DEPT. OF MATH. UNIVERSITY OF OSLO
PURE MATHEMATICS
ISBN 82–553–1364–8 no 12
ISSN 0806–2439 April 2003

Optimal Smooth Portfolio Selection for An Insider

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Last revised in July 2007

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Abstract

We study the optimal portfolio problem for an insider, in the case that the performance is measured in terms of the logarithm of the terminal wealth minus a term measuring the roughness and the growth of the portfolio. We give explicit solutions in some cases. Our method uses stochastic calculus of forward integrals.

1 Introduction

There has been an increasing interest in the insider trading in recent years (see for example [1]–[6] and [8]–[10] and the references therein). By an *insider* in a financial market we mean a certain investor who possesses more information than the information generated by the financial market itself. An insider may be for example an executive or simply an employee of a company. In probabilistic terminology information is generally represented by a filtration. Usually an investor can only use the filtration generated by the market to make a decision. We call such investors *honest*. An insider has a larger filtration (more information) available to him and can use this larger filtration to make his decision (for example to maximize his portfolio).

Mathematics Subject Classifications (2000). Primary 60H40, 60H05, 60G15; Secondary 91B28.
Key words and phrases: Insider trading, optimal portfolio, enlargement of filtration, log utility, information flow.
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To simplify our presentation we assume that the market consists of the following two assets over the time period $[0,T]$. The first one is a bond whose price is determined by a stochastic process

$$dS_0(t) = r(t)S_0(t)dt\,,\ 0 \le t \le T\,.$$

Another asset is the stock whose price follows the following geometric Brownian motion

$$dS_1(t) = S_1(t)\left[\mu(t)dt + \sigma(t)dB(t)\right]\,,\ 0 \le t \le T\,,$$

where $r(t)$, $\mu(t)$, and $\sigma(t)$ are deterministic functions and $B(t) = B_t(\omega)\,, 0 \le t \le T$, is a Brownian motion and $dB(t)$ denotes the Itô type stochastic differential. Denote $\mathcal{F}_t = \sigma(B_s, 0 \le s \le t)$, the information generated by the market. Assume for example that at the beginning ($t=0$) the insider knows in addition the future value of the underlying Brownian motion at time T_0, where $T_0 > T$. Then his information filtration is given by $\mathcal{G}_t = \sigma(B_s, 0 \le s \le t) \vee \sigma(B_{T_0})$, the filtration generated by the Brownian motion up to time t and B_{T_0}. The insider may use this filtration (rather than as usual use only the filtration $\mathcal{F}_t$) to optimize his portfolio.

More explicitly, let us express the portfolio in terms of the fraction $\pi(t)$ of the total wealth invested in the stocks at time t. Let $X^{(\pi)}(t)$ denote the corresponding wealth at time t. In [9] the problem of maximizing the expectation of the logarithmic utility of terminal wealth,

$$\Phi_{\mathcal{G}} := \sup_{\pi}\left\{\mathbb{E}\,\left[\log(X^{(\pi)}(T))\right]\right\} \tag{1.1}$$

is considered, where the supremum is taken over all $\mathcal{G}_t$-adapted portfolios $\pi(\cdot)$. They prove that in this case the optimal insider portfolio is

$$\pi^*(t) = \frac{\mu(t) - r(t)}{\sigma^2(t)} + \frac{B(T_0) - B(t)}{\sigma(t)(T_0 - t)}\,. \tag{1.2}$$

Moreover, the corresponding maximal expected utility $\Phi_{\mathcal{G}}$ is given by

$$\Phi_{\mathcal{G}} = \mathbb{E}\,\left[\int_0^T \left\{r(s) + \frac{1}{2}\frac{(\mu(s) - r(s))^2}{\sigma^2(s)} + \frac{1}{2(T_0 - s)}\right\} ds\right]\,;\quad T_0 \ge T\,.$$

In particular, if $T_0 = T$ we get

$$\Phi_{\mathcal{G}} = \infty\,.$$

This is clearly an unrealistic result. If $T_0 = T$ we see by (1.2) that the optimal portfolio π^* needed to achieve $\Phi_{\mathcal{G}} = \infty$ will converge towards the derivative of $B(t)$ at $t = T_0^-$. Thus $\pi^*(t)$ will consist of more and more wild fluctuations as $t \to T_0^-$. This is both practically impossible and also undesirable from the point of view of the insider: He does not want to expose a too conspicuous portfolio, compared to that of the honest trader, which in the optimal case is just

$$\pi^*_{\text{honest}}(t) = \frac{\mu(t) - r(t)}{\sigma^2(t)}\,.$$

To model this constraint we propose to modify the problem (1.1) to the following:

PROBLEM 1.1 Find $\pi^* \in \mathcal{A}_\mathcal{G}$ and Φ such that

$$\begin{aligned}\Phi &= \sup_{\pi \in \mathcal{A}_\mathcal{G}} \mathbb{E}\left[\log\left(X^{(\pi)}(T)\right) - \int_0^T |\mathbb{Q}\pi(s)|^2 ds\right] \\ &= \mathbb{E}\left[\log\left(X^{(\pi^*)}(T)\right) - \int_0^T |\mathbb{Q}\pi^*(s)|^2 ds\right],\end{aligned}$$

where $\mathcal{A}_\mathcal{G}$ is a suitable family of admissible $\mathcal{G}_t$-adapted portfolios π. Here $\mathbb{Q} : \mathcal{A}_\mathcal{G} \to \mathcal{A}_\mathcal{G}$ is some linear operator measuring the size and/or the fluctuations of the portfolio. For example we could have

$$\mathbb{Q}\pi(s) = \lambda_1(s)\pi(s)\,, \tag{1.3}$$

where $\lambda_1(s) \geq 0$ is some given weight function. This models the situation where the insider is penalized for large volumes of trade.

An alternative choice of $\mathbb{Q}$ would be

$$\mathbb{Q}\pi(s) = \lambda_2(s)\pi'(s)\,, \tag{1.4}$$

for some weight function $\lambda_2(s) \geq 0$. ($\pi'(s) = \frac{d}{ds}\pi(s)$.) In this case the insider is penalized for large trade fluctuations. Other choices of $\mathbb{Q}$ are also possible, including combinations of (1.2) and (1.3).

We will return to Problem 1.1 in Section 3, after giving a brief introduction to the forward integral.

2 The Forward Integral

In general $B(t)$ need not be a semimartingale with respect to a bigger filtration $\mathcal{G}_t \supset \mathcal{F}_t$. A simple example is

$$\mathcal{G}_t = \mathcal{F}_{t+\delta}; \qquad t \geq 0$$

where $\delta > 0$ is a constant.

Therefore, to be able to deal with corresponding (anticipating) $\mathcal{G}_t$-adapted integrands $\phi(t, \omega)$, we must go beyond the semimartingale integral context. Following [3] we propose to use *the forward* integral to model such situations. This integral extends the semimartingale integral in the sense that the two integrals coincide if $B(t)$ is a semimartingale with respect to $\mathcal{G}_t$.

In this section we briefly review some basic concepts and results on forward integrals. We refer to [3] for motivations for using forward integrals in insider trading, and to [12], [13] for more information about forward integrals.

Definition 2.1 ([12]) *Let $\phi(t,\omega)$ be a measurable process (not necessarily adapted). Then the forward stochastic integral of ϕ is defined as*

$$\int_0^\infty \phi(t,\omega)d^-B(t) = \lim_{\varepsilon\to 0}\int_0^\infty \phi(t,\omega)\frac{B(t+\varepsilon)-B(t)}{\varepsilon}dt$$

if the convergence is in probability.

Let $\pi : 0 = t_0 < t_1 < \cdots < t_n = t$ be a partition of $[0,T]$ and denote $|\pi| = \max_{0\le j\le n-1}(t_{j+1}-t_j)$. It is easy to see that if ϕ is càdlàg, then

$$\int_0^T \phi(t,\omega)d^-B(t) = \lim_{|\pi|\to 0}\sum_{j=0}^{n-1}\phi(t_j)(B(t_{j+1})-B(t_j))\,. \tag{2.1}$$

(See [3] for details.)

Definition 2.2 *By a (1-dimensional) forward process we mean a process $X(t) = X(t,\omega)$ of the form*

$$X(t) = x + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)d^-B(s)\,;\ t > 0\,, \tag{2.2}$$

where $u(s,\omega)$ and $v(s,\omega)$ are measurable processes (not necessarily $\mathcal{F}_t$-adapted) such that

$$\int_0^t |u(s,\omega)|ds < \infty\,, \quad a.s. \quad for\ all\ \ t > 0$$

and the Itô forward integral

$$\int_0^t v(s,\omega)d^-B(s)$$

exists for all $t > 0$.

In accordance with the classical Itô process notation, we use the short hand notation

$$d^-X(t) = u(t)dt + v(t)d^-B(t)$$

for the integral equation (2.2).

Theorem 2.3 ([13]) *(An Itô formula for forward processes) Let*

$$d^-X(t) = u(t)dt + v(t)d^-B(t)$$

be a forward process. Let $f \in C^2(\mathbb{R})$ and define

$$Y(t) = f(X(t))\,.$$

Then $Y(t)$ is also a forward process and

$$d^-Y(t) = f'(X(t))d^-X(t) + \frac{1}{2}f''(X(t))v^2(t)dt$$

As an application of the Itô formula for forward integrals we get

Corollary 2.4 ([3]) *Let $u(t)$, $v(t)$ be measurable processes such that the integrals*

$$\int_0^t (|u(s)|^2 + |v(s)|^2) ds \quad \textit{and} \quad \int_0^t v(s) d^- B(s) \quad \textit{exist for all} \;\; t > 0\,.$$

Then the forward stochastic differential equation

$$dX(t) = X(t) \left[u(t)dt + v(t)d^- B(t)\right] \,; \quad X(0) = x > 0$$

has the unique solution

$$X(t) = x \exp \left(\int_0^t \left(u(s) - \frac{1}{2} v^2(s) \right) ds + \int_0^t v(s) d^- B(s) \right) \,.$$

We also need the following, which follows easily from the definition:

Lemma 2.5 *Suppose $\phi(t)$ is forward integrable and G is an $\mathcal{F}_T$-measurable random variable. Then*

$$\int_0^T G\phi(t) d^- B(t) = G \int_0^T \phi(t) d^- B(t).$$

3 Optimal Smooth Portfolio for An Insider

We now return to Problem 1.1 in the introduction. So we assume the market consists of the two investment possibilities:
(i) A bond, with price given by

$$dS_0(t) = r(t) S_0(t) dt\,; \quad S_0(0) = 1\,; \quad 0 \le t \le T\,.$$

(ii) A stock, with price given by

$$dS_1(t) = S_1(t) \left[\mu(t)dt + \sigma(t) dB(t)\right] \,; \quad 0 \le t \le T\,,$$

where $T > 0$ is constant and $r(t)$, $\mu(t)$ and $\sigma(t)$ are given $\mathcal{F}_t$-adapted processes. We assume that

$$\mathbb{E} \left[\int_0^T \left\{ |\mu(t)| + |r(t)| + \sigma^2(t) \right\} dt \right] < \infty$$

$$\sigma(t) \neq 0 \quad \text{for a.a.} \quad (t,\omega) \in [0,T] \times \Omega\,.$$

Let $\mathcal{G}_t \supset \mathcal{F}_t$ be the information filtration available to the insider and let $\pi(t)$ be the portfolio chosen by the insider, measured in terms of the fraction of the total wealth $X(t) = X^{(\pi)}(t)$ invested in the stock at time $t \in [0,T]$. Then the corresponding wealth $X(t) = X^{(\pi)}(t)$ at time t is modeled by the forward differential equation

$$\begin{aligned} dX(t) &= (1 - \pi(t)) X(t) r(t) dt + \pi(t) X(t) \left[\mu(t)dt + \sigma(t) d^- B(t)\right] \\ &= X(t) \left[[r(t) + (\mu(t) - r(t))\pi(t)] dt + \sigma(t)\pi(t) d^- B(t) \right] \,. \end{aligned} \tag{3.1}$$

For simplicity we assume $X(0) = 1$. The motivation for using this forward integral model for the anticipating stochastic differential equation (3.1) is the formula (2.1), which expresses the forward integral as a limit of Riemann sums of the Itô type, i.e. where the i-th term has the form $\phi(t_i)(B(t_{i+1}) - B(t_i))$ with ϕ evaluated at the *left* end point t_i of the interval $[t_i, t_{i+1}]$. Moreover, if $B(t)$ happens to be a semimartingale with respect to $\mathcal{G}_t$, then indeed the forward integral coincides with the semimartingale integral. See [3] and [12], [13] for more details on this.

We now specify the set $\mathcal{A} = \mathcal{A}_{\mathcal{G}}$ of the admissible portfolios π as follows:

Definition 3.1 *In the following we let $\mathcal{A} = \mathcal{A}_{\mathcal{G}}$ denote a linear space of stochastic processes $\pi(t)$ such that (3.2)–(3.5) hold, where*

$$\begin{aligned} &\pi(t) \ \textit{ is } \mathcal{G}_t \ - \ \textit{adapted and the } \ \sigma - \ \textit{algebra generated by } \ \{\pi(t)\,;\pi\in\mathcal{A}\} \\ &\qquad\qquad \textit{is equal to } \ \mathcal{G}_t\,, \ \textit{for all } t\in[0,T], \end{aligned} \tag{3.2}$$

$$\pi \ \textit{ belongs to the domain of } \ \mathbb{Q}, \tag{3.3}$$

$$\sigma(t)\pi(t) \ \textit{ is forward integrable}, \tag{3.4}$$

$$\mathbb{E}\left[\int_0^T |\mathbb{Q}\pi(t)|^2 dt\right] < \infty\,. \tag{3.5}$$

With these definitions we can now specify Problem 1.1 as follows:

Problem 3.2 *Find Φ and $\pi^* \in \mathcal{A}$ such that*

$$\Phi = \sup_{\pi\in\mathcal{A}} J(\pi) = J(\pi^*)\,,$$

where

$$J(\pi) = \mathbb{E}\left[\log(X^{(\pi)}(T)) - \frac{1}{2}\int_0^T |\mathbb{Q}\pi(s)|^2 ds\right]\,,$$

$\mathbb{Q} : \mathcal{A} \to \mathcal{A}$ being a given linear operator ($\mathbb{E}$ denotes the expectation with respect to P). We call Φ the value of the insider and $\pi^* \in \mathcal{A}$ an optimal portfolio (if it exists).

We now proceed to solve Problem 3.2: Using Corollary 2.4 we get that the solution of (3.1) is

$$\begin{aligned} X(t) \;&=\; \exp\Bigg(\int_0^t \left\{r(s) + (\mu(s) - r(s))\pi(s) - \frac{1}{2}\sigma^2(s)\pi^2(s)\right\} ds \\ &\qquad + \int_0^t \sigma(s)\pi(s)d^-B(s)\Bigg). \end{aligned}$$

Therefore we get

$$
\begin{aligned}
J(\pi) &= \mathbb{E}\left[\int_0^T \left\{r(t) + (\mu(t) - r(t))\pi(t) - \frac{1}{2}\sigma^2(t)\pi^2(t)\right\} dt\right. \\
&\qquad \left. + \int_0^T \sigma(t)\pi(t) d^- B(t) - \frac{1}{2}\int_0^T |\mathbb{Q}\pi(t)|^2 dt\right]. \qquad (3.6)
\end{aligned}
$$

To maximize $J(\pi)$ we use a calculus of variation technique, as follows: Suppose an optimal insider portfolio $\pi = \pi^*$ exists (in the following we omit the *). Let $\theta \in \mathcal{A}$ be another portfolio. Then the function

$$f(y) := J(\pi + y\theta)\,;\ y \in \mathbb{R}$$

is maximal for $y = 0$ and hence

$$
\begin{aligned}
0 &= f'(0) = \frac{d}{dy}\left[J(\pi + y\theta)\right]_{y=0} \\
&= \mathbb{E}\left[\int_0^T \left\{(\mu(t) - r(t))\theta(t) - \sigma^2(t)\pi(t)\theta(t)\right\} dt\right. \\
&\qquad \left. + \int_0^T \sigma(t)\theta(t) d^- B(t) - \int_0^T \mathbb{Q}\pi(t)\mathbb{Q}\theta(t) dt\right]. \qquad (3.7)
\end{aligned}
$$

Let $\mathbb{Q}^*$ denote the adjoint of $\mathbb{Q}$ in the Hilbert space $L^2([0,T] \times \Omega)$, namely,

$$\mathbb{E}\left[\int_0^T \alpha(t)(\mathbb{Q}\beta)(t) dt\right] = \mathbb{E}\left[\int_0^T (\mathbb{Q}^*\alpha)(t)\beta(t) dt\right]$$

for all α and β in $\mathcal{A}$. Then we can rewrite (3.7) as

$$\mathbb{E}\left[\int_0^T \left\{\mu(t) - r(t) - \sigma^2(t)\pi(t) - \mathbb{Q}^*\mathbb{Q}\pi(t)\right\}\theta(t) dt + \int_0^T \sigma(t)\theta(t) d^- B(t)\right] = 0\,. \qquad (3.8)$$

Now we apply this to a special choice of θ: Fix $t \in [0,T]$ and $h > 0$ such that $t + h < T$ and choose

$$\theta(s) = \theta_0(t)\chi_{[t,t+h]}(s)\,;\ s \in [0,T]\,,$$

where $\theta_0(t)$ is $\mathcal{G}_t$-measurable. Then by Lemma 2.5 we have

$$
\begin{aligned}
\mathbb{E}\left[\int_0^T \sigma(s)\theta(s) d^- B(s)\right] &= \mathbb{E}\left[\int_t^{t+h} \sigma(s)\theta_0(t) d^- B(s)\right] \\
&= \mathbb{E}\left[\theta_0(t)\int_t^{t+h} \sigma(s) dB(s)\right].
\end{aligned}
$$

Combining this with (3.8) we get

$$\mathbb{E}\left[\left(\int_t^{t+h} \left\{\mu(s) - r(s) - \sigma^2(s)\pi(s) - \mathbb{Q}^*\mathbb{Q}\pi(s)\right\} ds + \int_t^{t+h} \sigma(s) dB(s)\right)\theta(t)\right] = 0\,.$$

Since this holds for all such $\theta(t)$ we conclude that

$$\mathbb{E}\ [M(t+h)-M(t)|\mathcal{G}_t]=0\,,$$

where

$$M(t):=\int_0^t\left\{\mu(s)-r(s)-\sigma^2(s)\pi(s)-\mathbb{E}\ [\mathbb{Q}^*\mathbb{Q}\pi(s)|\mathcal{G}_s]\right\}ds+\int_0^t\sigma(s)dB(s)\,. \tag{3.9}$$

Since $\sigma\neq 0$ this proves

Theorem 3.3 *Suppose an optimal insider portfolio $\pi\in\mathcal{A}$ for Problem 3.2 exists. Then*

$$dB(t)=d\hat{B}(t)-\frac{1}{\sigma(t)}\left\{\mu(t)-\rho(t)-\sigma^2(t)\pi(t)-\mathbb{E}\ [\mathbb{Q}^*\mathbb{Q}\pi(t)|\mathcal{G}_t]\right\}dt$$

where $\hat{B}(t):=\int_0^t\sigma^{-1}(s)dM(s)$ is a $\mathcal{G}_t$-Brownian motion. In particular,

$$B(t)\quad \textit{is a semimartingale with respect to}\quad \mathcal{G}_t\,.$$

We now use this to find an equation for an optimal portfolio π:

Theorem 3.4 *Assume that there exists a process $\gamma_t(s,\omega)$ such that $\gamma_t(s)$ is $\mathcal{G}_t$-measurable for all $s\le t$ and*

$$t\to\int_0^t\gamma_t(s)ds\quad \textit{is of finite variation a.s.}$$

and

$$N(t):=B(t)-\int_0^t\gamma_t(s)ds\quad \textit{is a martingale with respect to}\quad \mathcal{G}_t\,. \tag{3.10}$$

Assume that $\pi\in\mathcal{A}$ is optimal. Then

$$\sigma^2(t)\pi(t)+\mathbb{E}\ [\mathbb{Q}^*\mathbb{Q}\pi(t)|\mathcal{G}_t]=\mu(t)-r(t)+\sigma(t)\frac{d}{dt}\left(\int_0^t\gamma_t(s)ds\right)\,. \tag{3.11}$$

Proof By comparing (3.9) and (3.10) we get that

$$\sigma(t)dN(t)=dM(t),$$

i.e.

$$-\sigma(t)\frac{d}{dt}\left(\int_0^t\gamma_t(s)ds\right)=\mu(t)-r(t)-\sigma^2(t)\pi(t)-\mathbb{E}\ [\mathbb{Q}^*\mathbb{Q}\pi(t)|\mathcal{G}_t]\,.$$

□

Next we turn to a partial converse of Theorem 3.4:

Theorem 3.5 *Suppose (3.10) holds. Let $\pi(t)$ be a process solving the equation (3.11). Suppose $\pi\in\mathcal{A}$. Then π is optimal for Problem 3.2.*

Proof Substituting

$$dB(t) = dN(t) + \frac{d}{dt}\left(\int_0^t \gamma_t(s)ds\right)dt$$

and

$$\sigma(t)\pi(t)d^- B(t) = \sigma(t)\pi(t)dN(t) + \sigma(t)\pi(t)\frac{d}{dt}\left(\int_0^t \gamma_t(s)ds\right)dt$$

into (3.6) we get

$$\begin{aligned} J(\pi) &= \mathbb{E}\left[\int_0^T \Big\{r(t) + (\mu(t) - r(t))\pi(t) - \frac{1}{2}\sigma^2(t)\pi^2(t)\right. \\ &\qquad \left. + \sigma(t)\pi(t)\frac{d}{dt}\left(\int_0^t \gamma_t(s)ds\right) - \frac{1}{2}|\mathbb{Q}\pi(t)|^2\Big\}dt\right]. \end{aligned} \tag{3.12}$$

This is a concave functional of π, so if we can find $\pi = \pi^* \in \mathcal{A}$ such that

$$\frac{d}{dy}\left[J(\pi^* + y\theta)\right]_{y=0} = 0 \quad \text{for all} \quad \theta \in \mathcal{A}\,,$$

then π^* is optimal. By a computation similar to the one leading to (3.8) we get

$$\begin{aligned} \frac{d}{dy}\left[J(\pi^* + y\theta)\right]_{y=0} &= \mathbb{E}\left[\int_0^T \Big\{\mu(t) - r(t) - \sigma^2(t)\pi^*(t)\right. \\ &\qquad \left. + \sigma(t)\frac{d}{dt}\int_0^t \gamma_t(s)ds - \mathbb{Q}^*\mathbb{Q}\pi(t)\Big\}\,\theta(t)dt\right]. \end{aligned}$$

This is 0 if $\pi = \pi^*$ solves equation (3.11). □

We now apply this to some examples:

Example 3.6 *Choose*

$$\mathbb{Q}\pi(t) = \lambda_1(t)\sigma(t)\pi(t) \tag{3.13}$$

where $\lambda_1(t) \geq 0$ is deterministic.

Then (3.11) takes the form

$$\sigma^2(t)\pi(t) + \lambda_1^2(t)\sigma^2(t)\pi(t) = \mu(t) - r(t) + \sigma(t)\frac{d}{dt}\int_0^t \gamma_t(s)ds$$

or

$$\pi(t) = \pi^*(t) = \frac{\mu(t) - r(t) + \sigma(t)\frac{d}{dt}\int_0^t \gamma_t(s)ds}{\sigma^2(t)[1 + \lambda_1^2(t)]}\,. \tag{3.14}$$

Substituting this into the formula (3.12) for $J(\pi)$ we obtain

Theorem 3.7 *Suppose (3.10) and (3.13) hold. Let $\pi^*(t)$ be given by (3.14). If $\pi \in \mathcal{A}$ then π^* is optimal for Problem 3.2. Moreover, the insider value is*

$$\begin{aligned} \Phi &= J(\pi^*) \\ &= \mathbb{E}\left[\int_0^T \left\{ r(t) + \frac{1}{2}(1+\lambda_1^2(t))^{-1}\left(\frac{\mu(t)-r(t)}{\sigma(t)} + \frac{d}{dt}\int_0^t \gamma_t(s)ds\right)^2\right\} dt\right]. \end{aligned} \tag{3.15}$$

In particular, if we consider the case mentioned in the introduction, where

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(B(T_0)) \quad \text{for some } T_0 > T\,,$$

then, by a result of Itô [7],

$$\gamma_t(s) = \gamma(s) = \frac{B(T_0) - B(s)}{T_0 - s}$$

and (3.14) becomes

$$\pi^*(t) = \sigma^{-2}(t)\left[1+\lambda_1^2(t)\right]^{-1}\left[\mu(t) - r(t) + \frac{\sigma(t)}{T_0 - t}\left(B(T_0) - B(t)\right)\right].$$

The corresponding value is, by (3.15),

$$J(\pi^*) = \mathbb{E}\left[\int_0^T \left\{ r(t) + \frac{1}{2}(1+\lambda_1^2(t))^{-1}\left(\frac{\mu(t)-r(t)}{\sigma(t)} + \frac{B(T_0)-B(t)}{T_0 - t}\right)^2\right\} dt\right].$$

In particular, we see that if $\sigma(t) \geq \sigma_0 > 0$ and

$$\lambda_1(t) = (T_0 - t)^{-\beta} \quad \text{for some constant} \quad \beta > 0\,, \tag{3.16}$$

then

$$J(\pi^*) \leq C_1 + C_2 \int_0^T (T_0 - t)^{-1+2\beta} dt < \infty\,,$$

even if $T_0 = T$.

Thus if we penalize large investments near $t = T_0$ according to (3.16) the insider gets a finite value even if $T_0 = T$.

Example 3.8 *Next we put*

$$\mathbb{Q}\pi(t) = \pi'(t) \quad \left(= \frac{d}{dt}\pi(t)\right). \tag{3.17}$$

This means that the insider is being penalized for large portfolio fluctuations. Choose $\mathcal{A}$ to be the set of all continuously differentiable processes $\pi(t)$ satisfying (3.2)–(3.5) and in addition

$$\pi(0) = \pi(T) = 0 \quad a.s. \tag{3.18}$$

For simplicity assume that

$$\sigma(t) \equiv 1 .$$

Then (3.11) gets the form

$$\pi(t) - \pi''(t) = a(t) ,$$

where

$$a(t) = \mu(t) - r(t) + \frac{d}{dt}\left(\int_0^t \gamma_t(s)ds\right) .$$

Using the *variation of parameter* method we obtain the solution

$$\pi(t) = \int_0^t \sinh(t-s)a(s)ds + K\sinh(t) , \tag{3.19}$$

where, as usual, $\sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$, $x \in \mathbb{R}$, is the hyperbolic sinus function and the constant K is chosen such that $\pi(T) = 0$. In particular, if we again consider the case

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(B(T_0)) , \quad T_0 > T ,$$

so that

$$\gamma_t(s) = \gamma(s) = \frac{B(T_0) - B(s)}{T_0 - s} , \quad 0 \le s \le T .$$

we obtain, by (3.19),

$$\pi(t) = \int_0^t \sinh(t-s)\left[\mu(s) - r(s) + \frac{B(T_0) - B(s)}{T_0 - s}\right] ds + K\sinh(t) . \tag{3.20}$$

The corresponding value is by (3.12),

$$\begin{aligned} J(\pi) \;\; = \;\; \mathbb{E}\Bigg[\int_0^T \bigg\{ & r(t) + (\mu(t) - r(t))\,\pi(t) - \frac{1}{2}\pi^2(t) \\ & +\pi(t)\frac{B(T_0) - B(t)}{T_0 - t} - \frac{1}{2}\left(\pi'(t)\right)^2\bigg\} dt\Bigg] . \end{aligned}$$

Note that if $0 \le t \le T < T_0$ then

$$\begin{aligned} \mathbb{E}\left[\pi(t)\frac{B(T_0) - B(t)}{T_0 - t}\right] \;\; & \le \;\; \mathbb{E}\left[\int_0^t \sinh(t-s)\frac{(B(T_0) - B(s))\,(B(T_0) - B(t))}{(T_0 - s)\,(T_0 - t)}ds\right] \\ & = \;\; \int_0^t \sinh(t-s)\frac{ds}{T_0 - s} . \end{aligned}$$

Therefore

$$J(\pi) \le \int_0^T \Big(\int_0^t \sinh(t-s)\frac{ds}{T_0 - s}\Big)dt \le \int_0^T \frac{\cosh(T-s) - 1}{T - s}ds \quad \text{for all } T_0 > T.$$

We have proved:

Theorem 3.9 *Suppose $\mathbb{Q}\pi(t) = \pi'(t)$ and $\mathcal{A}$ is chosen as in (3.17), (3.18) and assume that $\sigma(t) = 1$. Then the optimal insider portfolio is given by (3.19). In particular, if we choose*

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(B(T_0)) \quad \text{with} \quad T_0 > T\,,$$

then the optimal portfolio π is given by (3.20) and the corresponding insider value $J(\pi)$ is uniformly bounded for $T_0 > T$.

Remark 3.10 *Both of Examples 3.6 and 3.8 yield ways to penalize the insider investors so that he would not obtain infinite utility. In Example 3.6, $\lambda_1(t) = (T_0 - t)^{-\beta}$ for some $\beta > 0$. To use this penalization, one needs to know T_0. In Example 3.8, T_0 is not required to be known.*

Acknowledgments: Y. Hu is supported in part by the National Science Foundation under grants no. DMS 0204613 and DMS 0504783.

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