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OPTIMAL SELECTION
FROM A FINITE SEQUENCE
WITH SAMPLING COST

by

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ABSTRACT

Two variations of the problem of choosing the largest of N independent and identically distributed random variables with sampling cost are studied. In the first case it is assumed that the underlying distribution is continuous and known, but the information obtained by sampling is whether the sampled variable is larger or smaller than some given level. In the second case it is assumed that the distribution of the random variables is continuous but unknown, and the information obtained is the rank of the sampled variable relative to the other variables already in the sample. In each case both the optimal strategy and the distribution of the stopping variable are discussed.

KEY WORDS AND PHRASES

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1. INTRODUCTION

Let $X_1, \ldots, X_N$ be independent and identically distributed continuous random variables which are to be sampled sequentially, where N is a known fixed positive integer. The aim is to stop and choose the largest one. Exactly one random variable is to be selected and if, after any draw, a random variable is rejected, it cannot be recalled at a later stage. A large number of variations are possible in framing this, the so-called "Secretary Problem", some of which can be found in the references listed at the end of this article. Our aim in this paper is to study the following two variations of the above problem with a decision-theoretic approach.

PROBLEM I. The random variables are not observed directly. Rather, for each X_i we observe whether $X_i \leq L_i$ or $X_i > L_i$, where L_i is a level set by the experimenter, $1 \leq i \leq N$, and we stop experimentation the first time we find an $X_j > L_j$ (and we then select X_j). With certain gain (negative loss) and cost functions defined later (Sections 2 and 3 below), the aim is to find the optimal values of $L_1, \ldots, L_N$, that is, the levels that maximize the expected gain. It will be assumed that the distribution of X_i is known and continuous.

Problem I is discussed in Section 2, where the form of the optimal strategy, the distribution of the stopping variable, and the optimum levels are defined. Optimal levels are numerically calculated for several different costs per observation and gain structures, for $N = 2(1)10$. Enns [3] studied this problem when the sampling cost is zero. Leonardz [6] studied it when one observes the random variables directly. When one wishes to choose the best of N items from available stock (e.g. for use in a military or space mission), testing may well have associated cost (e.g. \$ c. per test). In some applications

X_i may be a life-length such that the gain due to functioning for X_i time units is $aX_i + b$ (e.g., communications satellites or other equipment). Then Table II below would be used in practice.

Note that in this problem the $L_1,\ldots,L_N$ are levels fixed in advance, and not set sequentially. However since (e.g.) we select X_1 if $X_1 > L_1$ (and hence do not then need to use L_2), thus needing L_2 iff $X_1 \leq L_1$, the situation when L_2 will be needed is fully clear in advance of experimentation and it is also clear (since X_1 is not observed directly, but only whether it exceeds L_1 or not) that no gain can be realized by setting levels sequentially.

PROBLEM II. The random variables are observed directly but it is assumed that the distribution function is completely unknown. Also as each random variable is observed, only its rank relative to its predecessors is noted, or is able to be noted.

Problem II is discussed in Section 3, where the form of the optimal strategy and the distribution of the stopping variable are given. Optimal values are tabulated for several different costs per observation for values of $N = 3(1)50$, for two gain functions: gain $b > 0$ if the maximum of $X_1,\ldots,X_N$ is selected (0 gain otherwise); and, gain $b\,\pi(X_j)+a$ if X_j is selected, where $\pi(X_j)$ is the rank of X_j among $X_1,\ldots,X_N$. Gilbert and Mosteller [5] studied this problem, when the sampling cost is zero, for our first gain function. When one wishes to choose the best of several candidates for a position (e.g. a faculty or managerial position), interview cost is often measured in thousands of dollars. Table III allows one to rationally choose the number of interviewees in such settings. Similarly for a seller evaluating multivariate bids on a depreciating or appreciating asset.

2. CASE OF KNOWN DISTRIBUTION: RANDOM VARIABLES NOT OBSERVED DIRECTLY

The Optimal Strategy

When the distribution of X_i is known and continuous, it suffices to consider the sample as coming from a uniform distribution on the [0,1] interval (X_i is $U[0,1]$) because, if $F(x)$ is the distribution function of X_i, then $Y_i = F(X_i)$ is distributed uniformly on [0,1]. So if L_i is the level used for Y_i, then $F^{-1}(L_i)$, the L_i-th quantile of the distribution of X_i, is an equivalent level for X_i. Therefore, suppose that X_i are independent and identically distributed as $U[0,1]$, $i = 1,\ldots,N$.

Let us call a particular sequence of levels $\underset{\sim}{L} = (L_1, \ldots, L_N)$, used for making the selection, a <u>strategy</u>. Not all strategies are equally good. A strategy will be called <u>optimal</u> if it maximizes the expected gain (taking into account sampling cost and terminal decision gain) of the statistical decision problem.

Recall that a sequential decision problem consists of five elements: Θ, the space of the unknown parameter; $\mathcal{A}$, the space of terminal actions available to the statistician; L, the real-valued loss function on $\Theta \times \mathcal{A}$; $\underset{\sim}{X} = (X_1, X_2, \ldots)$, the random variables available to the statistician for observation; and $\{c_j(\theta, x_1, \ldots, x_j),\ j = 1, 2, \ldots\}$, the cost function, a sequence of real-valued functions with c_j defined on $\Theta \times \mathfrak{X}_1 \times \ldots \times \mathfrak{X}_j$, where $\mathfrak{X}_i$ is the sample space of X_i, $i = 1, \ldots, j$, and $c_j(\theta, x_1, \ldots, x_j)$ represents the cost of taking observations $X_1 = x_1, \ldots, X_j = x_j$ and then stopping, when θ is the true value of the parameter.

Here $\theta = \max(X_1, \ldots, X_N)$ and $\theta \in [0,1] = \Theta$. Also, since we are interested in selecting one of the random varaibles, let $\mathcal{A} = \{X_1, \ldots, X_N\}$. Let the cost per observation be c and let the loss function be $L(\theta, a) = -g_\theta(a)$, where $g_\theta(a)$ (henceforth denoted $g(a)$ for simplicity of notation), the gain function, is a non-decreasing function of a for each θ. Let the decision rule be

$$d_N(\underset{\sim}{L}, S) = \{d_j(X_1, \ldots, X_j), S(j), j = 1, \ldots, N\}$$

where

$$S(j) = \begin{cases} j, & \text{if } X_j > L_j \text{ and } X_i \leq L_i \ (i=1, \ldots, j-1) \\ 0, & \text{otherwise,} \end{cases}$$

and

$$d_j(X_1, \ldots, X_j) = X_j \ , \quad \text{when } S(j) = j.$$

Thus the expected gain conditional on stopping after the j-th draw is

$$E(g(X_j)|S=j)-cj.$$

Therefore, the expected gain in employing levels $\underset{\sim}{L}$ is

$$G_N(d|\underset{\sim}{L}) = \sum_{j=1}^{N} \{E(g(X_j)|S=j)-cj\}Pr(S=j) = \sum_{j=1}^{N} E(g(X_j)|S=j)Pr(S=j)-cE(S). \tag{2.1}$$

We now show that the optimal strategy must consist of a non-increasing sequence of levels.

PROPOSITION 2.1. For the sequential decision problem outlined above, the optimal strategy consists of a non-increasing sequence of levels, $L_1 \geq L_2 \geq \ldots \geq L_{N-1} \geq L_N$.

PROOF: Let $a_1, \ldots, a_N$ be any levels for Problem I,

$$0 \leq a_i \leq 1, \qquad i = 1, \ldots, N.$$

Since one of the random variables has to be accepted, one of the a_i's is zero. Let M_v denote the event that the random variable chosen is $\geq v$, and let S be the number of random variables sampled. Then

$$Pr(M_v, S = s|a_1,\ldots,a_s) = Pr(X_i \leq a_i, i=1, \ldots, s-1; X_s > a_s; X_s \geq v)$$

$$= \int_{\max(a_s,v)}^{1} \left[\prod_{i=1}^{s-1} \int_0^{a_i} dx_i\right] dx_s$$

$$\leq \int_{\max(a_{[s]},v)}^{1} \left[\prod_{i=1}^{s-1} \int_0^{a_{[i]}} dx_i\right] dx_s$$

$$= Pr(M_v, S = s|a_{[1]} \geq a_{[2]} \geq \cdots \geq a_{[s]}) \tag{2.2}$$

where $a_{[s]} \leq \cdots \leq a_{[1]}$ denote the ordered a_i's. Since (2.2) is true for all v and s, it follows that the risk (2.1) will be minimized when the stragey consists of non-increasing levels.

Thus, we may without loss of optimality consider only the strategies which form a monotone sequence. We compare X_i with L_i, $i = 1,\ldots,N$. If $X_i > L_i$, we stop sampling and accept X_i; if $X_i \leq L_i$, we sample X_{i+1} and compare it with L_{i+1}. Since one random variable has to be accepted, it must necessarily be true that $L_N = 0$. Let $L_0 = 1$. Then the optimal strategy forms a monotone sequence,

$$0 = L_N \leq L_{N-1} \leq \cdots \leq L_1 \leq L_0 = 1 .$$

The Stopping Variable

Let S denote the stopping variable, that is, the number of random variables sampled before one is accepted. Then $S \in \{1,2,\ldots,N\}$, and

$$\begin{aligned}\Pr(S=j) &= \Pr(X_1 \leq L_1,\ldots,X_{j-1} \leq L_{j-1}, X_j > L_j) \\ &= \left[\prod_{k=0}^{j-1} \int_0^{L_k} dx_k\right] \int_{L_j}^{1} dx_j = (1-L_j) \prod_{k=0}^{j-1} L_k, \end{aligned} \tag{2.3}$$

for $j = 1,\ldots,N$, and

$$E(S) = \sum_{j=1}^{N} j\Pr(S=j) = \sum_{j=1}^{N} j(1-L_j) \prod_{k=0}^{j-1} L_k = \sum_{j=1}^{N} \prod_{k=0}^{N-j} L_k . \tag{2.4}$$

The Optimal Levels

We consider two different gain functions g.

(i) Suppose that, for some constant $b > 0$,

$$g(X_j) = \begin{cases} b, & \text{if } X_j \text{ is maximum} \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$G_N(d|\underset{\sim}{L}) = b \sum_{j=1}^{N} \Pr(X_j \text{ is maximum and } S=j) - cE(S). \tag{2.5}$$

Now, from Enns [3] we have, denoting by $P_N(\underset{\sim}{L})$ the probability that the maximum is actually attained using levels $\underset{\sim}{L} = (L_1,\ldots,L_N)$,

$$P_N(\underset{\sim}{L}) = \sum_{j=1}^{N} \Pr(X_j \text{ is maximum and } S=j)$$

$$= \sum_{j=1}^{N} \frac{1}{j} \prod_{r=0}^{N-j} L_r - \frac{1}{N-1} \sum_{j=1}^{N} L_j^N - \sum_{j=1}^{N-2} \frac{1}{j(j+1)} \sum_{r=1}^{j} L_r^{j+1} \prod_{k=r+1}^{N-j+r-1} L_k \quad (N \geqslant 3), (2.6)$$

$$P_2(\underset{\sim}{L}) = \frac{1}{2} + L_1 - L_1^2 ,$$

$$P_1(\underset{\sim}{L}) = 1 .$$

Therefore

$$G_N(d|\underset{\sim}{L}) = \sum_{j=1}^{N} (\frac{b}{j} - c) \prod_{k=0}^{N-j} L_k - (\frac{b}{N-1}) \sum_{j=1}^{N} L_j^N$$

$$- b \sum_{j=1}^{N-2} \frac{1}{j(j+1)} \sum_{r=1}^{j} L_r^{j+1} \prod_{k=r+1}^{N-j+r-1} L_k \quad (N \geqslant 3), \qquad (2.7)$$

$$G_2(d|\underset{\sim}{L}) = b(\frac{1}{2} + L_1 - L_1^2) - c(L_1+1),$$

$$G_1(d|\underset{\sim}{L}) = b - c.$$

(ii) Next, suppose that, for some constant $b > 0$,

$$g(X_j) = \begin{cases} 0 , & \text{if } S \neq j \\ bX_j + a , & \text{if } S = j , \ j = 1,\dots,N. \end{cases}$$

Since the gain is now linear in the obeservations, it is more appropriate to consider the linear gain in the original observations, rather than in the transformed observations, because if the gain of accepting Y_j is taken as $bY_j + a$, then the gain of accepting $X_j = F(Y_j)$ is $bF^{-1}(X_j) + a$, which is linear in X_j if and only if Y_j has a uniform distribution. Let $Y_1,\dots,Y_N$ be the original independent and identically distributed random variables with distribution function $F(\cdot)$. Let the corresponding set of levels be

$$Q_N \leqslant Q_{N-1} \leqslant \dots \leqslant Q_1 \leqslant Q_0 ,$$

where Q_0 is the smallest x such that $F(x) = 1$ for every $x \geqslant Q_0$ and Q_N is the largest x such that $F(x) = 0$ for every $x < Q_N$. The gain function is (with $b > 0$)

$$g(Y_j) = \begin{cases} 0 , & \text{if } S \neq j \\ bY_j + a , & \text{if } S = j , \quad j = 1,\dots,N \end{cases}$$

$$= \begin{cases} 0 , & \text{if } Y_j \leqslant Q_j \\ bY_j + a , & \text{if } Y_j > Q_j , \quad j = 1,\dots,N. \end{cases}$$

PROPOSITION 2.2. If Y_j , $j = 1,\dots,N$, are independent and identically distributed as $F(\cdot)$, and if S is the stopping variable for the strategy consisting of levels $Q_N \leqslant Q_{N-1} \leqslant \dots \leqslant Q_1 \leqslant Q_0$, then

$$\Pr(S = j) = (1 - F(Q_j)) \prod_{k=0}^{j-1} F(Q_k), \quad j = 1,\dots,N \tag{2.8}$$

and

$$E(S) = \sum_{j=1}^{N} \prod_{k=0}^{N-j} F(Q_k). \tag{2.9}$$

This proposition's proof is trivial. Now to find the corresponding expected gain, note that the conditional distribution of Y_j given $Y_j > Q_j$ is

$$F^*(y) = \Pr(Y_j \leqslant y | Y_j > Q_j)$$

$$= \begin{cases} 0 , & \text{if } y \leqslant Q_j \\ \dfrac{F(y) - F(Q_j)}{1 - F(Q_j)} , & \text{if } y > Q_j . \end{cases}$$

Therefore,

$$E(Y_j | Y_j > Q_j) = \int_{Q_j}^{Q_0} y dF^*(y) = \frac{1}{1 - F(Q_j)} \int_{Q_j}^{Q_0} y dF(y).$$

Let $L_j = F(Q_j)$. Then the expected gain in employing levels

$$\underset{\sim}{Q} = (Q_1 = F^{-1}(L_1),\dots,Q_N = F^{-1}(L_N))$$

is

$$G_N(d|\underset{\sim}{L}) = \sum_{j=1}^{N} E(bY_j + a | Y_j > Q_j) \Pr(S = j) - c\, E(S)$$

$$= a + b \sum_{j=1}^{N} E(Y_j | Y_j > Q_j) \Pr(S = j) - c\, E(S)$$

$$= a + \sum_{j=1}^{N} \left[b\left(\int_{Q_j}^{Q_0} y_j dF(y_j) \right) \prod_{k=0}^{j-1} L_k - c \prod_{k=0}^{N-j} L_k \right]. \tag{2.10}$$

Special Cases

(1) (Uniform) Let Y_j be independent and identically distributed as $U[0,1]$, $j = 1,\dots,N$. Then $Q_N = 0$, $Q_0 = 1$, and $Q_j = L_j$. Also

$$\int_{Q_j}^{Q_0} y_j dF(y_j) = \int_{L_j}^{1} y_j dy_j = \frac{1}{2} - L_j^2/2 .$$

Therefore

$$G_N^U(d|\underset{\sim}{L}) = a + \sum_{j=1}^{N}\left[(b(1-L_j^2)/2)\prod_{k=0}^{j-1} L_k - c\prod_{k=0}^{N-j} L_k\right], \tag{2.11}$$

where $G_N^U(d|\underset{\sim}{L})$ denotes the expected gain when the underlying distribution is U[0,1].

(2) (Exponential) Let Y_j be independent and identically distributed as exponential ($\lambda = 1$). Then $Q_N = 0$, $Q_0 = \infty$, and $Q_j = -\log(1-L_j)$. Also

$$\int_{Q_j}^{Q_0} y_j dF(y_j) = [1 - \log(1-L_j)](1-L_j).$$

Therefore

$$G_N^{Ex}(d|\underset{\sim}{L}) = a + \sum_{j=1}^{N}\left[b(1-L_j)[1 - \log(1-L_j)]\prod_{k=0}^{j-1} L_k - c\prod_{k=0}^{N-j} L_k\right], \tag{2.12}$$

where $G_N^{Ex}(d|\underset{\sim}{L})$ denotes the expected gain when the underlying distribution is exponential.

(3) (Normal) Let Y_j be independent and identically distributed as $N(0,1)$. Then $Q_N = -\infty$, $Q_0 = \infty$, and $Q_j = \Phi^{-1}(L_j)$. Also

$$\int_{Q_j}^{Q_0} y_j dF(y_j) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\Phi^{-1}(L_j))^2} .$$

Therefore

$$G_N^N(d|\underset{\sim}{L}) = a + \sum_{j=1}^{N}\left[\frac{b}{\sqrt{2\pi}} e^{-\frac{1}{2}(\Phi^{-1}(L_j))^2}\prod_{k=0}^{j-1} L_k - c\prod_{k=0}^{N-j} L_k\right], \tag{2.13}$$

where $G_N^N(d|\underset{\sim}{L})$ denotes the risk when the underlying distribution is normal.

NOTE: We do not lose generality by assuming $\lambda = 1$ in case (2) or $(\mu = 0, \sigma^2 = 1)$ in case (3), since the gain function is linear in observations and a location and scale transformation does not change the linearity. The corresponding levels when $\lambda \neq 1$ or $\mu \neq 0$ or $\sigma^2 \neq 1$ can be obtained by suitable location and scale transformations.

Numerical Results

Table I below gives the optimum levels, $\underset{\sim}{L}^*$, and the corresponding maximum expected gains $G_N(d|\underset{\sim}{L}^*)$ for N = 2(1)10 in the case of (2.7). Tables II give the above quantities in case of (2.11), (2.12), and (2.13). [Note that all of the tables in this paper were obtained using the sequential simplex program for solving minimization problems which was developed by Olsson [7].] These tables show that for a given N and b (respectively, c) as c decreases (respectively, as b increases) the optimal levels $\underset{\sim}{L}^*$ increase componentwise. Therefore, if the gain is not much as compared to the cost, we stop and make the selection earlier.

TABLE I

Table showing the optimum levels $\underset{\sim}{L}^{**}$ and the corresponding maximum expected gain $G_N(d|\underset{\sim}{L}^*)$ for the gain function

$$g(X_j) = \begin{cases} b \text{ , if } X_j \text{ is maximum} \\ 0 \text{ , otherwise.} \end{cases}$$

b/c \ N	2	3	4	5	6	7	8	9	10
10.0	0.45000	0.60312 0.49053	0.67643 0.62017 0.52769	0.71792 0.64690 0.63728 0.62213	0.72314 0.72080 0.70012 0.68703 0.60219	0.76579 0.73113 0.72521 0.70221 0.67019 0.64642	0.77219 0.76444 0.76444 0.72743 0.68868 0.64061 0.52137	0.77882 0.76523 0.75891 0.75767 0.75554 0.68978 0.62930 0.59461	0.78586 0.78242 0.78242 0.72236 0.70993 0.70515 0.60501 0.51869 0.51249
$G_N(d\|\underset{\sim}{L}^{**})/c$	0.06025	0.04829	0.04021	0.03330	0.02783	0.02275	0.01786	0.01357	0.00863
100.0	0.49500	0.66586 0.54007	0.74985 0.68518 0.58167	0.80498 0.76346 0.66965 0.66965	0.80563 0.80344 0.79505 0.71604 0.70819	0.84756 0.84047 0.81064 0.77779 0.74998 0.69585	0.86880 0.86022 0.85994 0.79982 0.76796 0.74204 0.67593	0.87574 0.84946 0.84946 0.80733 0.77070 0.76226 0.68256 0.61307	0.88262 0.85555 0.85555 0.80187 0.76768 0.75554 0.67307 0.62563 0.55051
$G_N(d\|\underset{\sim}{L}^{**})/c$	0.73502	0.65952	0.62168	0.59420	0.57532	0.56575	0.55253	0.53281	0.49968
1000.0	0.49900	0.67193 0.54499	0.75671 0.69143 0.58697	0.79499 0.77831 0.70727 0.63063	0.83702 0.79560 0.78864 0.73228 0.68347	0.85166 0.85166 0.82509 0.77000 0.76024 0.69308	0.88056 0.86603 0.84864 0.82721 0.79955 0.76102 0.69780	0.88173 0.85570 0.85570 0.82322 0.79453 0.77461 0.68009 0.57061	0.89066 0.86033 0.85326 0.85326 0.78290 0.71930 0.66908 0.62724 0.51768
$G_N(d\|\underset{\sim}{L}^{**})/c$	7.88500	6.77807	6.44851	6.25160	6.11837	6.02855	5.97558	5.78685	5.44586

TABLE II

Table showing the optimum levels $\underset{\sim}{L}^*$ and the corresponding maximum expected gain $G_N(d|\underset{\sim}{L}^*)$ for the gain function

$$g(X_j) = \begin{cases} bX_j + a, & \text{if select } X_j \\ 0, & \text{otherwise.} \end{cases}$$

(a) UNIFORM DISTRIBUTION

b/c \ N	2	3	4	5	6	7	8	9	10
10.0	0.40000	0.48000 0.40000	0.51304 0.48390 0.39997	0.57110 0.47436 0.45051 0.45051	0.57171 0.49965 0.48796 0.47684 0.47607	0.57754 0.52758 0.50223 0.44602 0.40920 0.37782	0.63685 0.56511 0.52431 0.43842 0.43807 0.43807 0.38241	0.68067 0.56637 0.49128 0.45185 0.41946 0.34187 0.34169 0.33742	0.69992 0.53936 0.49121 0.44231 0.44231 0.43708 0.33606 0.28888 0.25114
$(G_N(d\|\underset{\sim}{L}^*)-a)/c$	0.04800	0.05152	0.05327	0.05404	0.05460	0.05483	0.05459	0.05491	0.05386
100.0	0.49000	0.61005 0.49000	0.67608 0.61005 0.49000	0.66910 0.64405 0.63940 0.63261	0.74368 0.74334 0.63546 0.62395 0.49598	0.75328 0.74235 0.71201 0.65936 0.64064 0.58473	0.76881 0.76461 0.74692 0.73697 0.64615 0.62646 0.58274	0.78753 0.78753 0.77364 0.72564 0.71920 0.71155 0.60470 0.58452	0.80762 0.80656 0.80284 0.76670 0.73794 0.71248 0.64733 0.56169 0.51806
$(G_N(d\|\underset{\sim}{L}^*)-a)/c$	0.61000	0.67608	0.71854	0.74360	0.76913	0.78525	0.79819	0.80850	0.81709
1000.0	0.49900	0.62350 0.49900	0.69338 0.62350 0.49900	0.72682 0.70840 0.61806 0.50525	0.74683 0.70337 0.69905 0.69149 0.68953	0.77511 0.76034 0.75050 0.64657 0.64560 0.61829	0.80826 0.77295 0.77257 0.72087 0.71757 0.63796 0.58825	0.85334 0.83200 0.79643 0.75974 0.75974 0.68147 0.67093 0.55926	0.85792 0.83698 0.82324 0.80308 0.77907 0.75584 0.70871 0.63003 0.52485
$(G_N(d\|\underset{\sim}{L}^*)-a)/c$	6.48500	6.93376	7.39385	7.72170	7.90007	8.14605	8.31469	8.44611	8.56349

(b) EXPONENTIAL DISTRIBUTION

b/c \ N	2	3	4	5	6	7	8	9	10
10.0	0.59343	0.70078	0.75159	0.76607	0.79372	0.80884	0.83827	0.84847	0.85658
		0.59343	0.66373	0.74607	0.78078	0.80365	0.83016	0.83893	0.83842
			0.65760	0.74583	0.77995	0.78342	0.82969	0.82667	0.83634
				0.67980	0.67989	0.78210	0.78364	0.81043	0.81748
					0.66979	0.73444	0.72587	0.78796	0.81261
						0.60462	0.64894	0.75483	0.78933
							0.62500	0.70078	0.71931
								0.59343	0.64552
									0.55287
$(G_N(d\|\underset{\sim}{L}^*)-a)/c$	0.05233	0.14058	0.15466	0.16529	0.17481	0.18232	0.18838	0.19385	0.19798
100.0	0.62842	0.74117	0.75111	0.80084	0.85332	0.87523	0.88875	0.89124	0.89334
		0.62842	0.73524	0.80064	0.80893	0.85756	0.87523	0.87331	0.86945
			0.72962	0.73693	0.80415	0.83341	0.85756	0.85826	0.84754
				0.73226	0.76926	0.79818	0.83341	0.85826	0.84752
					0.66519	0.74117	0.79818	0.79205	0.77617
						0.62842	0.74117	0.74739	0.75313
							0.62842	0.68211	0.70472
								0.58221	0.62105
									0.50299
$(G_N(d\|\underset{\sim}{L}^*)-a)/c$	0.66726	1.60041	1.77685	1.93818	2.07798	2.19603	2.29727	2.37640	2.42484
1000.0	0.63175	0.74494	0.77955	0.84353	0.85990	0.87947	0.89305	0.89489	0.89908
		0.63175	0.72135	0.79790	0.81121	0.86174	0.87947	0.87445	0.86948
			0.69876	0.71010	0.80769	0.83751	0.86174	0.87445	0.86948
				0.68904	0.75706	0.80216	0.83751	0.83224	0.83147
					0.71253	0.74494	0.80216	0.80195	0.76874
						0.63175	0.74494	0.73415	0.73767
							0.63175	0.66768	0.66367
								0.59072	0.59568
									0.54427
$(G_N(d\|\underset{\sim}{L}^*)-a)/c$	6.81658	16.20310	18.11419	19.74702	21.09834	22.35419	23.41366	24.18497	24.61120

(c) NORMAL DISTRIBUTION

b/c \ N	2	3	4	5	6	7	8	9	10
10.0	0.46017	0.59907 0.46017	0.66891 0.59907 0.46017	0.67516 0.65345 0.63549 0.55603	0.70503 0.69672 0.68684 0.59439 0.59164	0.73101 0.72150 0.72129 0.68782 0.59004 0.58886	0.76831 0.72946 0.72682 0.72605 0.66352 0.65914 0.54226	0.77543 0.77119 0.74456 0.73463 0.70389 0.68440 0.58834 0.56760	0.81043 0.77808 0.75107 0.75107 0.73430 0.66538 0.65004 0.56833 0.52834
$(G_N(d\|\underset{\sim}{L}^*)-a)/c$	0.02509	0.04017	0.05549	0.06302	0.06888	0.07344	0.07697	0.07973	0.08167
100.0	0.49601	0.64950 0.49601	0.72906 0.64950 0.49601	0.73872 0.70588 0.70009 0.65873	0.79105 0.73994 0.73917 0.65350 0.63464	0.80181 0.78099 0.77884 0.77440 0.65733 0.62209	0.82406 0.82284 0.80025 0.76907 0.76846 0.68949 0.62389	0.86846 0.85407 0.83572 0.81152 0.77814 0.72906 0.64950 0.49601	0.88060 0.86524 0.83634 0.81456 0.78246 0.78246 0.69721 0.62765 0.51229
$(G_N(d\|\underset{\sim}{L}^*)-a)/c$	0.38396	0.60997	0.76593	0.86173	0.96741	1.04390	1.11047	1.17520	1.22166
1000.0	0.49960	0.65448 0.49960	0.73492 0.65448 0.49960	0.74056 0.70499 0.69842 0.66002	0.77833 0.75738 0.75357 0.70884 0.57856	0.80647 0.80647 0.80047 0.73943 0.66811 0.62850	0.84552 0.82885 0.81632 0.76171 0.76171 0.65931 0.65835	0.87651 0.86182 0.84312 0.81852 0.78464 0.73492 0.65448 0.49960	0.88406 0.87484 0.84460 0.83914 0.81389 0 77324 0.69796 0.60004 0.47631
$(G_N(d\|\underset{\sim}{L}^*)-a)/c$	3.9743	6.27764	7.87950	8.89138	9.98878	10.79848	11.48759	12.17867	12.68490

3. CASE OF UNKNOWN DISTRIBUTION: RANDOM VARIABLES OBSERVED DIRECTLY

Now we consider the case when the distribution function is continuous but unknown. The random variables are observed directly. As each random variable is observed, only its rank relative to its predecessors is noted or able to be noted.

The Optimal Strategy

For choosing the maximum of a sequence of N random variables in this case the derivation of the form of the optimal strategy and terminology are well-known from [5]. Call X_i, the random variable drawn at the i-th draw, a "candidate" if $X_j < X_i$, $j = 1,\ldots,i-1$. The optimal strategy is to pass, say, r-1 random variables and then choose the first candidate. Thus we want to find the optimal value of r. (It is known that this strategy, optimal for gain functions as in (i) below, is not optimal for gain functions such as that in (ii) below. However the optimal r for this strategy is of interest in (ii), as is the effect of sampling cost, and these are studied below.)

The Stopping Variable

Let S denote the draw at which we stop after passing $r-1$ random variables. Then $S \in \{1,2,\ldots,N-r+1\}$. For $s = 1,\ldots,N-r$, we have

$$\Pr(S=s) = \frac{1}{s+r-1} \cdot \frac{r+1}{s+r-2} .$$

The Optimal Value of r

Suppose that the cost per observation is c and that the gain is $g(X_j)$ if we accept X_j. Then the expected gain in employing the optimal stategy conditional on stopping at $S=s$ is $E(g(X_{s+r-1})|S=s)-(r-1)c-sc$, and therefore the expected gain in using the above strategy is $E(g(X_{s+r-1})|S=s)-(r-1)c-sc$, and therefore the expected gain in using the above strategy is

$$G_N(r) = \sum_{s=1}^{N-r+1} E(g(X_{s+r-1})|S=s]\Pr(S=s)-c(r-1)-cE(S). \quad (3.2)$$

We now consider two different gain functions g.

(i) Suppose that, for some constant $b > 0$,

$$g(X_{s+r-1}) = \begin{cases} b, & \text{if } X_{s+r-1} \text{ is maximum} \\ 0, & \text{otherwise.} \end{cases}$$

Here it is well-known that

$$E(g(X_{x+r-1})|S=s) = b \cdot \frac{r-1}{N(s+r-2)} , \qquad (3.3)$$

hence in this case

$$G_N(r) = (\frac{b}{N}-c) \quad + \quad (r-1)(\frac{b}{N}-c) \quad (\frac{1}{r} + \frac{1}{r+1} + \dots + \frac{1}{N-1})-c(r-1). \qquad (3.4)$$

Therefore, the optimal value of r is the smallest r^* such that

$G_N(r^*) > G_N(r^*-1)$ and $G_N(r^*) > G_N(r^*+1)$:

$$\frac{1}{r^*} + \frac{1}{r^*+1} + \dots + \frac{1}{N-1} < \frac{b}{b-Nc} < \frac{1}{r^*-1} + \frac{1}{r^*} + \dots + \frac{1}{N-1} , \quad b > Nc . \qquad (3.5)$$

(ii) Since the distribution of the random varaibles is unknown, let us consider the gain function

$$g(X_{s+r-1}) = \begin{cases} b\mathcal{R}(X_{s+r-1}) + a , & \text{if } S = s \\ 0 , & \text{if } S \neq s, \quad s = 1,\dots,N-r+1 , \end{cases}$$

where $\mathcal{R}(X_{s+r-1})$ is the rank of X_{s+r-1} among $X_1,\dots,X_N$, and $b > 0$. (For $c=0$. this reduces to the problem of maximizing expected rank, which has been studied by Chow, Moriguti, Robbins, and Samuels [1] and De Groot [2]. More general functions of rank, but with $c = 0$ also, have been studied by Rasmussen [8].)

Here it is well-known that

$$E[\mathcal{R}(X_{s+r-1})|S=s] = \begin{cases} \frac{N(N+1)}{2(N-s-r+2)} - \frac{(s+r-2)(s+r-1)}{2(N-s-r+2)}, & s = 1,\dots,N-r \\ \frac{N+1}{2}, & s = N-r+1, \end{cases}$$

hence in this case

$$G_N(r) = a + (\frac{b}{2}-c) \quad (r-1)\left[1 + \frac{1}{r} + \dots + \frac{1}{N-1}\right] + \frac{b(N+1)}{2} - \frac{b(r-1)}{2} - c. \qquad (3.6)$$

Therefore, the optimal value of r is the smallest r^* such that

$$G_N(r^*) > G_N(r^*-1) \quad \text{and} \quad G_N(r^*) > G_N(r^*+1):$$

$$\frac{1}{r^*} + \frac{1}{r^*+1} + \dots + \frac{1}{N-1} < \frac{b}{b-2c} < \frac{1}{r^*-1} + \frac{1}{r^*} + \dots + \frac{1}{N-1}, \quad b > 2c. \qquad (3.7)$$

It is interesting to compare (3.5) and (3.7). Since $\frac{b}{b-2c} < \frac{b}{b-Nc}$, $N \geq 3$, (3.7) yields a smaller value of r^*. This is as one could expect on comparing the two gain functions.

Numerical Results

Table III below gives the optimum values r^* and the corresponding maximum expected gains $G_N(r^*)$ for $N = 3(1)50$ in case of (3.4) and (3.6). The table shows that if the gain is much more than the cost of sampling one should observe a larger number of random variables before making a final selection.

TABLE III

Table showing optimum r^* and the corresponding **maximum expected gain.**

	$g(X_{s+r-1}) = \begin{cases} b, & \text{if } X_{s+r-1} \text{ is max.} \\ 0, & \text{otherwise} \end{cases}$						$g(X_{s+r-1}) = \begin{cases} bR(X_{s+r-1}) + a, & \text{if } S = s \\ 0, & \text{otherwise} \end{cases}$					
N	b/c = 10.0		100.0		1000.0		b/c = 10.0		100.0		1000.0	
	r^*	$G_N(r^*)/c$	r^*	$G_N(r^*)/c$	r^*	$G_N(r^*)/c$	r^*	$(G_N(r^*)-a)/c$	r^*	$(G_N(r^*)-a)/c$	r^*	$(G_N(r^*)-a)/c$
3	2	0.02500	2	0.47500	2	4.97500	2	0.20000	2	2.22500	2	22.47499
4	2	0.01750	2	0.43000	2	4.55500	2	0.26333	2	2.88833	2	29.13832
5	2	0.01083	3	0.39167	3	4.29167	2	0.32333	3	3.54167	3	35.79166
6	3	0.03333	3	0.38211	3	4.23211	3	0.38267	3	4.23767	3	42.78764
7	3	0.04571	3	0.36529	3	4.09358	3	0.44600	3	4.90100	3	49.45097
8	3	0.05750	4	0.34704	4	4.03543	3	0.50743	4	5.57650	4	56.33005
9	4	0.06889	4	0.33942	4	3.99298	3	0.56743	4	6.25025	4	63.20129
10	4	0.08000	4	0.32882	4	3.91703	4	0.62948	4	6.92358	4	69.86462
11			4	0.31685	5	3.90030			5	7.60744	5	76.82883
12			5	0.30805	5	3.86768			5	8.28562	5	83.64337
13			5	0.29994	6	3.82161			5	8.94896	6	90.39590
14			5	0.29093	6	3.81230			6	9.63716	6	97.31514
15			5	0.28146	6	3.78568			6	10.31216	6	104.09727
16			6	0.27416	7	3.75876			7	10.98259	7	110.92470
17			6	0.26672	7	3.74731			7	11.66634	7	117.79596
18			6	0.25892	8	3.72469			7	12.33928	7	124.55713
19			6	0.25093	8	3.70724			8	13.01471	8	131.43558
20			7	0.24346	8	3.69524			8	13.69524	8	138.27399
21			7	0.23668	8	3.67545			8	14.36673	9	145.03487
22			7	0.22974	9	3.66308			9	15.04579	9	151.93581
23			7	0.22272	9	3.65114			9	15.72398	9	158.75037
24			7	0.21568	9	3.63346			9	16.39441	10	165.55797
25			7	0.20867	10	3.62393			10	17.07622	10	172.42924
26			8	0.20214	10	3.61229			10	17.75261	10	179.22563
27			8	0.19580	10	3.59625			11	18.42470	11	186.06995
28			8	0.18946	11	3.58835			11	19.10617	11	192.91808
29			8	0.18316	11	3.57708			11	19.78117	11	199.70027
30			8	0.17690	12	3.56291			12	20.45616	12	206.57397
31			8	0.17072	12	3.55538			12	21.13583	12	213.40356
32			8	0.16461	12	3.54451			12	21.80969	13	220 20093
33			8	0.15860	13	3.51215			13	22.48706	13	227.07227
34			8	0.15268	13	3.52439			13	23.16524	13	233.88672
35			8	0.14687	13	3.51391			13	23.83818	14	240.71289
36			8	0.14115	14	3.50278			14	24.51753	14	247.56616
37			8	0.13555	14	3.49492			14	25.19447	14	254.36816
38			8	0.13005	14	3.48480			15	25.86714	15	261.21851
39			8	0.12466	15	3.47453			15	26.54767	15	268.05688
40			8	0.11937	15	3.46665			15	27.22357	15	274.64814
41			8	0.11419	15	3.45688			16	27.89828	16	281.71924
42			8	0.10912	16	3.44719			16	28.57756	16	288.54492
43			8	0.10414	16	3.43935			16	29.25256	17	295.35962
44			8	0.09927	16	3.42990			17	29.92906	17	302.21631
45			8	0.09450	17	3.42060			17	30.60724	17	309.03076
46			8	0.08982	17	3.41283			17	31.28146	18	315.86597
47			8	0.08524	17	3.40367			18	31.95950	18	322.71021
48			8	0.08075	18	3.39464			18	32.63673	18	329.51489
49			8	0.07635	18	3.38696			18	33.31027	19	336.36841
50			8	0.07204	18	3.37807			19	33.98972	19	343.20142

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