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# OPTIMAL STOPPING AND A MARTINGALE APPROACH TO THE PENALTY METHOD

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**1. Introduction.** Let $(\Omega, F, P)$ be a complete probability space equipped with an increasing, right-continuous family of complete sub-$\sigma$-fields $(F_t)_{t\geqq 0}$ such that $F = \bigvee_{t\geqq 0} F_t$. Let $X$ be an $(F_t)$-adapted and right-continuous process such that

$$(1) \qquad E\left[\sup_t X_t^+\right] < \infty\,, \quad E[X_t^-] < \infty \quad (t \geqq 0)$$

where $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$. We define the following classes of stopping times:

$$(2) \qquad \begin{aligned} \bar{C}_t &= \{\tau \mid \text{stopping time},\ \tau \geqq t,\ E[X_\tau^-] < \infty\}\,, \\ C_t &= \{\tau \in \bar{C}_t \mid \text{finite}\}\,, \end{aligned}$$

for each $t \geqq 0$, where $X_\infty$ is interpreted as $\limsup_{t\to\infty} X_t$. A stopping time $\sigma \in \bar{C}_0$ is said to be optimal if $E[X_\sigma] = \sup_{\tau\in\bar{C}_0} E[X_\tau]$. Snell's envelopes $Y$ and $\bar{Y}$ are defined as follows:

$$Y_t = \operatorname*{ess\,sup}_{\tau\in C_t} E[X_\tau \mid F_t]\,, \quad \bar{Y}_t = \operatorname*{ess\,sup}_{\tau\in\bar{C}_t} E[X_\tau \mid F_t]\,.$$

Our aim is to give some extensions of the results in optimal stopping and the penalty method presented by Fakeev [4] and Stettner and Zabczyk [9]. In Section 2, we show that the right-continuity of $Y$ follows from that of $X$. This fact is not necessarily pointed out in other articles [4], [7] and [10], and it plays an important role in Sections 3 and 4. In Section 5, we introduce the generator $A$ associated with non-Markov processes. In Section 6, we assume that $X$ is of the form $X_t = e^{-\alpha t} f_t + \int_0^t e^{-\alpha s} g_s ds$ for some $f$, $g$, and we give a martingale treatment of variational inequalities presented by Bensoussan and Lions [2] and approximate Snell's envelope $Y$ by the penalty method in [9], whose results are reduced to the case of $g = 0$.

**2. Right-continuous Snell's envelope.**

LEMMA 1. *Let $t \geqq 0, \varepsilon > 0$ and $A \in F_t$ be arbitrary. Then there exists $\tau \in C_t$ such that $E[Y_t I_A] \leqq E[X_\tau I_A] + \varepsilon$.*

PROOF. Let $\nu, \mu \in C_t$ and define $\sigma = \nu I_{B^c} + \mu I_B$ with $B = \{E[X_\nu | F_t] < E[X_\mu | F_t]\}$. Then,

$$E[X_\sigma | F_t] = \text{ess sup}\,(E[X_\nu | F_t], E[X_\mu | F_t]) ,$$

that is, the class $\{E[X_\nu | F_t] \,|\, \nu \in C_t\}$ is closed under the operation sup. By Proposition VI-1-1 of [6], there exists a sequence $\tau_n \in C_t$ such that $E[X_{\tau_n} | F_t]$ is increasing and $\lim_{n\to\infty} E[X_{\tau_n} | F_t] = Y_t$ a.s. Since $\{E[X_{\tau_n} | F_t] \,|\, n = 1, 2, \cdots\}$ dominates the random variable $E[X_{\tau_1} | F_t]$, the monotone convergence theorem implies that $\lim_{n\to\infty} E[X_{\tau_n} I_A] = E[Y_t I_A]$ for each $A \in F_t$. Thus the lemma is proved.

THEOREM 1. *Let $X$ be the process satisfying* (1). *Then there exists a right continuous supermartingale $\hat{Y}$ majorizing $X$ and satisfying*:

(a) $\hat{Y}_t = Y_t = \bar{Y}_t$ *a.s.*, $t \geqq 0$,

(b) $\limsup_{t\to\infty} \hat{Y}_t = \limsup_{t\to\infty} X_t$,

(c) $E[\hat{Y}_t] = \sup_{\tau \in \bar{C}_t} E[X_\tau] = \sup_{\tau \in C_t} E[X_\tau]$,

(d) *For any $\sigma, \tau \in \bar{C}_0$ with $\sigma \leqq \tau$, $E[|\hat{Y}_\tau|] < \infty$ and $E[\hat{Y}_\tau | F_\sigma] \leqq \hat{Y}_\sigma$.*

PROOF. We first note that $X_t \leqq Y_t \leqq \bar{Y}_t$ a.s. for all $t$. By Lemma 1, for any $\varepsilon > 0$, $t \geqq s \geqq 0$ and $A \in F_s$, there exists $\tau \in C_t$ such that $E[Y_t I_A] \leqq E[X_\tau I_A] + \varepsilon$. Hence,

$$E[Y_t I_A] \leqq E[E[X_\tau | F_s] I_A] + \varepsilon \leqq E[Y_s I_A] + \varepsilon .$$

Letting $\varepsilon \to 0$, we see that $Y$ is a supermartingale. Define the process $\hat{Y}$ by $\hat{Y}_t = Y_{t+}$. Clearly, $\hat{Y}$ is a right-continuous supermartingale such that $X_t \leqq \hat{Y}_t \leqq Y_t$ a.s. for all $t$. For each $u \geqq 0$, $\sup_{s \geqq u} X_s$ belongs to $L^1$ by the inequalities $X_u \leqq \sup_{s\geqq u} X_s \leqq \sup_{s \geqq u} X_s^+$. For any $t \geqq u$ and $\tau \in C_t$,

$$E[X_\tau | F_t] \leqq E\left[\sup_{s \geqq t} X_s \middle| F_t\right] \leqq E\left[\sup_{s\geqq u} X_s \middle| F_t\right] ,$$

from which $\hat{Y}_t \leqq Y_t \leqq E[\sup_{s\geqq u} X_s | F_t]$. Letting $t \to \infty$, we see that

$$\limsup_{t\to\infty} \hat{Y}_t \leqq \sup_{s\geqq u} X_s < \infty \quad (u \geqq 0) ,$$

and then, letting $u \to \infty$, we obtain the inequality

$$\limsup_{t\to\infty} \hat{Y}_t \leqq \limsup_{t\to\infty} X_t < \infty .$$

Thus (b) follows from the fact that $X_t \leqq_t \hat{Y}$ for all $t$. For any $\tau \in \bar{C}_t$ and $A \in F_t$,

$$(3) \qquad \begin{aligned} E[\hat{Y}_{t\wedge\tau} I_A] &= E[\hat{Y}_\tau I_{A\cap(\tau<t)}] + E[\hat{Y}_t I_{A\cap(\tau\geqq t)}] \\ &\geqq E[X_\tau I_{A\cap(\tau<t)}] + E[E[X_\tau | F_t] I_{A\cap(\tau\geqq t)}] = E[E[X_\tau | F_t] I_A] , \end{aligned}$$

that is, we have $E[X_\tau | F_t] \leqq \hat{Y}_{t\wedge\tau}$. Applying the optional sampling theorem to the right continuous supermartingale $(\hat{Y}_{t\wedge\tau})$, we obtain (d). By (d), for any $\tau\in\bar{C}_t$, $E[X_\tau | F_t] \leqq E[\hat{Y}_\tau | F_t] \leqq \hat{Y}_t$. Thus we have $\bar{Y}_t \leqq \hat{Y}_t$ which implies (a). From (a) it follows that $\sup_{\tau\in\bar{C}_t} E[X_\tau] \leqq E[\hat{Y}_t]$. By Lemma 1, for any $\varepsilon > 0$, there exists $\tau\in C_t$ such that $E[\hat{Y}_t] \leqq E[X_\tau] + \varepsilon$. Thus we have $E[\hat{Y}_t] \leqq \sup_{\tau\in C_t} E[X_\tau] + \varepsilon$. Letting $\varepsilon\to 0$, we obtain (c). Consequently, the theorem is established.

## 3. Conditions for optimality.

THEOREM 2. *$\sigma\in\bar{C}_0$ is optimal if and only if $E[X_\sigma | F_t] = \hat{Y}_{t\wedge\sigma}$ for all $t$.*

PROOF. The sufficiency follows immediately from Theorem 1 (c). To prove the necessity, let us show that

$$E[X_\sigma | F_t] = \hat{Y}_t \quad \text{on} \quad \{t \leqq \sigma\} . \tag{4}$$

By Theorem 1 (d), $E[X_{\sigma\vee t}|F_t] \leqq E[\hat{Y}_{\sigma\vee t}|F_t] \leqq \hat{Y}_t$, from which $E[X_\sigma|F_t] \leqq \hat{Y}_t$ on $\{t\leqq\sigma\}$. On the other hand, set $B = \{E[X_\sigma | F_t] < \hat{Y}_t\}$ and $A = B\cap\{t\leqq\sigma\}$. Suppose that $P(A) > 0$. Then, by Lemma 1, for any $\varepsilon$ with $0<\varepsilon<(E[\hat{Y}_t I_A] - E[X_\sigma I_A])$, there exists $\tau\in C_t$ such that

$$E[\hat{Y}_t I_A] \leqq E[X_\tau I_A] + \varepsilon < E[X_\tau I_A] + E[\hat{Y}_t I_A] - E[X_\sigma I_A] ,$$

that is, $E[X_\sigma I_A] < E[X_\tau I_A]$. Define $\rho\in\bar{C}_0$ by

$$\rho = \tau I_A + \sigma I_{A^c} = \tau I_A + \sigma I_{B^c\cap(\sigma\geqq t)} + \sigma I_{(\sigma<t)} .$$

Then, $E[X_\rho] = E[X_\tau I_A] + E[X_\sigma I_{A^c}] > E[X_\sigma I_A] + E[X_\sigma I_{A^c}] = E[X_\sigma]$, which is a contradiction. Let $\sigma$ be optimal. Then we have $E[X_\sigma] = E[\hat{Y}_0] \geqq E[\hat{Y}_\sigma]$ by Theorem 1, and so $X_\sigma = \hat{Y}_\sigma$. By (4), for any $A\in F_t$,

$$\begin{aligned} E[\hat{Y}_{t\wedge\sigma} I_A] &= E[\hat{Y}_\sigma I_{A\cap(\sigma<t)}] + E[\hat{Y}_t I_{A\cap(\sigma\geqq t)}] \\ &= E[X_\sigma I_{A\cap(\sigma<t)}] + E[E[X_\sigma | F_t] I_{A\cap(\sigma\geqq t)}] = E[E[X_\sigma | F_t] I_A] . \end{aligned}$$

Consequently, we have $E[X_\sigma | F_t] = \hat{Y}_{t\wedge\sigma}$, completing the proof.

Since the process $(\hat{Y}_t - E[\sup_s X_s^+ | F_t])$ is a negative right-contionuous supermartingale, it belongs to the class (DL). By the Doob-Meyer theorem, $\hat{Y}$ has a unique decomposition

$$\hat{Y}_t = M_t - A_t ,$$

where $M$ is a martingale and $A$ is a predictable increasing process with $A_0 = 0$.

THEOREM 3. *In order that there exists an optimal stopping time $\sigma\in C_0$, it is necessary and sufficient that the stopping time $\theta = \inf\{t \mid X_t = M_t\}$ belongs to $C_0$. In this case $\theta$ is optimal in $C_0$.*

PROOF. By Theorem 2, we have

$$E[X_\sigma \mid F_t] = \hat{Y}_{t\wedge\sigma} = M_{t\wedge\sigma} - A_{t\wedge\sigma} .$$

Hence, $(A_{t\wedge\sigma})$ is a predictable increasing martingale, and so $A_{t\wedge\sigma} = 0$. Letting $t \to \infty$, we have $M_\sigma = \hat{Y}_\sigma = X_\sigma$. Therefore, $\theta \leqq \sigma < \infty$, and $X_\theta = \hat{Y}_\theta = E[X_\sigma \mid F_\theta] \in L^1$, i.e., $\theta \in C_0$. Conversely, since $X_\theta = \hat{Y}_\theta = M_\theta$, we have $A_\theta = 0$ and then $\hat{Y}_{t\wedge\theta} = M_{t\wedge\theta}$. By (3) and the definition of $\hat{Y}$,

$$E[X_\theta \mid F_t] \leqq \hat{Y}_{t\wedge\theta} \leqq E\left[\sup_s X_s^+ \middle| F_t\right] .$$

Thus, it is easy to see that $(\hat{Y}_{t\wedge\theta})$ is a uniformly integrable martingale. By the optional sampling theorem, we have

$$E[X_\theta \mid F_t] = E[\hat{Y}_\theta \mid F_t] = \hat{Y}_{t\wedge\theta} .$$

By Theorem 2, $\theta$ is optimal in $C_0$.

## 4. Existence of optimal stopping times.

THEOREM 4. *Suppose that for any sequence $\tau_n \in C_0$ increasing to $\tau$,*

$$(5) \qquad \limsup_{n\to\infty} X_{\tau_n} \leqq X_\tau \quad on \quad \{\tau < \infty\} .$$

*Then $\gamma = \inf\{t \mid X_t = \hat{Y}_t\}$ is optimal in $\bar{C}_0$, and there exists an optimal stopping time $\sigma \in C_0$ if and only if $\gamma < \infty$ a.s. If, in addition, $\lim_{t\to\infty} X_t = -\infty$, then $\gamma$ is optimal in $C_0$.*

PROOF. This is proved in [7], except the last assertion, but we briefly sketch its proof. By Theorem 1 (b), for any integer $n$, it is possible to show that $\tau_n = \inf\{t \mid X_t \geqq \hat{Y}_t - 1/n\}$ is finite a.s. and $X_{\tau_n} \geqq \hat{Y}_{\tau_n} - 1/n$. According to the same arguments as in [10, Chap. 3, Lemma 19], we can prove that $E[\hat{Y}_{\tau_n}] = E[\hat{Y}_0]$. Thus, it is clear that $\tau_n \in C_0$ and

$$(6) \qquad E[\hat{Y}_0] = \sup_{\tau \in C_0} E[X_\tau] \leqq E[X_{\tau_n}] + 1/n .$$

Let $\tau = \lim_{n\to\infty} \tau_n$. Then $\tau \leqq \gamma$ a.s. and by (5) and (6),

$$E[\hat{Y}_0] = \lim_{n\to\infty} E[X_{\tau_n}] \leqq E\left[\limsup_{n\to\infty} X_{\tau_n}\right] \leqq E[X_\tau] .$$

Clearly, $\tau \in \bar{C}_0$ and $X_\tau = \hat{Y}_\tau$. Therefore, $\tau = \gamma$ is optimal in $\bar{C}_0$ and if $\gamma < \infty$ a.s., then $\gamma$ is optimal in $C_0$. Conversely, if there exists an optimal stopping time $\sigma \in C_0$, then $X_\sigma = \hat{Y}_\sigma$ by Theorem 3 and thus $\gamma \leqq \sigma < \infty$ a.s. To prove the last assertion, let us assume that $P(\gamma = \infty) > 0$. By Theorem 2, $\lim_{t\to\infty} \hat{Y}_{t\wedge\gamma} = \lim_{t\to\infty} E[X_\gamma \mid F_t] = X_\gamma$. Hence, $\lim_{t\to\infty} \hat{Y}_t = X_\gamma$ a.s. on $\{\gamma = \infty\}$. If $\lim_{t\to\infty} X_t = -\infty$, it follows from Theorem 1 (b) that $\lim_{t\to\infty}\hat{Y}_t = -\infty$, which is a contradiction. Thus the theorem is established.

**5. Generator** $A$. For $1 < p \leqq \infty$ fixed, let $W^p$ be the Banach space of all right-continuous, $(F_t)$-adapted processes $x$ such that $\|x\|_p = \|\sup_t |x_t|\,\|_{L^p} < \infty$. We set $T_s x(t) = E[x(t+s)|F_t]$ for each $s \geqq 0$ and $x \in W^p$, and define the linear operators $\{G_\alpha\}_{\alpha>0}$ from $W^p$ into itself by

$$G_\alpha x(t) = \int_0^\infty e^{-\alpha s} T_s x(t) ds = E\left[\int_t^\infty e^{-\alpha(s-t)} x_s ds \middle| F_t\right].$$

Then, $G_\alpha$ is one to one and satisfies the resolvent equation

$$(7) \qquad G_\alpha - G_\beta + (\alpha - \beta) G_\alpha G_\beta = 0 \quad (\alpha, \beta > 0).$$

Indeed, interchanging the orders of integration, we obtain

$$\begin{aligned} G_\alpha G_\beta x(t) &= E\left[\int_t^\infty e^{-\alpha(s-t)} E\left[\int_s^\infty e^{-\beta(r-s)} x_r dr \middle| F_s\right] ds \middle| F_t\right] \\ &= E\left[\int_t^\infty \left(\int_s^\infty e^{-\alpha(s-t)} e^{-\beta(r-s)} x_r dr\right) ds \middle| F_t\right] \\ &= E\left[\int_t^\infty \left(\int_t^r e^{-\alpha(s-t)} e^{-\beta(r-s)} ds\right) x_r dr \middle| F_t\right] \\ &= E\left[\int_t^\infty (\beta-\alpha)^{-1}(e^{-\alpha(r-t)} - e^{-\beta(r-t)}) x_r dr \middle| F_t\right] \\ &= (\beta-\alpha)^{-1}(G_\alpha - G_\beta) x(t), \end{aligned}$$

which implies (7). Let $G_\alpha x(t) = 0$ for each $t \geqq 0$. Then $G_\beta x(t) = 0$ for all $\beta > 0$ by (7). Hence $T_s x(t) = 0$ for all $s \geqq 0$ by the right-continuity of the mapping $s \to T_s x(t)$. Thus, taking $s = 0$, we have $x_t = 0$. This implies that $G_\alpha$ is one to one. Therefore, $G_\alpha(W^p)$ and $\alpha - G_\alpha^{-1}$ are independent of $\alpha$. Consequently, we can define the subclass $D(A)$ of $W^p$ and the generator $A$ from $D(A)$ into $W^p$ by $D(A) = G_\alpha(W^p)$ and $A = \alpha - G_\alpha^{-1}$.

LEMMA 2. *Let $x, c \in W^p$ and $y \in D(A)$. Then we have:*

(i) $x_t = \int_0^t c_r dr$ *implies* $x \in D(A)$ *and* $Ax = c$.

(ii) $A(e^{-\alpha\cdot} y)(t) = e^{-\alpha t}(-\alpha + A) y(t)$.

PROOF. Interchanging the orders of integration, we have

$$\begin{aligned} \alpha G_\alpha x(t) &= \alpha e^{\alpha t} E\left[\int_t^\infty e^{-\alpha s}\left(\int_0^s c_r dr\right) ds \middle| F_t\right] \\ &= \alpha e^{\alpha t} E\left[\int_0^t \left(\int_t^\infty e^{-\alpha s} ds\right) c_r dr + \int_t^\infty \left(\int_r^\infty e^{-\alpha s} ds\right) c_r dr \middle| F_t\right] \\ &= x_t + G_\alpha c(t), \end{aligned}$$

which implies (i). Let $y = G_\alpha x$ for $x \in W^p$. Integrating by parts, we obtain

$$\begin{aligned} G_\alpha(e^{-\alpha\cdot}x)(t) &= e^{\alpha t}E\left[\int_t^\infty e^{-\alpha s}(e^{-\alpha s}x_s)ds \,\middle|\, F_t\right] \\ &= e^{\alpha t}E\left[\left[e^{-\alpha s}\left(-\int_s^\infty e^{-\alpha r}x_r dr\right)\right]_t^\infty + \alpha\int_t^\infty e^{-\alpha s}\left(-\int_s^\infty e^{-\alpha r}x_r dr\right)ds \,\middle|\, F_t\right] \\ &= e^{-\alpha t}G_\alpha x(t) - \alpha G_\alpha(e^{-\alpha\cdot}(G_\alpha x))(t) . \end{aligned}$$

Thus, $(\alpha G_\alpha - I)e^{-\alpha\cdot}y = G_\alpha(e^{-\alpha\cdot}(-\alpha + A)y)$, which implies (ii).

THEOREM 5. *Let $y \in D(A)$ and $s \geqq 0$. Then,*

$$T_t y(s) - y(s) = \int_0^t T_r Ay(s)dr \quad \textit{for all} \quad t \geqq 0 .$$

PROOF. Let $y = G_\alpha x$ for $x \in W^p$. Integrating by parts, we obtain

$$\begin{aligned} \int_0^t T_r AG_\alpha x(s)dr &= \alpha E\left[\int_0^t G_\alpha x(s+r)dr \,\middle|\, F_s\right] - E\left[\int_0^t x(s+r)dr \,\middle|\, F_s\right] \\ &= E\left[\int_s^{t+s} \alpha e^{\alpha r}\left(\int_r^\infty e^{-\alpha v}x_v dv\right)dr \,\middle|\, F_s\right] + E\left[\int_s^{t+s} e^{\alpha r}(-e^{-\alpha r}x_r)dr \,\middle|\, F_s\right] \\ &= E\left[\left[e^{\alpha r}\int_r^\infty e^{-\alpha v}x_v dv\right]_s^{t+s} \,\middle|\, F_s\right] \\ &= E\left[E\left[\int_{t+s}^\infty e^{-\alpha(r-(t+s))}x_r dr \,\middle|\, F_{t+s}\right] \,\middle|\, F_s\right] - G_\alpha x(s) . \end{aligned}$$

This completes the proof.

COROLLARY. *Let $x \in D(A)$ and $Ax \leqq 0$. Then $x$ is a supermartingale.*

PROOF. The proof is immediate from Theorem 5.

REMARK. Let $x \in W^p$. Then $x$ is a martingale if and only if $Ax = 0$. Indeed, the sufficiency is immediate from Theorem 5. Conversely, let $x \in W^p$ be a martingale. Then $x$ can be rewritten as $x_t = E[x_\infty \mid F_t]$ for some $x_\infty \in L^p$. Hence,

$$G_\alpha x(t) = E\left[\int_t^\infty e^{-\alpha(s-t)}E[x_\infty \mid F_s]ds \,\middle|\, F_t\right] = E\left[\int_t^\infty e^{-\alpha(s-t)}x_\infty ds \,\middle|\, F_t\right] = x_t/\alpha .$$

Thus, we have $x \in D(A)$ and $Ax = 0$.

**6. The penalty method.** Let $f, g \in W^\infty$ and set $X_t = e^{-\alpha t}f_t + \int_0^t e^{-\alpha s}g_s ds$ for $\alpha > 0$. Let $U$ be the class of all adapted and right-continuous processes $z$ such that

$$(8) \qquad e^{-\alpha\cdot}z \in W^\infty \quad \text{and} \quad \lim_{t\to\infty} e^{-\alpha t}z_t = 0 \quad \text{a.s.} ,$$

(9) $$f_t \leqq z_t \quad \text{for all} \quad t\ ,$$

(10) $$\left(e^{-\alpha t}z_t + \int_0^t e^{-\alpha s}g_s ds\right) \quad \text{is a supermartingale}\ .$$

We next consider the penalized problem, defined as follows: to find the solution $z^\varepsilon \in W^\infty$ of the following equation

(11) $$(\alpha - A)z^\varepsilon - \varepsilon^{-1}(f - z^\varepsilon)^+ = g\ , \quad \varepsilon > 0\ .$$

Then we can obtain the following theorem.

THEOREM 6. *The solution $z^\varepsilon$ of* (11) *converges to the minimal element $z^*$ of $U$ almost surely for each $t$ as $\varepsilon \downarrow 0$ and*

(12) $$z^*(t) = \operatorname*{ess\,sup}_{\tau \in \bar{C}_t} E\left[e^{-\alpha(\tau - t)}f_\tau + \int_t^\tau e^{-\alpha(s-t)}g_s ds \middle| F_t\right]\ .$$

*Furthermore, if $f$ satisfies* (5), *then $\zeta = \inf\{t \mid z^*(t) = f_t\}$ is an optimal stopping time in $\bar{C}_0$ with respect to $X$.*

For the proof, we need the following lemmas.

LEMMA 3. *Equation* (11) *has a unique solution $z^\varepsilon \in D(A)$.*

PROOF. Let $x \in W^\infty$, and define $z = T_\varepsilon x$ by

$$z(t) = E\left[\int_t^\infty e^{-(\alpha+\varepsilon^{-1})(s-t)}(g_s + \varepsilon^{-1}f \vee x(s))ds \middle| F_t\right]\ .$$

Then $T_\varepsilon$ maps $W^\infty$ into itself. Moreover, for $z_i = T_\varepsilon x_i$ with $x_i \in W^\infty$ ($i = 1, 2$), we have

$$\begin{aligned} |z_1(t) - z_2(t)| &\leqq \int_0^\infty e^{-(\alpha+\varepsilon^{-1})s}\varepsilon^{-1}E[|f \vee x_1(s+t) - f \vee x_2(s+t)| \,|\, F_t]ds \\ &\leqq (\alpha\varepsilon + 1)^{-1}E[\sup_r |x_1(r) - x_2(r)| \,|\, F_t]\ . \end{aligned}$$

Thus, $\|z_1 - z_2\|_\infty \leqq (\alpha\varepsilon + 1)^{-1}\|x_1 - x_2\|_\infty$, and so the map $T_\varepsilon$ is a contraction. A fixed point $z^\varepsilon$ of $T_\varepsilon$ satisfies

$$z^\varepsilon(t) = E\left[\int_t^\infty e^{-(\alpha+\varepsilon^{-1})(s-t)}(g_s + \varepsilon^{-1}(f - z^\varepsilon)^+(s) + \varepsilon^{-1}z^\varepsilon(s))ds \middle| F_t\right]\ .$$

By virtue of Lemma 1 of [9], this equality is equivalent to

(13) $$z^\varepsilon = G_\alpha(g + \varepsilon^{-1}(f - z^\varepsilon)^+)\ ,$$

which completes the proof.

Let $V_\varepsilon$ be the class of all progressively measurable processes $v = (v_t)$ satisfying the inequalities $0 \leqq v_t \leqq \varepsilon^{-1}$ for all $t$. For each $v \in V_\varepsilon$, we define

$$J_t(v) = E\left[\int_t^\infty \exp\left(-\int_t^s \alpha + v_r dr\right)(g_s + v_s f_s)ds \middle| F_t\right].$$

Then we can obtain the following lemma:

LEMMA 4. *Let* $v^\varepsilon(t) = \varepsilon^{-1}$ *if* $z^\varepsilon(t) \leqq f_t$, *and* $v^\varepsilon(t) = 0$ *if* $z^\varepsilon(t) > f_t$. *Then we have*

$$z^\varepsilon(t) = J_t(v^\varepsilon) = \operatorname*{ess\,sup}_{v \in V_\varepsilon} J_t(v) . \tag{14}$$

PROOF. By virtue of Lemma 1 of [9] and (13),

$$\begin{aligned} z^\varepsilon(t) &= E\left[\int_t^\infty \exp\left(-\int_t^s \alpha + v_r dr\right)(g_s + \varepsilon^{-1}(f - z^\varepsilon)^+(s) + v_s z^\varepsilon(s))ds \middle| F_t\right] \\ &= J_t(v) + E\left[\int_t^\infty \exp\left(-\int_t^s \alpha + v_r dr\right)(v_s(z^\varepsilon - f)(s) + \varepsilon^{-1}(f - z^\varepsilon)^+(s))ds \middle| F_t\right]. \end{aligned}$$

For any $v \in V_\varepsilon$, we have $v(z^\varepsilon - f) + \varepsilon^{-1}(f - z^\varepsilon)^+ \geqq 0$, and also $v^\varepsilon(z^\varepsilon - f) + \varepsilon^{-1}(f - z^\varepsilon)^+ = 0$. Thus we obtain (14).

LEMMA 5. *Let* $z \in U$ *and* $g \in W^\infty$. *Then*

$$\operatorname*{ess\,sup}_{v \in V_\varepsilon} J'_t(v) \leqq z_t \quad \textit{for all } t ,$$

*where* $J'_t(v) = E\left[\int_t^\infty \exp\left(-\int_t^s \alpha + v_r dr\right)(g_s + v_s z_s)ds \middle| F_t\right]$.

PROOF. We denote $y_t = e^{-\alpha t}(z_t - G_\alpha g(t))$. Since $(y_t)$ can be rewritten as

$$y_t = e^{-\alpha t} z_t + \int_0^t e^{-\alpha s} g_s ds - E\left[\int_0^\infty e^{-\alpha s} g_s ds \middle| F_t\right],$$

$(y_t)$ is a supermartingale such that $\lim_{t\to\infty} y_t = 0$. By virtue of Lemma 4 of [9], we have

$$E\left[\int_t^\infty \exp\left(-\int_t^s v_r dr\right)(v_s y_s)ds \middle| F_t\right] \leqq y_t .$$

Also, by virtue of Lemma 1 of [9], $G_\alpha g$ can be rewritten as

$$G_\alpha g(t) = E\left[\int_t^\infty \exp\left(-\int_t^s \alpha + v_r dr\right)(g_s + v_s G_\alpha g(s))ds \middle| F_t\right].$$

Hence,

$$\begin{aligned} J'_t(v) - G_\alpha g(t) &= e^{\alpha t} E\left[\int_t^\infty \exp\left(-\int_t^s v_r dr\right)(v_s y_s)ds \middle| F_t\right] \\ &\leqq e^{\alpha t} y_t = z_t - G_\alpha g(t) , \end{aligned}$$

which completes the proof.

PROOF OF THEOREM 6. By virtue of Theorem 1, the right hand side of (12) admits the right-continuous modification, denoted by $z'$. Right-continuous Snell's envelope $\hat{Y}$ of $X$ is of the form

$$\hat{Y}_t = \operatorname*{ess\,sup}_{\tau \in \bar{C}_t} E\left[ e^{-\alpha\tau} f_\tau + \int_0^\tau e^{-\alpha s} g_s ds \,\middle|\, F_t \right] = e^{-\alpha t} z'_t + \int_0^t e^{-\alpha s} g_s ds \,.$$

By using Theorem 1, it is easy to check that $z'$ belongs to $U$. For any $z \in U$, by (8)-(10),

$$\hat{Y}_t \leqq \operatorname*{ess\,sup}_{\tau \in \bar{C}_t} E\left[ e^{-\alpha\tau} z_\tau + \int_0^\tau e^{-\alpha s} g_s ds \,\middle|\, F_t \right] \leqq e^{-\alpha t} z_t + \int_0^t e^{-\alpha s} g_s ds \,.$$

This implies that $z'$ is a minimal element of $U$. By (9), (14) and Lemma 5,

$$\begin{aligned} z^\varepsilon(t) &= \operatorname*{ess\,sup}_{v \in V_\varepsilon} E\left[\int_t^\infty \exp\left(-\int_t^s \alpha + v_r dr\right)(g_s + v_s f_s) ds \,\middle|\, F_t\right] \\ &\leqq \operatorname*{ess\,sup}_{v \in V_\varepsilon} E\left[\int_t^\infty \exp\left(-\int_t^s \alpha + v_r dr\right)(g_s + v_s z^{s'}) ds \,\middle|\, F_t\right] \leqq z'_t \,, \end{aligned}$$

and $z^\varepsilon(t)$ is increasing as $\varepsilon \downarrow 0$. Thus we can define $z^*(t) = \lim_{\varepsilon \downarrow 0} z^\varepsilon(t)$ a.s., and we show that $z^*$ belongs to $U$. By Lemma 2,

$$\begin{aligned} A\left(e^{-\alpha t} z^\varepsilon(t) + \int_0^t e^{-\alpha s} g_s ds\right) &= -e^{-\alpha t}(\alpha - A) z^\varepsilon(t) + e^{-\alpha t} g_t \\ &= e^{-\alpha t}(-\varepsilon^{-1}(f - z^\varepsilon)^+(t)) \leqq 0 \,. \end{aligned}$$

Hence, by Corollary to Theorem 5, $\left(e^{-\alpha t} z^\varepsilon(t) + \int_0^t e^{-\alpha s} g_s ds\right)$ is a supermartingale. Thus, by the monotone convergence theorem and Theorem 16 of [5, Chap. VI], it is easily seen that $\left(e^{-\alpha t} z^*(t) + \int_0^t e^{-\alpha s} g_s ds\right)$ is a right-continuous supermartingale, i.e., $z^*$ satisfies (10). By the inequalities $z^\varepsilon \leqq z^* \leqq z'$, $z^*$ satisfies (8). By (11), it is clear that $G_\alpha(f - z^\varepsilon)^+ = \varepsilon(z^\varepsilon - G_\alpha g) \leqq \varepsilon(z' - G_\alpha g)$, which converges to zero as $\varepsilon \downarrow 0$. Hence, by the monotone convergence theorem, we have $G_\alpha(f - z^*)^+ = 0$, which implies that $z^*$ satisfies (9). Consequently, $z^* \in U$ and (12) follows from the minimality of $z'$. Finally, $\hat{Y}$ can be rewritten as

$$\hat{Y}_t = e^{-\alpha t} z^*(t) + \int_0^t e^{-\alpha s} g_s ds \,.$$

Therefore, we have $\zeta = \inf\{t \mid X_t = \hat{Y}_t\}$, which is optimal in $\bar{C}_0$ by Theorem 4. The theorem is established.

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