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OPTIMAL SCALING FOR THE TRANSIENT PHASE OF THE RANDOM WALK METROPOLIS ALGORITHM: THE MEAN-FIELD LIMIT[1]

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We consider the random walk Metropolis algorithm on $\mathbb{R}^n$ with Gaussian proposals, and when the target probability measure is the n-fold product of a one-dimensional law. In the limit $n \to \infty$, it is well known (see [*Ann. Appl. Probab.* **7** (1997) 110–120]) that, when the variance of the proposal scales inversely proportional to the dimension n whereas time is accelerated by the factor n, a diffusive limit is obtained for each component of the Markov chain if this chain starts at equilibrium. This paper extends this result when the initial distribution is not the target probability measure. Remarking that the interaction between the components of the chain due to the common acceptance/rejection of the proposed moves is of mean-field type, we obtain a propagation of chaos result under the same scaling as in the stationary case. This proves that, in terms of the dimension n, the same scaling holds for the transient phase of the Metropolis–Hastings algorithm as near stationarity. The diffusive and mean-field limit of each component is a diffusion process nonlinear in the sense of McKean. This opens the route to new investigations of the optimal choice for the variance of the proposal distribution in order to accelerate convergence to equilibrium (see [Optimal scaling for the transient phase of Metropolis–Hastings algorithms: The longtime behavior *Bernoulli* (2014) To appear]).

1. Introduction. Many Markov Chain Monte Carlo (MCMC) methods are based on the Metropolis–Hastings algorithm [11, 15]. Let us recall this well-known sampling technique. Let us consider a target probability distribution on $\mathbb{R}^n$ with density p. Starting from an initial random variable X_0, the Metropolis–Hastings algorithm generates iteratively a Markov chain $(X_k)_{k\geq 0}$ in two steps. At time k, given X_k, a candidate Y_{k+1} is sampled using a proposal distribution with density $q(X_k, y)$. Then the proposal Y_{k+1} is accepted with probability $\alpha(X_k, Y_{k+1})$, where

$$\alpha(x, y) = 1 \wedge \frac{p(y)q(y, x)}{p(x)q(x, y)}.$$

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Here and in the following, we use the standard notation $a \wedge b = \min(a, b)$. If the proposed value is accepted, then $X_{k+1} = Y_{k+1}$ otherwise $X_{k+1} = X_k$. The Markov chain $(X_k)_{k\geq 0}$ is by construction reversible with respect to the target density p, and thus admits $p(x)\,dx$ as an invariant distribution. The efficiency of this algorithm highly depends on the choice of the proposal distribution q. One common choice is a Gaussian proposal centered at the current position $x \in \mathbb{R}^n$ with variance $\sigma^2 \operatorname{Id}_{n\times n}$:

$$q(x, y) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{|x-y|^2}{2\sigma^2}\right).$$

Since the proposal is symmetric ($q(x, y) = q(y, x)$), the acceptance probability reduces to

$$\alpha(x, y) = 1 \wedge \frac{p(y)}{p(x)}. \tag{1.1}$$

Metropolis–Hastings algorithms with symmetric kernels are called random walk Metropolis (RWM) algorithms.

The choice of the variance σ^2 is crucial for the performance of the RWM algorithm. It should be sufficiently large to ensure a good exploration of the state space, but not too large otherwise the rejection rate becomes typically very high since the proposed moves fall in low probability regions, in particular in high dimension. It is expected that the higher the dimension, the smaller the variance of the proposal should be. The first theoretical results to optimize the choice of σ^2 in terms of the dimension n are due to Roberts, Gelman and Gilks in [21]. The authors study the RWM algorithm under two fundamental (and somewhat restrictive) assumptions: (i) the target probability distribution is the n-fold tensor product of a one-dimensional density:

$$p(x) = \prod_{i=1}^{n} \frac{\exp(-V(x_i))}{Z}, \tag{1.2}$$

where $x = (x_1, \dots, x_n)$ and $Z = \int_{\mathbb{R}} \exp(-V)$, and (ii) the initial distribution is the target probability:

$$X_0^n \sim p(x)\,dx.$$

The superscript n in the Markov chain $(X_k^n)_{k\geq 0}$ explicitly indicates the dependency on the dimension n. Then, under additional regularity assumptions on V, the authors prove that for a proper scaling of the variance as a function of the dimension, namely

$$\sigma_n^2 = \frac{l^2}{n},$$

where l is a fixed constant, the Markov process $(X^{1,n}_{\lfloor nt \rfloor})_{t\geq 0}$ (where $X^{1,n}_k \in \mathbb{R}$ denotes the first component of $X^n_k \in \mathbb{R}^n$) converges in law to a diffusion process:

$$dX_t = \sqrt{h(l)}\, dB_t - h(l)\frac{1}{2}V'(X_t)\, dt, \tag{1.3}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion,

$$h(l) = 2l^2\Phi\left(-\frac{l\sqrt{I}}{2}\right) \quad \text{and} \quad I = \int_{\mathbb{R}} (V')^2 \frac{\exp(-V)}{Z}. \tag{1.4}$$

Here and in the following, $\lfloor\cdot\rfloor$ denotes the integer part (for $y \in \mathbb{R}$, $\lfloor y \rfloor \in \mathbb{Z}$ and $\lfloor y \rfloor \leq y < \lfloor y \rfloor + 1$) and Φ is the cumulative distribution function of the normal distribution [$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp(-y^2/2)\, dy$]. The scaling as a function of the dimension of the variance and of the time are indications on how to make the RWM algorithm efficient in high dimension. Moreover, a practical counterpart of this result is that l should be chosen such that $h(l)$ is maximum (the optimal value of l is $l^* = \frac{2.38}{\sqrt{I}}$), in order to optimize the time scaling in (1.3). This optimal value of l corresponds equivalently to an average acceptance rate 0.234 (independently of the value of I): for $l = l^*$,

$$\iint \alpha(x, y)p(x)q(x, y)\, dx\, dy = 2\Phi\left(-\frac{l^*\sqrt{I}}{2}\right) \simeq 0.234.$$

Thus, the practical way to choose σ^2 is to scale it in such a way that the average acceptance rate is roughly $1/4$.

There exist several extensions of such results for various Metropolis–Hastings algorithms, see [3–5, 16, 17, 22, 23], and some of them relax in particular the first main assumption mentioned above about the product form of the target distribution; see [1, 2, 6–8]. Extensions to infinite-dimensional settings have also been explored; see [6, 14, 18].

All these results assume stationarity: the initial measure is the target probability measure. To the best of the authors' knowledge, the only works which deal with a nonstationary case are [9] where partial scaling results are obtained for the RWM algorithm with a Gaussian target and [19]. In the latter paper, the target measure is assumed to be absolutely continuous with respect to the law of an infinite-dimensional Gaussian random field and this measure is approximated in a space of dimension n where the MCMC algorithm is performed. The authors consider a modified RWM algorithm (called preconditioned Crank–Nicolson walk) started at a deterministic initial condition and prove that when σ_n tends to 0 as n tends to ∞ (with no restriction on the rate of convergence of σ_n to 0), the rescaled algorithm converges to a stochastic partial differential equation, started at the same initial condition.

The aim of the present article is to show that, for the RWM algorithm, using the same scaling for the variance and the time as in the stationary case [namely

$\sigma_n^2 = \frac{l^2}{n}$ and considering $(X^{1,n}_{\lfloor nt \rfloor})_{t\geq 0}$], one obtains in the limit n goes to infinity the nonlinear (in the sense of McKean) diffusion process:

$$
\begin{aligned}
dX_t &= \Gamma^{1/2}\bigl(\mathbb{E}\bigl[\bigl(V'(X_t)\bigr)^2\bigr], \mathbb{E}\bigl[V''(X_t)\bigr]\bigr)\, dB_t \\
&\quad - \mathcal{G}\bigl(\mathbb{E}\bigl[\bigl(V'(X_t)\bigr)^2\bigr], \mathbb{E}\bigl[V''(X_t)\bigr]\bigr) V'(X_t)\, dt,
\end{aligned}
\tag{1.5}
$$

where, for $a \in [0, +\infty]$ and $b \in \mathbb{R}$,

$$
\Gamma(a,b) = \begin{cases} l^2 \Phi\Bigl(-\dfrac{lb}{2\sqrt{a}}\Bigr) + l^2 e^{(l^2(a-b))/2} \Phi\Bigl(l\Bigl(\dfrac{b}{2\sqrt{a}} - \sqrt{a}\Bigr)\Bigr), & \\ \qquad\qquad \text{if } a \in (0, +\infty), & \\ \dfrac{l^2}{2}, & \text{if } a = +\infty, \\ l^2 e^{-(l^2 b^+)/2}, & \text{if } a = 0, \end{cases}
\tag{1.6}
$$

where $b^+ = \max(b, 0)$, and

$$
\mathcal{G}(a,b) = \begin{cases} l^2 e^{(l^2(a-b))/2} \Phi\Bigl(l\Bigl(\dfrac{b}{2\sqrt{a}} - \sqrt{a}\Bigr)\Bigr), & \text{if } a \in (0, +\infty), \\ 0, & \text{if } a = +\infty, \\ 1_{\{b>0\}} l^2 e^{-(l^2 b)/2}, & \text{if } a = 0. \end{cases}
\tag{1.7}
$$

Notice that we will assume V'' to be bounded, so that the coefficients in (1.5) are well defined. This convergence result is precisely stated in Theorem 1 below and can be seen as a mean-field limit combined with a diffusion approximation. We would like to mention that another (different in nature) mean-field limit is considered in [7] in the context of optimal scaling: the limit is obtained, under the stationarity assumption, for a target measure which admits some mean-field limit as $n \to \infty$.

Our convergence result generalizes the previous analysis in [21] which was limited to the stationary case [namely X_0^n is distributed according to $p(x)\,dx$]. In particular, in the stationary case, we recover the dynamics (1.3). It also generalizes results from [9] to non-Gaussian targets.

The proof is based on a classical technique to prove propagation of chaos [24]. We first show the tightness of the empirical distribution. Then we pass to the limit in a martingale problem, which is the weak formulation of (1.5). Notice that such a weak formulation has also recently been used in [14] to deal with the stationary case.

This new result opens the route to new investigations of the optimal choice for the variance of the proposal distribution, by precisely taking into account the transient regime (when the Markov chain is not yet at equilibrium). It shows, for example, how to scale properly the variance and the number of samples as a function of the dimension, at least for a product target. A more detailed analysis of the longtime behavior of the nonlinear diffusion (1.5) and of the practical counterparts of this convergence result are the subject of a companion paper [12].

The paper is organized as follows. In Section 2, we state our main convergence result, we present a formal derivation of the limiting diffusion process and we explain the three main steps of its rigorous proof. Sections 3, 4 and 5 are, respectively, devoted to each of these main steps: uniqueness for the stochastic differential equation (1.5) and its weak formulation as a martingale problem, tightness of the laws of the processes $(X^{1,n}_{\lfloor nt\rfloor})_{t\geq 0}$ and identification of the limit probability measures on the path space thanks to the martingale problem. Last, in Section 6, we prove the convergence of the acceptance probability in the RWM algorithm to $\frac{1}{l^2}\Gamma(\mathbb{E}[(V'(X_t))^2], \mathbb{E}[V''(X_t)])$.

2. The main convergence result. Let us first present the precise statement for the main convergence result. Then we will give a formal derivation of the limiting process before sketching the rigorous proof.

2.1. *Notation and convergence to the diffusion process.* We consider a random walk Metropolis algorithm using Gaussian proposal with variance $\sigma_n^2 = \frac{l^2}{n}$, and with target p defined by (1.2). The Markov chain generated using this algorithm writes

$$X^{i,n}_{k+1} = X^{i,n}_k + \frac{l}{\sqrt{n}} G^i_{k+1} 1_{\mathcal{A}_{k+1}}, \qquad 1 \leq i \leq n \tag{2.1}$$

with

$$\mathcal{A}_{k+1} = \{U_{k+1} \leq e^{\sum_{i=1}^n (V(X^{i,n}_k) - V(X^{i,n}_k + (l/\sqrt{n})G^i_{k+1}))}\},$$

where $(G^i_k)_{i,k\geq 1}$ is a sequence of independent and identically distributed (i.i.d.) normal random variables, independent from a sequence $(U_k)_{k\geq 1}$ of i.i.d. random variables uniform on [0, 1]. We assume that the initial positions $(X^{1,n}_0, \ldots, X^{n,n}_0)$ are exchangeable (namely the law of the vector is invariant under permutation of the indices) and independent from $(G^i_k)_{i,k\geq 1}$ and $(U_k)_{k\geq 1}$. Exchangeability is preserved by the dynamics: for all $k \geq 1$, $(X^{1,n}_k, \ldots, X^{n,n}_k)$ are exchangeable. We denote by $\mathcal{F}^n_k$ the sigma field generated by $(X^{1,n}_0, \ldots, X^{n,n}_0)$ and $(G^1_l, \ldots, G^n_l, U_l)_{1\leq l\leq k}$.

In all the following, we also assume that

$$\begin{cases} V \text{ is a } \mathcal{C}^3 \text{ function on } \mathbb{R} \\ \text{with bounded second- and third-order derivatives.} \end{cases} \tag{2.2}$$

For $t > 0$ and $i \in \{1, \ldots, n\}$, let

$$\begin{aligned} Y^{i,n}_t &= (\lceil nt\rceil - nt) X^{i,n}_{\lfloor nt\rfloor} + (nt - \lfloor nt\rfloor) X^{i,n}_{\lceil nt\rceil} \\ &= X^{i,n}_{\lfloor nt\rfloor} + (nt - \lfloor nt\rfloor) \frac{l}{\sqrt{n}} G^i_{\lceil nt\rceil} 1_{\mathcal{A}_{\lceil nt\rceil}} \end{aligned}$$

be the linear interpolation of the Markov chain obtained by rescaling time (the characteristic time scale is $1/n$, and $Y^{i,n}_{k/n} = X^{i,n}_k$, $\forall k \in \mathbb{Z}$). Here and in the following $\lceil \cdot \rceil$ is the upper integer part (for $y \in \mathbb{R}$, $\lceil y \rceil \in \mathbb{Z}$ and $\lceil y \rceil - 1 < y \leq \lceil y \rceil$).

Let us define the notion of convergence (namely the propagation of chaos) that will be useful to study the convergence of the interacting particle system $((Y^{1,n}_t, \dots, Y^{n,n}_t)_{t\geq 0})_{n\geq 1}$ in the limit n goes to infinity.

DEFINITION 1. Let E be a separable metric space. A sequence $(\chi^n_1, \dots, \chi^n_n)_{n\geq 1}$ of exchangeable E^n-valued random variables is said to be ν-chaotic where ν is a probability measure on E if for fixed $j \in \mathbb{N}^*$, the law of $(\chi^n_1, \dots, \chi^n_j)$ converges in distribution to $\nu^{\otimes j}$ as n goes to ∞.

We are now in position to state the main convergence result.

THEOREM 1. *Assume* (2.2) *and let m be a probability measure on $\mathbb{R}$ such that $\int_{\mathbb{R}} (V')^4(x) m(dx) < +\infty$. If the initial positions $(X^{1,n}_0, \dots, X^{n,n}_0)_{n\geq 1}$ are exchangeable, m-chaotic and such that* $\sup_n \mathbb{E}[(V'(X^{1,n}_0))^4] < +\infty$, *then the processes $((Y^{1,n}_t, \dots, Y^{n,n}_t)_{t\geq 0})_{n\geq 1}$ are P-chaotic where P denotes the law* [*on the space $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ of continuous functions with values in $\mathbb{R}$*] *of the solution to the nonlinear stochastic differential equation in the sense of McKean* (*for which strong and weak existence and uniqueness hold*)

$$X_t = \xi + \int_0^t \Gamma^{1/2}\big(\mathbb{E}\big[\big(V'(X_s)\big)^2\big], \mathbb{E}\big[V''(X_s)\big]\big)\, dB_s - \int_0^t \mathcal{G}\big(\mathbb{E}\big[\big(V'(X_s)\big)^2\big], \mathbb{E}\big[V''(X_s)\big]\big) V'(X_s)\, ds, \tag{2.3}$$

where Γ and $\mathcal{G}$ are, respectively, defined by (1.6) *and* (1.7) *and $(B_t)_{t\geq 1}$ is a Brownian motion independent from the initial position ξ distributed according to m.*

Let us make a few remarks on this result. First, concerning the assumption on the initial positions $(X^{1,n}_0, \dots, X^{n,n}_0)_{n\geq 1}$, we note that it is satisfied, for instance, when the random variables $X^{1,n}_0, \dots, X^{n,n}_0$ are i.i.d. according to the probability measure m on $\mathbb{R}$. Second, notice that the results of Theorem 1 do not require $\exp(-V)$ to be integrable. Finally, according to [10] (see Proposition 10.4, page 149 and Theorem 10.2, page 148), under the assumptions of Theorem 1, the piecewise constant processes $((X^{1,n}_{\lfloor nt \rfloor}, \dots, X^{n,n}_{\lfloor nt \rfloor})_{t\geq 0})_{n\geq 1}$ are also P-chaotic when the space of càdlàg sample paths from $[0, +\infty)$ is endowed with the topology of uniform convergence on compact sets.

In addition to the previous convergence result, we are able to identify the limiting average acceptance rate.

PROPOSITION 1. *Under the assumptions of Theorem 1, the function*

$$t \mapsto \mathbb{E}\left|\mathbb{P}(\mathcal{A}_{\lfloor nt\rfloor+1}|\mathcal{F}^n_{\lfloor nt\rfloor}) - \frac{1}{l^2}\Gamma\left(\mathbb{E}\left[\left(V'(X_t)\right)^2\right], \mathbb{E}\left[V''(X_t)\right]\right)\right|$$

converges locally uniformly to 0 *and in particular, the average acceptance rate* $t \mapsto \mathbb{P}(\mathcal{A}_{\lfloor nt\rfloor+1})$ *converges locally uniformly to* $t \mapsto \mathrm{acc}(\mathbb{E}[(V'(X_t))^2], \mathbb{E}[V''(X_t)])$ *where for any* $a \geq 0$ *and* $b \in \mathbb{R}$,

$$\mathrm{acc}(a, b) = \frac{\Gamma(a, b)}{l^2}. \tag{2.4}$$

In the following, we will also need the infinitesimal generator associated to (2.3). For a probability measure μ on $\mathbb{R}$, $\langle \mu, V''\rangle$ is well defined by boundedness of V'', and $\langle \mu, (V')^2\rangle$ is also well defined in $[0, +\infty]$. Here and in the following, the bracket notation refers to the duality bracket for probability measures on $\mathbb{R}$: for μ a probability measure and ϕ a bounded or positive measurable function,

$$\langle \mu, \phi\rangle = \int_{\mathbb{R}} \phi(x)\mu(dx).$$

The infinitesimal generator associated to (2.3) is L_μ defined by

$$\begin{aligned} L_\mu\varphi(x) &= \tfrac{1}{2}\Gamma\left(\langle \mu, (V')^2\rangle, \langle \mu, V''\rangle\right)\varphi''(x) \\ &\quad - \mathcal{G}\left(\langle \mu, (V')^2\rangle, \langle \mu, V''\rangle\right)V'(x)\varphi'(x). \end{aligned} \tag{2.5}$$

More precisely, if $(X_t)_{t\geq 0}$ satisfies (2.3) and P_t denotes the law of X_t, then

$$\begin{aligned} &\text{for any test function } \varphi, \\ &\left(\varphi(X_t) - \int_0^t L_{P_s}\varphi(X_s)\,ds\right)_{t\geq 0} \text{ is a martingale.} \end{aligned} \tag{2.6}$$

Equivalently, for any $s < t$,

$$\mathbb{E}\left(\varphi(X_t) - \int_s^t L_{P_r}\varphi(X_r)\,dr \,\middle|\, \mathcal{F}_s\right) = \varphi(X_s), \tag{2.7}$$

where $\mathcal{F}_s = \sigma(X_r, r \leq s)$. Actually, as explained in Section 3 below, this martingale representation characterizes the distribution [over $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$] of solutions to (2.3): probability measures under which (2.6) holds are distributions of solutions to (2.3), and reciprocally.

2.2. *Relation to previous results in the literature.* Let us discuss how this theorem is related to previous results in the literature. First, when $Z = \int_{\mathbb{R}} e^{-V(x)}\,dx < +\infty$, our convergence result generalizes the scaling limit for the random walk Metropolis–Hastings algorithm stated in the early paper [21] under the restrictive assumption that the vector of initial positions $(X_0^{1,n}, \dots, X_0^{n,n})$ is distributed according to the target distribution $p(x)\,dx$. In this case, it is clear that for all

$n, k \in \mathbb{N}$, $(X_k^{1,n}, \dots, X_k^{n,n})$ is distributed according to $p(x)\,dx$. Moreover, we have the following result.

LEMMA 1. *Assume that* (2.2) *holds, and that* $\int_{\mathbb{R}} e^{-V(x)}\,dx < \infty$. *Then*

$$\int_{\mathbb{R}} (V'(x))^2 e^{-V(x)}\,dx = \int_{\mathbb{R}} V''(x) e^{-V(x)}\,dx < +\infty.$$

PROOF. The integrability of e^{-V} implies that $\liminf_{|x|\to\infty} |x| e^{-V(x)} = 0$. Since $|V'(x)| \le |V'(0)| + \|V''\|_\infty |x|$, one deduces the existence of a sequence $(x_n)_n$ of negative numbers tending to $-\infty$ and a sequence $(y_n)_n$ of positive numbers tending to $+\infty$ such that $\lim_{n\to+\infty} |V'(x_n)|e^{-V(x_n)} + |V'(y_n)|e^{-V(y_n)} = 0$. By integration by parts,

$$\begin{aligned}\int_{x_n}^{y_n} (V'(x))^2 e^{-V(x)}\,dx \\ &= V'(x_n)e^{-V(x_n)} - V'(y_n)e^{-V(y_n)} + \int_{x_n}^{y_n} V''(x)e^{-V(x)}\,dx.\end{aligned}$$

Taking the limit $n \to \infty$ thanks to monotone convergence in the left-hand side and thanks to Lebesgue's theorem and boundedness of V'' in the integral in the right-hand side, one concludes that $\int_{\mathbb{R}} (V'(x))^2 e^{-V(x)}\,dx = \int_{\mathbb{R}} V''(x)e^{-V(x)}\,dx < +\infty$. □

One deduces that for each $t \ge 0$ the solution X_t of (2.3) is distributed according to $Z^{-1}\exp(-V(x))\,dx$ so that $(X_t)_{t\ge 0}$ also solves the stochastic differential equation (1.3)–(1.4) with time-homogeneous coefficients [here, we use the fact that $\Gamma(I, I) = 2\mathcal{G}(I, I) = h(l)$ where $I = \int_{\mathbb{R}} (V'(x))^2 e^{-V(x)} \frac{dx}{Z} = \int_{\mathbb{R}} V''(x) e^{-V(x)} \frac{dx}{Z}$]. Notice that our convergence result requires more regularity but less integrability than in [21], Theorem 1.1, where the log-density $-V$ is assumed to be $\mathcal{C}^2$ with a bounded second-order derivative and such that $\int_{\mathbb{R}} (V')^8 \exp(-V) < +\infty$.

Second, we also recover results from [9], where the authors consider a nonstationary case, but restrict their analysis to Gaussian distributions: $V(x) = \frac{x^2}{2}$. In this case, the function V'' is constant equal to 1 and, for X_t solution to (2.3), one obtains that

$$\begin{aligned}\frac{d}{dt}\mathbb{E}[X_t^2] &= \Gamma(\mathbb{E}[X_t^2], 1) - 2\mathbb{E}[X_t^2]\mathcal{G}(\mathbb{E}[X_t^2], 1) \\ &= l^2 \Phi\Bigl(-\frac{l}{2\sqrt{\mathbb{E}[X_t^2]}}\Bigr) \\ &\quad + (1 - 2\mathbb{E}(X_t^2)) l^2 e^{(l^2(\mathbb{E}[X_t^2]-1))/2} \Phi\Bigl(l\Bigl(\frac{1}{2\sqrt{\mathbb{E}[X_t^2]}} - \sqrt{\mathbb{E}[X_t^2]}\Bigr)\Bigr).\end{aligned}$$

This is indeed the ordinary differential equation satisfied by the deterministic function obtained as the limit (when $n \to \infty$) of the processes $(\frac{1}{n}\sum_{i=1}^n (X_{\lfloor nt \rfloor}^{i,n})^2)_{t\ge 0}$

in [9], Theorem 1. More precisely, the proof of our Proposition 1 ensures that $\mathbb{E}|\frac{1}{n}\sum_{i=1}^{n}(X_{\lfloor nt\rfloor}^{i,n})^2 - \mathbb{E}[X_t^2]|$ converges to 0 locally uniformly in t as $n \to \infty$.

2.3. *A formal derivation.* Before going into the details of a rigorous proof, let us explain how this limit diffusion process can be formally derived.

First, let us make precise how to choose the scaling of σ_n as a function of n. The idea (see [23]) is to choose σ_n in such a way that the limiting acceptance rate (when $n \to \infty$) is neither zero nor one. In the first case, this would mean that the variance of the proposal is too large, so that all proposed moves are rejected. In the second case, the variance of the proposal is too small, and the rate of convergence to equilibrium is thus not optimal. In particular, it is easy to check that σ_n should go to zero as n goes to infinity. Now, notice that the limiting acceptance rate is

$$
\begin{aligned}
\mathbb{E}(1_{\mathcal{A}_{k+1}}|\mathcal{F}_k^n) &= \mathbb{E}\big(e^{\sum_{i=1}^n (V(X_k^{i,n}) - V(X_k^{i,n}+\sigma_n G_{k+1}^i))} \wedge 1 | \mathcal{F}_k^n\big) \\
&= \mathbb{E}\big(e^{-\sum_{i=1}^n (V'(X_k^{i,n})\sigma_n G_{k+1}^i + V''(X_k^{i,n})(\sigma_n^2/2))} \wedge 1 | \mathcal{F}_k^n\big) \\
&\quad + \mathcal{O}(n\sigma_n^3) + \mathcal{O}(\sqrt{n}\sigma_n^2) \\
&= \exp\left(\frac{a_n - b_n}{2}\right)\Phi\left(\frac{b_n}{2\sqrt{a_n}} - \sqrt{a_n}\right) + \Phi\left(-\frac{b_n}{2\sqrt{a_n}}\right) \\
&\quad + \mathcal{O}(n\sigma_n^3) + \mathcal{O}(\sqrt{n}\sigma_n^2) \\
&= \frac{1}{l^2}\Gamma(a_n, b_n) + \mathcal{O}(n\sigma_n^3) + \mathcal{O}(\sqrt{n}\sigma_n^2),
\end{aligned}
\tag{2.8}
$$

where $a_n = \frac{\sigma_n^2}{l^2}\sum_{i=1}^n (V'(X_k^{i,n}))^2$ and $b_n = \frac{\sigma_n^2}{l^2}\sum_{i=1}^n V''(X_k^{i,n})$. To obtain (2.8), we used an explicit computation of the expectation with respect to the Gaussian measure; see (A.5) below (with $\alpha = 0$). From this expression, assuming a propagation of chaos (law of large number) result on the random variables $(X_k^{i,n})_{1\le i\le n}$, one can check that the correct scaling for the variance is $\sigma_n^2 = \frac{l^2}{n}$ in order to obtain a nontrivial limiting acceptance rate (see Proposition 1 above). More precisely, if $a_n \to 0$ and $b_n \to 0$, then the acceptance rate goes to 1 [by continuity of Γ at point $(0,0)$, see Lemma 2 below]. If $a_n \sim \alpha n^{\epsilon}$ and $b_n \sim \beta n^{\epsilon}$ (for some $\epsilon > 0$), then the acceptance rate goes to 0 if $\beta > 0$ and to 1 if $\beta < 0$.

Using the scaling $\sigma_n^2 = \frac{l^2}{n}$, we observe that, for a test function $\varphi : \mathbb{R} \to \mathbb{R}$,

$$
\begin{aligned}
\mathbb{E}(\varphi(X_{k+1}^{1,n})|\mathcal{F}_k^n) &= \mathbb{E}\left(\varphi\left(X_k^{1,n} + \frac{l}{\sqrt{n}}G_{k+1}^1 1_{\mathcal{A}_{k+1}}\right)\Big|\mathcal{F}_k^n\right) \\
&= \varphi(X_k^{1,n}) + \varphi'(X_k^{1,n})\frac{l}{\sqrt{n}}\mathbb{E}(G_{k+1}^1 1_{\mathcal{A}_{k+1}}|\mathcal{F}_k^n) \\
&\quad + \frac{l^2}{2n}\varphi''(X_k^{1,n})\mathbb{E}((G_{k+1}^1)^2 1_{\mathcal{A}_{k+1}}|\mathcal{F}_k^n) + \mathcal{O}(n^{-3/2}).
\end{aligned}
\tag{2.9}
$$

We compute

$$
\begin{aligned}
&\mathbb{E}(G^1_{k+1}1_{\mathcal{A}_{k+1}}|\mathcal{F}^n_k)\\
&\quad= \mathbb{E}\big(G^1_{k+1}\big(e^{\sum_{i=1}^n (V(X^{i,n}_k)-V(X^{i,n}_k+(l/\sqrt{n})G^i_{k+1}))}\wedge 1\big)|\mathcal{F}^n_k\big)\\
&\quad= \mathbb{E}\big(G^1_{k+1}\big(e^{-\sum_{i=1}^n (V'(X^{i,n}_k)(l/\sqrt{n})G^i_{k+1}+V''(X^{i,n}_k)(l^2/(2n)))}\wedge 1\big)|\mathcal{F}^n_k\big)\\
&\qquad+\mathcal{O}(n^{-1/2})\\
&\quad= -V'(X^{1,n}_k)\frac{1}{l\sqrt{n}}\mathcal{G}(\langle \nu^n_k,(V')^2\rangle,\langle \nu^n_k,V''\rangle)+\mathcal{O}(n^{-1/2}),
\end{aligned}
\tag{2.10}
$$

where

$$
\nu^n_k = \frac{1}{n}\sum_{i=1}^n \delta_{X^{i,n}_k}
$$

denotes the empirical distribution associated to the interacting particle system. Equation (2.10) is a consequence of (A.3) below. A more detailed analysis (see Lemma 5 below) shows that the remainder is of order $\mathcal{O}(n^{-3/4})$. This is one of the most crucial estimate to prove rigorously the convergence result. For the diffusion term, we get

$$
\begin{aligned}
&\mathbb{E}((G^1_{k+1})^2 1_{\mathcal{A}_{k+1}}|\mathcal{F}^n_k)\\
&\quad= \mathbb{E}\big((G^1_{k+1})^2\big(e^{\sum_{i=1}^n (V(X^{i,n}_k)-V(X^{i,n}_k+(l/\sqrt{n})G^i_{k+1}))}\wedge 1\big)|\mathcal{F}^n_k\big)\\
&\quad= \mathbb{E}\big((G^1_{k+1})^2\big(e^{-\sum_{i=1}^n (V'(X^{i,n}_k)(l/\sqrt{n})G^i_{k+1}+V''(X^{i,n}_k)(l^2/(2n)))}\wedge 1\big)|\mathcal{F}^n_k\big)\\
&\qquad+\mathcal{O}(n^{-1/2})\\
&\quad= \frac{1}{l^2}\Gamma(\langle \nu^n_k,(V')^2\rangle,\langle \nu^n_k,V''\rangle)+\mathcal{O}(n^{-1/2}).
\end{aligned}
\tag{2.11}
$$

To obtain (2.11), we again used an explicit computation; see (A.5) below.

By plugging (2.10) [with the remainder of order $\mathcal{O}(n^{-3/4})$] and (2.11) into (2.9), we see that the correct scaling in time is to consider $Y^{i,n}_t$ such that $Y^{i,n}_{k/n} = X^{i,n}_k$, and we get

$$
\begin{aligned}
&\mathbb{E}(\varphi(Y^{1,n}_{(k+1)/n})|\mathcal{F}^n_k)\\
&\quad= \varphi(Y^{1,n}_{k/n})-\varphi'(Y^{1,n}_{k/n})\frac{1}{n}V'(Y^{1,n}_{k/n})\mathcal{G}(\langle \mu^n_{k/n},(V')^2\rangle,\langle \mu^n_{k/n},V''\rangle)\\
&\qquad+\frac{1}{2n}\varphi''(Y^{1,n}_{k/n})\Gamma(\langle \mu^n_{k/n},(V')^2\rangle,\langle \mu^n_{k/n},V''\rangle)+\mathcal{O}(n^{-5/4})\\
&\quad= \varphi(Y^{1,n}_{k/n})+\frac{1}{n}(L_{\mu^n_{k/n}}\varphi)(Y^{1,n}_{k/n})+\mathcal{O}(n^{-5/4}),
\end{aligned}
\tag{2.12}
$$

where L_μ is defined by (2.5), and μ_t^n denotes the time-marginal of μ^n defined by (2.13) below (for $k \in \mathbb{N}$, $\mu_{k/n}^n = \nu_k^n$). This can be seen as a discrete-in-time version (over a timestep of size $1/n$) of the martingale property (2.7) [which is actually a characterization in law of a solution to (2.5), as explained below]. Thus, by sending n to infinity and assuming that a law of large number holds for the empirical measure ν_k^n, we expect $Y_t^{1,n}$ to converge to a solution to (2.3). The aim of Section 2.4 is to sketch the rigorous proof of this result.

2.4. *Sketch of the rigorous proof.* The next sections are, respectively, devoted to the three steps of the proof of Theorem 1. In Section 3, we first introduce a nonlinear martingale problem which is a weak formulation of (2.3): the law of any solution to this stochastic differential equation solves the martingale problem. We check uniqueness for the martingale problem by proving trajectorial uniqueness for the stochastic differential equation (2.3). Then, in Section 4, we check the tightness of the sequence of laws of the processes $(Y_t^{1,n})_{t\geq 0}$. Because of the exchangeability of the processes $((Y_t^{1,n}, \dots, Y_t^{n,n})_{t\geq 0})_{n\geq 1}$ and according to [24], this is equivalent to the tightness of the sequence $(\pi^n)_n$ of the laws of the empirical measures

$$\mu^n = \frac{1}{n}\sum_{i=1}^{n} \delta_{Y^{i,n}} \tag{2.13}$$

considered as random variables valued in the space $\mathcal{P}(\mathcal{C})$ of probability measure on the set $\mathcal{C}$ of continuous paths from $[0, +\infty)$ to $\mathbb{R}$. The space $\mathcal{C}$ is endowed with the topology of uniform convergence on compact sets and $\mathcal{P}(\mathcal{C})$ with the corresponding topology for convergence in distribution. The third and last step, performed in Section 5, consists in checking that the limit π^∞ of any convergent subsequence of $(\pi^n)_n$ is concentrated on the solutions of the martingale problem, which, in particular, provides existence of a solution P to this problem. A probability measure Q on $\mathcal{C}$ with initial marginal $Q_0 = m$ solves the martingale problem if and only if $F(Q) = 0$ for a countable set of functionals F of the form (5.1) below. Since the chaoticity of the initial conditions implies that $\pi^\infty(\{Q \in \mathcal{P}(\mathcal{C}) : Q_0 = m\}) = 1$, checking that $\mathbb{E}^{\pi^\infty}|F(Q)| = 0$ for all F in this countable set is enough to conclude that $\pi^\infty = \delta_P$. Combined with the results of the two first steps, this ensures that the whole sequence $(\pi^n)_n$ converges weakly to δ_P where P denotes the unique solution of the martingale problem, namely the law of the unique solution to the stochastic differential equation (2.3). According to [24], this is equivalent to the P-chaoticity of the processes $((Y_t^{1,n}, \dots, Y_t^{n,n})_{t\geq 0})_{n\geq 1}$ and this completes the proof of Theorem 1.

Finally, Section 6 is devoted to the proof of Proposition 1.

As already mentioned, our main result combines a diffusion approximation and a mean-field limit. Mean-field limits apply to systems of n interacting particles (here the components $Y^{i,n}$) when the interaction between two particles is of order $1/n$. At first sight, it is not obvious that this is the case for the system considered in the paper. Nevertheless, from the above formal computation of $\mathbb{E}(\varphi(Y^{1,n}_{(k+1)/n})|\mathcal{F}^n_k)$, we see in equation (2.12) that the interaction is actually of mean-field type: the other components influence the evolution of $Y^{1,n}_{(k+1)/n}$ only through the empirical measure $\mu^n_{k/n} = \frac{1}{n}\sum_{i=1}^n \delta_{Y^{i,n}_{k/n}}$. The mean-field limit is a law of large numbers for the empirical measure μ^n on the path-space: we prove that μ^n converges to the unique solution P of the martingale problem. In the same time, we have to deal with the diffusion approximation.

Notice that in previous scaling results given in the literature, the assumption that the vector of initial positions $(X^{1,n}_0, \dots, X^{n,n}_0)$ is distributed according to the target density makes the derivation of both the mean-field limit and the diffusion approximation much easier: since at subsequent times, $(X^{1,n}_k, \dots, X^{n,n}_k)$ remains distributed according to the target density, it is enough to identify the limiting infinitesimal generator at the initial time. Moreover, under this stationarity assumption and when the target density is the n-fold product of a fixed probability density, the mean-field limit is obtained by the standard law of large numbers.

We end this section with the following lemma which states some basic properties of the functions Γ and $\mathcal{G}$.

LEMMA 2. *The function* Γ *is continuous on* $[0,+\infty]\times\mathbb{R}$ *and such that*

$$\inf_{(a,b)\in[0,+\infty]\times[\inf V'',\sup V'']} \Gamma(a,b) > 0, \tag{2.14}$$

$$\begin{aligned}&\exists C<+\infty, \forall (a,b) \text{ and } (a',b')\in[0,+\infty]\times\mathbb{R},\\ &\quad |\Gamma(a,b)-\Gamma(a',b')| \le C(|b'-b|+|a'-a|+|\sqrt{a'}-\sqrt{a}|).\end{aligned} \tag{2.15}$$

The function $\mathcal{G}$ *is continuous on* $\{[0,+\infty]\times\mathbb{R}\}\setminus\{(0,0)\}$ *and such that*

$$\forall (a,b)\in[0,+\infty]\times\mathbb{R}, \qquad \sqrt{a}\mathcal{G}(a,b) \le \left(l^2\sqrt{b^+} \vee \frac{2l}{\sqrt{2\pi}}\right), \tag{2.16}$$

$$\begin{aligned}&\exists C<+\infty, \forall (a,b) \text{ and } (a',b')\in[0,+\infty]\times[\inf V'',\sup V''],\\ &\quad (\sqrt{a}\wedge\sqrt{a'})|\mathcal{G}(a,b)-\mathcal{G}(a',b')|\\ &\qquad \le C(|b'-b|+|a'-a|+|\sqrt{a'}-\sqrt{a}|).\end{aligned} \tag{2.17}$$

Last,

$$\forall (a,b)\in[0,+\infty]\times\mathbb{R}, \qquad 0\le \mathcal{G}(a,b)\le\Gamma(a,b)\le l^2. \tag{2.18}$$

Notice that $\mathcal{G}$ is indeed discontinuous at point $(0,0)$ since $\lim_{b\to 0^+}\mathcal{G}(0,b) \neq \mathcal{G}(0,0)$. The proof of this lemma is given in the Appendix.

3. Uniqueness for the limiting diffusion. In the present section, we are going to prove trajectorial uniqueness for the stochastic differential equation (2.3) nonlinear in the sense of McKean and deduce uniqueness for the following weak formulation of this dynamics.

DEFINITION 2. Let $(Y_t)_{t\geq 0}$ denote the canonical process on $\mathcal{C}$ and recall the definition (2.5) of L_μ. A probability measure $P \in \mathcal{P}(\mathcal{C})$ with time-marginals $(P_t)_{t\geq 0}$ solves the nonlinear martingale problem (MP) if $P_0 = m$ and for any $\varphi : \mathbb{R} \to \mathbb{R}$ C^2 with compact support,

$$\left(M_t^\varphi \stackrel{\text{def}}{=} \varphi(Y_t) - \int_0^t L_{P_s}\varphi(Y_s)\,ds\right)_{t\geq 0} \qquad \text{is a } P\text{-martingale.}$$

This martingale problem is the weak formulation of the nonlinear stochastic differential equation (2.3). Indeed, the law of any solution of (2.3) solves (MP). Conversely, when P solves (MP), one easily checks by Paul Lévy's characterization (see [13], Theorem 3.16, page 157) that

$$\left(\beta_t = \int_0^t \frac{dY_s + \mathcal{G}(\langle P_s, (V')^2\rangle, \langle P_s, V''\rangle)V'(Y_s)\,ds}{\sqrt{\Gamma(\langle P_s, (V')^2\rangle, \langle P_s, V''\rangle)}}\right)_{t\geq 0}$$

is a P-Brownian motion. Thus, this implies the existence of a weak solution with law P for the stochastic differential equation

$$\begin{aligned} X_t^P = \xi &+ \int_0^t \Gamma^{1/2}(\langle P_s, (V')^2\rangle, \langle P_s, V''\rangle)\,dB_s \\ &- \int_0^t \mathcal{G}(\langle P_s, (V')^2\rangle, \langle P_s, V''\rangle)V'(X_s^P)\,ds. \end{aligned} \tag{3.1}$$

For fixed time-dependent coefficients $\Gamma^{1/2}(\langle P_s, (V')^2\rangle, \langle P_s, V''\rangle)$ and $\mathcal{G}(\langle P_s, (V')^2\rangle, \langle P_s, V''\rangle)$, by boundedness of $\mathcal{G}$ on $[0, +\infty] \times [\inf V'', \sup V'']$ (see Lemma 2 above) and Lipschitz continuity of V', it is standard to check that trajectorial uniqueness holds for this (linear in the sense of McKean) stochastic differential equation. As a consequence, by the Yamada–Watanabe theorem (see [13], Proposition 3.20, page 309, Corollary 3.23, page 310), this linear stochastic differential equation admits a unique strong solution and the law of this solution is P. In conclusion, one may associate a strong solution to (2.3) with law P, to any solution P of the nonlinear martingale problem (MP).

Notice that the two next sections will ensure existence for (MP) and (2.3). Uniqueness is ensured by the following proposition.

PROPOSITION 2. *For any probability measure m on $\mathbb{R}$, uniqueness holds for the nonlinear martingale problem* (*MP*) *and trajectorial uniqueness holds for the stochastic differential equation* (2.3).

To prove Proposition 2, we need the following technical lemma.

LEMMA 3. *For any solution $(X_t)_{t\geq 0}$ of* (2.3),

$$\forall 0 \leq s \leq t, \qquad \mathbb{E}\big[(X_t - X_s)^2\big] \leq 2l^2\bigg[(t-s) + \bigg(l^2 \sup(V'')^+ \vee \frac{2}{\pi}\bigg)(t-s)^2\bigg].$$

Moreover, if $\langle m, (V')^2\rangle < +\infty$, then $t \mapsto \mathbb{E}[(V'(X_t))^2]$ is locally bounded. If $\langle m, (V')^2\rangle = +\infty$, then $\forall t \geq 0$, $\mathbb{E}[(V'(X_t))^2] = +\infty$.

PROOF. Let $(X_t)_{t\geq 0}$ solve (2.3). Then for $0 \leq s \leq t$,

$$\begin{aligned}
\mathbb{E}\big[(X_t - X_s)^2\big] &\leq 2\mathbb{E}\bigg[\bigg(\int_s^t \Gamma^{1/2}\big(\mathbb{E}\big[\big(V'(X_r)\big)^2\big], \mathbb{E}\big[V''(X_r)\big]\big)\, dB_r\bigg)^2\bigg] \\
&\quad + 2(t-s)\int_s^t \mathcal{G}^2\big(\mathbb{E}\big[\big(V'(X_r)\big)^2\big], \mathbb{E}\big[V''(X_r)\big]\big)\mathbb{E}\big[\big(V'(X_r)\big)^2\big]\, dr \\
&\leq 2l^2(t-s) + 2\bigg(l^4 \sup(V'')^+ \vee \frac{2l^2}{\pi}\bigg)(t-s)^2,
\end{aligned}$$

where we used the boundedness properties of Γ and $\sqrt{a}\mathcal{G}(a,b)$ stated in Lemma 2.

One easily deduces the properties of $t \mapsto \mathbb{E}[(V'(X_t))^2]$ since

$$\begin{aligned}
\big(V'(X_t)\big)^2 &\geq \tfrac{1}{2}\big(V'(X_0)\big)^2 - \big(V'(X_t) - V'(X_0)\big)^2 \\
&\geq \tfrac{1}{2}\big(V'(\xi)\big)^2 - \|V''\|_\infty^2 (X_t - X_0)^2, \\
\big(V'(X_t)\big)^2 &\leq 2\big(V'(X_0)\big)^2 + 2\big(V'(X_t) - V'(X_0)\big)^2 \\
&\leq 2\big(V'(\xi)\big)^2 + 2\|V''\|_\infty^2 (X_t - X_0)^2,
\end{aligned}$$

with ξ distributed according to m. □

We are now in position to prove Proposition 2.

PROOF OF PROPOSITION 2. By the discussion following Definition 2, we know that, for a given Brownian motion B_t and initial condition ξ, one may associate a strong solution to (2.3) with law P to any solution P of the nonlinear martingale problem (MP). Therefore, to get uniqueness of solutions to (MP), it is enough to prove trajectorial uniqueness for (2.3). Let $(X_t)_{t\geq 0}$ and $(\tilde{X}_t)_{t\geq 0}$ denote two solutions of this nonlinear stochastic differential equation, with the same initial condition, and driven by the same Brownian motion. If $\langle m, (V')^2\rangle = +\infty$, then by Lemma 3 and since $\Gamma(\infty, b) = \frac{l^2}{2}$ and $\mathcal{G}(\infty, b) = 0$, these two processes are equal to $(X_0 + \frac{lB_t}{\sqrt{2}})_{t\geq 0}$. This proves trajectorial uniqueness.

Let us now assume that $\langle m, (V')^2\rangle < +\infty$. By Lemma 3, $t \mapsto \mathbb{E}[(X_t - \tilde{X}_t)^2] = \mathbb{E}[(X_t - X_0 - (\tilde{X}_t - \tilde{X}_0))^2]$ and $t \mapsto \mathbb{E}[(V'(X_t))^2] \vee \mathbb{E}[(V'(\tilde{X}_t))^2]$ are locally bounded. In order to simplify the notation, let us denote

$$\Gamma_s = \Gamma\big(\mathbb{E}\big[\big(V'(X_s)\big)^2\big], \mathbb{E}\big[V''(X_s)\big]\big),$$
$$\tilde{\Gamma}_s = \Gamma\big(\mathbb{E}\big[\big(V'(\tilde{X}_s)\big)^2\big], \mathbb{E}\big[V''(\tilde{X}_s)\big]\big)$$

and

$$\mathcal{G}_s = \mathcal{G}\big(\mathbb{E}\big[\big(V'(X_s)\big)^2\big], \mathbb{E}\big[V''(X_s)\big]\big),$$
$$\tilde{\mathcal{G}}_s = \mathcal{G}\big(\mathbb{E}\big[\big(V'(\tilde{X}_s)\big)^2\big], \mathbb{E}\big[V''(\tilde{X}_s)\big]\big).$$

Computing $(X_t - \tilde{X}_t)^2$ by Itô's formula and taking expectations, one obtains

$$\begin{aligned} \mathbb{E}\big[(X_t - \tilde{X}_t)^2\big] &= \int_0^t \big(\Gamma_s^{1/2} - \tilde{\Gamma}_s^{1/2}\big)^2\, ds \\ &\quad + 2\mathbb{E}\bigg[\int_0^t \big(\mathcal{G}_s V'(X_s) - \tilde{\mathcal{G}}_s V'(\tilde{X}_s)\big)(\tilde{X}_s - X_s)\, ds\bigg]. \end{aligned} \tag{3.2}$$

One has, using (2.18) and the Cauchy–Schwarz inequality,

$$\begin{aligned} &\mathbb{E}\big[\big(\mathcal{G}_s V'(X_s) - \tilde{\mathcal{G}}_s V'(\tilde{X}_s)\big)(\tilde{X}_s - X_s)\big] \\ &\quad = \mathcal{G}_s \mathbb{E}\big[\big(V'(X_s) - V'(\tilde{X}_s)\big)(\tilde{X}_s - X_s)\big] + (\mathcal{G}_s - \tilde{\mathcal{G}}_s)\mathbb{E}\big[V'(\tilde{X}_s)(\tilde{X}_s - X_s)\big] \\ &\quad \le l^2 \|V''\|_\infty \mathbb{E}\big[(X_s - \tilde{X}_s)^2\big] + |\mathcal{G}_s - \tilde{\mathcal{G}}_s| \mathbb{E}^{1/2}\big[\big(V'(\tilde{X}_s)\big)^2\big]\mathbb{E}^{1/2}\big[(\tilde{X}_s - X_s)^2\big] \end{aligned}$$

which, combined with the similar inequality obtained by exchanging $\tilde{X}$ and X, yields

$$\begin{aligned} &\mathbb{E}\big[\big(\mathcal{G}_s V'(X_s) - \tilde{\mathcal{G}}_s V'(\tilde{X}_s)\big)(\tilde{X}_s - X_s)\big] \\ &\quad \le l^2 \|V''\|_\infty \mathbb{E}\big[(X_s - \tilde{X}_s)^2\big] \\ &\qquad + |\mathcal{G}_s - \tilde{\mathcal{G}}_s| \big(\mathbb{E}\big[\big(V'(X_s)\big)^2\big] \wedge \mathbb{E}\big[\big(V'(\tilde{X}_s)\big)^2\big]\big)^{1/2} \mathbb{E}^{1/2}\big[(X_s - \tilde{X}_s)^2\big]. \end{aligned}$$

Using this inequality to deal with the second term on the right-hand side of (3.2) and (2.14) to deal with the first one then using the boundedness of V'' and (2.15), (2.17) and Young's inequality, one obtains that

$$\begin{aligned} &\mathbb{E}\big[(X_t - \tilde{X}_t)^2\big] \\ &\quad \le \frac{1}{4 \inf_{a \ge 0, b \in [\inf V'', \sup V'']} \Gamma(a, b)} \int_0^t (\Gamma_s - \tilde{\Gamma}_s)^2\, ds \\ &\qquad + 2l^2 \|V''\|_\infty \int_0^t \mathbb{E}\big[(X_s - \tilde{X}_s)^2\big]\, ds \\ &\qquad + 2\int_0^t |\mathcal{G}_s - \tilde{\mathcal{G}}_s| \big(\mathbb{E}\big[\big(V'(X_s)\big)^2\big] \wedge \mathbb{E}\big[\big(V'(\tilde{X}_s)\big)^2\big]\big)^{1/2} \mathbb{E}^{1/2}\big[(X_s - \tilde{X}_s)^2\big]\, ds \end{aligned}$$

$$
\begin{aligned}
&\le C \int_0^t \mathbb{E}[(X_s - \tilde{X}_s)^2] + \mathbb{E}^2[V''(X_s) - V''(\tilde{X}_s)] \\
&\qquad + \mathbb{E}^2[(V'(X_s))^2 - (V'(\tilde{X}_s))^2] \\
&\qquad + (\mathbb{E}^{1/2}[(V'(X_s))^2] - \mathbb{E}^{1/2}[(V'(\tilde{X}_s))^2])^2 \, ds.
\end{aligned}
$$

Now, since

$$
\begin{aligned}
&|\mathbb{E}[V''(X_s) - V''(\tilde{X}_s)]| \le \|V^{(3)}\|_\infty \mathbb{E}^{1/2}[(X_s - \tilde{X}_s)^2], \\
&|\mathbb{E}[(V'(X_s))^2 - (V'(\tilde{X}_s))^2]| \\
&\qquad \le \|V''\|_\infty \mathbb{E}^{1/2}[(X_s - \tilde{X}_s)^2](\mathbb{E}^{1/2}[(V'(X_s))^2] + \mathbb{E}^{1/2}[(V'(\tilde{X}_s))^2]), \\
&|\mathbb{E}^{1/2}[(V'(X_s))^2] - \mathbb{E}^{1/2}[(V'(\tilde{X}_s))^2]| \\
&\qquad \le \mathbb{E}^{1/2}[(V'(X_s) - V'(\tilde{X}_s))^2] \\
&\qquad \le \|V''\|_\infty \mathbb{E}^{1/2}[(X_s - \tilde{X}_s)^2],
\end{aligned}
$$

the local boundedness of $t \mapsto \mathbb{E}[(V'(X_t))^2] \vee \mathbb{E}[(V'(\tilde{X}_t))^2]$, the local integrability of $t \mapsto \mathbb{E}[(X_t - \tilde{X}_t)^2]$ and Gronwall's lemma ensure that $\forall t \ge 0$, $\mathbb{E}[(X_t - \tilde{X}_t)^2] = 0$. □

REMARK 1. When $\langle m, (V')^2 \rangle = +\infty$, we have already shown uniqueness of solutions to (2.3), and it is actually easy to build a strong solution. Indeed, since

$$
\left(V'\left(\xi + \frac{lB_t}{\sqrt{2}}\right)\right)^2 \ge \frac{1}{2}(V'(\xi))^2 - \frac{l^2 \|V''\|_\infty^2 B_t^2}{2},
$$

one has $\mathbb{E}[(V'(\xi + \frac{lB_t}{\sqrt{2}}))^2] = +\infty$ for all $t \ge 0$. As a consequence $(\xi + \frac{lB_t}{\sqrt{2}})_{t \ge 0}$ solves (2.3).

4. Tightness.

According to [24], because of exchangeability, the tightness of the sequence $(\pi^n)_n$ is equivalent to the tightness of the laws of the processes $(Y_t^{1,n})_{t \ge 0}$. As a consequence, the following proposition ensures that the sequence $(\pi^n)_n$ is tight under the assumptions of Theorem 1.

PROPOSITION 3. *Assume that the laws of the random variables* $(X_0^{1,n})_{n \ge 1}$ *are tight and that* $\sup_n \mathbb{E}[(V'(X_0^{1,n})^4)] < +\infty$. *Then the laws of the linearly interpolated processes* $(Y_t^{1,n} = (\lceil nt \rceil - nt) X_{\lfloor nt \rfloor}^{1,n} + (nt - \lfloor nt \rfloor) X_{\lceil nt \rceil}^{1,n}, t \ge 0)_{n \ge 1}$ *are tight in* $\mathcal{C}$. *Moreover,*

$$
t \mapsto \sup_{n \ge 1} \mathbb{E}[(V'(Y_t^{1,n}))^4] \quad \text{is locally bounded.} \tag{4.1}
$$

The proof of this proposition relies on the following estimate; the proof of which is given after the one of the proposition.

LEMMA 4. *Assume that* $\sup_n \mathbb{E}[(V'(X_0^{1,n}))^4] < +\infty$. *Then there exists a finite constant C depending on this supremum but not on n such that*

$$\forall 0 \le \underline{k} \le \overline{k}, \qquad \mathbb{E}\big((X_{\overline{k}}^{1,n} - X_{\underline{k}}^{1,n})^4\big) \le C\bigg(\frac{(\overline{k}-\underline{k})^2}{n^2} + e^{C(\overline{k}^4/n^4)}\frac{(\overline{k}-\underline{k})^4}{n^4}\bigg). \tag{4.2}$$

PROOF OF PROPOSITION 3. Since the laws of the initial random variables $(X_0^{1,n})_{n\ge 1}$ are supposed to be tight, Kolmogorov criterion ensures the desired tightness property as soon as there exists a nondecreasing function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\forall n \ge 1, \forall 0 \le s \le t, \qquad \mathbb{E}\big((Y_t^{1,n} - Y_s^{1,n})^4\big) \le \gamma(t)(t-s)^2. \tag{4.3}$$

Combining this estimation with the inequality

$$\mathbb{E}\big[\big(V'(Y_t^{1,n})\big)^4\big] \le 8\mathbb{E}\big[\big(V'(X_0^{1,n})\big)^4\big] + 8\|V''\|_\infty^4 \mathbb{E}\big[(Y_t^{i,n} - Y_0^{i,n})^4\big]$$

one also easily checks that $t \mapsto \sup_{n\ge 1} \mathbb{E}[(V'(Y_t^{1,n}))^4]$ is locally bounded. Let us show how to deduce (4.3) from (4.2). For $t > s \ge 0$ with $\lfloor nt \rfloor \ge \lceil ns \rceil$, using (4.2) for the second inequality, one obtains

$$\begin{aligned}
&\mathbb{E}\big((Y_t^{1,n} - Y_s^{1,n})^4\big) \\
&\quad \le 27\mathbb{E}\bigg(\frac{(l(nt - \lfloor nt \rfloor)G^1_{\lceil nt \rceil})^4}{n^2} + (X^{1,n}_{\lfloor nt \rfloor} - X^{1,n}_{\lceil ns \rceil})^4 + \frac{(l(\lceil ns \rceil - ns)G^1_{\lceil ns \rceil})^4}{n^2}\bigg) \\
&\quad \le \tilde{C}\bigg(\frac{(nt - \lfloor nt \rfloor)^2}{n^2} \\
&\qquad + \bigg(\frac{(\lfloor nt \rfloor - \lceil ns \rceil)^2}{n^2} + e^{Ct^4}\frac{(\lfloor nt \rfloor - \lceil ns \rceil)^4}{n^4}\bigg) + \frac{(\lceil ns \rceil - ns)^2}{n^2}\bigg) \\
&\quad \le C\big(1 + t^2 e^{Ct^4}\big)(t-s)^2.
\end{aligned}$$

For $t > s \ge 0$ with $\lfloor ns \rfloor = \lfloor nt \rfloor$, one has $(nt - ns)^4 \le (nt - ns)^2$ and, therefore,

$$\mathbb{E}\big((Y_t^{1,n} - Y_s^{1,n})^4\big) = \frac{l^4(nt - ns)^4}{n^2}\mathbb{E}\big((G^1_{\lceil nt \rceil})^4\big) \le C(t-s)^2. \qquad \square$$

The proof of Lemma 4 relies on the second inequality in the next lemma, the proof of which is postponed to the Appendix.

LEMMA 5. *Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\nu_n = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$. There exists a finite constant C not depending on n and x such that*

$$\mathbb{E}\Big[\Big(e^{\sum_{i=1}^n (V(x_i)-V(x_i+(l/\sqrt{n})G^i))} \wedge 1 - e^{-\sum_{i=1}^n ((l/\sqrt{n})V'(x_i)G^i+(l^2/(2n))V''(x_i))} \wedge 1\Big)^2\Big] \le \frac{C}{n}, \tag{4.4}$$

$$\big|\mathbb{E}\big(G^1\big(1-e^{\sum_{i=1}^n (V(x_i)-V(x_i+(l/\sqrt{n})G^i))}\big)^+\big)\big| \le C\left(\frac{|V'(x_1)|}{\sqrt{n}}+\frac{1}{n}\right), \tag{4.5}$$

$$\begin{aligned}&\left|\mathbb{E}\big(G^1\big(e^{\sum_{i=1}^n (V(x_i)-V(x_i+(l/\sqrt{n})G^i))} \wedge 1\big)\big) + \frac{V'(x_1)}{l\sqrt{n}}\mathcal{G}\big(\langle \nu_n, (V')^2\rangle, \langle \nu_n, V''\rangle\big)\right| \\ &\quad \le C\left(\frac{1+|V'(x_1)|}{n} + \frac{|V'(x_1)|}{n^{3/4}\langle \nu_n, (V')^2\rangle^{1/4}} + \frac{|V'(x_1)|^{3/2}}{n^{3/4}\sqrt{\langle \nu_n, (V')^2\rangle}}\right).\end{aligned} \tag{4.6}$$

PROOF OF LEMMA 4. Let $\overline{k} > \underline{k} \ge 0$. One has

$$\begin{aligned}&\mathbb{E}\big((X^{1,n}_{\overline{k}} - X^{1,n}_{\underline{k}})^4\big) \\ &\quad \le \frac{8l^4}{n^2}\mathbb{E}\left(\left(\sum_{k=\underline{k}+1}^{\overline{k}} G^1_k\right)^4\right) + \frac{8l^4}{n^2}\mathbb{E}\left(\left(\sum_{k=\underline{k}+1}^{\overline{k}} G^1_k 1_{\mathcal{A}^c_k}\right)^4\right) \\ &\quad = \frac{24l^4(\overline{k}-\underline{k})^2}{n^2} + \frac{8l^4}{n^2}\sum_{\underline{k}+1\le k_1,k_2,k_3,k_4\le \overline{k}} \mathbb{E}\left(\prod_{j=1}^4 G^1_{k_j} 1_{\mathcal{A}^c_{k_j}}\right) \\ &\quad = \frac{24l^4(\overline{k}-\underline{k})^2}{n^2} + \frac{8l^4}{n^2}(T_{1,1,1,1}+T_{2,1,1}+T_{3,1}+T_{2,2}+T_4),\end{aligned} \tag{4.7}$$

where the sum has been separated into five disjoint terms:

- $T_{1,1,1,1}$ corresponds to the restriction of the summation to indexes k_1, k_2, k_3 and k_4 taking distinct values,
- $T_{2,1,1}$ to the restriction to indexes such that the cardinality of $\{k_1, k_2, k_3, k_4\}$ is equal to 3,
- $T_{3,1}$ to three indexes equal and the last one different,
- $T_{2,2}$ to two pairs of equal indexes taking different values,
- T_4 to four equal indexes.

One has

$$\begin{aligned}T_4 + T_{2,2} + T_{3,1} &\le (\overline{k}-\underline{k})\mathbb{E}\big((G^1_1)^4\big) + 3(\overline{k}-\underline{k})(\overline{k}-\underline{k}-1)\mathbb{E}\big((G^1_1)^2(G^1_2)^2\big) \\ &\quad + 4(\overline{k}-\underline{k})(\overline{k}-\underline{k}-1)\mathbb{E}\big(|G^1_1|^3\big)\mathbb{E}|G^1_2| \\ &= 3(\overline{k}-\underline{k})^2 + \frac{16(\overline{k}-\underline{k})(\overline{k}-\underline{k}-1)}{\pi}.\end{aligned} \tag{4.8}$$

Let us now estimate $T_{1,1,1,1}$ and $T_{2,1,1}$. For fixed k_1, k_2, k_3 and k_4 (four integers in $\{\underline{k}+1,\dots,\overline{k}\}$), let us define $(\tilde{X}_k^{i,n}, k \geq 0)_{1\leq i\leq n}$ such that $(\tilde{X}_0^{1,n},\dots,\tilde{X}_0^{n,n}) = (X_0^{1,n},\dots,X_0^{n,n})$ and, for $k \geq 0$ and $1 \leq i \leq n$,

$$\tilde{X}_{k+1}^{i,n} = \tilde{X}_k^{i,n} + 1_{\{k\notin\{k_1-1,k_2-1,k_3-1,k_4-1\}\}} \times \frac{l}{\sqrt{n}} G_{k+1}^i 1_{\{U_{k+1}\leq e^{\sum_{i=1}^n (V(\tilde{X}_k^{i,n})-V(\tilde{X}_k^{i,n}+(l/\sqrt{n})G_{k+1}^i))}\}}.$$

Let us also denote by $\mathcal{F}$ the sigma-field generated by these processes which are exchangeable, independent of $(U_k, (G_k^1,\dots,G_k^n))_{k\in\{k_1,k_2,k_3,k_4\}}$ and equal to the original processes $(X_k^{i,n}, k\geq 1)_{1\leq i\leq n}$ on the event

$$\bigcap_{j=1}^4 \mathcal{A}_{k_j}^c = \bigcap_{j=1}^4 \{U_{k_j} > e^{\sum_{i=1}^n (V(X_{k_j-1}^{i,n})-V(X_{k_j-1}^{i,n}+(l/\sqrt{n})G_{k_j}^i))}\}.$$

When the indices k_1, k_2, k_3 and k_4 are distinct (namely for $T_{1,1,1,1}$), by conditional independence of the vectors $((G_{k_j}^1,\dots,G_{k_j}^n, U_{k_j}))_{1\leq j\leq 4}$ given $\mathcal{F}$, one has

$$\begin{aligned}
&\left|\mathbb{E}\left(\prod_{j=1}^4 G_{k_j}^1 1_{\mathcal{A}_{k_j}^c}\right)\right| \\
&\quad = \left|\mathbb{E}\left(\prod_{j=1}^4 G_{k_j}^1 1_{\{U_{k_j}>e^{\sum_{i=1}^n (V(\tilde{X}_{k_j-1}^{i,n})-V(\tilde{X}_{k_j-1}^{i,n}+(l/\sqrt{n})G_{k_j}^i))}\}}\right)\right| \\
&\quad = \left|\mathbb{E}\left(\prod_{j=1}^4 \mathbb{E}\left(G_{k_j}^1 \left(1-e^{\sum_{i=1}^n (V(\tilde{X}_{k_j-1}^{i,n})-V(\tilde{X}_{k_j-1}^{i,n}+(l/\sqrt{n})G_{k_j}^i))}\right)^+ \middle| \mathcal{F}\right)\right)\right| \\
&\quad \leq \mathbb{E}\left[\prod_{j=1}^4 \left|\mathbb{E}\left(G_{k_j}^1 \left(1-e^{\sum_{i=1}^n (V(\tilde{X}_{k_j-1}^{i,n})-V(\tilde{X}_{k_j-1}^{i,n}+(l/\sqrt{n})G_{k_j}^i))}\right)^+ \middle| \mathcal{F}\right)\right|\right] \\
&\quad \leq C\mathbb{E}\left[\prod_{j=1}^4 \left(\frac{1}{n} + \frac{|V'(\tilde{X}_{k_j-1}^{1,n})|}{\sqrt{n}}\right)\right] \\
&\quad \leq C\left(\frac{1}{n^4} + \frac{1}{n^2}\mathbb{E}\left[\sum_{j=1}^4 |V'(\tilde{X}_{k_j-1}^{1,n})|^4\right]\right),
\end{aligned}$$

where we used (4.5) for the last but one inequality and Young's inequality for the last one. Now for $k_1 < k_2 < k_3 < k_4$, according to the above definition of $(\tilde{X}_k^{i,n}, k \geq 0)_{1\leq i\leq n}$, the random vector $(\tilde{X}_{k_j-1}^{1,n})_{1\leq j\leq 4}$ has the same distribution as

$(X^{1,n}_{k_j-j})_{1\le j\le 4}$. Therefore,

$$T_{1,1,1,1} \le 4!C \sum_{\underline{k}+1\le k_1<k_2<k_3<k_4\le\overline{k}} \left(\frac{1}{n^4}+\frac{1}{n^2}\mathbb{E}\left[\sum_{j=1}^4 \left|V'(X^{1,n}_{k_j-j})\right|^4\right]\right)$$
$$= 4!C\left(\frac{\binom{\overline{k}-\underline{k}}{4}}{n^4}+\frac{\binom{\overline{k}-\underline{k}}{3}}{n^2}\sum_{k=\underline{k}}^{\overline{k}-4}\mathbb{E}\left[\left|V'(X^{1,n}_k)\right|^4\right]\right).$$

To deal with $T_{2,1,1}$ we remark that if, for instance, k_2, k_3 and k_4 are distinct and $k_1 = k_2$, then reasoning like above, and using that $\mathbb{E}[(G^1_{k_1})^2 1_{\mathcal{A}^c_{k_1}}|\mathcal{F}] \le \mathbb{E}[(G^1_{k_1})^2|\mathcal{F}] = 1$, one obtains

$$\left|\mathbb{E}\left(\prod_{j=1}^4 G^1_{k_j} 1_{\mathcal{A}^c_{k_j}}\right)\right|$$
$$\le \mathbb{E}\left[\prod_{j=3}^4 \left|\mathbb{E}\left(G^1_{k_j}\left(1-e^{\sum_{i=1}^n (V(\tilde{X}^{i,n}_{k_j-1})-V(\tilde{X}^{i,n}_{k_j-1}+(l/\sqrt{n})G^i_{k_j}))}\right)^+\middle|\mathcal{F}\right)\right|\right]$$
$$\le C\left(\frac{1}{n^2}+\frac{1}{n}\mathbb{E}\left[\sum_{j=3}^4 \left|V'(\tilde{X}^{1,n}_{k_j-1})\right|^2\right]\right).$$

One deduces that

$$T_{2,1,1} \le C\binom{4}{2}\left(\frac{(\overline{k}-\underline{k})(\overline{k}-\underline{k}-1)(\overline{k}-\underline{k}-2)}{n^2}+\frac{4\binom{\overline{k}-\underline{k}}{2}}{n}\sum_{k=\underline{k}}^{\overline{k}-3}\mathbb{E}\left[\left(V'(X^{1,n}_k)\right)^2\right]\right).$$

By combining the estimations of $T_{3,1}+T_{2,2}+T_4$, $T_{1,1,1,1}$ and $T_{2,1,1}$ with Young's and Jensen's inequalities, one obtains that

$$\begin{aligned}
&\mathbb{E}\left(\left(X^{1,n}_{\overline{k}}-X^{1,n}_{\underline{k}}\right)^4\right)\\
&\quad\le C\left(\frac{(\overline{k}-\underline{k})^2}{n^2}+\frac{(\overline{k}-\underline{k})^4}{n^6}+\frac{(\overline{k}-\underline{k})^3}{n^4}\sum_{k=\underline{k}}^{\overline{k}-1}\mathbb{E}\left[\left(V'(X^{1,n}_k)\right)^4\right]\right).
\end{aligned}\tag{4.9}$$

For the choice $\underline{k}=0$ and using $\sup_n \mathbb{E}[(V'(X^{1,n}_0))^4] < +\infty$,

$$\mathbb{E}\left[\left(V'(X^{1,n}_k)\right)^4\right] \le 8\mathbb{E}\left[\left(V'(X^{1,n}_0)\right)^4\right]+8\|V''\|^4_\infty\mathbb{E}\left[\left(X^{1,n}_k-X^{1,n}_0\right)^4\right],\tag{4.10}$$

one obtains that

$$\mathbb{E}\left(\left(X^{1,n}_{\overline{k}}-X^{1,n}_0\right)^4\right) \le C\left(\frac{\overline{k}^2}{n^2}+\frac{\overline{k}^4}{n^4}+\frac{\overline{k}^3}{n^4}\sum_{k=0}^{\overline{k}-1}\mathbb{E}\left(\left(X^{1,n}_k-X^{1,n}_0\right)^4\right)\right).$$

By a discrete version of Gronwall's lemma, one deduces that

$$\forall k \geq 0, \qquad \mathbb{E}\big((X_k^{1,n} - X_0^{1,n})^4\big) \leq Ce^{C(k^4/n^4)}\left(\frac{k^2}{n^2} \vee \frac{k^4}{n^4}\right) \leq Ce^{C(k^4/n^4)}.$$

With (4.9) and (4.10), one concludes that (4.2) holds. □

5. Identification of the limits of converging subsequences of $(\pi^n)_{n\geq 1}$. From the previous section, we know that the sequence $(\pi^n)_n$ is tight. Let π^∞ denote the limit of a converging subsequence of $(\pi^n)_n$ that we still index by n for notational simplicity. We want to prove that π^∞ gives full weight to the solutions of the nonlinear martingale problem (MP) (see Definition 2). To do so, for $\varphi : \mathbb{R} \to \mathbb{R}$ C^3 with compact support, $p \in \mathbb{N}$, $g : \mathbb{R}^p \to \mathbb{R}$ continuous and bounded and $0 \leq s_1 \leq s_2 \leq \cdots \leq s_p \leq s \leq t$, we define

$$F : Q \in \mathcal{P}(\mathcal{C}) \mapsto \left\langle Q, \left(\varphi(Y_t) - \varphi(Y_s) - \int_s^t L_{Q_r}\varphi(Y_r)\,dr\right) g(Y_{s_1}, \ldots, Y_{s_p})\right\rangle. \tag{5.1}$$

Since the chaoticity of the initial conditions implies that $\pi^\infty(\{Q \in \mathcal{P}(\mathcal{C}) : Q_0 = m\}) = 1$, to prove that π^∞ gives full weight to the solutions of (MP), it is enough to check that $\mathbb{E}^{\pi^\infty}|F(Q)| = 0$. Indeed, taking g in a countable subset of the space of continuous functions with compact support on $\mathbb{R}^p$ dense for the uniform convergence and $(s_1, \ldots, s_p)$ in a countable dense subset of $[0, s]$, one obtains

$$\pi^\infty\left(\left\{Q \in \mathcal{P}(\mathcal{C}) : \mathbb{E}^Q\left(\varphi(Y_t) - \varphi(Y_s) - \int_s^t L_{Q_r}\varphi(Y_r)\,dr \,\Big|\, (Y_u)_{u\in[0,s]}\right) = 0\right\}\right) = 1.$$

Then taking s, t in a countable dense subset of $\mathbb{R}_+$ and φ in a countable subset of C^3 functions with compact support on $\mathbb{R}$ dense in the space $C_c^2(\mathbb{R})$ of C^2 functions with compact support on $\mathbb{R}$ for the uniform convergence of the function and its derivatives up to the order 2, one concludes that

$$\pi^\infty\left(\left\{Q : \forall \varphi \in C_c^2(\mathbb{R}), \left(\varphi(Y_t) - \int_0^t L_{Q_r}\varphi(Y_r)\,dr\right)_{t\geq 0} \text{ is a } Q\text{-martingale}\right\}\right) = 1.$$

In Section 5.1, we present the main steps of the proof. Then, in Sections 5.2 and 5.3, we provide the proofs of the technical propositions stated and used in Section 5.1.

5.1. *Proof of* $\mathbb{E}^{\pi^\infty}|F(Q)| = 0$. By combining the two next propositions, one first obtains the asymptotic behavior of $\mathbb{E}^{\pi^n}|F(Q)| = \mathbb{E}|F(\mu^n)|$ as $n \to \infty$.

PROPOSITION 4. *Let*

$$M_k^{i,n} = \frac{l}{\sqrt{n}} \sum_{j=0}^{k-1} \varphi'(X_j^{i,n})\big(G_{j+1}^i 1_{\mathcal{A}_{j+1}} - \mathbb{E}[G_{j+1}^i 1_{\mathcal{A}_{j+1}} | \mathcal{F}_j^n]\big)$$
$$+ \frac{l^2}{2n} \sum_{j=0}^{k-1} \varphi''(X_j^{i,n})\big((G_{j+1}^i)^2 1_{\mathcal{A}_{j+1}} - \mathbb{E}[(G_{j+1}^i)^2 1_{\mathcal{A}_{j+1}} | \mathcal{F}_j^n]\big).$$

Under the assumptions of Theorem 1, for all $s < t$, $\exists C < \infty, \forall n \geq 1$,

$$\sup_{1\leq i\leq n} \mathbb{E}\left|\varphi(Y_t^{i,n}) - \varphi(Y_s^{i,n}) - \int_s^t L_{\mu_r^n}\varphi(Y_r^{i,n})\,dr - (M_{\lceil nt\rceil}^{i,n} - M_{\lceil ns\rceil}^{i,n})\right| \leq \frac{C}{n^{1/4}},$$

where μ_r^n denotes the marginal at time r of μ^n [*defined by* (2.13)].

PROPOSITION 5. *Under the assumptions of Theorem 1,*

$$\exists C < \infty, \forall n \geq 1, \qquad \mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^n (M_{\lceil nt\rceil}^{i,n} - M_{\lceil ns\rceil}^{i,n}) g(Y_{s_1}^{i,n}, \ldots, Y_{s_p}^{i,n})\right)^2\right] \leq \frac{C}{\sqrt{n}}.$$

Since

$$F(\mu^n) = \frac{1}{n}\sum_{i=1}^n \left(\varphi(Y_t^{i,n}) - \varphi(Y_s^{i,n}) - \int_s^t L_{\mu_r^n}\varphi(Y_r^{i,n})\,dr\right) g(Y_{s_1}^{i,n}, \ldots, Y_{s_p}^{i,n}),$$

one has

$$\begin{aligned}
&\mathbb{E}|F(\mu^n)| \\
&\quad\leq \frac{\|g\|_\infty}{n}\sum_{i=1}^n \mathbb{E}\left|\varphi(Y_t^{i,n}) - \varphi(Y_s^{i,n}) - \int_s^t L_{\mu_r^n}\varphi(Y_r^{i,n})\,dr - (M_{\lceil nt\rceil}^{i,n} - M_{\lceil ns\rceil}^{i,n})\right| \\
&\qquad + \mathbb{E}^{1/2}\left[\left(\frac{1}{n}\sum_{i=1}^n (M_{\lceil nt\rceil}^{i,n} - M_{\lceil ns\rceil}^{i,n}) g(Y_{s_1}^{i,n}, \ldots, Y_{s_p}^{i,n})\right)^2\right].
\end{aligned}$$

One deduces that

$$\lim_{n\to\infty} \mathbb{E}^{\pi^n}|F(Q)| = 0. \tag{5.2}$$

Since g, $\mathcal{G}$, Γ and $V'\varphi'$ are bounded, the function F is bounded. Unfortunately, when V' is not bounded, the lack of continuity of $\mu \in \mathcal{P}(\mathbb{R}) \mapsto \langle \mu, (V')^2\rangle$ implies that F is not continuous and the weak convergence of π^n to π^∞ does not directly ensure that $\mathbb{E}^{\pi^\infty}|F(Q)| = 0$.

To overcome this difficulty, for $k \in \mathbb{N}$, we introduce the second-order differential operator L_μ^k defined like L_μ in (2.5) but with $\langle \mu, (V')^2 \wedge k\rangle$ replacing $\langle \mu, (V')^2\rangle$. We also define F_k like F but with L_{Q_r} replaced by $L_{Q_r}^k$. The functions F_k are uniformly bounded and converge pointwise to F by the properties of $\mathcal{G}$ and Γ stated in Lemma 2. Moreover, F_k is continuous. Indeed, to deal with the discontinuity of $\mathcal{G}$ at $(0, 0)$, it is enough to remark that for $\nu, \mu \in \mathcal{P}(\mathbb{R})$,

$$\begin{aligned}
&\langle \nu, |\mathcal{G}(\langle \nu, (V')^2 \wedge k\rangle, \langle \nu, V''\rangle) - \mathcal{G}(\langle \mu, (V')^2 \wedge k\rangle, \langle \mu, V''\rangle)| \times |V'\varphi'|\rangle \\
&\quad\leq 1_{\{\langle \mu, (V')^2\wedge k\rangle > 0\}}\|V'\varphi'\|_\infty \\
&\qquad \times |\mathcal{G}(\langle \nu, (V')^2 \wedge k\rangle, \langle \nu, V''\rangle) - \mathcal{G}(\langle \mu, (V')^2 \wedge k\rangle, \langle \mu, V''\rangle)| \\
&\qquad + 1_{\{\langle \mu, (V')^2\wedge k\rangle = 0\}} 2l^2\langle \nu - \mu, |V'\varphi'|\rangle,
\end{aligned}$$

where we used in the last line the fact that $1_{\{\langle \mu,(V')^2\wedge k\rangle=0\}}\langle \mu, |V'\varphi'|\rangle = 0$. As a consequence,

$$\mathbb{E}^{\pi^\infty}|F(Q)| = \lim_{k\to\infty}\mathbb{E}^{\pi^\infty}|F_k(Q)| = \lim_{k\to\infty}\lim_{n\to\infty}\mathbb{E}^{\pi^n}|F_k(Q)|$$
$$\le \limsup_{k\to\infty}\limsup_{n\to\infty}\mathbb{E}^{\pi^n}|F_k(Q) - F(Q)|,$$

where we used (5.2) for the inequality. One concludes that $\mathbb{E}^{\pi^\infty}|F(Q)| = 0$ by the next proposition.

PROPOSITION 6. *Under the assumptions of Theorem* 1,

$$\lim_{k\to\infty}\sup_{n\ge 1}\mathbb{E}|F_k(\mu^n) - F(\mu^n)| = 0.$$

5.2. *Proof of Proposition* 4. This section is devoted to the proof of Proposition 4. As already pointed out in Section 2.3, the main difficulty is the identification of the drift term.

PROOF OF PROPOSITION 4. One has $dY_t^{i,n} = l\sqrt{n}G^i_{\lceil nt\rceil}1_{\mathcal{A}_{\lceil nt\rceil}}\,dt$. As a consequence,

$$\varphi(Y_t^{i,n}) - \varphi(Y_s^{i,n}) = \int_s^t l\sqrt{n}\varphi'(Y_r^{i,n})G^i_{\lceil nr\rceil}1_{\mathcal{A}_{\lceil nr\rceil}}\,dr.$$

Using the Taylor expansion,

$$\varphi'(Y_r^{i,n}) = \varphi'(X^{i,n}_{\lfloor nr\rfloor}) + \varphi''(X^{i,n}_{\lfloor nr\rfloor})(nr - \lfloor nr\rfloor)\frac{l}{\sqrt{n}}G^i_{\lceil nr\rceil}1_{\mathcal{A}_{\lceil nr\rceil}}$$
$$+ \varphi^{(3)}(\chi_r^{i,n})(nr - \lfloor nr\rfloor)^2\frac{l^2}{2n}(G^i_{\lceil nr\rceil})^2 1_{\mathcal{A}_{\lceil nr\rceil}},$$

with $\chi_r^{i,n}\in[X^{i,n}_{\lfloor nr\rfloor}, Y_r^{i,n}]$, one deduces that

$$\varphi(Y_t^{i,n}) - \varphi(Y_s^{i,n})$$
$$- \int_s^t\left(l\sqrt{n}\varphi'(X^{i,n}_{\lfloor nr\rfloor})G^i_{\lceil nr\rceil}1_{\mathcal{A}_{\lceil nr\rceil}} + \frac{l^2}{2}\varphi''(X^{i,n}_{\lfloor nr\rfloor})(G^i_{\lceil nr\rceil})^2 1_{\mathcal{A}_{\lceil nr\rceil}}\right)dr$$
$$= \frac{l^3}{2\sqrt{n}}\int_s^t \varphi^{(3)}(\chi_r^{i,n})(nr - \lfloor nr\rfloor)^2(G^i_{\lceil nr\rceil})^3 1_{\mathcal{A}_{\lceil nr\rceil}}\,dr$$
$$+ \frac{l^2(ns - \lfloor ns\rfloor)(\lceil ns\rceil - ns)}{2n}\varphi''(X^{i,n}_{\lfloor ns\rfloor})(G^i_{\lceil ns\rceil})^2 1_{\mathcal{A}_{\lceil ns\rceil}}$$
$$- \frac{l^2(nt - \lfloor nt\rfloor)(\lceil nt\rceil - nt)}{2n}\varphi''(X^{i,n}_{\lfloor nt\rfloor})(G^i_{\lceil nt\rceil})^2 1_{\mathcal{A}_{\lceil nt\rceil}}.$$

By the boundedness of φ'' and $\varphi^{(3)}$, one easily concludes that

$$\mathbb{E}\bigg|\varphi(Y_t^{i,n})-\varphi(Y_s^{i,n})-\int_s^t l\sqrt{n}\varphi'(X_{\lfloor nr\rfloor}^{i,n})G_{\lceil nr\rceil}^i 1_{\mathcal{A}_{\lceil nr\rceil}} + \frac{l^2}{2}\varphi''(X_{\lfloor nr\rfloor}^{i,n})(G_{\lceil nr\rceil}^i)^2 1_{\mathcal{A}_{\lceil nr\rceil}}\,dr\bigg| \le \frac{C}{\sqrt{n}}. \tag{5.3}$$

To complete the proof, we now consider the decomposition

$$\begin{aligned}&\int_s^t l\sqrt{n}\varphi'(X_{\lfloor nr\rfloor}^{i,n})G_{\lceil nr\rceil}^i 1_{\mathcal{A}_{\lceil nr\rceil}} + \frac{l^2}{2}\varphi''(X_{\lfloor nr\rfloor}^{i,n})(G_{\lceil nr\rceil}^i)^2 1_{\mathcal{A}_{\lceil nr\rceil}}\,dr\\ &\quad -\int_s^t L_{\mu_r^n}\varphi(Y_r^{i,n})\,dr-(M_{\lceil nt\rceil}^{i,n}-M_{\lceil ns\rceil}^{i,n})\\ &\quad = T_1^{i,n}+T_2^{i,n}+T_3^{i,n}-T_4^{i,n}+T_5^{i,n},\end{aligned} \tag{5.4}$$

where

$$\begin{aligned}T_1^{i,n}&=\int_s^t \varphi'(X_{\lfloor nr\rfloor}^{i,n})\big(l\sqrt{n}\mathbb{E}[G_{\lceil nr\rceil}^i 1_{\mathcal{A}_{\lceil nr\rceil}}|\mathcal{F}_{\lfloor nr\rfloor}^n]\\ &\qquad + \mathcal{G}(\langle\mu_{\lfloor nr\rfloor/n}^n,(V')^2\rangle,\langle\mu_{\lfloor nr\rfloor/n}^n,V''\rangle)V'(X_{\lfloor nr\rfloor}^{i,n})\big)\,dr,\\ T_2^{i,n}&=\frac{1}{2}\int_s^t \varphi''(X_{\lfloor nr\rfloor}^{i,n})\big(l^2\mathbb{E}[(G_{\lceil nr\rceil}^i)^2 1_{\mathcal{A}_{\lceil nr\rceil}}|\mathcal{F}_{\lfloor nr\rfloor}^n]\\ &\qquad -\Gamma(\langle\mu_{\lfloor nr\rfloor/n}^n,(V')^2\rangle,\langle\mu_{\lfloor nr\rfloor/n}^n,V''\rangle)\big)\,dr,\\ T_3^{i,n}&=\int_s^t L_{\mu_{\lfloor nr\rfloor/n}^n}\varphi(Y_{\lfloor nr\rfloor/n}^{i,n})-L_{\mu_r^n}\varphi(Y_r^{i,n})\,dr,\\ T_4^{i,n}&=\bigg(\frac{l(\lceil nt\rceil-nt)}{\sqrt{n}}\varphi'(X_{\lfloor nt\rfloor}^{i,n})(G_{\lceil nt\rceil}^i 1_{\mathcal{A}_{\lceil nt\rceil}}-\mathbb{E}[G_{\lceil nt\rceil}^i 1_{\mathcal{A}_{\lceil nt\rceil}}|\mathcal{F}_{\lfloor nt\rfloor}^n])\\ &\qquad+\frac{l^2(\lceil nt\rceil-nt)}{2n}\varphi''(X_{\lfloor nt\rfloor}^{i,n})((G_{\lceil nt\rceil}^i)^2 1_{\mathcal{A}_{\lceil nt\rceil}}-\mathbb{E}[(G_{\lceil nt\rceil}^i)^2 1_{\mathcal{A}_{\lceil nt\rceil}}|\mathcal{F}_{\lfloor nt\rfloor}^n])\bigg)\end{aligned}$$

and

$$\begin{aligned}T_5^{i,n}&=\bigg(\frac{l(\lceil ns\rceil-ns)}{\sqrt{n}}\varphi'(X_{\lfloor ns\rfloor}^{i,n})(G_{\lceil ns\rceil}^i 1_{\mathcal{A}_{\lceil ns\rceil}}-\mathbb{E}[G_{\lceil ns\rceil}^i 1_{\mathcal{A}_{\lceil ns\rceil}}|\mathcal{F}_{\lfloor ns\rfloor}^n])\\ &\qquad+\frac{l^2(\lceil ns\rceil-ns)}{2n}\varphi''(X_{\lfloor ns\rfloor}^{i,n})((G_{\lceil ns\rceil}^i)^2 1_{\mathcal{A}_{\lceil ns\rceil}}-\mathbb{E}[(G_{\lceil ns\rceil}^i)^2 1_{\mathcal{A}_{\lceil ns\rceil}}|\mathcal{F}_{\lfloor ns\rfloor}^n])\bigg).\end{aligned}$$

The boundedness of φ' and φ'' implies that

$$\mathbb{E}(|T_4^{i,n}|+|T_5^{i,n}|)\le\frac{C}{\sqrt{n}}. \tag{5.5}$$

By (4.6), Hölder's inequality and the equality

$$\mathbb{E}\left[\frac{(V'(Y^{i,n}_{\lfloor nr\rfloor/n}))^2}{\langle \mu^n_{\lfloor nr\rfloor/n}, (V')^2\rangle}\right] = 1 \tag{5.6}$$

deduced from exchangeability, one obtains

$$\begin{aligned}
\mathbb{E}|T_1^{i,n}| &\le C\int_s^t \frac{1+\mathbb{E}|V'(Y^{i,n}_{\lfloor nr\rfloor/n})|}{\sqrt{n}} + \frac{1}{n^{1/4}}\mathbb{E}\left|\frac{V'(Y^{i,n}_{\lfloor nr\rfloor/n})}{\langle \mu^n_{\lfloor nr\rfloor/n}, (V')^2\rangle^{1/4}}\right| \\
&\qquad + \frac{1}{n^{1/4}}\mathbb{E}\left(\frac{|V'(Y^{i,n}_{\lfloor nr\rfloor/n})|^{3/2}}{\langle \mu^n_{\lfloor nr\rfloor/n}, (V')^2\rangle^{1/2}}\right) dr \\
&\le C\int_s^t \frac{1+\mathbb{E}|V'(Y^{i,n}_{\lfloor nr\rfloor/n})|}{\sqrt{n}} + \frac{\mathbb{E}^{3/4}(|V'(Y^{i,n}_{\lfloor nr\rfloor/n})|^{2/3})}{n^{1/4}} \\
&\qquad + \frac{\mathbb{E}^{1/2}|V'(Y^{i,n}_{\lfloor nr\rfloor/n})|}{n^{1/4}}\, dr.
\end{aligned} \tag{5.7}$$

Concerning $T_2^{i,n}$, by Cauchy–Schwarz inequality and (4.4), one easily checks that

$$\begin{aligned}
&\big|\mathbb{E}\big[(G^i_{\lceil nr\rceil})^2 1_{\mathcal{A}_{\lceil nr\rceil}}|\mathcal{F}^n_{\lfloor nr\rfloor}\big] \\
&\quad - \mathbb{E}\big[(G^i_{\lceil nr\rceil})^2\big(e^{-\sum_{l=1}^n (V'(X^{l,n}_{\lfloor nr\rfloor})(l/\sqrt{n})G^l_{\lceil nr\rceil} + (l^2/(2n))V''(X^{l,n}_{\lfloor nr\rfloor}))} \wedge 1\big)|\mathcal{F}^n_{\lfloor nr\rfloor}\big]\big| \\
&\quad \le \frac{C}{\sqrt{n}}.
\end{aligned}$$

Moreover, by (A.5) and (A.6),

$$\begin{aligned}
&\mathbb{E}\big[G^i_{\lceil nr\rceil}G^j_{\lceil nr\rceil}\big(e^{-\sum_{l=1}^n (V'(X^{l,n}_{\lfloor nr\rfloor})(l/\sqrt{n})G^l_{\lceil nr\rceil} + (l^2/(2n))V''(X^{l,n}_{\lfloor nr\rfloor}))} \wedge 1\big)|\mathcal{F}^n_{\lfloor nr\rfloor}\big] \\
&\quad = \frac{1_{\{i=j\}}}{l^2}\Gamma\big(\langle \mu^n_{\lfloor nr\rfloor/n}, (V')^2\rangle, \langle \mu^n_{\lfloor nr\rfloor/n}, V''\rangle\big) \\
&\qquad + \frac{V'(X^{i,n}_{\lfloor nr\rfloor})V'(X^{j,n}_{\lfloor nr\rfloor})}{n}\Big(\mathcal{G}\big(\langle \mu^n_{\lfloor nr\rfloor/n}, (V')^2\rangle, \langle \mu^n_{\lfloor nr\rfloor/n}, V''\rangle\big) \\
&\qquad\qquad - \frac{l^2 e^{-(((l/2)\langle \mu^n_{\lfloor nr\rfloor/n}, V''\rangle)^2)/(2\langle \mu^n_{\lfloor nr\rfloor/n}, (V')^2\rangle)}}{\sqrt{2\pi l^2\langle \mu^n_{\lfloor nr\rfloor/n}, (V')^2\rangle}}\Big).
\end{aligned} \tag{5.8}$$

(We will need this expression for $i \neq j$ below.) With the boundedness of $\mathcal{G}$ and (5.6), this implies that

$$\mathbb{E}|T_2^{i,n}| \le \frac{C}{\sqrt{n}} + \frac{C}{n}\int_s^t \mathbb{E}\big[\big(V'(Y^{i,n}_{\lfloor nr\rfloor/n})\big)^2\big] + \mathbb{E}^{1/2}\big[\big(V'(Y^{i,n}_{\lfloor nr\rfloor/n})\big)^2\big]\, dr. \tag{5.9}$$

To deal with $T_3^{i,n}$, one remarks that by exchangeability, boundedness of $\mathcal{G}$, φ' and $(V'\varphi')'$, then by (2.17)

$$
\begin{aligned}
&\mathbb{E}\big|\mathcal{G}\big(\langle\mu_r^n,(V')^2\rangle,\langle\mu_r^n,V''\rangle\big)V'\varphi'(Y_r^{i,n})\\
&\quad-\mathcal{G}\big(\langle\mu_{\lfloor nr\rfloor/n}^n,(V')^2\rangle,\langle\mu_{\lfloor nr\rfloor/n}^n,V''\rangle\big)V'\varphi'(Y_{\lfloor nr\rfloor/n}^{i,n})\big|\\
&\qquad\le\mathbb{E}\big(\big|\mathcal{G}\big(\langle\mu_r^n,(V')^2\rangle,\langle\mu_r^n,V''\rangle\big)\\
&\qquad\qquad-\mathcal{G}\big(\langle\mu_{\lfloor nr\rfloor/n}^n,(V')^2\rangle,\langle\mu_{\lfloor nr\rfloor/n}^n,V''\rangle\big)\big|\big(\langle\mu_r^n,|V'\varphi'|\rangle\wedge\langle\mu_{\lfloor nr\rfloor/n}^n,|V'\varphi'|\rangle\big)\big)\\
&\qquad\quad+C\mathbb{E}\big|Y_r^{i,n}-Y_{\lfloor nr\rfloor/n}^{i,n}\big|\\
&\qquad\le C\mathbb{E}\big(\big|\langle\mu_r^n-\mu_{\lfloor nr\rfloor/n}^n,V''\rangle\big|+\big|\langle\mu_r^n-\mu_{\lfloor nr\rfloor/n}^n,(V')^2\rangle\big|\\
&\qquad\qquad+\big|\langle\mu_r^n-\mu_{\lfloor nr\rfloor/n}^n,(V')^2\rangle\big|^{1/2}+\big|Y_r^{i,n}-Y_{\lfloor nr\rfloor/n}^{i,n}\big|\big).
\end{aligned}
$$

By exchangeability, $\mathbb{E}|\langle\mu_r^n-\mu_{\lfloor nr\rfloor/n}^n,V''\rangle|\le\|V^{(3)}\|_\infty\mathbb{E}|Y_r^{i,n}-Y_{\lfloor nr\rfloor/n}^{i,n}|$. Moreover, $|Y_r^{i,n}-Y_{\lfloor nr\rfloor/n}^{i,n}|\le\frac{l}{\sqrt{n}}|G_{\lceil nr\rceil}^i|$. Dealing in the same way with the diffusion term by boundedness of Γ and $\varphi^{(3)}$ and (2.15), one deduces that

$$
\begin{aligned}
\mathbb{E}|T_3^{i,n}|\le\frac{C}{\sqrt{n}}+\int_s^t\mathbb{E}\big|\langle\mu_r^n-\mu_{\lfloor nr\rfloor/n}^n,(V')^2\rangle\big|\\
+\mathbb{E}^{1/2}\big|\langle\mu_r^n-\mu_{\lfloor nr\rfloor/n}^n,(V')^2\rangle\big|\,dr.
\end{aligned}
\tag{5.10}
$$

One has

$$
\begin{aligned}
&\mathbb{E}\big|\langle\mu_r^n-\mu_{\lfloor nr\rfloor/n}^n,(V')^2\rangle\big|\\
&\qquad\le\sqrt{2}\|V''\|_\infty\mathbb{E}^{1/2}\big[\big(V'(Y_r^{i,n})\big)^2+\big(V'(Y_{\lfloor nr\rfloor/n}^{i,n})\big)^2\big]\\
&\qquad\quad\times\mathbb{E}^{1/2}\big[\big(Y_r^{i,n}-Y_{\lfloor nr\rfloor/n}^{i,n}\big)^2\big]\\
&\qquad\le\frac{C}{\sqrt{n}}\mathbb{E}^{1/2}\big[\big(V'(Y_r^{i,n})\big)^2+\big(V'(Y_{\lfloor nr\rfloor/n}^{i,n})\big)^2\big].
\end{aligned}
\tag{5.11}
$$

Plugging this inequality in (5.10) and inserting the resulting inequality together with (5.5), (5.7) and (5.9) into (5.4), one concludes with (5.3) and the local boundedness of $r\mapsto\sup_{n\ge1}\sup_{1\le i\le n}\mathbb{E}[(V'(Y_r^{i,n}))^2]$ deduced from (4.1) and exchangeability. □

This completes the proof of Proposition 4.

5.3. *Proofs of Propositions* 5 *and* 6. Finally, it remains to prove Propositions 5 and 6.

PROOF OF PROPOSITION 5. Since for $1 \le i \le n$, $(M_k^{i,n})$ is a $\mathcal{F}_k^n$-martingale and $g(Y_{s_1}^{i,n}, \dots, Y_{s_p}^{i,n})$ is $\mathcal{F}_{\lceil ns \rceil}^n$-measurable, one has

$$\mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n}(M_{\lceil nt \rceil}^{i,n} - M_{\lceil ns \rceil}^{i,n})g(Y_{s_1}^{i,n}, \dots, Y_{s_p}^{i,n})\right)^2\right]$$

$$\begin{aligned}= \frac{1}{n^2}\sum_{i,j=1}^{n}\sum_{k=\lceil ns \rceil}^{\lceil nt \rceil - 1}\mathbb{E}\big[\mathbb{E}\big[(M_{k+1}^{i,n} - M_k^{i,n})(M_{k+1}^{j,n} - M_k^{j,n})|\mathcal{F}_k^n\big] \\ \times g(Y_{s_1}^{i,n}, \dots, Y_{s_p}^{i,n})g(Y_{s_1}^{j,n}, \dots, Y_{s_p}^{j,n})\big]. \end{aligned} \tag{5.12}$$

Using the boundedness of φ' and φ'', then (4.4), (5.8) and the equality

$$\begin{aligned}&\mathbb{E}\big[G_{k+1}^i\big(e^{-\sum_{l=1}^n (V'(X_k^{l,n})(l/\sqrt{n})G_{k+1}^l + (l^2/(2n))V''(X_k^{l,n}))} \wedge 1\big)|\mathcal{F}_k^n\big] \\ &\quad = -\frac{V'(X_k^{i,n})}{l\sqrt{n}}\mathcal{G}\big(\langle \mu_{k/n}^n, (V')^2\rangle, \langle \mu_{k/n}^n, V''\rangle\big)\end{aligned}$$

deduced from (A.3), one obtains

$$\begin{aligned}&\big|\mathbb{E}\big[(M_{k+1}^{i,n} - M_k^{i,n})(M_{k+1}^{j,n} - M_k^{j,n})|\mathcal{F}_k^n\big]\big| \\ &\quad \le \frac{C}{n}\big|\mathbb{E}\big[G_{k+1}^i G_{k+1}^j 1_{\mathcal{A}_{k+1}}|\mathcal{F}_k^n\big] - \mathbb{E}\big[G_{k+1}^i 1_{\mathcal{A}_{k+1}}|\mathcal{F}_k^n\big]\mathbb{E}\big[G_{k+1}^j 1_{\mathcal{A}_{k+1}}|\mathcal{F}_k^n\big]\big| \\ &\quad \le \frac{C}{n^{3/2}} \\ &\qquad + \frac{C}{n}\Big|\mathbb{E}\big[G_{k+1}^i G_{k+1}^j\big(e^{-\sum_{l=1}^n (V'(X_k^{l,n})(l/\sqrt{n})G_{k+1}^l + (l^2/(2n))V''(X_k^{l,n}))} \wedge 1\big)|\mathcal{F}_k^n\big] \\ &\qquad\quad - \mathbb{E}\big[G_{k+1}^i\big(e^{-\sum_{l=1}^n (V'(X_k^{l,n})(l/\sqrt{n})G_{k+1}^l + (l^2/(2n))V''(X_k^{l,n}))} \wedge 1\big)|\mathcal{F}_k^n\big] \\ &\qquad\quad\quad \times \mathbb{E}\big[G_{k+1}^j\big(e^{-\sum_{l=1}^n (V'(X_k^{l,n})(l/\sqrt{n})G_{k+1}^l + (l^2/(2n))V''(X_k^{l,n}))} \wedge 1\big)|\mathcal{F}_k^n\big]\Big| \\ &\quad \le C\left(\frac{1}{n^{3/2}} + \frac{1_{\{i=j\}}}{n} + \frac{|V'(X_k^{i,n})V'(X_k^{j,n})|}{n^2} + \frac{|V'(X_k^{i,n})V'(X_k^{j,n})|}{n^2\sqrt{\langle \mu_{k/n}^n, (V')^2\rangle}}\right).\end{aligned}$$

Plugging this estimate into (5.12) and using the boundedness of g and (5.6), one concludes that

$$\begin{aligned}&\mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n}(M_{\lceil nt \rceil}^{i,n} - M_{\lceil ns \rceil}^{i,n})g(Y_{s_1}^{i,n}, \dots, Y_{s_p}^{i,n})\right)^2\right] \\ &\quad \le C\left(\frac{\lceil nt \rceil - \lceil ns \rceil}{n^{3/2}} + \frac{1}{n^2}\sum_{k=\lceil ns \rceil}^{\lceil nt \rceil - 1}\Big(\mathbb{E}\big[\big(V'(Y_{k/n}^{i,n})\big)^2\big] + \sqrt{\mathbb{E}\big[\big(V'(Y_{k/n}^{i,n})\big)^2\big]}\Big)\right).\end{aligned}$$

One concludes with the local boundedness of $r \mapsto \sup_{n\geq 1}\sup_{1\leq i\leq n}\mathbb{E}[(V'(Y_r^{i,n}))^2]$ deduced from (4.1) and exchangeability. □

PROOF OF PROPOSITION 6. Since the function φ is compactly supported and V' is continuous, one may suppose that k is large enough so that $\forall x\in\mathbb{R}$, $|V'\varphi'(x)|\leq\|\varphi'\|_\infty\sqrt{(V'(x))^2\wedge k}$ and, therefore,

$$\langle\mu_r^n, |V'\varphi'|\rangle\leq\|\varphi'\|_\infty\sqrt{\langle\mu_r^n,(V')^2\wedge k\rangle}.$$

By boundedness of g and φ'', then using (2.15) and (2.17), one deduces

$$\begin{aligned}
&\mathbb{E}|F_k(\mu^n)-F(\mu^n)|\\
&\quad\leq C\int_s^t\mathbb{E}\Big[\big|\Gamma\big(\langle\mu_r^n,(V')^2\wedge k\rangle,\langle\mu_r^n,V''\rangle\big)-\Gamma\big(\langle\mu_r^n,(V')^2\rangle,\langle\mu_r^n,V''\rangle\big)\big|\\
&\qquad\qquad+\big|\mathcal{G}\big(\langle\mu_r^n,(V')^2\wedge k\rangle,\langle\mu_r^n,V''\rangle\big)-\mathcal{G}\big(\langle\mu_r^n,(V')^2\rangle,\langle\mu_r^n,V''\rangle\big)\big|\\
&\qquad\qquad\times\sqrt{\langle\mu_r^n,(V')^2\wedge k\rangle}\Big]\,dr\\
&\quad\leq C\int_s^t\mathbb{E}\Big[\sqrt{\langle\mu_r^n,((V')^2-k)^+\rangle}+\langle\mu_r^n,((V')^2-k)^+\rangle\Big]\,dr.
\end{aligned}\tag{5.13}$$

Since $|V'(Y_r^{1,n})|\leq|V'(X_0^{1,n})|+\|V''\|_\infty|Y_r^{1,n}-Y_0^{1,n}|$, using the Cauchy–Schwarz and the Markov inequalities, one obtains that

$$\begin{aligned}
&\mathbb{E}[\langle\mu_r^n,((V')^2-k)^+\rangle]\\
&\quad\leq\mathbb{E}[(V'(Y_r^{1,n}))^2 1_{\{|V'(Y_r^{1,n})|\geq\sqrt{k}\}}]\\
&\quad\leq 2\mathbb{E}[((V'(X_0^{1,n}))^2+\|V''\|_\infty^2|Y_r^{1,n}-Y_0^{1,n}|^2)\\
&\qquad\quad\times(1_{\{|V'(X_0^{1,n})|\geq(\sqrt{k}/2)\}}+1_{\{|Y_r^{1,n}-Y_0^{1,n}|\geq(\sqrt{k}/(2\|V''\|_\infty))\}})]\\
&\quad\leq\frac{C}{k}(\mathbb{E}[(V'(X_0^{1,n}))^4]\\
&\qquad\quad+\mathbb{E}^{1/2}[|Y_r^{1,n}-Y_0^{1,n}|^4]\mathbb{E}^{1/2}[(V'(X_0^{1,n}))^4]+\mathbb{E}[|Y_r^{1,n}-Y_0^{1,n}|^4]).
\end{aligned}$$

Therefore, by (4.3),

$$\lim_{k\to\infty}\sup_{n\geq 1}\sup_{r\in[0,t]}\mathbb{E}[\langle\mu_r^n,((V')^2-k)^+\rangle]=0.\tag{5.14}$$

One concludes by plugging this result into (5.13). □

6. Proof of Proposition 1. By (4.4) and [21], Proposition 2.4, which is also a consequence of (A.5) for the choice $\alpha=0$, there is a finite deterministic constant C not depending on t such that

$$\left|\mathbb{P}(\mathcal{A}_{\lfloor nt\rfloor+1}|\mathcal{F}^n_{\lfloor nt\rfloor})-\frac{1}{l^2}\Gamma\big(\langle\mu^n_{\lfloor nt\rfloor/n},(V')^2\rangle,\langle\mu^n_{\lfloor nt\rfloor/n},V''\rangle\big)\right|\leq\frac{C}{\sqrt{n}}.$$

With (2.15), one deduces that

$$\begin{aligned}
&\mathbb{E}\left|\mathbb{P}(\mathcal{A}_{\lfloor nt\rfloor+1}|\mathcal{F}^n_{\lfloor nt\rfloor}) - \frac{1}{l^2}\Gamma\big(\mathbb{E}[(V'(X_t))^2], \mathbb{E}[V''(X_t)]\big)\right| \\
&\quad \le C\bigg(\frac{1}{\sqrt{n}} + (\mathbb{E}+\mathbb{E}^{1/2})|\langle\mu^n_{\lfloor nt\rfloor/n}, (V')^2\rangle - \mathbb{E}[(V'(X_t))^2]| \\
&\qquad\qquad + \mathbb{E}|\langle\mu^n_{\lfloor nt\rfloor/n}, V''\rangle - \mathbb{E}[V''(X_t)]|\bigg).
\end{aligned} \tag{6.1}$$

One has for $k \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{E}|\langle\mu^n_{\lfloor nt\rfloor/n}, (V')^2\rangle - \mathbb{E}[(V'(X_t))^2]| \\
&\quad \le \mathbb{E}|\langle\mu^n_{\lfloor nt\rfloor/n} - \mu^n_t, (V')^2\rangle| + \mathbb{E}\langle\mu^n_t, ((V')^2 - k)^+\rangle \\
&\qquad + \mathbb{E}|\langle\mu^n_t, (V')^2 \wedge k\rangle - \mathbb{E}[(V'(X_t))^2 \wedge k]| + \mathbb{E}[((V')^2 - k)^+(X_t)].
\end{aligned}$$

By the end of the proof of Proposition 4 [see in particular (5.11)], the first term in the right-hand side converges to 0 locally uniformly in t as $n \to \infty$. By (5.14) and Theorem 1, the sum of the second and last terms in the right-hand side converges to 0 as $k \to \infty$ uniformly in n and locally uniformly in t. Last, for fixed k, the third term converges to 0 as $n \to \infty$ locally uniformly in t by Theorem 1. One deduces that $\mathbb{E}|\langle\mu^n_{\lfloor nt\rfloor/n}, (V')^2\rangle - \mathbb{E}[(V'(X_t))^2]|$ converges to 0 as $n \to \infty$ locally uniformly in t. Dealing with the other expectation in the right-hand side of (6.1) in a similar but easier way (since V'' is bounded), one completes the proof.

APPENDIX: PROOFS OF TECHNICAL RESULTS

In this section, we first give a proof of Lemma 2 which gives basic properties of the functions Γ and $\mathcal{G}$. Then we give some explicit formulas for some expectations involving Gaussian random variables.

PROOF OF LEMMA 2. The functions $\mathcal{G}$ and Γ are clearly continuous on $(0, +\infty) \times \mathbb{R}$. We recall the usual tail estimate for the Normal law: $\forall x > 0$,

$$\Phi(-x) = \int_x^{+\infty} e^{-y^2/2}\frac{dy}{\sqrt{2\pi}} \le \int_x^{+\infty} \frac{y}{x}e^{-y^2/2}\frac{dy}{\sqrt{2\pi}} = \frac{e^{-x^2/2}}{x\sqrt{2\pi}}. \tag{A.1}$$

One deduces that for $a > b^+$,

$$\begin{aligned}
&\Phi\left(l\left(\frac{b}{2\sqrt{a}} - \sqrt{a}\right)\right) \le \frac{2}{l\sqrt{2\pi a}}e^{-(l^2(b-2a)^2)/(8a)} \quad \text{and} \\
&\mathcal{G}(a,b) \le \frac{2l}{\sqrt{2\pi a}}e^{-(l^2b^2)/(8a)}.
\end{aligned} \tag{A.2}$$

Since for $0 \le a \le b$, $\mathcal{G}(a,b) \le l^2 \times 1 \times 1$, one deduces (2.16). Moreover, (A.2) implies that $\mathcal{G}$ is continuous on $\{(0, +\infty] \times \mathbb{R}\} \cup \{\{0\} \times (-\infty, 0)\}$.

With the continuity of $(a,b) \mapsto \frac{b}{\sqrt{a}}$ on $(0,+\infty] \times \mathbb{R}$ under the convention $\frac{b}{\sqrt{\infty}} = 0$, one deduces that Γ is continuous on $(0,+\infty] \times \mathbb{R}$. For $\beta > 0$, $\lim_{a\to 0^+, b\to\beta} \Phi(\frac{b}{2\sqrt{a}} - \sqrt{a}) = 1$ and, therefore, $\lim_{a\to 0^+, b\to\beta} \mathcal{G}(a,b) = \mathcal{G}(0,\beta)$, which completes the proof of the continuity properties of $\mathcal{G}$. Since for $(a,b) \in (0,+\infty) \times \mathbb{R}$, $\partial_b \Gamma(a,b) = -\frac{l^4}{2} e^{(l^2(a-b))/2} \Phi(l(\frac{b}{2\sqrt{a}} - \sqrt{a})) < 0$, for fixed $a \in (0,+\infty)$, the function $b \mapsto \Gamma(a,b)$ is decreasing. One easily checks that for fixed $b < 0$, $\lim_{a\to 0^+} \Gamma(a,b) = l^2 + 0 = \Gamma(0,b)$ and for fixed $b > 0$, $\lim_{a\to 0^+} \Gamma(a,b) = 0 + l^2 e^{-(l^2 b)/2} = \Gamma(0,b)$. With the previous monotonicity property, one deduces that $\lim_{a\to 0^+} \Gamma(a,0) = l^2 = \Gamma(0,0)$. The continuity of $b \mapsto \Gamma(0,b)$ and Dini's lemma implies that $b \mapsto \Gamma(a,b)$ converges locally uniformly to $b \mapsto \Gamma(0,b)$ as $a \to 0^+$ and that Γ is continuous on $[0,+\infty] \times \mathbb{R}$. Since Γ is positive on $[0,+\infty] \times \mathbb{R}$, one deduces that (2.14) holds. For $a > 0$, by (A.2), $\lim_{b\to-\infty} \mathcal{G}(a,b) = 0$. Since $\lim_{b\to-\infty} \Phi(-\frac{lb}{2\sqrt{a}}) = 1$, one deduces that $\lim_{b\to-\infty} \Gamma(a,b) = l^2$. By monotonicity of $b \mapsto \Gamma(a,b)$, one deduces that $\forall (a,b) \in (0,+\infty) \times \mathbb{R}$, $\Gamma(a,b) \le l^2$. This bound still holds for $a \in \{0,+\infty\}$ by continuity (or using the explicit expression of Γ). For $(a,b) \in (0,+\infty) \times \mathbb{R}$, one has

$$
\begin{aligned}
\partial_b \Gamma(a,b) &= -\frac{l^2}{2} \mathcal{G}(a,b), \\
\partial_a \Gamma(a,b) &= \frac{l^2}{2} \mathcal{G}(a,b) - \frac{l^3}{2\sqrt{2\pi a}} e^{-(l^2 b^2)/(8a)}, \\
\partial_b \mathcal{G}(a,b) &= -\frac{l^2}{2} \mathcal{G}(a,b) + \frac{l^3}{2\sqrt{2\pi a}} e^{-(l^2 b^2)/(8a)}, \\
\partial_a \mathcal{G}(a,b) &= \frac{l^2}{2} \mathcal{G}(a,b) - \frac{l^3}{2\sqrt{2\pi}} \left(\frac{1}{\sqrt{a}} + \frac{b}{2a^{3/2}} \right) e^{-(l^2 b^2)/(8a)}.
\end{aligned}
$$

The boundedness of $\mathcal{G}$ then implies (2.15). Concerning (2.17), let us give some details for the inequality

$$
(\sqrt{a} \wedge \sqrt{a'}) |\mathcal{G}(a,b) - \mathcal{G}(a',b)| \le C(|a'-a| + |\sqrt{a'} - \sqrt{a}|).
$$

Let us assume that $0 \le a < a'$ and $b \in [\inf V'', \sup V'']$. Then we have

$$
\begin{aligned}
&(\sqrt{a} \wedge \sqrt{a'}) |\mathcal{G}(a,b) - \mathcal{G}(a',b)| \\
&\quad = \sqrt{a} \left| \int_a^{a'} \partial_a \mathcal{G}(x,b)\, dx \right| \\
&\quad = \sqrt{a} \left| \int_a^{a'} \frac{l^2}{2} \mathcal{G}(x,b) - \frac{l^3}{2\sqrt{2\pi}} \left(\frac{1}{\sqrt{x}} + \frac{b}{2x^{3/2}} \right) e^{-(l^2 b^2)/(8x)}\, dx \right|
\end{aligned}
$$

$$\leq C\sqrt{a}\int_a^{a'}\left(\frac{1}{\sqrt{x}}+\frac{1}{x}\right)dx \leq C\left((a'-a)+\int_a^{a'}\frac{\sqrt{a}}{x}\,dx\right)$$
$$\leq C\left((a'-a)+\int_a^{a'}\frac{1}{\sqrt{x}}\,dx\right) \leq C\left((a'-a)+(\sqrt{a'}-\sqrt{a})\right),$$

where we used (2.16) and the boundedness of $(x,b) \in (0,+\infty] \times \mathbb{R} \mapsto \frac{b}{2\sqrt{x}}e^{-(l^2b^2)/(8x)}$ for the first inequality. □

LEMMA 6. *For $\alpha,\beta,\gamma,\delta \in \mathbb{R}$ and independent normal random variables G, $\tilde{G}$ and $\hat{G}$, one has*

$$\begin{aligned}&\mathbb{E}\big(G\big(e^{\alpha G+\beta\tilde{G}+\gamma}\wedge 1\big)\big)\\ &\quad=\alpha e^{\gamma+((\alpha^2+\beta^2)/2)}\Phi\left(-\frac{\gamma+\alpha^2+\beta^2}{\sqrt{\alpha^2+\beta^2}}\right)=\frac{\alpha}{l^2}\mathcal{G}\left(\frac{\alpha^2+\beta^2}{l^2},-\frac{2\gamma}{l^2}\right),\end{aligned}\tag{A.3}$$

$$\big|\mathbb{E}\big(G\big(1-e^{\alpha G+\beta\tilde{G}+\gamma}\big)^+\big)\big| \leq \left(\sqrt{\frac{2}{\pi}}+\sqrt{2\gamma^-}\right)\sqrt{\frac{\alpha^2}{\alpha^2+\beta^2}},\tag{A.4}$$

$$\begin{aligned}&\mathbb{E}\big(G^2\big(e^{\alpha G+\beta\tilde{G}+\gamma}\wedge 1\big)\big)\\ &\quad=\big(1+\alpha^2\big)e^{\gamma+((\alpha^2+\beta^2)/2)}\Phi\left(-\frac{\gamma+\alpha^2+\beta^2}{\sqrt{\alpha^2+\beta^2}}\right)\\ &\qquad+\Phi\left(\frac{\gamma}{\sqrt{\alpha^2+\beta^2}}\right)-\frac{\alpha^2}{\sqrt{2\pi(\alpha^2+\beta^2)}}e^{-(\gamma^2/(2(\alpha^2+\beta^2))},\end{aligned}\tag{A.5}$$

$$\begin{aligned}&\mathbb{E}\big(G\hat{G}\big(e^{\alpha G+\beta\tilde{G}+\delta\hat{G}+\gamma}\wedge 1\big)\big)\\ &\quad=\alpha\delta\left(e^{\gamma+((\alpha^2+\beta^2+\delta^2)/2)}\Phi\left(-\frac{\gamma+\alpha^2+\beta^2+\delta^2}{\sqrt{\alpha^2+\beta^2+\delta^2}}\right)\right.\\ &\qquad\qquad\left.-\frac{e^{-(\gamma^2/(2(\alpha^2+\beta^2+\delta^2)))}}{\sqrt{2\pi(\alpha^2+\beta^2+\delta^2)}}\right),\end{aligned}\tag{A.6}$$

$$\forall a \in [0,+\infty), \qquad \mathbb{E}\big(\mathcal{G}(a,\alpha G+\beta)\big)=\mathcal{G}\left(a+\frac{l^2\alpha^2}{4},\beta\right).\tag{A.7}$$

PROOF. In this proof, the identity $\mathbb{E}(f(G)e^{\alpha G-\alpha^2/2})=\mathbb{E}(f(\alpha+G))$ is repeatedly used. Let us start with (A.3). By the symmetry of the normal law, $\alpha \mapsto \mathbb{E}(G(e^{\alpha G+\beta\tilde{G}+\gamma}\wedge 1))$ is an odd function and we only need to check (A.3) for $\alpha > 0$. Conditioning by $\tilde{G}$ for the third equality, we get

$$\begin{aligned}
&\mathbb{E}\big(G\big(e^{\alpha G+\beta\tilde{G}+\gamma}\wedge 1\big)\big)\\
&\quad= \mathbb{E}\big(e^{\gamma+(\alpha^2/2)}e^{\alpha G-(\alpha^2/2)}e^{\beta\tilde{G}}G1_{\{G\le -(\gamma+\beta\tilde{G})/\alpha\}} + G1_{\{G>(\gamma+\beta\tilde{G})/\alpha\}}\big)\\
&\quad= e^{\gamma+(\alpha^2/2)}\mathbb{E}\big(e^{\beta\tilde{G}}(\alpha+G)1_{\{\alpha+G\le -(\gamma+\beta\tilde{G})/\alpha\}}\big) + \mathbb{E}\big(G1_{\{G>(\gamma+\beta\tilde{G})/\alpha\}}\big)\\
&\quad= \alpha e^{\gamma+((\alpha^2+\beta^2)/2)}\mathbb{P}\biggl(\frac{\alpha G+\beta(\beta+\tilde{G})}{\sqrt{\alpha^2+\beta^2}} \le -\frac{\gamma+\alpha^2}{\sqrt{\alpha^2+\beta^2}}\biggr)\\
&\qquad - \frac{e^{\gamma+(\alpha^2/2)}}{\sqrt{2\pi}}\mathbb{E}\big(e^{\beta\tilde{G}}e^{-(\gamma+\alpha^2+\beta\tilde{G})^2/(2\alpha^2)}\big)\\
&\qquad + \frac{1}{\sqrt{2\pi}}\mathbb{E}\big(e^{-(\gamma+\beta\tilde{G})^2/(2\alpha^2)}\big).
\end{aligned}$$

We deduce (A.3) by remarking that the two last terms compensate each other since

$$\gamma + \frac{\alpha^2}{2} + \beta\tilde{G} - \frac{(\gamma+\alpha^2+\beta\tilde{G})^2}{2\alpha^2} = -\frac{(\gamma+\beta\tilde{G})^2}{2\alpha^2}.$$

To obtain the inequality (A.4), we notice that

$$\begin{aligned}
&\mathbb{E}\big(G\big(1-e^{\alpha G+\beta\tilde{G}+\gamma}\big)^+\big)\\
&\quad= \mathbb{E}\big(G\big(1-e^{\alpha G+\beta\tilde{G}+\gamma}\big)^+\big) - \mathbb{E}(G)\\
&\quad= -\mathbb{E}\big(G\big(e^{\alpha G+\beta\tilde{G}+\gamma}\wedge 1\big)\big)\\
&\quad= -\frac{\alpha}{l\sqrt{\alpha^2+\beta^2}} \times \sqrt{\frac{\alpha^2+\beta^2}{l^2}}\mathcal{G}\biggl(\frac{\alpha^2+\beta^2}{l^2}, -\frac{2\gamma}{l^2}\biggr)
\end{aligned}$$

and conclude using (2.16). To derive (A.5), one obtains by conditioning by G for the second equality

$$\begin{aligned}
&\mathbb{E}\big(G^2\big(e^{\alpha G+\beta\tilde{G}+\gamma}\wedge 1\big)\big)\\
&\quad= e^{((\alpha^2+\beta^2)/2)+\gamma}\mathbb{E}\big(G^2 e^{\alpha G+\beta\tilde{G}-((\alpha^2+\beta^2)/2)}1_{\{\alpha G+\beta\tilde{G}\le -\gamma\}}\big)\\
&\qquad + \mathbb{E}\biggl(G^2\Phi\biggl(\frac{\gamma+\alpha G}{|\beta|}\biggr)\biggr) \qquad\qquad \text{(A.8)}\\
&\quad= e^{((\alpha^2+\beta^2)/2)+\gamma}\mathbb{E}\biggl((G^2+2\alpha G+\alpha^2)\Phi\biggl(-\frac{\gamma+\alpha G+\alpha^2+\beta^2}{|\beta|}\biggr)\biggr)\\
&\qquad + \mathbb{E}\biggl(G^2\Phi\biggl(\frac{\gamma+\alpha G}{|\beta|}\biggr)\biggr).
\end{aligned}$$

By integration by parts,

$$
\begin{aligned}
&\mathbb{E}\Big(G^2\Phi\Big(\frac{\gamma+\alpha G}{|\beta|}\Big)\Big)\\
&\quad=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} x^2\Phi\Big(\frac{\gamma+\alpha x}{|\beta|}\Big)e^{-x^2/2}\,dx\\
&\quad=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \Phi\Big(\frac{\gamma+\alpha x}{|\beta|}\Big)e^{-x^2/2}\,dx\\
&\qquad+\frac{\alpha}{2\pi|\beta|}\int_{\mathbb{R}} xe^{-(x^2/2)-((\gamma+\alpha x)^2/(2\beta^2))}\,dx \qquad\text{(A.9)}\\
&\quad=\mathbb{P}(|\beta|\tilde{G}-\alpha G\le\gamma)\\
&\qquad+\frac{\alpha e^{-\gamma^2/(2(\alpha^2+\beta^2))}}{2\pi|\beta|}\int_{\mathbb{R}} xe^{-((\alpha^2+\beta^2)(x+((\gamma\alpha)/(\alpha^2+\beta^2)))^2)/(2\beta^2)}\,dx\\
&\quad=\Phi\Big(\frac{\gamma}{\sqrt{\alpha^2+\beta^2}}\Big)-e^{-\gamma^2/(2(\alpha^2+\beta^2))}\frac{\alpha^2\gamma}{\sqrt{2\pi(\alpha^2+\beta^2)^3}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}\Big(G\Phi\Big(-\frac{\gamma+\alpha G+\alpha^2+\beta^2}{|\beta|}\Big)\Big)\\
&\quad=-\frac{\alpha}{2\pi|\beta|}\int_{\mathbb{R}} e^{-(x^2/2)-((\alpha x+\gamma+\alpha^2+\beta^2)^2/(2\beta^2))}\,dx \qquad\text{(A.10)}\\
&\quad=-\frac{\alpha}{\sqrt{2\pi(\alpha^2+\beta^2)}}e^{-(\gamma+\alpha^2+\beta^2)^2/(2(\alpha^2+\beta^2))}.
\end{aligned}
$$

One obtains (A.5) by plugging this last equality together with (A.9) also written with (α,γ) replaced by $(-\alpha,-(\gamma+\alpha^2+\beta^2))$ in (A.8).

To prove (A.6), conditioning by $\hat{G}$, using (A.3) and then (A.10), one obtains

$$
\begin{aligned}
&\mathbb{E}(G\hat{G}(e^{\alpha G+\beta\tilde{G}+\delta\hat{G}+\gamma}\wedge 1))\\
&\quad=\alpha e^{\gamma+((\alpha^2+\beta^2)/2)}\mathbb{E}\Big(\hat{G}e^{\delta\hat{G}}\Phi\Big(-\frac{\gamma+\delta\hat{G}+\alpha^2+\beta^2}{\sqrt{\alpha^2+\beta^2}}\Big)\Big)\\
&\quad=\alpha e^{\gamma+((\alpha^2+\beta^2+\delta^2)/2)}\mathbb{E}\Big((\hat{G}+\delta)\Phi\Big(-\frac{\gamma+\delta\hat{G}+\alpha^2+\beta^2+\delta^2}{\sqrt{\alpha^2+\beta^2}}\Big)\Big)\\
&\quad=\alpha\delta e^{\gamma+((\alpha^2+\beta^2+\delta^2)/2)}\\
&\qquad\times\Big(\Phi\Big(-\frac{\gamma+\alpha^2+\beta^2+\delta^2}{\sqrt{\alpha^2+\beta^2+\delta^2}}\Big)-\frac{e^{-(\gamma+\alpha^2+\beta^2+\delta^2)^2/(2(\alpha^2+\beta^2+\delta^2))}}{\sqrt{2\pi(\alpha^2+\beta^2+\delta^2)}}\Big).
\end{aligned}
$$

Last,

$$\begin{aligned}
&\frac{1}{l^2}\mathbb{E}\big(\mathcal{G}(a,\alpha G+\beta)\big)\\
&\qquad = e^{(l^2(a+l^2\alpha^2/4-\beta))/2}\mathbb{P}\bigg(\tilde{G}\le l\bigg(\frac{\alpha G-l^2\alpha^2/2+\beta}{2\sqrt{a}}-\sqrt{a}\bigg)\bigg)\\
&\qquad = e^{(l^2(a+l^2\alpha^2/4-\beta))/2}\mathbb{P}\bigg(\frac{\sqrt{a+l^2\alpha^2/4}}{\sqrt{a}}\hat{G}\le l\frac{\beta-2(a+l^2\alpha^2/4)}{2\sqrt{a}}\bigg),
\end{aligned}$$

which yields (A.7). □

To prove Lemma 5, we need the following lemma.

LEMMA 7. *Let X, Y denote two real random variables with respective cumulative distribution functions F_X and F_Y and $f:\mathbb{R}\to\mathbb{R}$ be a bounded function, Lipschitz continuous with constant $L(f)$ outside $[-\varepsilon,\varepsilon]$ for some constant $\varepsilon>0$. If X admits a bounded density p_X with respect to the Lebesgue measure on $\mathbb{R}$, then*

$$\begin{aligned}
&\big|\mathbb{E}[f(X)]-\mathbb{E}[f(Y)]\big|\\
&\qquad \le L(f)W_1(X,Y)+2(\sup f-\inf f)\big(\sqrt{2\|p_X\|_\infty W_1(X,Y)}+\|p_X\|_\infty\varepsilon\big),
\end{aligned}$$

where $W_1(X,Y)=\inf_{(Z,W):Z\overset{(d)}{=}X,W\overset{(d)}{=}Y}\mathbb{E}|Z-W|$ denotes the Wasserstein distance between the laws of X and Y.

PROOF. Let for $u\in(0,1)$, $F_X^{-1}(u)=\inf\{x\in\mathbb{R}:F_X(x)\ge u\}$ denote the càg pseudo-inverse of F_X and F_Y^{-1} be defined in the same way. Then $\forall x\in\mathbb{R}$, $\forall u\in(0,1)$, $F_X^{-1}(u)\le x\Leftrightarrow u\le F_X(x)$. Moreover, if U is uniformly distributed on $[0,1]$, then $F_X^{-1}(U)\overset{(d)}{=}X$, $F_Y^{-1}(U)\overset{(d)}{=}Y$ and according to [20], pages 107–109, $W_1(X,Y)=\mathbb{E}|F_X^{-1}(U)-F_Y^{-1}(U)|$. As a consequence,

$$\begin{aligned}
&\big|\mathbb{E}[f(X)]-\mathbb{E}[f(Y)]\big|\\
&\quad =\big|\mathbb{E}\big[f\big(F_X^{-1}(U)\big)-f\big(F_Y^{-1}(U)\big)\big]\big|\\
&\quad \le\big|\mathbb{E}\big[\big(f\big(F_X^{-1}(U)\big)-f\big(F_Y^{-1}(U)\big)\big)\\
&\qquad\qquad \times(1_{\{F_X^{-1}(U)\vee F_Y^{-1}(U)\le-\varepsilon\}}+1_{\{F_X^{-1}(U)\wedge F_Y^{-1}(U)>\varepsilon\}})\big]\big|\\
&\qquad +\big|\mathbb{E}\big[\big(f\big(F_X^{-1}(U)\big)-f\big(F_Y^{-1}(U)\big)\big)\\
&\qquad\qquad \times(1_{\{F_X^{-1}(U)\le-\varepsilon<F_Y^{-1}(U)\}}+1_{\{F_X^{-1}(U)>\varepsilon\ge F_Y^{-1}(U)\}})\big]\big|\\
&\qquad +\big|\mathbb{E}\big[\big(f\big(F_X^{-1}(U)\big)-f\big(F_Y^{-1}(U)\big)\big)1_{\{-\varepsilon<F_X^{-1}(U)\le\varepsilon\}}\big]\big|
\end{aligned}$$

$$\begin{aligned}
&\le L(f)\mathbb{E}\big|F_X^{-1}(U) - F_Y^{-1}(U)\big| \\
&\quad + (\sup f - \inf f)\big(\mathbb{P}\big(F_Y(-\varepsilon) < U \le F_X(-\varepsilon)\big) \\
&\qquad\qquad + \mathbb{P}\big(F_X(\varepsilon) < U \le F_Y(\varepsilon)\big) \\
&\qquad\qquad + \mathbb{P}\big(F_X(-\varepsilon) < U \le F_X(\varepsilon)\big)\big) \\
&= L(f)W_1(X,Y) \\
&\quad + (\sup f - \inf f)\bigg(\big(F_X(-\varepsilon) - F_Y(-\varepsilon)\big)^+ + \big(F_Y(\varepsilon) - F_X(\varepsilon)\big)^+ \\
&\qquad\qquad + \int_{-\varepsilon}^{\varepsilon} p_X(x)\,dx\bigg).
\end{aligned}$$

One concludes by using the inequality

$$\sup_{x\in\mathbb{R}}\big|F_X(x) - F_Y(x)\big| \le \sqrt{2\|p_X\|_\infty W_1(X,Y)}.$$

This inequality is stated in [14], Lemma 5.4, with the factor 2 replaced by 4 but a careful look at the proof of this lemma shows that it holds with the factor 2. □

PROOF OF LEMMA 5. By Lipschitz continuity of $x \mapsto e^x \wedge 1$ and the Taylor expansion

$$V\left(x_i + \frac{l}{\sqrt{n}}G^i\right) = V(x_i) + \frac{lV'(x_i)}{\sqrt{n}}G^i + \frac{l^2V''(x_i)}{2n}(G^i)^2 + \frac{l^3V^{(3)}(\chi_i)}{6n^{3/2}}(G^i)^3$$

with $\chi_i \in [x_i, x_i + \frac{l}{\sqrt{n}}G^i]$, one obtains

$$\begin{aligned}
&\mathbb{E}\big[\big(e^{\sum_{i=1}^n(V(x_i)-V(x_i+(l/\sqrt{n})G^i))} \wedge 1 - e^{-\sum_{i=1}^n((l/\sqrt{n})V'(x_i)G^i+(l^2/(2n))V''(x_i))} \wedge 1\big)^2\big] \\
&\quad \le \mathbb{E}\left[\left(\sum_{i=1}^n\left(\frac{l^2V''(x_i)}{2n}((G^i)^2-1) + \frac{l^3V^{(3)}(\chi_i)}{6n^{3/2}}(G^i)^3\right)\right)^2\right].
\end{aligned}$$

Developing the square and remarking that for $i \neq j$, $\mathbb{E}[((G^i)^2-1)((G^j)^2-1)] = 0 = \mathbb{E}[((G^i)^2-1)V^{(3)}(\chi_j)(G^j)^3]$, one easily deduces (4.4) using the boundedness of V'' and $V^{(3)}$.

The proof of the two other inequalities is inspired by [14], Section 5, where the authors first replace $V(x_1) - V(x_1 + \frac{l}{\sqrt{n}}G^1)$ by $-\frac{lV'(x_1)}{\sqrt{n}}G^1$ in the exponential factor at a cost $\mathcal{O}(\frac{1}{n})$. Then they explicitly compute the conditional expectation given $(G^2, \dots, G^n)$ to improve the regularity of the function in the expectation. Next, they replace $\sum_{i=2}^n(V(x_i + \frac{l}{\sqrt{n}}G^i) - V(x_i))$ by the Gaussian random variable $\sum_{i=2}^n(\frac{lV'(x_i)}{\sqrt{n}}G^i + \frac{l^2V''(x_i)}{2n})$ and control the resulting error by some Wasserstein distance estimate between these two random variables. To preserve symmetry in the estimate and in particular to obtain $\langle \nu_n, (V')^2\rangle$ instead of $\frac{1}{n}\sum_{i=2}^n (V'(x_i))^2$ in the denominators, we write G_1 as the sum of two independent variables distributed according to $\mathcal{N}(0, \frac{1}{2})$.

Let $\tilde{G}^1 = \frac{G^1}{\sqrt{2}}$, $\tilde{G}^i = G^i$ for $i \geq 2$ and $\hat{G}^1 \sim \mathcal{N}(0, \frac{1}{2})$ be independent from $(G^1, \dots, G^n)$. One has

$$\begin{aligned}
&\mathbb{E}\big(G^1\big(e^{\sum_{i=1}^n (V(x_i)-V(x_i+(l/\sqrt{n})G^i))} \wedge 1\big)\big)\\
&\quad = 2\mathbb{E}\big(\hat{G}^1\big(e^{V(x_1)-V(x_1+(l/\sqrt{n})(\tilde{G}^1+\hat{G}^1))+\sum_{i=2}^n (V(x_i)-V(x_i+(l/\sqrt{n})\tilde{G}^i))} \wedge 1\big)\big).
\end{aligned}$$

As in the above derivation of (4.4), one deduces from the Lipschitz continuity of $y \mapsto e^y \wedge 1$ and the boundedness of V'' that

$$\big|\mathbb{E}\big(G^1\big(e^{\sum_{i=1}^n (V(x_i)-V(x_i+(l/\sqrt{n})G^i))} \wedge 1\big)\big) - E\big| \leq \frac{C}{n},$$

where, by conditioning by $(\tilde{G}^1, \dots, \tilde{G}^n)$ and using (A.3),

$$\begin{aligned}
E &\stackrel{\text{def}}{=} 2\mathbb{E}\big(\hat{G}^1\big(e^{-((lV'(x_1))/\sqrt{n})\hat{G}^1+\sum_{i=1}^n (V(x_i)-V(x_i+(l/\sqrt{n})\tilde{G}^i))} \wedge 1\big)\big)\\
&= -\frac{V'(x_1)}{l\sqrt{n}}\mathbb{E}\left[\mathcal{G}\left(\frac{(V'(x_1))^2}{2n}, \frac{2}{l^2}\sum_{i=1}^n \left(V\left(x_i + \frac{l}{\sqrt{n}}\tilde{G}^i\right) - V(x_i)\right)\right)\right].
\end{aligned}$$

By boundedness of $\mathcal{G}$ and since

$$\begin{aligned}
&\mathbb{E}\big[G^1\big\{\big(e^{\sum_{i=1}^n (V(x_i)-V(x_i+(l/\sqrt{n})G^i))} \wedge 1\big) + \big(1 - e^{\sum_{i=1}^n (V(x_i)-V(x_i+(l/\sqrt{n})G^i))}\big)^+\big\}\big]\\
&\quad = \mathbb{E}[G_1] = 0,
\end{aligned}$$

one deduces (4.5).

Moreover, when $V'(x_1) = 0$, $E = 0$ and (4.6) holds. To deal with the case $V'(x_1) \neq 0$, we let

$$\begin{aligned}
X &\stackrel{\text{def}}{=} \sum_{i=1}^n \left(\frac{lV'(x_i)}{\sqrt{n}}\tilde{G}^i + \frac{l^2V''(x_i)}{2n}\right)\\
&\sim \mathcal{N}\left(\frac{l^2}{2}\langle \nu_n, V''\rangle, l^2\langle \nu_n, (V')^2\rangle - l^2\big(V'(x_1)\big)^2/2\right),\\
Y &\stackrel{\text{def}}{=} \sum_{i=1}^n \left(V\left(x_i + \frac{l}{\sqrt{n}}\tilde{G}^i\right) - V(x_i)\right)\\
&= X + \frac{l^2}{2n}\sum_{i=1}^n V''(x_i)\big((\tilde{G}^i)^2 - 1\big) + \frac{l^3}{6n^{3/2}}\sum_{i=1}^n V^{(3)}(\chi_i)\big(\tilde{G}^i\big)^3
\end{aligned}$$

with $\chi_i \in [x_i, x_i + \frac{l}{\sqrt{n}}\tilde{G}^i]$. By boundedness of V'' and $V^{(3)}$ and since $\mathbb{E}[((\tilde{G}^i)^2 - 1)((\tilde{G}^j)^2 - 1)] = 0$ as soon as $j \neq i$ and $\mathbb{E}[V^{(3)}(\chi_i)(\tilde{G}^i)^3((\tilde{G}^j)^2 - 1)] = 0$ as soon as $j \notin \{1, i\}$, $\mathbb{E}[(X - Y)^2] \le \frac{C}{n}$ which implies that $W_1(X, Y) \le \frac{C}{\sqrt{n}}$. The density of X is bounded by $(l^2\pi\langle \nu_n, (V')^2\rangle)^{-1/2}$. By Lemma 2, the function $\mathcal{G}$ takes its values in $[0, l^2]$. Moreover,

$$\partial_b \mathcal{G}(a, b) = -\frac{l^2}{2}\mathcal{G}(a, b) + \frac{l^3}{2\sqrt{2\pi a}} e^{-(l^2 b^2)/(8a)}$$

which ensures that $\sup_{(a,b):|b|\ge a^{1/4}} |\partial_b \mathcal{G}(a, b)| < +\infty$. Lemma 7 applied with $\varepsilon = \frac{\sqrt{|V'(x_1)|}}{(2n)^{1/4}}$ implies that

$$\begin{aligned}
&\left|\mathbb{E}\left[\mathcal{G}\left(\frac{(V'(x_1))^2}{2n}, \frac{2}{l^2}\sum_{i=1}^{n}\left(V\left(x_i + \frac{l}{\sqrt{n}}\tilde{G}^i\right) - V(x_i)\right)\right)\right]\right. \\
&\qquad\qquad\qquad\qquad \left. - \mathbb{E}\left[\mathcal{G}\left(\frac{(V'(x_1))^2}{2n}, \frac{2X}{l^2}\right)\right]\right| \\
&\quad \le \frac{C}{\sqrt{n}} + \frac{C}{(n\langle \nu_n, (V')^2\rangle)^{1/4}} + \frac{C\sqrt{|V'(x_1)|}}{n^{1/4}\sqrt{\langle \nu_n, (V')^2\rangle}},
\end{aligned}$$

where C depends neither on x nor on n. One concludes by remarking that, by (A.7),

$$\mathbb{E}\left[\mathcal{G}\left(\frac{(V'(x_1))^2}{2n}, \frac{2X}{l^2}\right)\right] = \mathcal{G}(\langle \nu_n, (V')^2\rangle, \langle \nu_n, V''\rangle). \qquad \square$$

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