Applied Mathematical Sciences, Vol. 1, 2007, no. 22, 1069 - 1082

Optimal Selection of Blocked Two-Level Fractional Factorial Designs

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Abstract

Blocked two-level fractional factorial designs are very useful for efficient data collection in industry experiments and other areas of scientific research. In some experiments, in addition to the main effects, some two-factor interactions may be important and should be estimated. In this article, we propose and study a method to select the best blocked two-level fractional factorial designs when some two-factor interactions are important. We then discuss how to search for the best designs according to this method and present some results for designs of 8 and 16 runs.

Mathematics Subject Classification: 62K05, 62K10, 62K15

Keywords: blocking factor; defining contrast subgroup; defining words; minimum aberration; resolution; word-length pattern; confounding pattern

1. Introduction

Two-level fractional factorial (2^{m-p}) designs allow us to study many factors with relatively small run size. They are very useful for identifying important factors and are widely used in many areas of scientific investigation. The practical and theoretical importance of this class of designs has long been established by Box, Hunter, and Hunter [3]. Since the fraction can be chosen in many different ways, a key concern is how to choose a fraction of the full factorial design for a given run size and number of factors. The most commonly used criterion for 2^{m-p} design selection is the minimum aberration criterion proposed by Fries and Hunter [10]. For small number of factors, Franklin [9] provided a useful catalogue of 2^{m-p} designs with minimum aberration. A more complete catalogue of 2^{m-p} designs which were ordered by the minimum aberration criterion was provided by Chen, Sun, and Wu [5].

In some experiments, it is often desirable to group the experimental units into blocks such as different days or batches of raw material. Hence blocked 2^{m-p} designs are often used in order to make data collection more efficient. Much work has been done on the criteria for optimal blocking schemes. This includes Bisgaard [1], Sun, Wu, and Chen [14], Sitter, Chen, and Feder [13], Chen and Cheng [4], Cheng and Wu [8], and Cheng, Li, and Ye [7]. All of these authors discussed blocking schemes for the models containing main effects and blocking effects. Their discussions were based on the schemes that have optimal estimation of the main effects and the blocking effects of the experiments.

In this article, we consider how to select blocked 2^{m-p} designs when some 2-factor interactions are presumably important. When some 2-factor interactions are important, the postulated model should consist of all main effects, blocking effects and these important 2-factor interactions. If the effects not in the postulated model cannot be completely ignored, they will bias the estimates of the effects in the model. To solve this problem, the key issues are to permit estimation of the main effects, blocking effects, and important 2-factor interactions in the postulated model and to minimize the bias caused by the other effects not included in the model. In this article, we propose and study a method to select the best blocked 2^{m-p} designs when some two-factor interactions are important. We then discuss how to search for the best designs according to this method and present some results for designs of 8 and16 runs.

Section 2 of the paper introduces two-level fractional factorial designs and blocking. Section 3 studies a method for selecting blocked 2^{m-p} designs for the models containing some important 2-factor interactions. Section 4 examines how to search for best designs based on this method and present some results for designs of 8 and 16 runs. Section 5 concludes the paper with an illustrative example.

2. Two-level fractional factorial designs and blocking

2.1 Two-level fractional factorial designs

A regular two-level fractional factorial design is commonly referred to as a 2^{m-p} design. It has m two-level factors with 2^{m-p} runs, and is completely determined by p independent *defining relations*. When $p = 0$, a 2^{m-p} design is reduced to a full factorial a 2^m design. A *defining relation* is given by a word of letters which are labels of factors denoted by 1, 2, …, m. The number of letters in a word is called its *word-length*. The group of *defining words* generated by the p independent *defining words* is called the *defining contrast subgroup* of the design. The length of the shortest word in the *defining contrast subgroup* is called the *resolution* of a design. The vector $W(d) = (A_1(d), A_2(d), \ldots, A_m(d))$ is called the *word-length pattern* of the design d, where $A_i(d)$ is the number of words of length i in the *defining contrast subgroup*. The *resolution* of a design is the smallest r satisfying $A_r \geq 1$. The *resolution* criterion proposed by Box and Hunter [2] select the 2^{m-p} designs that has higher *resolution*. Since two designs having the same *resolution* may have different *word-length pattern* and may not be equally good, Fries and Hunter [10] proposed the *minimum aberration* criterion to further discriminate 2^{m-p} designs. For two designs d_1 and d_2, suppose r is the smallest value such that $A_r(d_1) \neq A_r(d_2)$. We say that d_1 has less aberration than d_2 if $A_r(d_1) < A_r(d_2)$. If no design has less aberration than d_1, then d_1 is said to have *minimum aberration*. The *minimum aberration* criterion which selects 2^{m-p} designs with *minimum aberration* is most commonly used for selecting 2^{m-p} designs.

In what follows, we use an example to illustrate 2^{m-p} designs and minimum aberration. Suppose we wish to perform an experiment with eight runs and several factors at two levels, labeled +1 and −1. Table 1 gives the columns of an 8-run saturated design with its columns arranged in Yates's order, with the generating independent columns 1, 2, and 4, in boldface. The Yates's order of the columns of an 8-run saturated design can be written as ($\boldsymbol{a_1}$, $\boldsymbol{a_2}$, a_1a_2, $\boldsymbol{a_3}$, a_{13}, a_{23}, $a_1a_2a_3$) where $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, and $\boldsymbol{a_3}$ are three independent columns. Hence columns 3, 5, 6, and 7 can be obtained by multiplying columns 1 and 2, 1 and 4, 2 and 4, and 1, 2, and 4, respectively.

Table 1. Columns of an 8-run saturated design in Yates's order

Run	**1**	**2**	3 (3=12)	**4**	5 (5=14)	6 (6=24)	7 (7=124)	Response
1	−1	−1	+1	−1	+1	+1	−1	y_1
2	+1	−1	−1	−1	−1	+1	+1	y_2
3	−1	+1	−1	−1	+1	−1	+1	y_3
4	+1	+1	+1	−1	−1	−1	−1	y_4
5	−1	−1	+1	+1	−1	−1	+1	y_5
6	+1	−1	−1	+1	+1	−1	−1	y_6
7	−1	+1	−1	+1	−1	+1	−1	y_7
8	+1	+1	+1	+1	+1	+1	+1	y_8

If we use columns 1, 2, and 4 of Table 1 to set the levels of three factors A, B, and C, respectively, then y_1 through y_8 represent the responses at the $2^3 = 8$ possible combinations of factor settings. This gives a $2^3 = 8$ run, two-level, three-factor, full factorial design. By using this 2^3 full factorial design, the main effects of A, B, and C, as well as their interactions *AB*, *AC*, *BC*, and *ABC* can be estimated. If we would like to study one more factor *D* using the 8-run design, we have different choices. For design d_1, we assign the levels of factor *D* to the column 7. This gives a 2^{4-1} fraction factorial design. Since 7 = 124 or *I* = 1247 where *I* denotes the column of +1's, the estimate for the main effect *D* could not be separated from the effect of the interaction between *A*, *B*, and *C*. That is *D* = *ABC*, or *I* = *ABCD*. Here *I* = 1247 is the *defining relation* or *defining word* of the 2^{4-1} design. The resolution of the design is 4 and the *word-length pattern* $W(d_1) = (0, 0, 0, 1)$. For design d_2, we assign factor *D* to column 6 =24. The *defining word* is *I* = 246 and the resolution is 3. The *word-length pattern* $W(d_2) = (0, 0, 1, 0)$. Obviously, d_1 is better than d_2 because it has higher resolution and minimum aberration. Both definition of resolution and minimum aberration are based on the hierarchical assumption: (i) lower order interactions are more important than higher order interactions, (ii) effects of the same order are equally important. The advantage of d_1 is obvious based on the principle because the main effects in d_1 are confounded with three-factor interactions and the main effects in d_2 are confounded with two-factor interactions. The 8-run 2^{m-p} designs can be used to study up to seven factors by assigning these factors to all the seven columns. This design is denoted by 2^{7-4} and called saturated design.

2.2 blocking and its application

Blocking is a commonly used technique to control systematic noises in experiments. Such noises might come from day-to-day variation or batch-to-batch variation. Without blocking, the systematic noises can influence the accuracy and efficiency of effect estimation. Blocking can effectively eliminate the systematic variance by grouping the runs of an experiment into blocks. In a blocked design, the variance due to blocks is removed from the variance due to the treatments, thereby effectively reducing the magnitude of the estimated experimental error. Criteria for the choice of blocks are most frequently different settings or environments for the conduct of the experiment. In any case, blocks should be chosen so that the units within blocks are as homogeneous as possible. Blocking can be accomplished through the used of blocking factors in a design. For 2^{m-p} designs, since there are many choices to assign blocking factors to the unused columns of a saturated design, blocking schemes are needed to select the columns to best reduce. In blocked 2^{m-p} designs, interactions between treatment and blocking factors are assumed to be non-existent, which is necessary for the effectiveness of blocking. See Box, Hunter, & Hunter [1978] for principles and assumptions in the construction of blocked designs. For the models including only main effects and blocking effects, many blocking schemes were discussed in the literature. In this paper, we discuss the blocking schemes for the models containing some important two-factor interactions in addition to the main effects and the blocking effects.

3. A method of selecting blocked 2^{m-p} designs

3.1 The criterion for selecting blocked 2^{m-p} designs

For unblocked 2^{m-p} designs, Ke and Tang [11] propose a minimum *N*-aberration criterion to select designs by systematically minimizing the bias. This criterion was further studied and summarized by Cheng and Tang [6]. For further discussion about this issue, the reader is referred to Tang and Deng [17, 18], Tang [15, 16], and Ke, Tang, and Wu [12]. In this paper, we consider blocked 2^{m-p} designs with some 2-factor interactions needing to be estimated. Suppose that we wish to estimate main effects and some important 2-factor interactions using a blocked design. The fitted model should include all main effects, important 2-factor interactions, and blocking effects. When the effects not in the postulated model are not negligible, they will bias the estimates of the effects in the model. The optimal design should be selected such that to minimize the contamination of these non-negligible effects on the model. We propose a design selection criterion as follows:

Design selection criterion: Let N_j, j = 2, 3, …, m be the number of j-factor interactions not in the model confounded with the effects in the postulated model including main effects, blocking effects, and some important two-factor interactions. We select optimal blocked 2^{m-p} designs by sequentially minimizes N_2,…, N_m.

To gain further insight into the criterion, we now examine the criterion in detail. The postulated model consists of all the main effects, important 2-factor interactions, and blocking effects. Those 2-factor interactions not in the model and other higher-order interactions generally cause a bias on the estimation of the effects in the model. The measure of this bias, as given by N_j, is the number of the j-factor interactions outside the model that are confounded with the effects in the model. Under the hierarchical assumption that lower-order effects are more important than higher-order effects (Wu and Hamada [19]), to minimize the bias, we should sequentially minimizing N_2, N_3, …, N_m. The vector (N_2, N_3, …, N_m) is called the confound pattern of a design. Hence this criterion selects optimal design that has minimum N_2. If several designs have the same number of N_2, we select optimal design that has minimum N_3 among the designs that have minimum N_2, and so on.

We now look at an example. Suppose that we want to study four factors *A*, *B*, *C*, and *D*, and two-factor interactions *AB* and *AC* by using a blocked design of 8 runs. We consider two designs D_1 and D_2. Based on Table 1, for D_1, we assign the four treatment factors *A*, *B*, *C*, and *D* to columns 1, 4, 7, and 2 and the blocking factor to column 3_b (we use '*b*' to indicate blocking factor). The interactions to be estimated should be 14 and 17. Since 7 = 124 and 3_b =12, the *defining contrast subgroup* of the design is given by I = 1247 = 123_b = 3_b47. Hence we have 14=27, 17=24, 3_b=12, 3_b=47, 1=247, 2=147, 4=127, and 7=124. Therefore N_2 =4 and N_3 =4. Note that the interactions between blocking factors and treatment factors are assumed not existent and are not counted here. The confounding pattern of D_1 is (4, 4, 0) which means that Four 2-factor interactions and four 3-factor interactions not included in the model are confounded with the effects in the model. For D_2, we assign the four treatment factors A, B, C, and D to columns 4, 2, 3, and 1 and the blocking factor to column 5_b. The interactions to be estimated should be 42 and 43. Since 3 = 12 and 5_b =14, the *defining contrast subgroup* of design D_2 is given by I = 123 = 145_b = 2345_b. Hence we have 1=23, 2=13, 3=12, 5_b=14, 5_b=234, 24=134, 34=124, and 4=1234. Hence the confounding pattern of D_2 is (4, 3, 1) which means that four 2-factor interactions, three 3-factor interactions, and one 4-factor interaction not included in the model are confounded with the effects in the model. Based on our design selection criterion, D_2 is better than D_1 because $N_2(D_1) = N_2(D_2)$ and $N_3(D_1) > N_3(D_2)$ where $N_2(D_1)$, $N_2(D_2)$, $N_3(D_1)$, and $N_3(D_2)$ denote the N_2 and N_3 for D_1 and D_2 respectively.

3.2 Theoretical justification of the criterion

Suppose that we are interested in estimating all main effects, a set of important two-factor interactions by using a blocked 2^{m-p} designs. Then the fitted model is given by

$$Y = \alpha_0 I + W_1\gamma_1 + \varepsilon \qquad (1)$$

where Y denotes the vector of n observations, α_0 is the grand mean, I denotes the vector of n ones, γ_1 is the vector of parameters containing all main effects, a set of important two-factor interactions, and blocking effects, W_1 is the corresponding design matrix, and ε is the vector of uncorrelated random errors, assumed to have mean 0 and a constant variance. Since other interactions among treatment factors may not be negligible, the true model can be written as

$$Y = \alpha_0 I + W_1\gamma_1 + X_2\alpha_2 + X_3\alpha_3 + \cdots + X_m\alpha_m + \varepsilon, \qquad (2)$$

where α_2 is the vector of remaining two-factor interactions and X_2 is the corresponding design matrix, α_j is the vector of j factors interactions and X_j is the corresponding matrix. The least square estimator $\hat{\gamma}_1$ = $(W_1^T W_1)^{-1} W_1^T Y$ = $n^{-1} W_1^T Y$ from the fitted model in (1) has expectation, taken under the true model in (2), of $E(\hat{\gamma}_1)$ = γ_1 + $P\alpha_2$ + $P_3\alpha_3$ + $\cdots$ + $P_m\alpha_m$, where P_2 = $n^{-1}W_1^T X_2$ and P_j = $n^{-1}W_1^T X_j$. So the bias of $\hat{\gamma}_1$ for estimating γ_1 is given by

$$\text{Bias}(\hat{\gamma}_1, \gamma_1) = P_2\alpha_2 + P_3\alpha_3 + \cdots + P_m\alpha_m . \tag{3}$$

Note $P_2\alpha_2$ is the contribution of α_2 to the bias, and $P_j\,\alpha_j$ is the contribution of α_j to the bias. Because α_j is unknown, we have to work with P_j. One size measure for a matrix $P = (p_{ij})$ is given by $||P||^2 \overset{def}{=} \text{trace}(P^{\mathrm{T}}P) = \sum_{i,j} p_{ij}^2$. Under the hierarchical assumption that lower order effects are more important than higher order effects, to minimize the bias of $\hat{\gamma}_1$ we should sequentially minimize $||P_2||^2, \ldots, ||P_m||^2$. For regular 2^{m-p} designs, the entries of P_j are 0 or 1, and thus $||P_j||^2$ is simply the number of j-factor interactions aliased with the effects in the postulated model in (1). Now let $N_j = ||P_j||^2$. Based on the above results, we can select blocked 2^{m-p} designs by sequentially minimizing $N_2, \ldots, N_m$ where N_j is the number of j-factor interactions not in the model aliased with the effects in the postulated model.

4. Searching for the best blocked 2^{m-p} designs

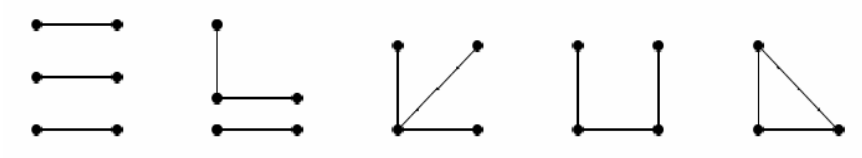
4.1 Search method

In this paper, we consider blocked 2^{m-p} designs of 8 runs and 16 runs. For designs of 8 runs, there are only 7 columns in a saturated design. The choice of the blocked 2^{m-p} designs is limited and the optimal design is easy to select according to our criterion. For designs of 16 run, there are 15 columns in a saturated design. There are a lot of choices for the blocked 2^{m-p} designs and optimal design is not easy to select. We use a computer program to calculate the confounding pattern for each choice of the designs for the given number of treatment factors, important two-factor interactions and blocking factors. Then select the best one that has minimum N_2. If two designs have same N_2, we select the one that has the minimum N_3, and so on. Through our search effort, we have found optimal blocked 2^{m-p} designs of 8 runs for all the existent models and 16 runs for the models containing up to three 2-factor interactions and two blocking factors. Let k be the number of important 2-factor interactions. For $k = 1$, there is only one model, as represented by Figure 1. For $k = 2$, 3, the number of models is 2 and 5, and the graphs for these models are given in Figure 2 and 3 respectively. In our search effort, we have used (N_2, N_3, N_4) instead of the entire vector (N_2, …, N_m) to reduce the computing burden. Actually five-factor and higher order interactions are very small and usually negligible in practice.

Figure 1. Graph for model with one 2-factor interaction.

(a) (b)

Figure 2. Graphs for models with two 2-factor interactions.



(a) (b) (c) (d) (e)

Figure 3. Graphs for models with three 2-factor interactions

4.2 Optimal blocked 2^{m-p} designs of 8 and 16 runs

Table 2 presents the optimal designs of 8 runs for all the existent models. Tables 3, through 8 present the optima designs of 16 runs for the models that have one, two, and three important two-factor interactions with one and two blocking factors respectively. In these tables, the entries under "$m_t + m_b$" give the number of treatment factors plus the number of blocking factors, the entries under "model" indicate which model is under consideration, and for example, an entry of 2(a) denotes the model represented by Figure 2(a). The entries under "treatment factor" give the additional columns in addition to the independent columns (which are 1, 2, and 4 for 8-run designs and 1, 2, 4, and 8 for 16-runs designs) for the main effects in the fitted model. The entries under "block factor" give the columns of the blocking factors. Column j in these tables denotes the j-th column in the saturated design with its columns arranged in Yates order. The Yates's order of the columns of a 16-run saturated design can be written as (a_1, a_2, a_1a_2, a_3, a_{13}, a_{23}, $a_1a_2a_3$, a_4, a_1a_4, a_2a_4, $a_1a_2a_4$, a_3a_4, $a_1a_3a_4$, $a_2a_3a_4$, $a_1a_2a_3a_4$) where other columns can be generated from the four independent columns a_1, a_2, a_3, and a_4. The entries under "2-f interaction" show how to assign the factors involved in the important 2-factor interactions. The last column in these tables gives (N_2, N_3, N_4).

Table 2. Optimal blocked designs of 8 runs for the models containing some 2-f interactions and one block factor

$m_t + m_b$	treatment factor	2-f interaction	block factor	(N_2, N_3, N_4)
4 + 1	7	(1, 4)	3	(3, 4, 0)
5 + 1	3 5	(2, 5)	6	(9, 8, 4)
4 + 1	3	(2, 4)(3, 4)	5	(4, 3, 1)

Table 3. Optimal blocked designs of 16 runs for the model containing one 2-f interaction and one block factor

$m_t + m_b$	treatment factor	2-f interaction	block factor	(N_2, N_3, N_4)
5 + 1	7	(1, 8)	11	(0, 6, 1)
6 + 1	7 11	(1, 4)	13	(1, 16, 2)
7 + 1	7 11 13	(1, 2)	14	(2, 37, 4)
8 + 1	7 11 13 14	(1, 2)	5	(7, 56, 16)
9 + 1	3 5 9 14 15	(2, 4)	7	(19, 64, 80)
10 + 1	3 5 6 9 14 15	(2, 8)	11	(31, 88, 160)
11 + 1	3 5 6 9 10 13 14	(1, 6)	15	(44, 129, 272)
12 + 1	1 3 5 6 9 10 13 14 15	(1, 6)	11	(59, 188, 432)
13 + 1	3 5 6 9 10 11 13 14 15	(1, 6)	12	(77, 264, 660)

Table 4. Optimal blocked designs of 16 runs for the model containing one 2-f interaction and two block factors

$m_t + m_b$	treatment factor	2-f interaction	block factor	(N_2, N_3, N_4)
5 + 2	7	(1, 8)	3 13	(2, 8, 1)
6 + 2	7 11	(1, 4)	3 13	(4, 20, 2)
7 + 2	7 11 13	(1, 2)	5 9	(11, 28, 16)
8 + 2	7 11 13 14	(1, 2)	5 9	(15, 56, 32)
9 + 2	3 5 9 14 15	(2, 4)	7 10	(27, 72, 96)
10 + 2	3 5 6 9 14 15	(2, 8)	7 11	(40, 104, 180)
11 + 2	3 5 6 9 10 13 14	(1,14)	7 11	(54, 153, 304)

Table 5. Optimal blocked designs of 16 runs for the models containing two 2-f interactions, as in Figure 2, and one block factor

$m_t + m_b$	model	treatment factor	2-f interaction	block factor	(N_2, N_3, N_4)
5 + 1	2(a)	15	(1, 2)(4, 8)	5	(1, 3, 5)
5 + 1	2(b)	7	(1, 8)(2, 8)	11	(0, 6, 2)
6 + 1	2(a)	7 11	(1, 4)(2, 8)	13	(2, 16, 4)
6 + 1	2(b)	7 11	(1, 4)(2, 4)	13	(2, 16, 4)
7 + 1	2(a)	7 11 13	(1, 2)(4, 8)	14	(4, 35, 8)
7 + 1	2(b)	7 11 13	(1, 2)(1, 4)	14	(4, 35, 8)
8 + 1	2(a)	7 11 13 14	(1, 2)(4, 8)	5	(10, 56, 24)
8 + 1	2(b)	7 11 13 14	(1, 2)(1, 4)	6	(10, 56, 24)
9 + 1	2(a)	3 5 9 14 15	(2, 4)(3, 8)	7	(22, 68, 88)
9 + 1	2(b)	3 5 9 14 15	(2, 4)(3, 4)	10	(22, 68, 88)
10 + 1	2(a)	3 5 6 9 14 15	(2, 8)(3,14)	11	(34, 96, 172)
10 + 1	2(b)	3 5 6 9 14 15	(2, 8)(3, 8)	12	(34, 96, 172)
11 + 1	2(a)	3 5 6 9 10 13 14	(1, 10)(2, 5)	15	(48, 141, 288)
11 + 1	2(b)	3 5 6 9 10 13 14	(1,13)(2,13)	7	(48, 141, 288)
12 + 1	2(a)	3 5 6 7 9 10 11 12	(1,12)(4,10)	15	(64, 203, 457)
12 + 1	2(b)	3 5 6 7 9 10 11 12	(1,12)(2,12)	15	(64, 203, 457)

Table 6. Optimal blocked designs of 16 runs for the models containing two 2-f interactions, as in Figure 2, and two block factors

$m_t + m_b$	model	treatment factor	2-f interaction	block factor	(N_2, N_3, N_4)
5 + 2	2(a)	15	(1, 2)(4, 8)	5 11	(3, 5, 5)
5 + 2	2(b)	7	(1, 8)(2, 8)	3 13	(2, 8, 2)
6 + 2	2(a)	7 11	(1, 4)(2, 8)	3 13	(5, 20, 4)
6 + 2	2(b)	7 11	(1, 4)(2, 4)	3 13	(5, 20, 4)
7 + 2	2(a)	3 5 14	(1, 8)(2,14)	7 10	(13, 25, 28)
7 + 2	2(b)	3 5 14	(1, 8)(1,14)	7 10	(13, 25, 28)
8 + 2	2(a)	7 11 13 14	(1, 2)(4, 8)	5 10	(18, 56, 40)
8 + 2	2(b)	7 11 13 14	(1, 2)(1, 4)	6 9	(18, 56, 40)
9 + 2	2(a)	3 5 9 14 15	(2, 4)(3, 8)	7 10	(30, 76, 104)
9 + 2	2(b)	3 5 9 14 15	(2, 4)(2, 8)	7 11	(30, 76, 104)
10 + 2	2(a)	3 5 6 9 10 13	(2,13)(4,10)	7 11	(43, 111, 194)
10 + 2	2(b)	3 5 6 9 10 13	(2,13)(3,13)	7 11	(43, 111, 194)

Table 7. Optimal blocked designs of 16 runs for the models containing three 2-f interactions, as in Figure 3, and one block factor

$m_t + m_b$	model	treatment factor	2-f interaction	block factor	(N_2, N_3, N_4)
5 + 1	3(a)	---	---	---	---
5 + 1	3(b)	15	(1, 2)(4, 8)(4,15)	5	(1, 4, 5)
5 + 1	3(c)	7	(1, 8)(2, 8)(4, 8)	11	(0, 6, 3)
5 + 1	3(d)	15	(1, 4)(4, 8)(8, 2)	3	(1, 4, 5)
5 + 1	3(e)	15	(1, 2)(1, 4)(2, 4)	7	(1, 4, 5)
6 + 1	3(a)	7 11	(1, 4)(2, 8)(7,11)	13	(3, 16, 6)
6 + 1	3(b)	7 11	(1, 4)(2, 8)(7, 8)	13	(3, 16, 6)
6 + 1	3(c)	7 11	(1, 4)(2, 4)(4, 8)	13	(3, 16, 6)
6 + 1	3(d)	7 11	(1, 4)(4, 8)(8, 2)	13	(3, 16, 6)
6 + 1	3(e)	7 11	(1, 4)(1, 8)(4, 8)	13	(3, 16, 6)
7 + 1	3(a)	7 11 13	(1, 4)(2, 8)(7,11)	14	(6, 35, 12)
7 + 1	3(b)	7 11 13	(1, 2)(4, 8)(7, 8)	14	(6, 35, 12)
7 + 1	3(c)	7 11 13	(1, 2)(1, 4)(1, 7)	14	(6, 35, 12)
7 + 1	3(d)	7 11 13	(1, 4)(4, 8)(8, 2)	14	(6, 35, 12)
7 + 1	3(e)	7 11 13	(1, 2)(1, 4)(2, 4)	14	(6, 35, 12)
8 + 1	3(a)	7 11 13 14	(1, 4)(2, 8)(7,11)	3	(13, 56, 32)
8 + 1	3(b)	7 11 13 14	(1, 2)(4, 8)(7, 8)	5	(13, 56, 32)
8 + 1	3(c)	7 11 13 14	(1, 2)(1, 4)(1, 7)	9	(13, 56, 32)
8 + 1	3(d)	7 11 13 14	(1, 4)(4, 8)(8, 2)	3	(13, 56, 32)
8 + 1	3(e)	7 11 13 14	(1, 2)(1, 4)(2, 4)	9	(13, 56, 32)
9 + 1	3(a)	3 5 9 14 15	(2, 4)(3, 8)(5, 9)	7	(25, 72, 96)
9 + 1	3(b)	3 5 9 14 15	(2, 4)(3, 8)(5, 8)	7	(25, 72, 96)
9 + 1	3(c)	3 5 9 14 15	(2, 4)(3, 4)(4, 8)	10	(25, 72, 96)
9 + 1	3(d)	3 5 9 14 15	(2, 4)(4, 8)(8, 3)	7	(25, 72, 96)
9 + 1	3(e)	3 5 9 14 15	(2, 4)(2, 8)(4,8)	7	(25, 72, 96)
10 + 1	3(a)	3 5 6 9 14 15	(2, 9)(3,14)(4, 8)	10	(37, 104, 184)
10 + 1	3(b)	3 5 6 9 14 15	(4, 8)(2, 9)(3, 9)	13	(37, 104, 184)
10 + 1	3(c)	3 5 6 9 14 15	(2, 8)(3, 8)(4, 8)	13	(37, 104, 184)
10 + 1	3(d)	3 5 6 9 14 15	(2,14)(14,3)(3,8)	10	(37, 104, 184)
10 + 1	3(e)	3 5 6 9 14 15	(2, 5)(2, 8)(5, 8)	11	(38, 104, 180)
11 + 1	3(a)	3 5 6 7 9 10 11	(4, 9)(5,10)(6, 8)	12	(52, 152, 304)
11 + 1	3(b)	3 5 6 7 9 10 11	(6, 8)(4, 9)(5,9)	15	(52, 152, 304)
11 + 1	3(c)	3 5 6 7 9 10 11	(4, 8)(5, 8)(6, 8)	15	(52, 152, 304)
11 + 1	3(d)	3 5 6 7 9 10 11	(4,10)(10,5)(5,8)	12	(52, 152, 304)
11 + 1	3(e)	3 5 6 9 10 13 14	(1, 6)(1,10)(6,10)	15	(52, 153, 304)

Table 8. Optimal blocked designs of 16 runs for the models containing three 2-f interactions, as in Figure 3, and two block factors

$m_t + m_b$	model	treatment factor	2-f interaction	block factor	(N_2, N_3, N_4)
5 + 2	3(a)	---	---	---	---
5 + 2	3(b)	15	(1, 2)(4, 8)(4,15)	7 9	(3, 6, 5)
5 + 2	3(c)	7	(1, 8)(2, 8)(4, 8)	3 13	(2, 8, 3)
5 + 2	3(d)	15	(1, 4)(4, 8)(8, 2)	3 13	(3, 6, 5)
5 + 2	3(e)	15	(1, 2)(1, 4)(2, 4)	7 9	(3, 6, 5)
6 + 2	3(a)	7 11	(1, 4)(2, 8)(7,11)	3 13	(6, 20, 6)
6 + 2	3(b)	7 11	(1, 4)(2, 8)(7, 8)	3 13	(6, 20, 6)
6 + 2	3(c)	7 11	(1, 4)(2, 4)(4, 8)	3 13	(6, 20, 6)
6 + 2	3(d)	7 11	(1, 4)(4, 8)(8, 2)	3 13	(6, 20, 6)
6 + 2	3(e)	7 11	(1, 4)(1, 8)(4, 8)	3 13	(6, 20, 6)
7 + 2	3(a)	3 5 14	(2, 5)(3, 8)(4,14)	6 9	(14, 29, 28)
7 + 2	3(b)	3 5 14	(2, 8)(1,14)(3,14)	7 11	(14, 27, 30)
7 + 2	3(c)	3 5 14	(1, 8)(2, 8)(5, 8)	7 11	(14, 27, 30)
7 + 2	3(d)	3 5 14	(2,14)(14,1)(1,8)	7 10	(13, 29, 30)
7 + 2	3(e)	3 5 14	(2, 5)(2, 8)(5, 8)	6 9	(14, 29, 28)
8 + 2	3(a)	7 11 13 14	(1, 4)(2, 8)(7,11)	6 9	(21, 56, 48)
8 + 2	3(b)	7 11 13 14	(1, 8)(2, 4)(2, 7)	3 12	(21, 56, 48)
8 + 2	3(c)	7 11 13 14	(1, 2)(1, 4)(1, 8)	6 10	(21, 56, 48)
8 + 2	3(d)	7 11 13 14	(1, 4)(4, 8)(8, 2)	6 9	(21, 56, 48)
8 + 2	3(e)	3 5 10 12	(3, 4)(3,10)(4,10)	6 11	(23, 48, 60)
9 + 2	3(a)	3 5 6 9 10	(4, 9)(5,10)(6, 8)	7 11	(33, 77, 117)
9 + 2	3(b)	3 5 6 9 14	(2, 8)(1,14)(3,14)	7 11	(33, 78, 116)
9 + 2	3(c)	3 5 6 9 14	(1,14)(2,14)(5,14)	7 10	(33, 78, 116)
9 + 2	3(d)	3 5 6 9 10	(4,10)(10,5)(5, 8)	7 11	(33, 77, 117)
9 + 2	3(e)	3 5 10 12 15	(1,10)(1,12)(10,12)	7 9	(33, 78, 117)

5. An illustrative example

The optimal blocked 2^{m-p} designs of 8 runs and 16 runs are listed in tables Table 2 through 8. When we plan to study several treatment factors and some important 2-factor interaction by using a blocked 2^{m-p} designs of 8 runs or 16 runs, we can choose an optimal design directly from these tables to satisfy our needs. Now we use an example to illustrate how to use these optimal design tables.

Suppose that in an experiment, the experimenter want to study six factors: *Nitrogen*, *Phosphorus*, *Potassium*, *temperature*, *moisture*, and *light*. She would like to use a blocked 2^{m-p} design of 16 runs with one blocking factor which is the subdivision of the field. In addition to the main effects of these factors, she also wants to estimate the three two-factor interactions that are between *Nitrogen* and *Phosphorus,* between *Nitrogen* and *Potassium*, and between *Phosphorus* and *Potassium*. The graph for this model is 3(e) as in Figure 3. Also she would like to use one blocking factor which is the subdivision of the field. The optimal design for this model can be found in Table 7. Now let us look at the row for $m_t + m_b = 6 + 1$ and model 3(e) in Table 7. We see that the additional columns for treatment factors are 7 and 11 in additional to columns 1, 2, 4, and 8 and the column for blocking factor is 13. To complete the specification of the optimal design, we need to appropriately assign the six treatment factors to the six columns 1, 2, 4, 8, 7, and 11. The "2-f interaction" column in Table 7 says that we should assign *Nitrogen*, *Phosphorus*, and *Potassium* to columns 1, 4, and 8 and assign other treatment factors to columns 2, 7, and 11. This design has $N_2 = 3$, meaning that three 2-factor interactions not in the model are confounded with the effects in the model. This is the minimum number of N_2 compared to other designs of this model.

Acknowledgments

The author would like to thank Professor Boxin Tang for his kindly help. This material is based upon work supported by the National Science Foundation/EPSCoR Grant No. 0091948 and by the State of South Dakota.

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Received: October 19, 2006