OPTIMAL REPLACEMENT TIMES – A GENERAL SET-UP

by

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Abstract

It turns out that for a large class of replacement models for stochastically deteriorating systems the optimality criteria total expected discounted cost and long run (expected) average cost per unit time have a common structure. In the present paper a formal description of this structure is given and the optimal rule is determined. A socalled "λ-minimization technique" is applied. This method is discussed in general terms.

REPLACEMENT; AVERAGE COST; EXPECTED DISCOUNTED COST; STOPPING TIME; OPTIMIZATION; λ-PROBLEMS

1 Introduction

Lately there has been a large research activity in the area of optimal replacement/maintenance of stochastically deteriorating system. In particular the interest has been focused on the problem of optimal replacement when there is information available about the underlying condition of the system. A stochastic process is usually assumed to describe this information.
Some examples on works in this direction are Taylor (1975), Zuckerman (1978a,b, 1979), Bergman (1978), Nummelin (1980), Yamada (1980), and Aven (1981).
Usually the optimal replacement rule is determined by minimizing the total expected discounted cost or the long run (expected) average cost per unit time. With exception of Yamade (1980) this is done in all the articles above.
Now these optimality criteria often, see e.g. the above mentioned papers, can be written in the common form

$$(1.1) \qquad \frac{E[\int_0^T a(t)h(t)dt+c(0)]}{E[\int_0^T h(t)dt+p(0)]},$$

where T is a stopping time based on the information about the condition of the system (and the results of some randomizing experiments), $\{a(t)\}$ a non-decreasing stochastic process, $\{h(t)\}$ a non-negative stochastic process and $c(0)$ and $p(0)$ non-negative random variables; all variables are adapted to the information about the condition of the system.

This means that T is the only control variable and that each replacement rule is identified by a stopping time T. A T minimizing (1.1) therefore determines the optimal rule.

It should be noted that if the optimality criterion is the total expected discounted cost, then the numerator in (1.1) represents the expected discounted cost associated with one replacement cycle and the denominator equals $E[1-e^{-\alpha\tau_T}]$, where α is a positive discount factor and τ_T the stochastic time to replacement. On the other hand if the long run (expected) average cost criterion is considered, then the numerator in (1.1) represents the expected cost associated with one replacement cycle and the denominator the stochastic time to replacement.

It should also be noted that criteria of the form (1.1) may arise in other types of regenerative stopping problems (i.e. stopping problems which recommence from the initial state upon stopping) than those which are considered in standard replacement/maintenance models, see e.g. Ross (1971) Section 4.

In most of the models it is given or assumed that the denominator in (1.1) is bounded in T, i.e.

(1.2) $$E[\int_0^\infty h(t)dt+p(0)] < \infty ;$$

this is clearly the case if the optimality criterion is the total expected discounted cost. We remark that (1.2) does not hold for the model of Aven (1982).

In our set-up we do not assume (1.2). The following assumption is made, however,

$$E[\int_0^{T_\lambda} h(t)dt+p(0)] < \infty ,$$

where $T_\lambda=\inf\{t\geq 0;\ a(t)\geq\lambda\}$, $\lambda\in(-\infty,\infty)$.

(Only stopping times for which the denominator in (1.1) is finite are considered.)

In Section 2 of this paper we study the problem of minimizing (1.1).

Although this problem has been investigated in special cases before, we think it is important, also having future work in this area in mind, to have at hand a general set-up where the conditions and assumptions are formulated independently of particular models. We also strongly feel the necessity of a more thorough analysis of the minimizing problem (1.1) than those given in special cases up to now.

The minimization problem (1.1) is solved by minimizing λ-functions

$$C_\lambda^T = E[\int_0^T a(t)h(t)dt+c(0)]-\lambda E[\int_0^T h(t)dt+p(0)]$$

$$= E[\int_0^T (a(t)-\lambda)h(t)dt+c(0)-\lambda p(0)]$$

(since $a(\cdot)$ is non-decreasing and h is non-negative, it follows that T_λ minimizes C_λ^T).
A similar indirect approach has been used earlier in special cases, see e.g. Ross (1971), Taylor (1975), Zuckerman (1978a,b, 1979), Bergman (1980) and Aven (1980), Section 4.
In the appendix we study in general terms how the minimization of a function $B^T = M^T/S^T$, T element of some set, can be carried out by first minimizing λ-functions $C_\lambda^T = M^T-\lambda S^T$. Our analysis here is much inspired by Bergman (1980). Using the results obtained in the appendix we can conclude that there always exists a λ^* such that T_{λ^*} minimizes (1.1).
The λ-minimization technique is very suitable in situations where it is easy to find solutions to λ-problems; in replacements/maintenance applications this is often the case.

Nummelin (1980) uses a different technique than the λ-technique in order to solve a minimizing problem of the form (1.1). His analysis is quite general, but is based on the assumption that (1.2) holds.

In Section 3 a discrete version of the minimization problem (1.1) is considered.
In a coming work we shall present some replacement models where the optimality criterion has the form described in Section 3.

2. Continuous time

2.1. The set-up. Let (Ω,Σ,P) be a complete probability space and let $\{F_t, t\in[0,\infty)\}$ be a non-decreasing family of sub-σ-fields of Σ which satisfies the usual condition, i.e. $\{F_t\}$ is right-continuous ($F_t = \bigcap_{s>t} F_s$) and F_0 contains all the negligable sets of Σ. (The σ-field F_t represents the information about the condition of the system at time t.)

We shall denote by F the smallest σ-field containing F_t for all t.

Let T' be the set of stopping times relative to $\{F_t\}$. (A random variable T taking values in $[0,\infty]$ is called a stopping time relative to $\{F_t\}$ if the event $\{T\leq t\}\in F_t$.)

Assume that the following random variables and stochastic processes are given:

$c(0)$ and $p(0)$, F_0-measurable non-negative random variables; $\{a(t), t\in[0,\infty)\}$, a progressively measurable stochastic process adapted to $\{F_t\}$ taking values in $(-\infty,\infty]$, i.e. the map $(t,\omega) \to a(t,\omega)$ from $[0,s]\times\Omega$ to $(-\infty,\infty]$ is $B[0,s]\times F_s$-measurable for all $s < \infty$ ($B[0,s]$ is the σ-field of Borel sets on $[0,s]$); $\{h(t), t\in[0,\infty)\}$; a progressively measurable process adapted to $\{F_t\}$ taking values in $[0,\infty]$.

Define the stopping times relative to $\{F_t\}$, T_λ, $\lambda \in (-\infty,\infty)$ by

(2.1) $$T_\lambda = \inf\{t\geq 0,\ a(t)\geq\lambda\}$$

(by convention $T_\lambda=\infty$ if $a(t)<\lambda$ for all $t\geq 0$)

and for each $\omega \in \Omega$ the measure μ by

$$\mu(B,\omega) = \int_B h(t,\omega)dt \ , \quad B \in B[0,\infty)$$

(formally $d\mu(t) = h(t)dt$).

We now make up a list of basic assumptions:

<u>Basic assumptions</u>:

(2.2) $0 < Ec(0) < \infty$,

(2.3) $0 \leq Ep(0) < \infty$,

(2.4) $\mu([0,s],\omega) > 0$, $s > 0$ for all ω ,

(2.5) $a(\cdot,\omega)$ is non-decreasing for all ω ,

(2.6) $E \int_0^{T_0} a(t)d\mu(t) > -\infty$,

(2.7) $E \int_0^{T_\lambda} d\mu(t) < \infty$, $\lambda \in (-\infty,\infty)$.

(In (2.4) and (2.5) the statement "for all ω" can be replaced by "for almost all ω".)

Let now for each $T \in \mathcal{T}'$,

$$M^T = E[\int_0^T a(t)d\mu(t)+c(0)]$$

$$S^T = E[\int_0^T d\mu(t)+p(0)] \ .$$

Consider the subclass $\mathcal{T}$ of $\mathcal{T}'$ defined by

$$\mathcal{T} = \{T\in\mathcal{T}' : M^T<\infty \text{ and } S^T<\infty\},$$

and let

(2.8) $\qquad B^T = M^T/S^T \ , \qquad T \in \mathcal{T} \ .$

We observe that $T_\lambda \in \mathcal{T}$ by (2.2), (2.3), the definition of T_λ and (2.7).

The problem is to find a $T \in \mathcal{T}$ minimizing $B^{\cdot}$.

2.2. <u>The optimal stopping time</u>. We will show that the minimization problem (2.8) is a special case of the minimization problem discussed in the appendix.

Using the assumptions (2.2)-(2.6) it is easily seen that

(2.9) $\qquad -\infty < M^T < \infty \qquad$ for all $T \in \mathcal{T}$,

(2.10) $\qquad 0 \leq S^T < \infty \qquad$ for all $T \in \mathcal{T}$,

(2.11) $\qquad S^T = 0 \Rightarrow T = 0$ almost surely (a.s.) $\Rightarrow M^T > 0$,

(2.12) $\qquad \lambda > \inf_{t>0}\{\text{ess inf } a(t)\} \Leftrightarrow P\{T_\lambda>0\} > 0$.

It follows form (2.9)-(2.12) that the conditions (a)-(d) stated in the appendix hold.

Let now

$$C_\lambda^T = M^T-\lambda S^T = E[\int_0^T (a(t)-\lambda)d\mu(t)+c(0)-\lambda p(0)], \quad T\in \mathcal{T} \ , \ \lambda\in(-\infty,\infty).$$

We assert that T_λ (defined by (2.1) minimizes C_λ^T , $T \in \mathcal{T}$. This is seen by observing that

$$a(t) < \lambda \quad \text{if} \quad t < T_\lambda \quad \text{and}$$

$$a(t) \geq \lambda \quad \text{if} \quad t > T_\lambda \ .$$

It follows that we here have a special case of the minimization problem discussed in the appendix. Besides, we have $\lambda^* \overset{\text{def}}{=} \inf B^T > -\infty$. We sketch the proof of this assertion. Using the extended monotone convergence theorem, see Ash (1972) page 47 (remember (2.6)), we find that $\lim_{\lambda\to-\infty} E \int_0^{T_\lambda} a(t)d\mu(t) = 0$ and thus $M^{T_\lambda} = E \int_0^{T_\lambda} a(t)d\mu(t)+Ec(0) \geq 0$ for λ less than some finite λ' by (2.2). It follows from A.4. of the appendix that $\lambda^* > -\infty$. Now A.6 and A.7 of the appendix give us the following results.

<u>Theorem 2.1.</u> The stopping time

T_{λ^*}, where $\lambda^* = \inf_{T\in\mathcal{T}} B^T$, minimizes B^T, $T \in \mathcal{T}$. The value λ^* is given as the unique solution of the equation $\lambda = B(\lambda) \overset{\text{def}}{=} B^{T_\lambda}$.
Moreover,
if $\lambda > \lambda^*$, then $\lambda > B(\lambda)$ whereas
if $\lambda < \lambda^*$, then $\lambda < B(\lambda)$.

<u>Proposition 2.2.</u> (See the appendix: Some final remarks, (iii))
Choose any λ_1 such that $S^{T_{\lambda_1}} > 0$ and set iteratively

$$\lambda_{n+1} = B(\lambda_n), \quad n = 1,2,\ldots .$$

Then

$$\lim_{n\to\infty} \lambda_n = \lambda^* .$$

2.3 Some further properties of the function $B(\lambda) = B^{T_\lambda}$

<u>Proposition 2.3.</u> $B(\cdot)$ is left-continuous.

<u>Proof.</u> Let $\lambda_n \uparrow \lambda$. Then if we can prove that $T_{\lambda_n} \uparrow T_\lambda$, it follows from the monotone convergence theorem that $S^{T_{\lambda_n}} \uparrow S^{T_\lambda}$, $M^{T_{\lambda_n}} \to M^{T_\lambda}$ (if $\lambda \leq 0$, then $M^{T_{\lambda_n}} \downarrow M^{T_\lambda}$ whereas if $\lambda > \lambda_n > 0$ then $M^{T_{\lambda_n}} \uparrow M^{T_\lambda}$).
Consider a fixed $\omega \in \Omega$. Clearly $T_{\lambda_n} \leq T_{\lambda_{n+1}} \leq \cdots \leq T_\lambda$. Hence $T_{\lambda_n} \uparrow y$, say, and we have $y \leq T_\lambda$. Suppose $y < T_\lambda$. Then we can find an $\varepsilon > 0$ such that $y+\varepsilon < T_\lambda$. It follows that $a(y+\varepsilon) < \lambda$.

On the other hand we have by (2.5) and the definition of T_{λ_n},

$$a(y+\varepsilon) \geq a(T_{\lambda_n}+\varepsilon) \geq \lambda_n$$

for each n. Letting $n \to \infty$ we get $a(y+\varepsilon) \geq \lambda$ and a contradiction is obstained. Thus $y = T_\lambda$ and the proposition is proved.

Proposition 2.4. The function $B(\lambda)$ is non-increasing in λ for $\lambda \leq \lambda^*$ and non-decreasing in λ for $\lambda \geq \lambda^*$.

Proof. Let $\lambda_2 \leq \lambda_1 \leq \lambda^*$. Then $T_{\lambda_2} \leq T_{\lambda_1} \leq T_{\lambda^*}$. We shall prove that $B(\lambda_1) \leq B(\lambda_2)$. If $S^{T_{\lambda_2}} = 0$, then clearly $B(\lambda_1) \leq B(\lambda_2)$ by (2.11). Assume therefore that $S^{T_{\lambda_2}} > 0$ (which implies $S^{T_{\lambda_1}} > 0$). Writing

$$S^{T_{\lambda_2}}(B(\lambda_2)-B(\lambda_1)) = M^{T_{\lambda_2}}-B(\lambda_1)S^{T_{\lambda_2}} = M^{T_{\lambda_2}}-M^{T_{\lambda_1}}$$
$$+ B(\lambda_1)(S^{T_{\lambda_1}}-S^{T_{\lambda_2}}) = E\int_{T_{\lambda_2}}^{T_{\lambda_1}}(B(\lambda_1)-a(t)d\mu(t)$$

and noting that

$$a(t) < \lambda_1 \leq B(\lambda_1) \quad \text{for} \quad t < T_{\lambda_1}$$

by the definition of T_{λ_1} and Theorem 2.1, it is seen that $B(\lambda_1) \leq B(\lambda_2)$.

Now let $\lambda^* \leq \lambda_1 \leq \lambda_2$. We shall prove that $B(\lambda_1) \leq B(\lambda_2)$. First note that $S^{T_{\lambda_i}} > 0$, $i = 1,2$ since $0 < S^{T_{\lambda^*}} \leq S^{T_{\lambda_i}}$ $i = 1,2$. The inequality $0 < S^{T_{\lambda^*}}$ is easily seen to hold by (2.11) and Theorem 2.1. As a consequence of the definition of T_{λ_1}, (2.5) and Theorem 2.1 we have

$$a(t) \geq \lambda_1 \geq B(\lambda_1) \quad \text{if} \quad T_{\lambda_1} < t,$$

and it follows that

$$S^{T_{\lambda_2}}(B(\lambda_2)-B(\lambda_1)) = E\int_{T_{\lambda_1}}^{T_{\lambda_2}}(a(t)-B(\lambda_1))d\mu(t) \geq 0.$$

Hence $B(\lambda_2) \geq B(\lambda_1)$. This completes the proof.

3. "Discrete time"

Let (Ω,Σ,P) be a complete probability space, and let $\{F_n\}_{n=0}^{\infty}$ be a non-decreasing sequence of sub-σ-fields of Σ such that F_0 includes all the negligable sets of Σ. Denote by F the smallest σ-field containing F_n for all n.

Let N' be the set of stopping times relative to $\{F_n\}$ (N is called a stopping time relative to $\{F_n\}$ if the event $\{N=n\} \in F_n$ $n = 1,2,\dots$). Assume the following random variables and sequences of random variables are given:

c_0 and p_0, F_0-measurable non-negative random variables; $\{a_n\}_{n=0}^{\infty}$, a sequence of random variables adapted to $\{F_n\}$ (i.e. a_n is F_n-measurable) with values in $(-\infty,\infty]$;

$\{h_n\}$, a sequence of random variables adapted to $\{F_n\}$ with values in $[0,\infty]$.

Define the stopping times relative to $\{F_n\}$,

$$N_\lambda = \inf\{n\geq 0,\ a_n\geq\lambda\},\ \lambda \in (-\infty,\infty)$$

and make the following assumptions:

$$0 < Ec_0 < \infty ,$$

$$0 \leq Ep_0 < \infty ,$$

$$h_0(\omega) > 0 \quad \text{for all (almost all) } \omega,$$

$$a_n(\omega) \quad \text{is non-decreasing for all (almost all) } \omega ,$$

$$E \sum_{n=0}^{N_0-1} a_n h_n > -\infty ,$$

$$E \sum_{n=0}^{N_\lambda-1} h_n < \infty , \quad \lambda \in (-\infty,\infty) .$$

The problem is to find an

$$N' \in N \overset{\text{def}}{=} \{N\in N' :\ M^N<\infty \text{ and } S^N<\infty\} ,$$

where

$$M^N = E[\textstyle\sum_{n=0}^{N-1} a_n h_n + c_0]$$

and

$$S^N = E[\textstyle\sum_{n=0}^{N-1} h_n + p_0] ,$$

such that N' minimizes

$$B^N = \frac{M^N}{S^N} .$$

Let

$$B(\lambda) = B^{N_\lambda} .$$

By proceeding in the same manner as in Section 2 we can prove the following theorem.

Theorem 3.1

(i) The stopping time N_{λ^*}, where $\lambda^* = \inf_{N \in N} B^N$, minimizes B^N, $N \in N$. The value λ^* is given as the unique solution of the equation $\lambda = B(\lambda)$.

(ii) If $\lambda > \lambda^*$, then $\lambda > B(\lambda)$ whereas if $\lambda < \lambda^*$, then $\lambda < B(\lambda)$.

(iii) Let λ_1 be such that $S^{N_{\lambda_1}} > 0$ and set iteratively

$$\lambda_{n+1} = B(\lambda_n), \quad n = 1,2,\ldots .$$

Then

$$\lim_{n\to\infty} \lambda_n = \lambda^* .$$

(iv) $B(\cdot)$ is left-continuous.

(v) $B(\lambda)$ is non-increasing in λ for $\lambda \leq \lambda^*$.

(vi) $B(\lambda)$ is non-decreasing in λ for $\lambda \geq \lambda^*$.

Acknowledgement

The author is grateful to Bent Natvig for valuable comments.

Appendix.

Let T be a set and $M^{\cdot}$ and $S^{\cdot}$ functions defined on T such that

(a) $-\infty < M^T < \infty$ for all $T \in T$,

(b) $0 \leq S^T < \infty$ for all $T \in T$,

(c) $S^T = 0 \Rightarrow M^T > 0$,

(d) there exists a $T \in T$ such that $S^T > 0$.

Define the function $B^{\cdot}$ from T to $(-\infty,\infty]$ by

$$B^T = \frac{M^T}{S^T}$$

and let

$$\lambda^* = \inf_{T \in T} B^T .$$

Clearly $-\infty \leq \lambda^* < \infty$. If there exists a $T \in T$ such that $B^T = \lambda^*$, we say that T is optimal.

Let the λ-functions C_λ^T, $\lambda \in (-\infty,\infty)$ from $\mathcal{T}$ to $(-\infty,\infty)$ be defined by

$$C_\lambda^T = M^T - \lambda S^T.$$

We shall see how the problem of minimizing B^T can be solved by minimizing the λ-functions $C_\lambda^\bullet$. A result which has been used by e.g. Brender (see Barlow and Proschan (1965) page 115-116), Ross (1971) and Taylor (1975) in special cases is the following:

A.1. If $T_\lambda \in \mathcal{T}$ minimizes $C_\lambda^\bullet$ and $C_\lambda^{T_\lambda} = 0$, then T_λ minimizes $B^\bullet$ (and $\lambda = B^{T_\lambda} = \lambda^*$).

(The proof of this result is left to the reader.)

It should be noted that if there exists a T which is optimal, then T minimizes $C_{\lambda^*}^\bullet$ and $C_{\lambda^*}^T = 0$.

We now make the following assumption.

(e) There exists a $T_\lambda \in \mathcal{T}$ such that T_λ minimizes $C_\lambda^\bullet$ for each $\lambda \in (-\infty,\infty)$.

We shall deduce some consequences of this assumption. The main result we prove is the following:

If $\lambda^* > -\infty$, then $C_{\lambda^*}^{T_{\lambda^*}} = 0$, (cf- A.1). Thus we can conclude that if (a)-(e) hold and $\lambda^* > -\infty$, then T_{λ^*} is optimal. In particular it follows that T_{λ^*} is optimal if (a)-(e) hold and $0 < M^T < \infty$, $T \in \mathcal{T}$ since then $\lambda^* > 0$.

Denote by

$$B(\lambda) = B^{T_\lambda}$$

and observe that

$$B(\lambda) \lesseqgtr \lambda \Leftrightarrow C_\lambda^{T_\lambda} \lesseqgtr 0.$$

We now prove the following results.

A.2.

(i) For each λ such that $\lambda^* < \lambda$ we have

$$C_\lambda^{T_\lambda} < 0 \text{ or equivalently } B(\lambda) < \lambda.$$

(ii) For each λ such that $\lambda < \lambda^*$ we have

$$C_\lambda^{T_\lambda} > 0 \text{ or equivalently } B(\lambda) > \lambda.$$

(iii) $C_\lambda^{T_\lambda}$ is a non-increasing function in λ.

Proof

(i) Let T be such that $\lambda^* \leq B^T < \lambda$. Then we get $C_\lambda^{T_\lambda} \leq C_\lambda^T \leq M^T - B^T S^T = 0$, which proves (i).

(ii) The result follows by noting that $B(\lambda) \geq \lambda^* > \lambda$.

(iii) Let $\lambda_1 \leq \lambda_2$. By (e) we have

$$C_{\lambda_2}^{T_{\lambda_2}} \leq C_{\lambda_2}^{T_{\lambda_1}} = M^{T_{\lambda_1}} - \lambda_2 S^{T_{\lambda_1}} ,$$

and so since $\lambda_1 \leq \lambda_2$ we get $C_{\lambda_2}^{T_{\lambda 2}} \leq C_{\lambda_1}^{T_{\lambda 1}}$,

and the statement follows.

A.3.

(i) Let $\lambda^* < \lambda \leq \lambda_1 < \lambda_2$. Then we have

$$S^{T_\lambda} < - \frac{C_{\lambda_2}^{T_{\lambda_2}}}{\lambda_2 - \lambda_1}$$

(ii) $M^{T_0} \leq M^T$ for all $T \in \mathcal{T}$.

Proof

(i) By (e), the fact that $\lambda \leq \lambda_1$ and A.2 (i) we get

$$\begin{aligned} C_{\lambda_2}^{T_{\lambda 2}} \leq C_{\lambda_2}^{T_\lambda} &= M^{T_\lambda} - \lambda_1 S^{T_\lambda} - (\lambda_2 - \lambda_1) S^{T_\lambda} \\ &\leq C_\lambda^{T_\lambda} - (\lambda_2 - \lambda_1) S^{T_\lambda} < -(\lambda_2 - \lambda_1) S^{T_\lambda} , \end{aligned}$$

and (i) follows.

(ii) The result is an immidiate consequence of assumption (e) (let $\lambda = 0$).

A.4. $\lambda^* = -\infty$ if and only if

$$M^{T_\lambda} < 0 \text{ for } \lambda \leq 0 \quad \text{and} \quad S^{T_\lambda} \to 0 \text{ as } \lambda \to -\infty .$$

Proof. "only if" follows from A.2 (i) and A.3 (i). "If" is trivial.

A.5. If $\lambda^* > -\infty$, then $C_{\lambda^*}^{T_{\lambda^*}} = 0$ (or equivalently $B(\lambda^*) = \lambda^*$).

Proof. Let $0 < \varepsilon < h$. Then from A.2 (i) we have

$$B(\lambda) < \lambda^*+\varepsilon \qquad \text{for all } \lambda \text{ such that } \lambda^*<\lambda<\lambda^*+\varepsilon .$$

This gives $M^{T_\lambda}-\lambda^* S^{T_\lambda} < \varepsilon S^{T_\lambda}$ and it follows that

$$C_{\lambda^*}^{T_{\lambda^*}} \leqslant C_{\lambda^*}^{T_\lambda} < \varepsilon S^{T_\lambda} , \qquad \lambda^*<\lambda<\lambda^*+\varepsilon .$$

Since S^{T_λ} is bounded by $-C_{\lambda^*+2h}^{T_{\lambda^*+2h}}/h < \infty$ for $\lambda^*<\lambda<\lambda^*+h$ (A.3(i)), we obtain by letting $\varepsilon \to 0$ that $C_{\lambda^*}^{T_{\lambda^*}} \leqslant 0$. But from the definition of λ^* we have that $C_{\lambda^*}^{T_{\lambda^*}} \geqslant 0$, so $C_{\lambda^*}^{T_{\lambda^*}} = 0$ and the proof is completed.

We summarize.

A.6. If (a)-(e) hold and $\lambda^* > -\infty$, then T_{λ^*} is optimal and λ^* is given as the unique solution of the equation $B(\lambda) = \lambda$.
Moreover, if $\lambda > \lambda^*$, then $\lambda > B(\lambda)$ whereas
if $\lambda < \lambda^*$, then $\lambda < B(\lambda)$.

Now we give an algorithm which always produces a sequence converging to λ^* (cf. Bergman (1980)).

A.7. Algorithm. Choose any λ_1 such that $S^{T_{\lambda_1}} > 0$, and set iteratively

$$\lambda_{n+1} = B(\lambda_n) , \qquad n = 1,2,3,\ldots .$$

Then

$$\lim_{n\to\infty} \lambda_n = \lambda^* .$$

Proof. Since $\infty > \lambda_2 = B(\lambda_1) \geqslant \lambda^*$ we have from A.6 that $\lambda^* \leqslant \lambda_3 = B(\lambda_2) \leqslant \lambda_2$. So by induction $\lambda^* \leqslant \lambda_{n+1} = B(\lambda_n) \leqslant \lambda_n < \infty$, $n \geqslant 2$. Hence there exists a $\lambda' \geqslant \lambda^*$ such that $\lim \lambda_n = \lambda'$.
We must show that $\lambda' = \lambda^*$. Assume first that $\lambda^* > -\infty$. It then sufficies to prove that $\lambda' \leqslant B(\lambda')$ or equivalently $C_{\lambda'}^{T_{\lambda'}} \geqslant 0$ since then $\lambda' = \lambda^*$ by A.2(i). Now writing

$$C_{\lambda_n}^{T_{\lambda_n}} = M^{T_{\lambda_n}}-\lambda_n S^{T_{\lambda_n}} = \lambda_{n+1} S^{T_{\lambda_n}}-\lambda_n S^{T_{\lambda_n}} = (\lambda_{n+1}-\lambda_n) S^{T_{\lambda_n}}$$

and noting that $S^{T_{\lambda_n}}$ is bounded (A.3(i)) and $\lambda_{n+1}-\lambda_n \to 0$, we see that $C_{\lambda_n}^{T_{\lambda_n}} \to 0$ as $n \to \infty$. Since $C_{\lambda'}^{T_{\lambda'}} \geqq C_{\lambda_n}^{T_{\lambda_n}}$ (A.2(iii)) we can conclude that $C_{\lambda'}^{T_{\lambda'}} \geqslant 0$. This proves the convergence if $\lambda^* > -\infty$. Assume now that $\lambda^* = -\infty$. We must show that $\lambda' = -\infty$. Suppose $\lambda' > -\infty$. Then by the proof above we have $C_{\lambda'}^{T_{\lambda'}} \geq 0$. From A.2 (i) we can conclude that $\lambda^* = \lambda'$. But this is a contradiction since $\lambda^* = -\infty$. Thus $\lambda' = -\infty$. The proof is now complete.

<u>Observation</u>: It is easily seen from the above proof that $\lambda_{n+1} < \lambda_n$ unless $\lambda_n = \lambda^*$.

<u>Some final remarks</u>

(i) Suppose that assumption (e) is replaced by

(e') $T_\lambda \in \mathcal{T}$ minimizes $\dot{C}_\lambda$ for $\lambda \in [u,v) \cap (-\infty,\infty)$,

where $-\infty \leq u \leq \lambda^* < v \leq \infty$. Then the results, A.1-A.7 (A.3(ii) if $u < 0$ and A.7 if $\lambda_2 = B(\lambda_1) < v$) are still valid if we restrict the λ-analysis to $[u,v) \cap (-\infty,\infty)$.

(ii) Suppose that (a)-(d) hold, $S^T \leq r < \infty$ for all $T \in \mathcal{T}$, $\lambda^* > -\infty$ and $T_{\lambda^*} \in \mathcal{T}$ minimizes $\dot{C}_{\lambda^*}$ (the assumption (e) is dropped). Then we can prove that $C_{\lambda^*}^{T_{\lambda^*}} = 0$, i.e. T_{λ^*} is optimal. Let $\varepsilon > 0$ be given and choose $T \in \mathcal{T}$ such that $B^T < \lambda^* + \varepsilon$. Then we have $0 \leq C_{\lambda^*}^{T_{\lambda^*}} \leq C_{\lambda^*}^T < \varepsilon S^T \leq \varepsilon r$. It follows that $C_{\lambda^*}^{T_{\lambda^*}} = 0$.

(iii) Standard numerical iterative methods, for example the bisection method or modified regula falsi (see e.g. Conte and de Boor (1972) Section 2) can in addition to A.7 be used to locate λ^*. We must then start with $\lambda_a \leq \lambda_b$ such that $\lambda_a \leq B(\lambda_a)$ and $\lambda_b \geq B(\lambda_b)$ (then $\lambda_a \leq \lambda^* \leq \lambda_b$).

<u>References</u>

Abdel-Hameed, M. and Shimi, I.N. (1978), Optimal replacement of damaged devices. J.Appl.Prob. 15, 153-161.

Ash, R.B. (1972), Real Analysis and Probability. Academic Press, New York.

Aven, T. (1980), Optimal replacement policies for a multistate system, Statistical Res. Report, Inst. of Math., Univ. of Oslo.

Aven, T. (1981), Optimal replacement under a minimal repair strategy - a general failure model. To appear in Adv. Appl. Prob.

Aven, T. (1982), Optimal replacement under a minimal repair strategy using average cost criterion - the general failure model. To appear as a Statistical Research Report, Inst. of Math., Univ. of Oslo.

Barlow, R.E. and Proschan, R. (1965), Math. Theory of Reliability, Wiley New York.

Bergman, B. (1978), Optimal replacement under a general failure model, Adv. Appl. Prob. 10, 431-451.

Bergman, B. (1980), On some recent advances in replacement theory. From the Society of Reliability Engineers symposium "Tillforlighetsdagar på Chalmars 5.-6. mai 1980", Chalmars Institute of Technology, Gothenburg.

Conte, S.D. and de Boor, C. (1972), Elementary Numerical Analysis. McGraw-Hill Kogakusha.

Dellacherie, C. and Meyer, P.A. (1978), Probabilities and Potential Mathematics studies, North-Holland Publishing Comp., Amsterdam.

Meyer, P.A. (1966), Probability and Potentials, Baisdell Publishing Comp., Waltham, Mass.

Nummelin (1979), The general failure model: Optimal replacement using discounted costs. Report-HTKK-MAT-A 144, Institute of Mathematics, Helsinki University of Technology.

Nummelin (1979), The general failure model: The discrete-time case. Report-HTKK-MAT-A 146, Institute of Mathematics, Helsinki University of Technology.

Nummelin (1979), A note on the general failure model: Non-negligabel replacement times. Report-HTKK-MAT-A 145, Institute of Mathematics, Helsinki University of Technology.

Nummelin, E. (1980), A general failure model: Optimal replacement with state dependent replacement and failure costs. Math. Operat. Res. 3, 381-387.

Ross, S. (1971), Infinitesimal look-ahead stopping rules, Ann. Math. Statist. 42, 297-303.

Taylor, H.M. (1975), Optimal replacement under additive damage and other failure models. Naval Res. Logist. Quart. 22, 1-18.

Yamada, K. (1980), Explicit formula of optimal replacement under additive shock processes. Stoch. Proc. Appl. 9, 193-208.

Zuckerman, D. (1978a), Optimal replacement policy for the case where damage process is a one-sided Lévy process. Stoch. Proc. Appl. 7, 141-151.

Zuckerman, D. (1978b), Optimal stopping in a Semi-Markow shock model. J. Appl. Prob. 15, 629-634.

Zuckerman, D. (1979), Optimal replacement rule-discounted cost criterion, R.A.I.R.O. 1, 67-74.