 - H_{PP}H_{WW}}(-\delta\lambda F_P - F^2 G_Q)G. \qquad (27)$$

For the case where the discount rate $\delta > 0$ and there is a learning effect on production cost, using Lemmas 1, 2 and 3 and Equations (27) and (4), we can calculate the optimal policies given in Table 1.

When the discount rate is zero, i.e., $\delta = 0$, which can be an approximation in situations where the discount rate is relatively low, we can obtain the results in Table 2.

Table 1. Optimal policies for a positive discount rate and learning effect on production cost

Conditions	$\delta\lambda\beta < k_1 \frac{(W(t)+k_2)^{\alpha}}{P(t)^{1-\beta}} \frac{(1-\psi)Q_M - 2Q}{Q_M^2}$	$\delta\lambda\beta > k_1 \frac{(W(t)+k_2)^{\alpha}}{P(t)^{1-\beta}} \frac{(1-\psi)Q_M - 2Q}{Q_M^2}$
$H_{WW} + \omega_W H_{PW} > 0$	$\dot{P} < 0$ and $\dot{W} < 0$	$\dot{P} > 0$ and $\dot{W} > 0$
$H_{PW} + \omega_W H_{PP} > 0$	$\dot{P} > 0$ and $\dot{W} > 0$	$\dot{P} < 0$ and $\dot{W} < 0$
$H_{PW} = H_{WP} < 0$	$\dot{P} > 0$ and $\dot{W} < 0$	$\dot{P} < 0$ and $\dot{W} > 0$

Table 2. Optimal policies for a zero discount rate

Conditions	*Diffusion phase* ($q_Q > 0$)	*Saturation phase* ($q_Q < 0$)
$H_{WW} + \omega_W H_{PW} > 0$	$\dot{P} < 0$ and $\dot{W} < 0$	$\dot{P} > 0$ and $\dot{W} > 0$
$H_{PW} + \omega_W H_{PP} > 0$	$\dot{P} > 0$ and $\dot{W} > 0$	$\dot{P} < 0$ and $\dot{W} < 0$
$H_{PW} = H_{WP} < 0$	$\dot{P} > 0$ and $\dot{W} < 0$	$\dot{P} < 0$ and $\dot{W} > 0$

5. Numerical example

Let the product failure distribution be an exponential distribution with parameter θ, so that $F(t, \theta) = 1 - e^{-\theta t}$. We consider that failed items under warranty are repaired minimally and let the expected cost of each minimal repair c_r=30. Thus we have that $\omega(W, \theta) = c_r\theta W = 30\,\theta W$.

We assume that the research and development cost is given by $c(\theta) = 50\,000 + 50\,000\exp((0.4 - \theta)/(\theta - 0.1))$. Let the initial unit manufacturing cost be given by $c_{m0}(\theta) = 10 + 30\theta^{-0.5}$, $0.1 < \theta \le 0.4$, indicating the limits of achievable reliability. We assume past production experience modeled (through Q_0) is given by $Q_0 = 2000$. We consider the production cost to follow a curve corresponding to a 25% decline every time the accumulated volume, $Q(t)$, doubles. This corresponds to $\mu = 0.4$. As a result, the manufacturing cost model is given by

$$c(t, \theta) = \frac{5\times 10^4 + 5\times 10^4\exp((0.4-\theta)/(\theta-0.1))}{Q_L} + (10 + 30\theta^{-0.5})\left[\frac{2000}{Q(t)}\right]^{0.4}.$$

For the sales model given by Equation (4), let $\alpha = 0.10$, $\beta = 2$, $k_1 = 3\times 10^9$, $k_2 = 0.1$, $Q_M = 25\,000$. This implies:

$$q(t) = 3\times 10^9 (W + 0.1)^{0.10} P^{-2} \times \left[1 - \frac{Q(t)}{25\,000}\right]\left[\psi + \frac{Q(t)}{25\,000}\right].$$

Finally, the discount rate $\delta = 0.15$. The expected discounted integral profit is given by

$$\pi_L = \int_0^L [P(t) - c(t,\theta) - \omega(W,\theta)]q(t)e^{-\delta t}\,\mathrm{d}t$$
$$= \int_0^L \left[P(t) - \frac{50\,000 + 50\,000\exp((0.4-\theta)/(\theta-0.1))}{Q_L}\right.$$

Table 3. Optimal price, warranty and reliability for different L (scenario 1)

L	P_0^*	W^*	θ^*	$\omega(W^*, \theta^*)$	$Q(L)$	π_L
2	231.9	0.87	0.40	10.40	22 031	3567 000
4	334.7	0.385	1.35	15.58	22 035	4653 000
6	418.4	2.09	0.319	19.97	22 033	5073 500

Table 4. Optimal solutions for different α values (scenario 1, $L =$ 2 years)

α	P^*	W^*	θ^*	$\omega(W^*, \theta^*)$	$Q(L)$	π_L
0.10	231.9	0.87	0.40	10.40	22 031	3567 000
0.15	243.2	1.61	0.355	17.17	21 902	3591 300
0.20	263.2	3.00	0.283	25.47	21 758	3693 600

$$- (10 + 30\theta^{-0.5})\left[\frac{2000}{Q(t)}\right]^{0.4} - 30[\theta W]\Bigg]$$
$$\times 3 \times 10^9 (W + 0.1)^{0.10} P^{-2}\left[1 - \frac{Q(t)}{25\,000}\right]$$
$$\times \left[\psi + \frac{Q(t)}{25\,000}\right] e^{-0.15t}\,\mathrm{d}t.$$

Given the complex nature of the integrand, numerical integration methods are used to evaluate the optimal price, warranty and reliability using the MATLAB program.

Scenario 1: Stable market ($P(t)$, $W(t)$ constant)

We consider $\psi = 0$. Table 3 gives the optimal price, warranty and reliability for $L = 2, 4$ and 6 years. As can be seen, when L is short, W^* is low and θ^* high. This is to be expected, since the manufacturer aims to maximize profits based solely on sales over a short period. θ^* is high in order to reduce the manufacturing cost. As a consequence, W^* must be low to reduce the warranty cost. As L increases, in order to attract more consumers W^* must increase. As a result θ^* must decrease (implying a more reliable product) to reduce warranty cost and maximize profits. As a result we see that W^* increases with L whereas θ^* decreases.

We now carry out a sensitivity analysis for the case where $L = 2$ years by varying one parameter at a time (for the following parameters α, β) whilst holding the remaining parameters at their nominal values for scenario 1 (stable market). Table 4 shows the results for variations in α (0.10, 0.15, 0.20), Table 5 shows the results for variations in β (1.8, 2.0, 2.1). From Table 4, we see that as α increases, because the warranty is more efficient than price in increasing profits, W^* increases to attract more consumers. As a result θ^* must decrease (implying a more reliable product) to reduce warranty cost and P^* increases. From Table 5, we see that as β increases, the optimal price is more efficient. So the price decreases to increase the sales. As a result warranty period decreases and θ^* increases to reduce the warranty cost.

Table 5. The optimal solutions for different β values (scenario 1, $L = 2$ years)

β	P^*	W^*	θ^*	$\omega(W^*, \theta^*)$	$Q(L)$	π_L
1.8	463.2	2.89	0.287	24.87	21 508	7364 000
2.0	231.9	0.87	0.40	10.40	22 031	3567 000
2.2	136.0	0.5	0.40	6.00	22 451	1877 300

Table 6. Optimal price, warranty and reliability for $L = 5$ (scenario 2)

	θ^*	P_i^*	W_i^*	$\omega(W^*, \theta^*)$	$c(t, \theta)$	$Q(t)$	$\Delta\pi(t) \times 10^5$
$i = 1$	0.16	404.0	4.11	19.73	182.3	3573	3.06
$i = 3$		355.3	3.60	17.28	160.4	9552	5.08
$i = 5$		334.8	3.39	16.27	151.1	16 847	4.16
$i = 7$		327.3	3.31	15.89	147.7	21 542	1.96
$i = 9$		324.5	3.28	15.74	146.5	23 676	0.71

Scenario 2: Dynamic market ($P(t)$ changes over time)

We consider $\psi = 0.03$. We assume that the price and warranty period change every half-year and consider the case $L = 5$ years. The price and warranty period are constant over each half-year period. Let $P_0, P_1, \ldots, P_9$, $W_0, W_1, \ldots, W_9$ denote the prices and warranty periods in the ten half-year periods. The optimal reliability is determined before the product is launched and it does not change over the life cycle. In this case we have an optimization problem involving 21 variables: the $P_0, P_1, \ldots, P_9, \theta$ and $W_0, W_1, \ldots, W_9$ that need to be selected optimally. We obtain the optimal values using a two-stage process. First we fix θ, then use a dynamic programming approach to get the optimal dynamic pricing and warranty. Note that there is need for only one state variable—the accumulated sales at the start of each period. The optimal decision variables (sale price and warranty length in the period) are obtained as a function of this taking into account the optimal prices and warranty length for the later periods. We assume some accumulated volume at the end of the planning period, $Q(5)$. We use Equation (11) to determine P_9^* and W_9^*, which optimizes the discounted profit obtained in the last half-year, $\Delta\pi_L(5)$. We then have $Q(4.5) = Q(5) - q(5)/2$. The process is repeated to obtain the price P_8^* and W_8^*, to optimize the integral profit for the last two half years, $\Delta\pi_L(4.5) + \Delta\pi_L(5)$. This process is repeated until the entire optimum solution corresponding to a given final accumulated volume, $Q(5)$, has been obtained. The entire procedure can be repeated for many values of $Q(5)$ until we obtain the optimum scenario corresponding to $Q(0) = 2000$. This is repeated for different values of θ to yield the optimal value for the parameter. This can be carried out by MATLAB programming. The values displayed in Table 6 are the optimal values at the start of the second half of each year over the life cycle. As the sales volume increases with time, the price $P^*(t)$decreases in order to attract more consumers as is to be expected. To decrease the product cost, the warranty period decreases to reduce the warrant cost.

This basic procedure can also be used to seek the optimal scenario for general failure distribution and any constraint which a manager may wish to impose.

6. Conclusions

In this paper, we have developed a model which integrates marketing aspects into the design phase for new products. The model involves optimally selecting the waranty period and the price (marketing variables) and product reliability (engineering design variable) to maximize the discounted expected profit for a product sold with a FRW policy. We have looked at two scenarios for price and warranty—a stable market (with constant price and warranty period) and a dynamic market (with the price and warranty period changing over the product life cycle) and incorporated the effect of the manufacturing cost decreasing with the volume produced (learning curve).

The model can be extended in several different directions. The first direction is to incorporate repeat purchases by satisfied customers. This will involve building models for customer satisfaction and the resulting repeat purchase. Another extension is to incorporate the effects of warranty as an advertising tool so that it affects some of the parameters (such as ψ in the diffusion model). The effects of market factors (such as competitor strategies) are important for products sold in competitive markets. The proposed model needs to be modified to take these into account. Finally, the study of these problems with different types of warranties is another topic for future research.

Acknowledgements

The authors gratefully acknowledge the helpful comments of four anonymous referees. This research was partially supported by the National Natural Science Foundation of China under the contract 50175010.

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Biographies

Dr. Hong-Zhong Huang is a full professor and the Dean of the School of Mechanical, Electronic, and Industrial Engineering, University of Electronic Science and Technology of China, Chengdu, Sichuan, China. He has held visiting appointments at several universities in the Canada, USA, and Asia. He received a Ph.D. degree in Reliability Engineering from Shanghai Jiaotong University, China. He has published 120 journal papers and 5 books in fields of reliability engineering, optimization design, fuzzy sets theory, and product development. He is a (senior) member of several professional societies, and has served on the boards of professional societies. He received the Golomski Award from the Institute of Industrial Engineers in 2006. His current research interests include system reliability analysis, warranty, maintenance planning and optimization, computational intelligence in product design.

Zhi-Jie Liu is a Ph.D. Candidate of School of Mechanical Engineering, Dalian University of Technology, China. His research interests include reliability engineering, product warranty, design optimization, total quality management, production and engineering management.

D.N.P. Murthy obtained B.E. and M.E. degrees from Jabalpur University and the Indian Institute of Science in India and M.S. and Ph.D. degrees from Harvard University. He is currently Research Professor in the Division of Mechanical Engineering at the University of Queensland and Senior Scientific Advisor to the Norwegian University of Science and Technology. He has held visiting appointments at several universities in the USA, Europe and Asia. His current research interests include various aspects of technology management (new product development, strategic management of technology), operations management (lot sizing, quality, reliability, maintenance), and post-sale support (warranties, service contracts). He has authored or co-authored over 150 journal papers and 140 conference papers. He is co-author of five books and co-editor of two books. He is a member of several professional societies and is on the editorial boards of seven international journals.