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OPTIMAL SELECTION OF SINGLE ARRAYS FOR PARAMETER DESIGN EXPERIMENTS

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Abstract: An outstanding issue in robust parameter design is the choice of experimental plans. Single arrays were proposed as an alternative to the inner-outer arrays advocated by Taguchi. Because factorial effects in parameter design experiments have properties distinctly different from those in traditional fractional factorial experiments, new principles on the relative importance of effects need to be considered. Based on them a new criterion is developed to discriminate among different single arrays. Search methods are proposed to find "optimal" single arrays with run size 8, 16 and 32.

Key words and phrases: Effect ordering principle, fractional factorial design, minimum J-aberration criterion.

1. Introduction

Robust parameter design (or briefly parameter design) is an important method for variation reduction in industrial processes and products. The quality of a system (a product or a process) is mainly affected by two types of factors: *control factors* are the variables whose values can be adjusted but remain fixed once they are chosen; *noise factors* are the variables which are hard to control in a system's normal production and use environments. When a parameter design experiment is conducted, both the control factors and noise factors are varied systematically. The basic idea of parameter design is to explore the effects of control factors, noise factors and their interactions on the performance of a system, and to exploit these effects, by choosing optimal control factor settings, to bring the system mean response on target and reduce the performance variation due to noise factors. For a comprehensive review, see Chap. 10 and Chap. 11 of Wu and Hamada (2000).

1.1. Planning and modeling techniques

Taguchi (1986) proposed to use cross arrays (or *inner-outer arrays* in his terminology) for parameter design experiments. Two separate arrays are generated for control factors and noise factors. They are called the control array (denoted by CA) and the noise array (denoted by NA), respectively. A *cross array* consists of all the combinations of the settings of CA and the settings of NA. Suppose CA and NA have run size m_1 and m_2 correspondingly. Then the run size of the cross array is $m_1 m_2$. Let $y_{i,j}$ be the response for the combination of the i^{th} control setting and the j^{th} noise setting. At any fixed control setting i, there are m_2 responses, $\{y_{i,j}\}_{1\leq j\leq m_2}$ across NA. The sample mean and sample variance, $\overline{y}_i = 1/m_2 \sum_{j=1}^{m_2} y_{i,j}$ and $s_i^2 = 1/(m_2-1)\sum_{j=1}^{m_2}(y_{i,j}-\overline{y}_i)^2$, are the summary statistics for the i^{th} control setting. Row-summary modeling is to model these summary statistics, or some functions depending on them, in terms of the control factors. Two examples are signal-to-noise ratio modeling and location-dispersion modeling (Myers and Montgomery (1995) and Wu and Hamada (2000)).

When the number of factors is large, cross arrays become costly. Single arrays proposed by Welch, Yu, Kang and Sacks (1990) and Shoemaker, Tsui and Wu (1991), are an economical alternative to cross arrays. Instead of using two arrays, a *single array* is employed with some of its columns assigned to control factors and others to noise factors. With the crossing structure ignored, a cross array can be viewed as a special case of a single array.

In the row-summary modeling approach, the responses across the noise array for any fixed control setting are considered as the noise replicates. The response y, in fact, can be modeled as a function of control and noise factors (Vining and Myers (1990), Welch et al. (1990) and Shoemaker et al. (1991)). This approach is called the response modeling approach and the fitted model $\hat{y}$ the response model. Based on $\hat{y}$, the mean and variance of the response can also be estimated, so that a two-step procedure can be employed for parameter design optimization. Unlike the row-summary modeling, the response modeling is especially suitable for single arrays. It provides flexibility to accommodate effects with different degrees of importance.

The problem of selecting optimal single arrays has not been properly addressed in the literature. Our idea, primarily motivated by Shoemaker et al. (1991) and Wu and Hamada (2000), is to consider all possible general single arrays, investigate their estimation capacity for the purpose of parameter design and select optimal arrays according to some overall criteria.

An interesting extension of cross array is the compound orthogonal array proposed by Rosenbaum (1994, 1996). A *compound orthogonal array* with parameters N_1, N_2, k_1, k_2, t_1 and t_2 is an $N_1N_2 \otimes (k_1+k_2)$ orthogonal array with the following structure: the first k_1 columns form N_2 identical copies of an $OA(N_1, k_1, 2, t_1)$ and, for each fixed setting of the first k_1 columns, the corresponding settings for the remaining k_2 columns form an $OA(N_2, k_2, 2, t_2)$. If the first k_1 columns are assigned to control factors and the remaining k_2 columns to noise factors, it is said that the strength among the control factors is t_1 and the strength for the noise factors is t_2. The strength of the whole compound array is denoted by t_3. See also Hedayat and Stufken (1999) for the construction of optimal compound orthogonal arrays. In order to estimate all main effects, control-by-control interactions and control-by-noise interactions, Borkowski and Lucas (1997) and Box and Jones (1993) suggested using designs with mixed-resolution. A mixed-resolution design is a second-order design for control effects and control-by-noise interactions.

1.2. Basics of two-level fractional factorial designs

Suppose there are l factors in an experiment. The factors are denoted by $1, \ldots, l$, called *letters* in design theory. The generalized interaction among factors $i_1, \ldots, i_k$ is denoted by $i_1 \ldots i_k$ called a *word*. The generalized interactions are also called factorial effects. A 2^{l-p} fractional factorial design, which has 2^r runs with $r = l - p$, is determined by r independent factors and p independent defining words. The *defining contrast subgroup* $\mathcal{G}$ consists of all possible combinations of the independent defining words. For two fractional factorial designs d_1 and d_2, if d_2 can be derived from d_1 by relabeling letters and/or changing signs, d_1 and d_2 are said to be *isomorphic.* The number of letters in a word is the wordlength, and the vector $W = (A_1, \ldots, A_l)$ is called the *wordlength pattern*, where A_i denotes the number of words of length i in $\mathcal{G}$. *Resolution* is defined as the smallest r such that $A_r \geq 1$. For two designs d_1 and d_2, d_1 is said to have less aberration than d_2 if $A_{i_0}(d_1) < A_{i_0}(d_2)$, where i_0 is the smallest value such that $A_{i_0}(d_1) \neq A_{i_0}(d_2)$. If there is no design with less aberration than d_1, then d_1 is said to have *minimum aberration* (Fries and Hunter (1980)).

Clear effects and eligible effects (Wu and Chen (1992)) are another two important concepts. A main effect or a two-factor interaction (henceforth abbreviated as $2fi$) is *clear* if it is not aliased with any other main effects or $2fi$'s, and is *eligible* if it is not clear but only aliased with some other $2fi$. The *number of clear effects* can be used as a supplementary criterion to minimum aberration.

The paper is organized as follows. In Section 2, single arrays are formally defined, their basic structure and property discussed, and several examples given. In Section 3, a new principle about factorial effects in parameter design is proposed. In Section 4, several criteria for selecting optimal single arrays are proposed. A search method for single arrays is presented in Section 5. In Section 6, various single arrays with small run size are discussed in detail. Good single arrays with run size 8, 16 and 32 are included in Appendices C.1−C.3.

2. General Single Arrays: Construction and Properties

Control factors are denoted by capital letters A, B and C, etc.; noise factors by lower case letters a, b and c, etc. The letters C and n are used generically to represent a control factor and a noise factor, respectively.

Suppose there are k_C control factors and k_n noise factors, each at two levels. A general single array is a 2^{l-p} fractional factorial design with k_C columns assigned to the control factors and k_n columns assigned to the noise factors, where $l = k_C + k_n$, and p is the fraction index. Single arrays do not require any *a priori* structures such as "crossing" in cross arrays. For a given run size, cross arrays and compound orthogonal arrays may not exist for certain numbers of control factors and of noise factors.

Lemma 1. *The smallest cross array for k_C control factors and k_n noise factors requires $2^{\lceil \log_2(k_C+1)\rceil + \lceil \log_2(k_n+1)\rceil}$ runs, where $\lceil x\rceil$ denotes the smallest integer greater than or equal to x.*

Proof. Suppose the run size of CA is $m_1 = 2^{n_1}$. A necessary and sufficient condition that the CA can accommodate k_C control factors is $2^{n_1} - 1 \geq k_C$, i.e., $n_1 \geq \log_2(k_C + 1)$, $n_1 \geq \lceil \log(k_C + 1)\rceil$. Therefore, the smallest CA has run size $2^{\lceil \log_2(k_C+1)\rceil}$. Similarly, the smallest NA has run size $2^{\lceil \log_2(k_n+1)\rceil}$. The lemma follows by taking the product of these two numbers.

For convenience, $S(k_C, k_n, p)$ is used to denote a single array with k_C control factors, k_n noise factors and $2^{(k_C+k_n)-p}$ runs. Suppose S_1 and S_2 are two single arrays. If S_1 can be derived from S_2 by the relabeling of control factors, of noise factors, or by change of signs, S_1 and S_2 are said to be *isomorphic*. If the control and noise factors are not distinguished, a single array becomes an ordinary fractional factorial plan. This fractional factorial plan is called the *basic frame* of the single array. Since control and noise factors play different roles in parameter design, different ways to assign the columns of a basic frame to control and noise factors can generate non-isomorphic single arrays. The distinction between control and noise factors also induces a partition of the columns of the basic frame into two subgroups. The columns assigned to the control factors are called the *control columns* and those to the noise factors the *noise columns*. Hence a single array is determined by its basic frame and the column partition. Obviously, if two single arrays have non-isomorphic basic frames, they are non-isomorphic.

In the following, the single array $S(3, 3, 2)$ is used to illustrate the structure and properties of single arrays. The three control factors are denoted by A, B and C, and the three noise factors by a, b and c. There are altogether four nontrivial and non-isomorphic 2^{6-2} basic frames given by the following defining relations (Chen, Sun and Wu (1993)):

$$I = 123 = 1456 = 23456, \tag{1}$$

$$I = 123 = 456 = 123456, \tag{2}$$

$$I = 1234 = 1256 = 3456, \tag{3}$$

and

$$I = 123 = 156 = 2356. \tag{4}$$

According to the minimum aberration criterion, (3) with the wordlength pattern W=(0, 0, 0, 3, 0, 0) is the best and (4) is the worst with the wordlength pattern W=(0, 0, 2, 1, 0, 0). Based on (1), there are six different ways to assign the columns to the control factors and the noise factors. For example, assigning columns 1, 2 and 3 to A, B and C, and columns 4, 5 and 6 to a, b and c produces a single array with the defining relation

$$S_1: \qquad I = ABC = Aabc = BCabc. \tag{5}$$

Assigning 1, 2 and 3 to a, b and c and 4, 5 and 6 to A, B and C leads to a different (and non-isomorphic) single array with the defining relation

$$S_2: \qquad I = abc = ABCa = ABCbc. \tag{6}$$

The other single arrays based on (1) are

$$S_3: \qquad I = Aab = BCac = ABCbc, \tag{7}$$

$$S_4: \qquad I = ABa = ACbc = BCabc, \tag{8}$$

$$S_5: \qquad I = Aab = ABCc = BCabc, \tag{9}$$

$$S_6: \qquad I = ABa = Cabc = ABCbc. \tag{10}$$

Based on the basic frame (2), there are eight non-isomorphic single arrays. Among them, one is given by

$$S_7: \qquad I = abc = ABC = ABCabc. \tag{11}$$

It is easy to see that S_7 is a $2^{3-1} \times 2^{3-1}$ cross array. The basic frame (3) is the 2^{6-2} minimum aberration design and generates two non-isomorphic single arrays:

$$S_8: \qquad I = ABab = ACac = BCbc, \tag{12}$$

$$S_9: \qquad I = ABCa = Aabc = BCbc. \tag{13}$$

Notice that S_2, S_4, S_5 and S_9 all have one defining word which consists of some control factors and only one noise factor. This implies that when the setting of the control factors is fixed, the level of the noise factor that appears in the defining word is also fixed. For instance, in S_2, the defining word $ABCa$ implies the aliasing of a with ABC. If the levels of A, B and C are chosen, so is a's. This implies that the corresponding noise array has strength 0, because the level of the noise factor a does not vary. Hence, S_2, S_4, S_5 and S_9 are not compound orthogonal arrays according to the definition.

Let N_C denote the number of clear control main effects, N_n the number of clear noise main effects, N_{CC} the number of clear control-by-control interactions, N_{Cn} the number of clear control-by-noise interactions (henceforth abbreviated as Cn effects), and N_{nn} the number of clear noise-by-noise interactions. The estimation capacity of single arrays S_1 to S_9 in terms of the numbers of eligible effects and clear effects is summarized in Table 1.

Table 1. Comparison of Estimation Capacities for S_1 to S_9.

Design	Eligible effects	Clear Effects	N_C	N_n	N_{CC}	N_{Cn}	N_{nn}
S_1	$A, B, C, Aa, Ab, Ac, ab, ac, bc$	$a, b, c, Ba, Bb, Bc, Ca, Cb, Cc$	0	3	0	6	0
S_2	$a, b, c, AB, AC, BC, Aa, Ba, Ca$	$A, B, C, Ab, Ac, Bb, Bc, Cb, Cc$	3	0	0	6	0
S_3	$A, a, b, Ba, Bc, Ca, Cc, BC, ac$	$B, C, c, AB, AC, Ac, Bb, Cb, bc$	2	1	2	3	1
S_4	$A, B, a, AC, Ab, Ac, Cb, Cc, b, c$	$C, b, c, BC, Bb, Bc, Ca, ac, bc$	1	2	1	3	2
S_5	$A, a, b, AB, AC, BC, Ac, Bc, Cc$	$B, C, c, Ba, Bb, Ca, Cb, ac, bc$	2	1	0	4	2
S_6	$A, B, a, Ca, Cb, Cc, ab, ac, bc$	$C, b, c, AC, BC, Ab, Ac, Bb, Bc$	1	2	2	4	0
S_7	A, B, C, a, b, c	$Aa, Ab, Ac, Ba, Bb, Bc, Ca, Cb, Cc$	0	0	0	9	0
S_8	all 2fi's	A, B, C, a, b, c	3	3	0	0	0
S_9	all 2fi's	A, B, C, a, b, c	3	3	0	0	0

Define

$$\alpha = (N_C, N_n, N_{CC}, N_{Cn}, N_{nn}) \tag{14}$$

for a single array and call it the *clear estimation index vector*. For single arrays with a given basic frame, the total numbers of clear main effects and of clear $2fi$'s are fixed, i.e., $N_C + N_n$ and $N_{CC} + N_{Cn} + N_{nn}$ are constants. But the distribution across N_C, N_n, N_{CC}, N_{Cn} and N_{nn} varies. This is transparent by comparing the single arrays S_1 to S_6 which share the basic frame (1). In parameter design, C and Cn are most important, because they can be used to adjust the responses on target and to reduce response variation. From Table 1, S_2 appears to be the best among S_1 to S_6 . If the experimenters can assume that CC's are negligible, then the eligible Cn effects, Aa, Ab and Ac, are also estimable. S_7 is a cross array. An important property for cross arrays is that all the Cn effects can be clearly estimated (see Theorem 10.1 of Wu and Hamada (2000)). So S_7 has all Cn effects clear, but its main effects are only eligible. If response adjustment is not important, S_7 may be preferred. For S_8 and S_9, all the main effects are clear, but none of the $2fi$'s are clear. Note that S_8 and S_9 are based on the basic frame (3), which has minimum aberration. Hence minimum aberration designs do not necessarily provide good basic frames for single arrays.

In general, for any fixed k_C, k_n and run size N, there are many non-isomorphic single arrays. The choice of optimal single arrays is a challenging problem. Standard criteria like maximum resolution and minimum aberration are not suitable for parameter design, because they do not recognize the different roles played by the control and noise factors. Although a compound orthogonal array makes a distinction between control and noise factors, its orthogonality requirement rules out such interesting designs as S_2, S_4, S_5 and S_9 in the previous example. The strengths t_1, t_2 and t_3 are only a rough description of the structure and properties of a compound orthogonal array. For example, for both S_1 and S_7, $t_1 = 2$, $t_2 = 2$ and $t_3 = 2$, but S_1 and S_7 are still different in terms of aliasing and estimation capacity. Mixed-resolution is another attempt to address this question, but a mixed resolution array requires the length of any defining words involving control factors to be at least 5, and the length of any defining words not involving control factors to be at least 3. This is a strong condition, even stronger than the crossing structure. As a result, the required run size is large. For example, for k_C=3, $k_n = 3$ and $N = 16$, no single arrays satisfy the mixed resolution criterion. The smallest mixed resolution array for the case is a 32-run 2^{6-1} plan with I=$ABCabc$ (Borkowski and Lucas (1997)).

A systematic approach is developed to address this problem. First, a new effect ordering principle is proposed. Based on this principle, optimality criteria are derived.

3. Effect Ordering Principle for Parameter Design

The minimum aberration criterion is based on the hierarchical ordering principle (HOP): (i) lower order effects are more important than higher order effects, (ii) effects of the same order are equally important. The factorial effects in parameter design have more complicated interpretations than those in ordinary fractional factorial design, because parameter design has two objectives, response mean optimization and variation reduction. If a factorial effect consists of i control factors and j noise factors, it is of *type* $e_{i,j} = \overbrace{C..C}^{i}\overbrace{n..n}^{j}$. Since control factors are not further distinguished among each other, the hierarchical ordering principle can be applied to control effects, that is, lower-order control effects are more important than higher-order control effects; control effects of the same order are equally important. The same can be said about noise effects. Notice that $\{e_{i,0}\}_{i\geq 0}$ is the collection of all types of control effects and $\{e_{0,j}\}_{j\geq 0}$ the collection of all types of noise effects. According to the HOP, control effects and noise effects can be rank-ordered as $e_{0,0} > e_{0,1} > e_{0,2} > \cdots > e_{0,j} > e_{0,j+1} > \cdots$, and $e_{0,0} > e_{1,0} > e_{2,0} > \cdots > e_{i,0} > e_{i+1,0} > \cdots$. It is not appropriate to directly apply the HOP to Cn effects, because HOP would find the four most important groups of effects to be $\{C, n\}$, $\{CC, Cn, nn\}$, $\{CCC, CCn, Cnn, nnn\}$ and $\{CCCC, CCCn, CCnn, Cnnn, nnnn\}$. In parameter design, the Cn effects are more likely to be present because engineering knowledge and experience may suggest that the selected noise factors are expected to interact with some control factors. Since Cn can often be used to achieve robustness without incurring more cost, priority should be given to these interactions so as not to miss any opportunities. Hence, C, n and Cn should be considered to be equally important, wherein C is crucial for mean response adjustment, and n and Cn are useful for variation reduction. Then the second set consists of CC and nn, wherein CC affects the response mean, and nn affects the response variation (but its contribution cannot be controlled or changed). Further opportunities for variation reduction appear in the third group which contains CCn and Cnn. Because Cnn involves more noise factors than CCn, CCn is considered to be more important than Cnn. Following a similar argument, all the factorial effects in parameter design can be rank-ordered. A numerical rule can be used to help define the ranking. In general, if an effect is of type $e_{i,j}$, its weight is defined to be

$$W(e_{i,j}) = \begin{cases} 1 & \text{if } \max(i,j) = 1, \\ i & \text{if } i > j \text{ and } i > 1, \\ j + \frac{1}{2} & \text{if } i \leq j \text{ and } j \geq 2. \end{cases}$$

For any w in $\{1, 2, 2.5, 3, 3.5, \ldots\}$, $\mathcal{E}_w$ is the set of effects with weight w. Sometimes, $\mathcal{E}_w$ can also be viewed as the set of effect types with weight w. The first seven $\mathcal{E}_w$'s are listed in Table 2. The previous discussion can be summarized by the following *Effect Ordering Principle* (EOP):

(i). Effects with smaller weight are more important than effects with larger weight.

(ii). Effects with same weight are equally important.

High order factorial effects are usually insignificant. In practice, the experimenters are seldom interested in effects of order higher than 5. Additional assumptions can also be considered.

(A.1) All effects with order higher than or equal to 4 are negligible.

(A.2) All effects with order higher than or equal to 3 are negligible.

Table 2. Factorial effects in parameter designs rank-ordered by EOP.

Weight	Factorial Effect
1	C, Cn, n
2	CC,CCn
2.5	$CCnn$, Cnn, nn
3	CCC, $CCCn$, $CCCnn$
3.5	$CCCnnn$, $CCnnn$, $Cnnn$, nnn
4	$CCCC$, $CCCCn$, $CCCCnn$, $CCCCnnn$
4.5	$CCCCnnnn$, $CCCnnnn$, $CCnnnn$, $Cnnnn$, $nnnn$
...	....

Applying (A.1) and the EOP leads to five groups of effects in the descending order of importance

$$\begin{aligned}\mathcal{E}_1 = \{C, Cn, n\} > \mathcal{E}_2 = \{CC, CCn\} > \mathcal{E}_{2.5} = \{Cnn, nn\} \\ > \mathcal{E}_3 = \{CCC\} > \mathcal{E}_{3.5} = \{nnn\}.\end{aligned} \tag{15}$$

Based on a different argument and weight assignment, Bingham and Sitter (2000) rank-ordered the factorial effects with order less than 4 as follows:

$$\begin{aligned}\mathcal{E}'_1 = \{C, n\} > \mathcal{E}'_{1.5} = \{Cn\} > \mathcal{E}'_2 = \{CC, nn\} \\ > \mathcal{E}'_{2.5} = \{CCn, Cnn\} > \mathcal{E}'_3 = \{CCC, nnn\}.\end{aligned} \tag{16}$$

The major difference concerns the control-by-noise interactions, Cn, CCn and Cnn, which are ranked higher in our approach. Ours is based on a different ordering of effects from theirs. We take the further step of using the numbers of pairs of aliased effects to measure the aliasing severity and to select optimal arrays. This approach is more flexible and statistically justifiable.

4. Criteria for Selecting Single Arrays

4.1. Optimality criteria for fractional factorial design revisited

For a given run size and fraction index, fractional factorial designs with less severe effect aliasing are considered to be better. A formal measure of the aliasing severity is thus needed. Suppose the number of factors is l. The *aliasing type* $i \sim j$ refers to the aliasing between an effect of order i and another effect of order j, where $1 \leq i \leq j \leq l$. The type $l \sim l$ is not possible. The types $1 \sim 1$, $l-1 \sim l$, $l-1 \sim l-1$ and $l-2 \sim l$ do not appear in designs with resolution III or higher, because these types lead to defining words of length one or two. If $i_1 \sim j_1$ is considered to be more severe than $i_2 \sim j_2$, it is written as $i_1 \sim j_1 > i_2 \sim j_2$. It is helpful to rank all the aliasing types in the order of severity. Clearly $1 \sim 2$ is the

most severe type, followed by $2 \sim 2$ and $1 \sim 3$. Arguably, $2 \sim 2$ is more severe than $1 \sim 3$. Two ordering schemes are considered below.
Scheme 1: (i) $i_1 \sim j_1 > i_2 \sim j_2$, if $i_1 + j_1 < i_2 + j_2$; (ii) $i_1 \sim j_1 > i_2 \sim j_2$, if $i_1 + j_1 = i_2 + j_2$ and $j_1 - i_1 < j_2 - i_2$. For $l = 6$, the aliasing types can be rank-ordered as follows:

$$\begin{aligned} &1 \sim 2 > 2 \sim 2 > 1 \sim 3 > 2 \sim 3 > 1 \sim 4 > 3 \sim 3 > 2 \sim 4 > 1 \sim 5 > 3 \sim 4 \\ &\quad > 2 \sim 5 > 1 \sim 6 > 4 \sim 4 > 3 \sim 5 > 2 \sim 6 > 4 \sim 5 > 3 \sim 6. \end{aligned} \tag{17}$$

Scheme 2: (i) $i_1 \sim j_1 > i_2 \sim j_2$, if $j_1 < j_2$; (ii) $i_1 \sim j_1 > i_2 \sim j_2$, if $j_1 = j_2$ and $i_1 < i_2$. For $l = 6$, the aliasing types can be rank-ordered as follows:

$$\begin{aligned} &1 \sim 2 > 2 \sim 2 > 1 \sim 3 > 2 \sim 3 > 3 \sim 3 > 1 \sim 4 > 2 \sim 4 > 3 \sim 4 > 4 \sim 4 \\ &\quad > 1 \sim 5 > 2 \sim 5 > 3 \sim 5 > 4 \sim 5 > 1 \sim 6 > 2 \sim 6 > 3 \sim 6. \end{aligned} \tag{18}$$

Let $N_{i\sim j}$ denote the number of pairs of aliased effects of the type $i \sim j$. Noting that a pair of aliased effects of a given type can be derived from various defining words in the defining contrast subgroup, $N_{i\sim j}$ is related to wordlength pattern in the following.

$$N_{i\sim j} = \sum_{k>0} \binom{[l-(i+j-2k)]^+}{k} d(i-k, j-k) A_{i+j-2k} + d(i,j) A_{i+j}, \tag{19}$$

where $d(i,j) = \binom{i+j}{i}$ for $i \neq j$, $= 1/2\binom{i+j}{i}$ for $i = j \neq 0$, and $d(0,0) = 0$. A derivation of (19) is given in Appendix A. Imposing an aliasing severity order by either scheme will result in a numerical summary of the aliasing severity of the corresponding design. To identify designs with least aliasing severity is equivalent to sequentially minimizing $N_{i\sim j}$. Equation (19) shows that the $N_{i\sim j}$ are functions of the wordlength pattern $W = (A_1, \ldots, A_l)$. Hence, the procedure is to sequentially minimize certain functions of the wordlength patterns. Applying mathematical induction, it can be easily shown that sequentially minimizing $N_{i\sim j}$ according to ordering scheme 1 or 2 is equivalent to sequentially minimizing A_i, which leads to the minimum aberration criterion. For example, if the total number of factors is 6, $N_{i\sim j}$ can be calculated from $W = (A_1, A_2, A_3, A_4, A_5, A_6)$ as follows: $N_{1\sim 2} = 3A_3$, $N_{2\sim 2} = 3A_4$ $N_{1\sim 3} = 4A_4$, $N_{2\sim 3} = 9A_3 + 10A_5$, $N_{1\sim 4} = 3A_3 + 5A_5$ $N_{3\sim 3} = 6A_4 + 10A_6$ and $N_{2\sim 4} = 8A_4 + 15A_6$. Sequentially minimizing $N_{1\sim 2}$, $N_{2\sim 2}$, $N_{1\sim 3}$, $N_{2\sim 3}$, $N_{1\sim 4}$, $N_{3\sim 3}$ and $N_{2\sim 4}$ based on either scheme leads to the minimum aberration criterion that sequentially minimizes A_3, A_4, A_5 and A_6.

Minimizing the number of aliased pairs does not necessarily result in maximizing the number of clear effects, these concepts are very different. Many supporting examples can be found in Appendix 4A of Wu and Hamada (2000).

4.2. Criteria for selecting optimal single arrays

The single array $S(k_C, k_n, p)$ is uniquely determined by its defining contrast subgroup $\mathcal{G}$. In parameter design, defining words of the same length cannot be treated equally because they may belong to different types. In general, for $1 \leq k \leq k_C + k_n$, a word of length k can be one of the types in $\{e_{i,j} : i + j = k, 0 \leq i \leq k_C, 0 \leq j \leq k_n\}$. Let $A_{i,j}$ be the number of effects of the type $e_{i,j}$ in $\mathcal{G}$, and $A = (A_{i,j})$ a matrix with entries $A_{i,j}$, where $0 \leq i \leq k_C$ and $0 \leq j \leq k_n$. A is called the *wordtype pattern* for $S(k_C, k_n, p)$. Based on the wordtype pattern A, general criteria for selecting single arrays will be developed along the lines of the minimum aberration criterion.

For simplicity, we write (i, j) instead of $e_{i,j}$. If two effects (i_1, j_1) and (i_2, j_2) are aliased, we write $(i_1, j_1) \sim (i_2, j_2)$. Define $N_{(i_1,j_1)\sim(i_2,j_2)}$ to be the number of pairs of aliased effects of the type $(i_1, j_1) \sim (i_2, j_2)$. Straightforward extension of (19) leads to

$$N_{(i_1,j_1)\sim(i_2,j_2)} = \sum_{k_1=0}^{i_1\wedge i_2} \sum_{k_2=0}^{j_1\wedge j_2} \binom{k_C + 2k_1 - i_1 - i_2}{k_1} \binom{k_n + 2k_2 - j_1 - j_2}{k_2}$$

$$d(i_1 - k_1, i_2 - k_1; j_1 - k_2, j_2 - k_2) A_{i_1+i_2-2k_1, j_1+j_2-2k_2}, \qquad (20)$$

where $i \wedge j = \min(i, j)$ for integers i and j, and $d(0, 0, 0, 0) = 0$, $d(x, y; u, v) = 1/2\binom{x+y}{x}\binom{u+v}{u}$ for $x = y$, $u = v$, and $x^2+y^2+u^2+v^2 \neq 0$; otherwise, $d(x, y; u, v) = \binom{x+y}{x}\binom{u+v}{u}$. The *group aliasing type* $i \approx j$ is defined to be the aliasing between an effect in $\mathcal{E}_i$ and an effect in $\mathcal{E}_j$ where i and j are from $\{1, 2, 2.5, 3, \ldots, l\}$ and $i \leq j$. Two schemes are considered for ordering the group aliasing types:

$$\begin{aligned} i_1 \approx j_1 > i_2 \approx j_2 \quad &\text{if} \quad i_1 + j_1 < i_2 + j_2, \\ \text{or} \;\; j_1 - i_1 < j_2 - i_2 \quad &\text{when} \quad i_1 + j_1 = i_2 + j_2; \end{aligned} \qquad (21)$$

$$i_1 \approx j_1 > i_2 \approx j_2 \quad \text{if} \quad j_1 < j_2 \quad \text{or} \quad i_1 < i_2 \quad \text{when} \quad j_1 = j_2. \qquad (22)$$

Let $N_{i\approx j}$ denote the number of aliased pairs of the type $i \approx j$. It can be easily calculated from $N_{(i_1,j_1)\sim(i_2,j_2)}$, as $N_{i\approx j} = \sum_{(i_1,j_1)\in\mathcal{E}_i,(i_2,j_2)\in\mathcal{E}_j} N_{(i_1,j_1)\sim(i_2,j_2)}$. For example, since $\mathcal{E}_1 = \{C, Cn, n\}$ and $\mathcal{E}_2 = \{CC, CCn\}$,

$$N_{1\approx 2} = N_{(1,0)\sim(2,0)} + N_{(1,0)\sim(2,1)} + N_{(1,1)\sim(2,0)} + N_{(0,1)\sim(2,0)} + N_{(0,1)\sim(2,1)}.$$

Based on (20), $N_{i\approx j}$ can be calculated from the wordtype pattern $(A_{i,j})_{0\leq i\leq k_C, 0\leq j\leq k_n}$. Applying the ordering scheme in (21) or (22), $N_{i\approx j}$ can be rank-ordered based on their indices. By sequentially minimizing $N_{i\approx j}$, we can obtain single arrays with minimum aliasing severity in terms of the number of aliased pairs of effects. Because the $N_{i\approx j}$ are functions of $A_{i,j}$, sequentially minimizing $N_{i\approx j}$ is equivalent to minimizing a sequence of functions of $A_{i,j}$. Therefore, general criteria based on $(A_{i,j})$ can be proposed to distinguish different single arrays. A complete development would take much effort and is left for future research. Here, a simplified yet practically important case is considered. Under the assumption (A.2) in Section 3, there are only three groups of effects,

$$\mathcal{E}_1 = \{C, Cn, n\} > \mathcal{E}_2 = \{CC\} > \mathcal{E}_{2.5} = \{nn\}. \tag{23}$$

According to (21), the group aliasing types involving $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_{2.5}$ are rank-ordered as

$$1 \approx 1 > 1 \approx 2 > 1 \approx 2.5 > 2 \approx 2 > 2 \approx 2.5 > 2.5 \approx 2.5; \tag{24}$$

and, according to (22), as

$$1 \approx 1 > 1 \approx 2 > 2 \approx 2 > 1 \approx 2.5 > 2 \approx 2.5 > 2.5 \approx 2.5. \tag{25}$$

Notice that (24) and (25) are slightly different. The relative positions of $1 \approx 2.5$ and $2 \approx 2$ are switched in (25). In the following, only (24) will be used. Define $J = (J_1, J_2, J_3, J_4, J_5, J_6)$ as follows:

$$J_1 = N_{1\approx 1} = 4A_{2,1} + 4A_{1,2} + 4A_{2,2}, \tag{26}$$
$$J_2 = N_{1\approx 2} = 3A_{3,0} + 3A_{3,1} + A_{2,1}, \tag{27}$$
$$J_3 = N_{1\approx 2.5} = A_{1,2} + 3A_{1,3} + 3A_{0,3}, \tag{28}$$
$$J_4 = N_{2\approx 2} = 6A_{4,0}, \tag{29}$$
$$J_5 = N_{2\approx 2.5} = A_{2,2}, \tag{30}$$
$$J_6 = N_{2.5\approx 2.5} = 6A_{0,4}. \tag{31}$$

J is called the *aliasing index vector*. If two single arrays have the same J, they are said to be J-equivalent. Based on J, a minimum J-aberration criterion can be defined.

Definition 1. (Minimum J-aberration) For two non-equivalent single arrays S_1 and S_2 which are not J-equivalent, let i_0 be the smallest i such that $J_i(S_1) \neq J_i(S_2)$. If $J_{i_0}(S_1) < J_{i_0}(S_2)$, then S_1 is said to have less J-aberration than S_2. If there are no other single arrays with less J-aberration than S_1, S_1 is said to have minimum J-aberration.

The simplicity of the aliasing index vector J is due to the assumpiton (A.2). First, the defining words with length 5 or higher are not considered. Second, the induced aliasing patterns from the defining words with length less than or equal to 4 do not need to be considered either. For instance, suppose there is a defining word $C_1C_2n_1$, then all possible basic aliasing pairs are $C_1 = C_2n_1$, $C_2 = C_1n_1$ and $n_1 = C_1C_2$. For any other control factor C_3 and noise factor n_2, there are 6 induced confounded pairs: $C_1C_3 = C_2C_3n_1$, $C_2C_3 = C_1C_3n_1$, $n_1C_3 = C_1C_2C_3$, $C_1n_2 = C_2n_1n_2$, $C_2n_2 = C_1n_1n_2$ and $n_1n_2 = C_1C_2n_2$. Each of them involves effects with order at least 3 and are assumed to be negligible. Therefore, these induced pairs are not counted.

Because of combinatorial complexity, it is not advisable to employ only one criterion, especially when no model is specified. The clear estimation index α defined in (14) can be used as an alternative for the evaluation of a single array.

Definition 2. (α-admissibility). A single array S_1 is said to be α-inadmissible if there exists another single array S_2 such that $\alpha^1(i) \le \alpha^2(i)$ for $1 \le i \le 5$, and at least one of the inequalities is strict. Otherwise S_1 is said to be α-admissible.

J-aberration and α-admissibility will be used to measure the goodness of single arrays.

5. Search for Optimal Arrays

Single arrays with 8, 16, 32 and 64 runs are of practical importance. Overall good single arrays based on the criteria proposed in Section 4.2 need to be selected and tabulated. All non-isomorphic single arrays need to be constructed and compared so as not to miss any good candidate. Recall that a necessary condition for two single arrays to be isomorphic is that their basic frames are isomorphic fractional factorial designs. For a given basic frame, the columns can be assigned to the control factors and the noise factors in $\binom{l}{k_C}$ different ways, where $l = k_C + k_n$. Therefore the classification of $S(k_C, k_n, p)$ can be divided into two steps: (1) construct all non-isomorphic 2^{l-p} designs as non-isomorphic basic frames; (2) for each basic frame, construct non-isomorphic single arrays from all possible candidates generated by different column assignments.

The non-isomorphic 8-, 16- and 32-run fractional factorial designs are available from Chen, Sun and Wu (1993). Only Step 2 needs to be carried out for these cases. For 64-run fractional factorial designs, Chen, Sun and Wu (1993) only keep designs with resolution IV or higher. For single arrays, designs with resolution III may be good basic frames, so Step 1 needs to be carried out. By definition, single arrays with different wordtype matrices are non-isomorphic, but single arrays with the same wordtype matrix are not necessarily isomorphic. A counterexample can be produced by modifying the work in Chen and Lin (1991). Thus a complete isomorphism check is required to discriminate arrays with the same wordtype matrix. The algorithm proposed in Chen, Sun and Wu (1991) was generalized and used to check isomorphism between arrays where control factors and noise factors must be distinguished.

6. Highlights on the Tables of Single Arrays

Since noise factors are hard to control, the number of noise factors included in parameter design experiments is often small. In the paper, we only consider $k_n \leq 3$. Applying the procedure discussed in the previous section, complete tables of non-isomorphic single arrays of 8, 16 and 32 runs are obtained. For fixed k_C and k_n, good single arrays based on J and α are included in Appendices C.1−C.3. In each case, only a few single arrays are selected due to space limitation. Some 64-run single arrays are available on the websites of both authors. More extensive tables are available in Zhu (2000). In each table, the first three columns are k_C, k_n and p, which correspond to the number of control factors, the number of noise factors and the fraction index. The column denoted by DC gives the p independent defining words in terms of their positions in the basic design matrix in Appendix B; N indicates the noise columns in the basic frame generated by the independent defining words. For the 8- and 16-run tables, the aliasing index vector J is included. For most 32-run single arrays, J becomes too large to be included in the table. By applying the formulae in the definition of J, it can be calculated from the wordtype matrix. The column A lists part of the wordtype pattern matrix, ($A_{3,0}$, $A_{2,1}$, $A_{1,2}$, $A_{0,3}$, $A_{4,0}$, $A_{3,1}$, $A_{2,2}$, $A_{1,3}$). The last column of each table reports the clear estimation index, $\alpha = (N_C, N_n, N_{CC}, N_{Cn}, N_{nn})$. For given k_C, k_n and p, the corresponding single arrays are listed in the order of the J-aberration criterion. The first or the first few arrays are minimum J-aberration single arrays, because different single arrays may share the same J. According to Lemma 1, for run size 2^k, cross arrays do not exsit for all possible k_C and k_n. For $k_n = 1$, they exist for $k_C \leq 2^{k-1} - 1$; for k_n=2 or 3, they exist for $k_c \leq 2^{k-2} - 1$. These conditions explain why cross arrays are not listed in some part of the tables. Cross arrays are marked by $*$ in the tables.

We use the following example to illustrate the usage of the tables. Suppose a 32-run single array is needed to study seven control factors and two noise factors, i.e., $k_C = 7$, $k_n = 2$ and $p = 4$. There are three corresponding single arrays listed in the table in Appendix C.3. Suppose the first one is chosen. Since there are nine factors and 32 runs, the basic frame is a 2^{9-4} design. The nine columns are denoted by the letters 1, 2, 3, 4, 5, 6, 7, 8, 9. The first five columns are independent, and the remaining four columns are generated by the four defining words given in DC, which correspond to the columns 7, 11, 13 and 30 in the basic design matrix in Appendix B. Since the columns are $(1, 1, 1, 0, 0)^t$, $(1, 1, 0, 1, 0)^t$, $(1, 0, 1, 1, 0)^t$ and $(0, 1, 1, 1, 1)^t$, the defining words for these four columns are 6=123, 7=124, 8=134 and 9=2345. In the N column, (5,9) indicates that columns 5 and 9 of the basic frame are assigned to the two noise factors; $\alpha = (7, 2, 0, 14, 1)$ reports that all the seven control main effects, the two noise main effects and the 14 Cn effects are clear, but none of the control-by-control interactions are clear.

The wordtype matrices and clear estimation indices listed in the tables reflect the complexity in classifying single arrays. For example, for $k_C = 6$, $k_n = 3$ and $p = 4$, the following non-isomorphic single arrays are given in Appendix C.3:

S_1 : $6 = 123, 7 = 124, 8 = 134, 9 = 2345$, noise columns: 1, 5, 9;

S_2 : $6 = 12, 7 = 13, 8 = 23, 9 = 12345$, noise columns: 4, 5, 9;

S_3 : $6 = 12, 7 = 13, 8 = 23, 9 = 45$, noise columns: 4, 5, 9.

S_1 is listed as the first single array according to the aliasing index vector J, which is (0, 12, 0, 18, 0, 0). All its control and noise main effects are clear. Twelve of the 18 Cn effects are clear, these are {25, 29, 35, 39, 45, 49, 56, 57, 58, 69, 79, 89}, and the other Cn effects are eligible. The eligible sets that include at least one Cn effect are

$$12 = 36 = 47, 13 = 26 = 48, 14 = 27 = 38,$$

$$16 = 23 = 78, 17 = 24 = 68, 18 = 34 = 67.$$

In addition, three noise-by-noise interactions $\{15, 19, 59\}$ are clear. The aliasing index J of S_2 is also $(0, 12, 0, 18, 0, 0)$, but S_2 is quite different from S_1 in terms of α. All its noise main effects, noise-by-noise and control-by-noise interactions are clear. The six control main effects are only eligible. The eligible sets are

$$1 = 26 = 37, 16 = 2 = 38, 17 = 28 = 3,$$

$$12 = 6 = 78, 13 = 68 = 7, 23 = 67 = 8.$$

It is easy to show that S_3 is a cross array, i.e., $S_3 = 2^{6-3} \otimes 2^{3-1}$. The crossing structure guarantees that all the Cn effects are clear but in S_3, the control and noise main effects are only eligible. Its vector J is $(0, 12, 3, 18, 0, 0)$. Compared to S_1 and S_2, S_3 should be viewed as inferior.

Several important issues will be briefly discussed here. As indicated earlier, minimum aberration designs do not necessarily provide the best basic frames for single arrays. This is evident for single arrays with large fraction index p or a large number of noise factors (i.e., close values of k_n and k_C). For small p and k_n, minimum aberration designs lead to minimum J-aberration single arrays. For example, minimum J-aberration 32-run single arrays $S(6, 1, 2)$, $S(5, 2, 2)$, $S(4, 3, 2)$, $S(7, 1, 3)$, $S(6, 2, 3)$ and $S(5, 3, 3)$ use the corresponding minimum aberration designs as the basic frames. But the minimum J-aberration single arrays $S(7, 2, 4)$, $S(6, 3, 4)$ and $S(8, 2, 5)$ are not based on the corresponding minimum aberration designs.

The inconsistency between minimum aberration and the maximum number of clear effects carries over to the minimum J-aberration single arrays. There are many cases in which the minimum J-aberration single arrays are also optimal in terms of the clear estimation index α. Minimum J-aberration single arrays are α-admissible in most cases, but there are exceptions. For example, the first and second arrays for $k_C = 7$, $k_n = 3$ and $p = 5$ have $\alpha^1 = (4, 0, 0, 6, 0)$ and $\alpha^2 = (7, 0, 0, 14, 0)$. Their aliasing index vectors are $J^1 = (0, 21, 3, 6, 0, 0)$ and $J^2 = (0, 24, 3, 42, 0, 0)$. Though the first array has minimum J-aberration, obviously it is α-inadmissible.

Cross arrays are often not good according to the minimum J-aberration criterion and can also be α-inadmissible. Because cross arrays guarantee that all the Cn effects are clear, they are usually ranked among the top 10 to 20 based on J, but many better single arrays are available. Two examples are given for illustration. For $k_C = 6$, $k_n = 2$ and $p = 3$, the minimum J-aberration single array, denoted by S_1, has $\alpha = (6, 2, 0, 12, 1)$. The cross array S_3 has $\alpha = (0, 2, 0, 12, 1)$. In both arrays, all the Cn effects are clear. All the control and noise main effects are clear in S_1, while they are only eligible in S_3. Another example is for $k_C = 7$, $k_n = 2$ and $p = 4$. Denote the first and the third reported arrays by S_1 and S_3, where S_1 has $\alpha = (7, 2, 0, 14, 1)$, and S_3 has $\alpha = (0, 2, 0, 14, 1)$. The former has minimum J-aberration while the latter is a cross array. From the two α vectors, it is clear that S_1 is much better than S_3. There are cases in which cross arrays are winners in terms of the number of clear Cn effects. When the fraction index p is large, the capacity of a fractional factorial design is limited and balancing estimation among different effects becomes difficult. The crossing structure puts one type of effects, namely Cn effects, as the top priority for estimation. For example, for $k_C = 11$, $k_n = 1$ and $p = 7$, the listed arrays are S_1, S_2 and S_3 with $\alpha = (0, 1, 1, 0, 0)$, $\alpha = (11, 1, 0, 0, 0)$ and $\alpha = (0, 1, 0, 11, 0)$, respectively. S_1 is a minimum J-aberration array, S_2 is based on the 2^{12-7} minimum aberration design, and S_3 is a cross array. Only the cross array can guarantee that all the Cn effects are clear.

7. Summary

Based on the argument that control-by-noise interactions play a pivotal role in parameter design experiments, a new effect ordering principle is proposed for ranking the relative importance of factorial effects. This principle, together with the concepts of aliasing type and wordtype pattern, leads to the minimum J-aberration criterion, which is an extension of the minimum aberration criterion for regular fractional factorial designs. Good single arrays can be chosen based on the J-aberration criterion and the clear estimation index vector α. The collection of useful single arrays given in Appendices C.1.−C.3. can aid experimenters in choosing appropriate experimental plans. In this paper, only two-level regular fractions are considered. Extensions to more than two levels and to nonregular fractions would be of interest.

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Appendix A. Derivation of (19)

Let $N_{i\sim j}$ denote the number of pairs of aliased effects of type $i \sim j$. A pair of aliased effects of type $i \sim j$ can be derived from different defining words in the defining contrast subgroup. Let (E_1, E_2) denote a pair of aliased effects of type $i \sim j$, where E_1 has order i, E_2 has order j and $i \leq j$. Define Θ_k to be the collection of (E_1, E_2) such that E_1 and E_2 have exactly k factors in common, where $0 \leq k \leq i$. For $k = 0$, suppose (E_1, E_2) is an arbitrary pair in Θ_0. It is induced from a defining word of length $i+j$. Every defining word of length $i+j$ can induce $d(i,j)$ different pairs of aliased effects of type $i \sim j$ which belong to Θ_0, where

$$d(i,j) = \begin{cases} \binom{i+j}{i} & \text{if } i \neq j; \\ \frac{1}{2}\binom{i+j}{i} & \text{if } i = j \neq 0. \end{cases}$$

In addition, define $d(0,0) = 0$. If (E_1, E_2) and (E_1', E_2') are induced from two different defining words of length $i+j$, they must be different. Therefore, $\mid \Theta_0 \mid= d(i,j)A_{i+j}$. For $k > 0$, Θ_k contains the pairs of aliased effects which share exactly k factors. Suppose $(E_1, E_2) \in \Theta_k$, which is induced from a defining word of length $i+j-2k$. Every defining word of length $i+j-2k$ can generate $\binom{[l-(i+j-2k)]^+}{k} d(i-k, j-k)$ pairs of $(E_1, E_2) \in \Theta_k$. Different defining words of the same length $i+j-2k$ must generate different pairs of aliased effects belonging to Θ_k. Therefore,

$$\mid \Theta_k \mid= \binom{[l-(i+j-2k)]^+}{k} d(i-k, j-k) A_{i+j-2k}.$$

Since $\Theta_0, \ldots, \Theta_i$ are mutually exclusive, one has (19).

Appendix B

(Design matrices for 16, 32 and 64-run designs. For 16-run designs, take the first 4 rows and 15 columns; for 32-run designs, take the first 5 rows and 31 columns; and for 64-run designs, take the whole matrix. Independent columns are numbered 1, 2, 4, 8, 16 and 32 and in bold face.)

1	**2**	3	**4**	5	6	7	**8**	9	10	11	12	13	14	15	**16**	17	18	19	20	21
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1
0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
22	23	24	25	26	27	28	29	30	31	**32**	33	34	35	36	37	38	39	40	41	42
0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1
1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0
0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1
43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Appendix C

(k_C=number of control factors, k_n=number of noise factors, p=fraction index, DC=defining columns, J=aliasing index vector, A=($A_{3,0}$, $A_{2,1}$, $A_{1,2}$, $A_{0,3}$, $A_{4,0}$, $A_{3,1}$, $A_{2,2}$, $A_{1,3}$), α=clear estimation index; a cross array is indicated by *.)

C.1. 8-run single arrays

k_c	k_n	p	DC	N	J	A	α
*3	1	1	3	3	0 3 0 0 0 0	1 0 0 0 0 0 0 0	0 1 0 3 0
3	1	1	7	4	0 3 0 0 0 0	0 0 0 0 0 1 0 0	3 1 0 0 0
3	1	1	3	4	4 1 0 0 0 0	0 1 0 0 0 0 0 0	1 0 2 1 0
2	2	1	7	3 4	4 0 0 0 1 0	0 0 0 0 0 0 1 0	2 2 0 0 0
2	2	1	3	1 2	4 0 1 0 0 0	0 0 1 0 0 0 0 0	0 0 1 2 0
2	2	1	3	3 4	4 1 0 0 0 0	0 1 0 0 0 0 0 0	0 0 0 2 1
4	1	2	3 5	1	4 1 0 6 0 0	0 1 0 0 1 0 0 0	0 0 0 0 0
4	1	2	3 5	5	4 4 0 0 0 0	1 1 0 0 0 1 0 0	0 0 0 0 0
3	2	2	3 5	1 2	8 4 1 0 0 0	0 1 1 0 0 1 0 0	0 0 0 0 0
3	2	2	3 5	4 5	12 2 0 0 1 0	0 2 0 0 0 0 1 0	0 0 0 0 0
5	1	3	3 5 6	1	8 14 0 6 0 0	2 2 0 0 1 2 0 0	0 0 0 0 0
4	2	3	3 5 6	1 2	16 11 1 0 1 0	1 2 1 0 0 2 1 0	0 0 0 0 0
4	2	3	3 5 6	3 4	24 4 0 6 2 0	0 4 0 0 1 0 2 0	0 0 0 0 0
3	3	3	3 5 6	1 3 4	20 5 5 0 1 0	0 2 2 0 0 1 1 1	0 0 0 0 0
3	3	3	3 5 6	1 2 4	24 3 3 0 3 0	0 3 0 1 0 0 3 0	0 0 0 0 0
3	3	3	3 5 6	1 2 3	24 3 3 0 3 0	1 0 3 0 0 0 3 0	0 0 0 0 0
6	1	4	3 5 6 7	1	12 25 0 18 0 0	4 3 0 0 3 4 0 0	0 0 0 0 0
5	2	4	3 5 6 7	1 2	28 22 1 6 2 0	2 1 1 0 1 4 2 0	0 0 0 0 0
4	3	4	3 5 6 7	1 2 3	36 15 6 0 3 0	1 3 3 0 0 3 3 1	0 0 0 0 0
4	3	4	3 5 6 7	1 2 4	48 6 3 6 6 0	0 6 0 1 1 0 6 0	0 0 0 0 0

C.2. 16-run single arrays

k_c	k_n	p	DC	N	J	A	α
4	1	1	15	1	0 0 0 0 0 0	0 0 0 0 0 0 0 0	4 1 6 4 0
3	2	1	15	1 2	0 0 0 0 0 0	0 0 00 0 0 0 0	3 2 3 6 1
∗3	2	1	3	3 4	0 3 0 0 0 0	1 0 0 0 0 0 0 0	0 2 0 6 1
2	3	1	15	1 2 3	0 0 0 0 0 0	0 0 0 0 0 0 0 0	2 3 1 6 3
∗2	3	1	3	1 2 5	0 0 3 0 0 0	0 0 0 1 0 0 0 0	2 0 1 6 0
5	1	2	3 13	3	0 6 0 0 0 0	1 0 0 0 0 1 0 0	2 1 4 2 0
∗5	1	2	3 5	4	0 6 0 6 0 0	2 0 0 0 1 0 0 0	0 1 0 5 0
5	1	2	7 11	1	0 6 0 6 0 0	0 0 0 0 1 2 0 0	5 1 0 0 0
5	1	2	3 13	2	4 1 0 6 0 0	0 1 0 0 1 0 0 0	3 0 3 3 0
4	2	2	3 13	2 5	4 0 1 6 0 0	0 0 1 0 1 0 0 0	3 0 0 6 0
4	2	2	7 11	1 3	4 6 0 0 1 0	0 0 0 0 0 2 1 0	4 2 0 0 0
3	3	2	3 13	1 2 5	0 3 3 0 0 0	0 0 0 1 0 1 0 0	3 0 0 6 0
∗3	3	2	3 12	1 2 5	0 3 3 0 0 0	1 0 0 1 0 0 0 0	0 0 0 9 0
3	3	2	7 11	1 2 3	4 3 3 0 1 0	0 0 0 0 0 1 1 1	3 3 0 0 0
6	1	3	3 5 14	4	0 12 0 6 0 0	2 0 0 0 1 2 0 0	1 1 1 1 0
6	1	3	7 11 13	1	0 12 0 18 0 0	0 0 0 0 3 4 0 0	6 1 0 0 0
∗6	1	3	3 5 6	4	0 12 0 18 0 0	4 0 0 0 3 0 0 0	0 1 0 6 0
5	2	3	3 5 14	4 7	8 6 0 6 2 0	2 0 0 0 1 0 2 0	0 2 0 2 0
5	2	3	7 11 13	1 2	8 12 0 6 2 0	0 0 0 0 1 4 2 0	5 2 0 0 0
5	2	3	3 5 6	1 4	8 14 0 6 0 0	2 2 0 0 1 2 0 0	0 1 0 5 1
4	3	3	3 5 14	1 2 5	8 7 3 0 1 0	0 1 0 1 0 2 1 0	2 0 0 2 0
4	3	3	3 5 10	1 3 6	8 7 3 0 1 0	1 1 0 1 0 1 1 0	0 0 0 4 0
4	3	3	7 11 13	1 2 3	12 9 3 0 3 0	0 0 0 0 0 3 3 1	4 3 0 0 0
7	1	4	3 5 9 14	8	0 21 0 18 0 0	3 0 0 0 3 4 0 0	0 1 0 1 0
7	1	4	7 11 13 14	1	0 21 0 42 0 0	0 0 0 0 7 7 0 0	7 1 0 0 0
∗7	1	4	3 5 6 7	4	0 21 0 42 0 0	7 0 0 0 7 0 0 0	0 1 0 7 0
6	2	4	3 5 6 15	4 8	12 12 0 18 3 0	4 0 0 0 3 0 3 0	0 2 0 0 0
6	2	4	7 11 13 14	1 2	12 24 0 18 3 0	0 0 0 0 3 8 3 0	6 2 0 0 0
6	2	4	3 5 6 7	1 4	12 27 0 18 0 0	4 3 0 0 3 4 0 0	0 1 0 6 1
5	3	4	3 5 10 12	1 2 5	16 14 3 0 2 0	1 2 0 1 0 3 2 0	0 0 0 0 0
5	3	4	3 5 6 9	1 4 8	16 14 3 6 2 0	2 2 0 1 1 2 2 0	0 0 0 2 0
8	1	5	3 5 6 9 14	8	4 31 0 30 0 0	5 1 0 0 5 5 0 0	0 0 0 0 0
7	2	5	3 5 6 7 9	4 9	16 21 1 42 3 0	7 0 1 0 7 0 3 0	0 0 0 0 0
6	3	5	3 5 10 12 15	1 2 5	24 27 3 0 3 0	2 3 0 1 0 6 3 0	0 0 0 0 0
9	1	6	3 5 6 9 10 13	7	8 44 0 54 0 0	7 2 0 0 9 7 0 0	0 0 0 0 0
8	2	6	3 5 6 7 9 10	4 9	24 41 1 42 3 0	7 2 1 0 7 6 3 0	0 0 0 0 0
7	3	6	3 5 6 9 14 15	1 2 6	36 43 5 18 3 0	2 4 2 0 3 11 3 1	0 0 0 0 0
10	1	7	3 5 6 9 10 13 14	1	12 60 0 96 0 0	9 3 0 0 16 10 0 0	0 0 0 0 0
9	2	7	3 5 6 7 9 10 12	4 9	32 64 1 60 3 0	8 4 1 0 10 12 3 0	0 0 0 0 0
8	3	7	3 5 6 9 10 13 14	1 3 7	52 63 5 30 5 0	4 6 2 0 5 15 5 1	0 0 0 0 0
11	1	8	3 5 6 9 10 13 14 15	1	16 79 0 156 0 0	12 4 0 0 26 13 0 0	0 0 0 0 0
10	2	8	3 5 6 9 10 13 14 15	1 2	40 93 1 96 3 0	9 6 1 0 16 20 3 0	0 0 0 0 0
9	3	8	3 5 6 7 9 10 11 12	1 3 7	68 91 6 54 7 0	7 7 3 0 9 21 7 1	0 0 0 0 0
12	1	9	3 5 6 7 9 10 11 12 13	2	20 107 0 228 0 0	17 5 0 0 38 17 0 0	0 0 0 0 0
11	2	9	3 5 6 7 9 10 11 12 13	2 3	52 125 1 150 4 0	13 8 1 0 25 26 4 0	0 0 0 0 0
10	3	9	3 5 6 7 9 10 11 12 13	2 3 8	84 129 6 90 9 0	10 9 3 0 15 30 9 1	0 0 0 0 0
13	1	10	3 5 6 7 9 10 11 12 13 14	1	24 138 0 330 0 0	22 6 0 0 55 22 0 0	0 0 0 0 0
12	2	10	3 5 6 7 9 10 11 12 13 14	1 2	64 163 1 228 5 0	17 10 1 0 38 34 5 0	0 0 0 0 0
11	3	10	3 5 6 7 9 10 11 12 13 14	1 2 3	108 168 6 150 12 0	13 12 3 0 25 39 12 1	0 0 0 0 0
14	1	11	3 5 6 7 9 10 11 12 13 14 15	1	28 175 0 462 0 0	28 7 0 0 77 28 0 0	0 0 0 0 0
13	2	11	3 5 6 7 9 10 11 12 13 14 15	1 2	76 210 1 330 6 0	22 12 1 0 55 44 6 0	0 0 0 0 0
12	3	11	3 5 6 7 9 10 11 12 13 14 15	1 2 3	132 219 6 228 15 0	17 15 3 0 38 51 15 1	0 0 0 0 0

C.3. 32-run single arrays

k_C	k_n	p	DC	N	A	α
5	1	1	31	1	0 0 0 0 0 0 0 0	5 1 10 5 0
*5	1	1	15	5	0 0 0 0 0 0 0 0	5 1 10 5 0
4	2	1	31	1 2	0 0 0 0 0 0 0 0	4 2 6 8 1
*4	2	1	7	4 5	0 0 0 0 1 0 0 0	4 2 0 8 1
3	3	1	31	1 2 3	0 0 0 0 0 0 0 0	3 3 3 9 3
*3	3	1	3	1 2 6	0 0 0 1 0 0 0 0	3 0 3 9 0
6	1	2	7 27	4	0 0 0 0 1 0 0 0	6 1 9 6 0
*6	1	2	7 11	5	0 0 0 0 3 0 0 0	6 1 0 6 0
6	1	2	7 27	1	0 0 0 0 0 1 0 0	6 1 12 3 0
5	2	2	7 27	4 5	0 0 0 0 1 0 0 0	5 2 4 10 1
5	2	2	3 29	3 4	1 0 0 0 0 0 0 0	2 2 7 10 1
*5	2	2	3 5	4 5	2 0 0 0 1 0 0 0	0 2 0 10 1
4	3	2	7 27	4 5 7	0 0 0 0 1 0 0 0	4 3 0 12 3
4	3	2	3 29	1 2 6	0 0 0 1 0 0 0 0	4 0 6 12 0
*4	3	2	3 28	1 2 6	0 0 0 1 1 0 0 0	4 0 0 12 0
7	1	3	7 11 29	5	0 0 0 0 3 0 0 0	7 1 6 7 0
*7	1	3	7 11 13	5	0 0 0 0 7 0 0 0	7 1 0 7 0
7	1	3	7 11 29	1	0 0 0 0 1 2 0 0	7 1 11 2 0
6	2	3	7 11 29	5 8	0 0 0 0 3 0 0 0	6 2 0 12 1
6	2	3	7 11 29	1 5	0 0 0 0 1 2 0 0	6 2 5 7 1
*6	2	3	3 5 6	4 5	4 0 0 0 3 0 0 0	0 2 0 12 1
6	2	3	3 12 21	2 6	1 0 1 0 1 0 0 0	2 0 6 10 0
5	3	3	7 11 29	1 5 8	0 0 0 0 1 2 0 0	5 3 0 10 3
5	3	3	3 5 30	4 5 8	2 0 0 0 1 0 0 0	0 3 0 15 3
*5	3	3	3 5 24	4 5 8	2 0 0 1 1 0 0 0	0 0 0 15 0
8	1	4	7 11 19 29	9	0 0 0 0 6 0 0 0	8 1 0 8 0
8	1	4	7 11 13 30	5	0 0 0 0 7 0 0 0	8 1 7 8 0
*8	1	4	7 11 13 14	5	0 0 0 0 14 0 0 0	8 1 0 8 0
7	2	4	7 11 13 30	5 9	0 0 0 0 7 0 0 0	7 2 0 14 1
7	2	4	7 11 13 30	1 5	0 0 0 0 3 4 0 0	7 2 6 8 1
*7	2	4	3 5 6 7	4 5	7 0 0 0 7 0 0 0	0 2 0 14 1
6	3	4	7 11 13 30	1 5 9	0 0 0 0 3 4 0 0	6 3 0 12 3
6	3	4	3 5 6 31	4 5 9	4 0 0 0 3 0 0 0	0 3 0 18 3
*6	3	4	3 5 6 24	4 5 9	4 0 0 1 3 0 0 0	0 0 0 18 0
9	1	5	7 11 19 29 30	1	0 0 0 0 6 4 0 0	9 1 0 0 0
9	1	5	3 5 14 22 25	10	2 0 0 0 6 2 0 0	4 1 0 4 0
*9	1	5	3 5 9 14 15	5	4 0 0 0 14 0 0 0	0 1 0 9 0
8	2	5	3 13 21 25 28	2 6	0 0 1 0 14 0 0 0	7 0 0 14 0
8	2	5	7 11 19 29 30	1 3	0 0 0 0 3 6 1 0	8 2 0 0 0
7	3	5	3 12 21 26 31	1 2 6	1 0 0 1 1 6 0 0	4 0 0 6 0
7	3	5	3 13 21 25 28	1 2 6	0 0 0 1 7 7 0 0	7 0 0 14 0
*7	3	5	3 5 6 7 24	4 5 10	7 0 0 1 7 0 0 0	0 0 0 21 0
10	1	6	3 5 14 22 24 31	11	4 0 0 0 6 4 0 0	0 1 1 0 0
10	1	6	7 11 13 14 19 21	5	0 0 0 0 18 8 0 0	10 1 0 0 0
*10	1	6	3 5 6 9 14 15	5	8 0 0 0 18 0 0 0	0 1 0 10 0
9	2	6	3 5 9 14 15 18	5 11	4 0 1 0 14 0 1 0	0 0 0 12 0
9	2	6	3 5 14 22 26 28	1 6	0 1 1 0 14 4 0 0	6 0 0 6 0
8	3	6	3 5 14 22 24 31	1 2 6	2 1 0 1 1 8 1 0	1 0 0 1 0
8	3	6	3 5 14 22 26 29	1 2 6	0 1 0 1 5 10 1 0	6 0 0 6 0
8	3	6	3 5 6 9 14 25	5 9 11	5 1 0 1 5 5 1 0	0 0 0 12 0
11	1	7	3 5 10 12 19 21 30	12	6 0 0 0 10 6 0 0	0 1 1 0 0
11	1	7	7 11 13 14 19 21 25	5	0 0 0 0 26 12 0 0	11 1 0 0 0
*11	1	7	3 5 6 9 10 13 14	5	12 0 0 0 26 0 0 0	0 1 0 11 0
10	2	7	3 5 6 9 14 15 18	5 12	8 0 1 0 18 0 2 0	0 0 0 10 0
10	2	7	3 5 9 14 22 26 28	1 6	0 2 1 0 18 8 0 0	5 0 0 5 0
9	3	7	3 5 10 12 19 21 30	2 4 8	3 2 0 1 3 11 2 0	1 0 0 1 0
9	3	7	3 5 6 9 22 26 29	1 4 9	3 2 0 1 5 11 2 0	2 0 0 4 0
12	1	8	3 5 10 12 19 21 25 30	13	8 0 0 0 15 8 0 0	0 1 0 0 0
12	1	8	7 11 13 14 19 21 22 25	13	0 0 0 0 39 16 0 0	12 1 0 0 0
*12	1	8	3 5 6 9 10 13 14 15	5	16 0 0 0 39 0 0 0	0 1 0 12 0
11	2	8	3 5 6 9 10 13 14 31	5 13	12 0 0 0 26 0 4 0	0 2 0 6 0
11	2	8	3 5 6 9 10 13 14 17	5 13	12 0 1 0 26 0 3 0	0 0 0 8 0
11	2	8	3 5 9 14 17 22 26 28	1 6	0 3 1 0 26 12 0 0	4 0 0 4 0
10	3	8	3 5 10 12 19 21 25 30	1 2 6	4 3 0 1 5 15 3 0	1 0 0 0 0
13	1	9	3 5 9 14 18 20 23 24 27	9	10 0 0 0 23 12 0 0	0 1 0 0 0
13	1	9	7 11 13 14 19 21 22 25 26	14	0 0 0 0 55 22 0 0 0	13 1 0 0 0
*13	1	9	3 5 6 7 9 10 11 12 13	5	22 0 0 0 55 0 0 0	0 1 0 13 0
12	2	9	3 5 6 9 10 13 14 15 17	5 14	16 0 1 0 39 0 4 0	0 0 0 6 0
11	3	9	3 5 9 14 18 20 26 29 31	1 2 6	5 4 0 1 10 20 4 0	0 0 0 0 0
14	1	10	3 5 9 14 18 20 23 24 27 29	9	12 0 0 0 33 16 0 0	0 1 0 0 0
14	1	10	7 11 13 14 19 21 22 25 26 28	1	0 0 0 0 77 28 0 0	14 1 0 0 0
*14	1	10	3 5 6 7 9 10 11 12 13 14	5	28 0 0 0 77 0 0 0	0 1 0 14 0
13	2	10	3 5 6 7 9 10 11 12 13 30	5 15	22 0 0 0 55 0 6 0	0 2 0 2 0
12	3	10	3 5 6 9 14 18 23 25 29 30	1 4 9	8 5 0 1 14 26 5 0	0 0 0 0 0
15	1	11	3 5 9 14 18 20 23 24 27 29 31	9	15 0 0 0 45 20 0 0	0 1 0 0 0
15	1	11	7 11 13 14 19 21 22 25 26 28 31	1	0 0 0 0 105 35 0 0	15 1 0 0 0
*15	1	11	3 5 6 7 9 10 11 12 13 14 15	5	35 0 0 0 105 0 0 0	0 1 0 15 0
14	2	11	3 5 6 7 9 10 11 12 13 14 31	5 16	28 0 0 0 77 0 7 0	0 2 0 0 0
13	3	11	3 5 6 9 10 12 17 18 21 30 31	1 15 16	11 6 0 1 22 33 6 0	0 0 0 0 0

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